



HAESE MATHEMATICS

Mathematics

Analysis and Approaches HL



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for use with

IB Diploma Programme

REVISION GUIDE

MATHEMATICS: ANALYSIS AND APPROACHES HL REVISION GUIDE

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FOREWORD

The aim of this Guide is to help you prepare for tests and the final examination for the Mathematics: Analysis and Approaches HL course.

This Guide covers all five Topics in the Mathematics: Analysis and Approaches HL syllabus. All of the relevant material from the Mathematics: Core Topics HL and the Mathematics: Analysis and Approaches HL textbooks is covered. However, it is a resource that can be used by any student, regardless of their main textbook.

For each Topic, there is a theory summary and a set of skill builder questions.

- The theory summaries highlight the important facts and concepts. They are intended to complement your textbook and International Baccalaureate formula booklet. When a formula can be found in the formula booklet, it may not be repeated in this Guide.
- The set of skill builder questions are designed to help consolidate your understanding of each Topic. They should be used to reinforce key ideas, and to identify any areas of weakness. Within each Topic, the questions are logically ordered according to the chapters of the textbook, so they can be used for test preparation.

Following the coverage of all five Topics, the Guide has 20 mixed questions sets, each containing 10 questions. Each set contains questions from every Topic, as well as cross-topic questions. It is recommended that you attempt all of the questions in a mixed questions set in one sitting, as this will give you practice in answering questions from a range of topics in a short time frame.

The Guide contains five trial examinations, written by IB teachers from around the world. Each trial examination contains three papers: Paper 1, where calculators are not permitted, and Papers 2 and 3, where calculators are required. This format is consistent with the Mathematics: Analysis and Approaches HL final examination. Full solutions are provided, but it is recommended that you work through a complete paper before checking the solutions.

The Guide concludes with some extra Paper 3 questions. These have been included to give you more practice at answering the extended, investigation-style questions you will encounter in your Paper 3 exam.

We recommend completing each trial examination under exam conditions. You are encouraged to print the formulae summary (see page 5), and have it alongside you as you complete the trial examinations.

- If you are having trouble with a question, it is often a good strategy to move on to other questions, and return to it later. Time management is very important during the examination, and too much time spent on a difficult question may mean that you do not leave yourself sufficient time to complete other questions.
- Set out your work clearly with full explanations. A correct answer with no working will not necessarily receive full marks.

- If you make a mistake, draw a single line through the work you want to replace. Do not cross out work until you have replaced it with something you consider better.
- Diagrams and graphs should be sufficiently large, well labelled, and clearly drawn.
- Remember to leave answers correct to three significant figures unless an exact answer is more appropriate or a different level of accuracy is requested in the question.
- Check for key words. If the word “hence” appears, then you must use the result you have just obtained. “Hence, or otherwise” means that you can use any method you like, although it is likely that the best method uses the previous result.
- It is important to read the question carefully. Rushing into a question may mean that you miss subtle points. Underlining key words may help.
- Remember that questions in the examination are often set so that, even if you cannot complete one part, the question can still be picked up in a later part.

After completing a trial examination, you should identify areas of weakness.

- Return to your notes or textbook and review any material you found challenging.
- Ask your teacher or a friend for help if further explanation is needed.
- Summarise each Topic. Summaries that you make yourself are the most valuable.
- If you have had difficulty with a question, try it again later. Do not just assume that you know how to do it once you have read the solution.

In addition to the formula booklet, your graphics display calculator is an essential aid.

- Make sure you are familiar with the model you will be using.
- In trigonometry questions, remember to check whether the graphics calculator should be in degrees or radians.
- Important features of graphs may be revealed by zooming in or out.
- When using your graphics calculator, it is always important to reflect on the reasonableness of the results.

We hope this Guide will help you structure your revision program effectively. Remember that good examination techniques will come from good examination preparation.

We welcome your feedback:

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FORMULAE SUMMARY



PRIOR LEARNING

Area of a parallelogram	$A = bh$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(bh)$, where b is the base, h is the height
Area of a trapezoid	$A = \frac{1}{2}(a + b)h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a cuboid	$V = lwh$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Volume of a prism	$V = Ah$, where A is the area of the cross-section, h is the height
Area of the curved surface of a cylinder	$A = 2\pi r h$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

TOPIC 1: NUMBER AND ALGEBRA

ARITHMETIC SEQUENCES

$$u_n = u_1 + (n - 1)d$$

$$S_n = \frac{n}{2}(2u_1 + (n - 1)d) \quad \text{or} \quad S_n = \frac{n}{2}(u_1 + u_n)$$

GEOMETRIC SEQUENCES

$$u_n = u_1 r^{n-1}$$

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$$

$$S_\infty = \frac{u_1}{r - 1}, \quad |r| < 1$$

COMPOUND INTEREST

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}, \quad \text{where}$$

FV is the future value

PV is the present value

n is the number of years

k is the number of compounding periods per year

$r\%$ is the nominal annual rate of interest

EXPONENTS AND LOGARITHMS

$$a^x = b \Leftrightarrow x = \log_a b, \text{ where } a > 0, b > 0, a \neq 1$$

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^m = m \log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

COMBINATIONS AND PERMUTATIONS

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$${}^nP_r = \frac{n!}{(n-r)!}$$

BINOMIAL THEOREM

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \dots + {}^nC_r a^{n-r}b^r + \dots + b^n$$

COMPLEX NUMBERS

$$z = a + bi$$

$$= r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta \quad \{\text{Modulus-argument (polar) form}\}$$

$$= re^{i\theta} \quad \{\text{Exponential (Euler) form}\}$$

De Moivre's theorem

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta) = r^n e^{in\theta} = r^n \operatorname{cis} n\theta$$

TOPIC 2: FUNCTIONS**STRAIGHT LINES**

$$y = mx + c \quad \text{or} \quad ax + by + d = 0 \quad \text{or} \quad y - y_1 = m(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

QUADRATIC FUNCTIONS AND EQUATIONS

$$f(x) = ax^2 + bx + c \Rightarrow \text{axis of symmetry is } x = -\frac{b}{2a}$$

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$$

$$\Delta = b^2 - 4ac$$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

$$a^x = e^{x \ln a} \quad \text{and} \quad \log_a a^x = x = a^{\log_a x} \quad \text{where } a, x > 0, a \neq 1$$

POLYNOMIAL EQUATIONS

For polynomial equations of the form $\sum_{r=0}^n a_r x^r = 0$:

- the sum of the roots is $-\frac{a_{n-1}}{a_n}$

- the product of the roots is $\frac{(-1)^n a_0}{a_n}$

TOPIC 3: GEOMETRY AND TRIGONOMETRY

MEASUREMENT

Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
Volume of a right-pyramid	$V = \frac{1}{3}Ah$, where A is the area of the base, h is the height
Volume of a right cone	$V = \frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
Area of the curved surface of a cone	$A = \pi r l$, where r is the radius, l is the slant height
Volume of a sphere	$V = \frac{4}{3}\pi r^3$, where r is the radius
Surface area of a sphere	$A = 4\pi r^2$, where r is the radius
Length of an arc	$l = r\theta$, where r is the radius, θ is the angle measured in radians
Area of a sector	$A = \frac{1}{2}r^2\theta$, where r is the radius, θ is the angle measured in radians

TRIGONOMETRY

Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule	$c^2 = a^2 + b^2 - 2ab \cos C$ and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
Area of a triangle	$A = \frac{1}{2}ab \sin C$
Identity for $\tan \theta$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$
Reciprocal trigonometric identities	$\sec \theta = \frac{1}{\cos \theta}$ $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$
Pythagorean identities	$\cos^2 \theta + \sin^2 \theta = 1$ $1 + \tan^2 \theta = \sec^2 \theta$ $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$
Double angle identities	$\sin 2\theta = 2 \sin \theta \cos \theta$ $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ $\quad = 2 \cos^2 \theta - 1$ $\quad = 1 - 2 \sin^2 \theta$ $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
Compound angle identities	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

VECTORS

Magnitude of a vector	$ \mathbf{v} = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
Scalar product	$\mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\mathbf{v} \bullet \mathbf{w} = \mathbf{v} \mathbf{w} \cos \theta$, where θ is the angle between \mathbf{v} and \mathbf{w}
Angle between two vectors	$\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{ \mathbf{v} \mathbf{w} }$
Vector equation of a line	$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$
Parametric form of the equation of a line	$x = x_0 + \lambda l, y = y_0 + \lambda m, z = z_0 + \lambda n$
Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
Vector product	$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $ \mathbf{v} \times \mathbf{w} = \mathbf{v} \mathbf{w} \sin \theta$, where θ is the angle between \mathbf{v} and \mathbf{w}
Area of a parallelogram	$A = \mathbf{v} \times \mathbf{w} $ where \mathbf{v} and \mathbf{w} form two adjacent sides of a parallelogram
Vector equation of a plane	$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$
Equation of a plane (using the normal vector)	$\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$
Cartesian equation of a plane	$ax + by + cz = d$

TOPIC 4: STATISTICS AND PROBABILITY

Interquartile range = $Q_3 - Q_1$

Mean of a set of data $\bar{x} = \frac{\sum_{i=1}^k f_i x_i}{n}$, where $n = \sum_{i=1}^k f_i$

Standard deviation of a set of data $\sigma = \sqrt{\frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n}}$

Variation $\sigma^2 = \frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - \mu^2$

PROBABILITY

Probability of an event A $P(A) = \frac{n(A)}{n(U)}$

$P(A) + P(A') = 1$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B)$ for mutually exclusive events

$P(A | B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(A) P(B)$ for independent events

Bayes' theorem

$$P(B | A) = \frac{P(B) P(A | B)}{P(B) P(A | B) + P(B') P(A | B')}$$

$$P(B_i | A) = \frac{P(B_i) P(A | B_i)}{P(B_1) P(A | B_1) + P(B_2) P(A | B_2) + P(B_3) P(A | B_3)}$$

RANDOM VARIABLES

Expected value of a discrete random variable X	$E(X) = \sum x P(X = x)$
Expected value of a continuous random variable X	$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$
Variance	$\text{Var}(X) = E[(X - \mu)^2] = E(X^2) - [E(X)]^2$
Variance of a discrete random variable X	$\text{Var}(X) = \sum (x - \mu)^2 P(X = x) = \sum x^2 P(X = x) - \mu^2$
Variance of a continuous random variable X	$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$
Linear transformation of a single random variable	$E(aX + b) = a E(X) + b$ $\text{Var}(aX + b) = a^2 \text{Var}(X)$

BINOMIAL DISTRIBUTION

For $X \sim B(n, p)$:

- Mean $E(X) = np$
- Variance $\text{Var}(X) = np(1 - p)$

STANDARDISED NORMAL VARIABLE

$$z = \frac{x - \mu}{\sigma}$$

TOPIC 5: CALCULUS

DERIVATIVE OF $f(x)$ FROM FIRST PRINCIPLES

If $y = f(x)$, then $\frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$

DIFFERENTIATION RULES

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\cos x$	$-\sin x$	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x (\ln a)$	$\tan x$	$\sec^2 x$	$\arctan x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\sec x$	$\sec x \tan x$		
$\log_a x$	$\frac{1}{x \ln a}$	$\text{cosec } x$	$-\text{cosec } x \cot x$		
		$\cot x$	$-\text{cosec}^2 x$		

Chain rule	$y = g(u)$, where $u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Product rule	$y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

INTEGRATION

$$\begin{aligned}\int x^n dx &= \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \\ \int \frac{1}{x} dx &= \ln|x| + C \\ \int \sin x dx &= -\cos x + C \\ \int \cos x dx &= \sin x + C \\ \int e^x dx &= e^x + C \\ \int a^x dx &= \frac{1}{\ln a} a^x + C \\ \int \frac{1}{a^2 + x^2} dx &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \\ \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \arcsin\left(\frac{x}{a}\right) + C, \quad |x| < a\end{aligned}$$

Area between a curve $y = f(x)$, and the x -axis, where $f(x) > 0$	$A = \int_a^b y dx$
Area of region enclosed by a curve and x -axis	$A = \int_a^b y dx$
Area of region enclosed by a curve and y -axis	$A = \int_a^b x dy$
Volume of revolution about the x or y -axes	$V = \int_a^b \pi y^2 dx \quad \text{or} \quad V = \int_a^b \pi x^2 dy$

Integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \quad \text{or} \quad \int u dv = uv - \int v du$$

KINEMATICS

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Distance travelled from t_1 to $t_2 = \int_{t_1}^{t_2} |v(t)| dt$

Displacement from t_1 to $t_2 = \int_{t_1}^{t_2} v(t) dt$

DIFFERENTIAL EQUATIONS

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n); \quad x_{n+1} = x_n + h$, where h is a constant (step length)
Integrating factor for $y' + P(x)y = Q(x)$	$e^{\int P(x) dx}$

MACLAURIN SERIES

$$\begin{aligned}f(x) &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots \\ e^x &= 1 + x + \frac{x^2}{2!} + \dots \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\end{aligned}$$

TOPIC 1: NUMBER AND ALGEBRA

SCIENTIFIC NOTATION (STANDARD FORM)

A number is in **scientific notation** if it is written in the form $a \times 10^k$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

SEQUENCES AND SERIES

A **number sequence** is a set of numbers defined by a rule. Often, the rule is a formula for the **general term** or **n th term** of the sequence.

A sequence which continues forever is called an **infinite sequence**. A sequence which terminates is called a **finite sequence**.

Arithmetic sequences

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$ for all $n \in \mathbb{Z}^+$, where d is a constant called the **common difference**.

For an arithmetic sequence with first term u_1 and common difference d , the n th term is $u_n = u_1 + (n - 1)d$.

Geometric sequences

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant, called the **common ratio** r .

$u_{n+1} = ru_n$, so we can find $r = \frac{u_{n+1}}{u_n}$ for all $n \in \mathbb{Z}^+$.

For a geometric sequence with first term u_1 and common ratio r , the n th term is $u_n = u_1 r^{n-1}$.

Series

A **series** is the sum of the terms of a sequence.

For a finite sequence with n terms, the corresponding series is $S_n = u_1 + u_2 + \dots + u_n$.

For an infinite sequence, the corresponding series $u_1 + u_2 + \dots + u_n + \dots$ can only be calculated in some cases.

Using **sigma notation** or **summation notation** we write $u_1 + u_2 + u_3 + \dots + u_n$ as $\sum_{k=1}^n u_k$.

For a **finite arithmetic series**, $S_n = \frac{n}{2}(u_1 + u_n)$ or $S_n = \frac{n}{2}(2u_1 + (n - 1)d)$.

For a **finite geometric series** with $r \neq 1$, $S_n = \frac{u_1(r^n - 1)}{r - 1}$.

The sum of an **infinite geometric series** is $S = \frac{u_1}{1 - r}$ provided $|r| < 1$.

If $|r| > 1$ the series is **divergent**.

Compound interest

The value of a compound interest investment after n time periods is

$$u_n = u_0(1 + i)^n$$

where u_0 is the initial value of the investment

and i is the interest rate per compounding period.

To find the **real value** of the investment, we divide by the inflation multiplier each year.

Depreciation

Depreciation is the loss in value of an item over time.

The value of an item after n years is $u_n = u_0(1 - d)^n$

where u_0 is the initial value of the item

and d is the rate of depreciation per year.

EXPONENTIALS AND LOGARITHMS

Laws of exponents	
$a^m \times a^n = a^{m+n}$	$a^0 = 1, a \neq 0$
$\frac{a^m}{a^n} = a^{m-n}$	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
$(a^m)^n = a^{mn}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$
$(ab)^n = a^n b^n$	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	

If $a^x = a^k$ then $x = k$. So, if the base numbers are the same, we can **equate indices**.
If $a^x = b$, $a \neq 1$, $a, b > 0$, we say that x is the **logarithm** of b in base a , and that $a^x = b \Leftrightarrow x = \log_a b$.
 $x = \log_a(a^x)$ and $x = a^{\log_a x}$ provided $x > 0$.
The **natural logarithm** is the logarithm in base e . $\ln x \equiv \log_e x$

Laws of logarithms	
Base a , $a \neq 1, a > 0$	Base e
$\log_a xy = \log_a x + \log_a y$	$\ln xy = \ln x + \ln y$
$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\ln \left(\frac{x}{y}\right) = \ln x - \ln y$
$\log_a (x^m) = m \log_a x$	$\ln (x^m) = m \ln x$
$\log_a 1 = 0$	$\ln 1 = 0$
$\log_a a = 1$	$\ln e = 1$

To change the base of a logarithm, use the rule $\log_a x = \frac{\log_b x}{\log_b a}$.

PROOF

A **mathematical proof** is a correct argument which establishes the truth of a mathematical statement.
In a **deductive proof**, we start with a **hypothesis**, and then perform a series of **implications** to arrive at the **conclusion**.
To prove that two statements A and B are **equivalent**, we must show that $A \Rightarrow B$ and $B \Rightarrow A$. We write $A \Leftrightarrow B$ to indicate that A and B are equivalent.
We can **disprove** a statement by finding a single **counter example** to the statement.
In a **proof by contrapositive**, we prove the implication $A \Rightarrow B$ by proving its **contrapositive** $\neg B \Rightarrow \neg A$.
In a **proof by contradiction**, we start by assuming the negation of the conclusion, and prove that this leads to a contradiction. We therefore deduce the assumption was false, so the conclusion must be true.

Mathematical induction

Suppose P_n is a proposition which is defined for every integer $n \geq a, a \in \mathbb{Z}$.
To construct a formal proof by mathematical induction:

- Prove the initial case P_a is true.
- Prove that if P_k is true then P_{k+1} is true.
- State your conclusion clearly.

COUNTING AND THE BINOMIAL THEOREM

Counting principles

If there are m different ways of performing an operation and n different ways of performing a second *independent* operation, there are mn ways of performing the two operations in succession.
In counting processes, the word:

- **and** suggests multiplying the number of possibilities
- **or** suggests adding the number of possibilities.

Factorial notation

$n! = n(n-1)(n-2) \dots \times 3 \times 2 \times 1$ for all integers $n \geq 1$.

$0! = 1$

Permutations and combinations

A **permutation** of a group of objects is *any arrangement* of those objects in a *definite order*.

The number of permutations on n distinct objects taken r at a time is $\frac{n!}{(n-r)!}$.

A **combination** is a selection of objects *without* regard to order or arrangement.

The number of combinations on n distinct objects taken r at a time is ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

The binomial theorem

- For $n \in \mathbb{Z}^+$, $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$

$$= \sum_{r=0}^n \binom{n}{r}a^{n-r}b^r$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the **binomial coefficient** for $n \in \mathbb{Z}^+$, $r \in \mathbb{N}$, $r \leq n$.

- For $n \in \mathbb{Q}$, $(a+bx)^n = a^n \left(1 + \frac{bx}{a}\right)^n = a^n \sum_{r=0}^{\infty} \binom{n}{r} \left(\frac{bx}{a}\right)^r$

where $\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$ is the **binomial coefficient** for $n \in \mathbb{Q}$, $r \in \mathbb{Z}^+$, and $\binom{n}{0} = 1$.

The **interval of convergence** for which the expansion converges is $\left|\frac{bx}{a}\right| < 1$, which means $-\left|\frac{a}{b}\right| < x < \left|\frac{a}{b}\right|$.

PARTIAL FRACTIONS

To decompose a rational function into the sum of **partial fractions**, we:

- Factorise the denominator.
- Write the rational function as a sum of partial fractions with the factors as denominators and unknown numerators.
- Multiply both sides by the denominator.
- Substitute values for the variable, usually the zeros of the denominators, to find the unknown numerators.

COMPLEX NUMBERS

Any number of the form $a+bi$ where a and b are real and $i = \sqrt{-1}$ is called a **complex number**.

If $z = a+bi$ where a and b are real then:

- a is the **real part** of z , written $\mathcal{Re}(z)$
- b is the **imaginary part** of z , written $\mathcal{Im}(z)$.

Two complex numbers are equal if their real parts are equal *and* their imaginary parts are equal

$$a+bi = c+di \Leftrightarrow a=c \text{ and } b=d.$$

The **complex conjugate** of $z = a+bi$ is $z^* = a-bi$.

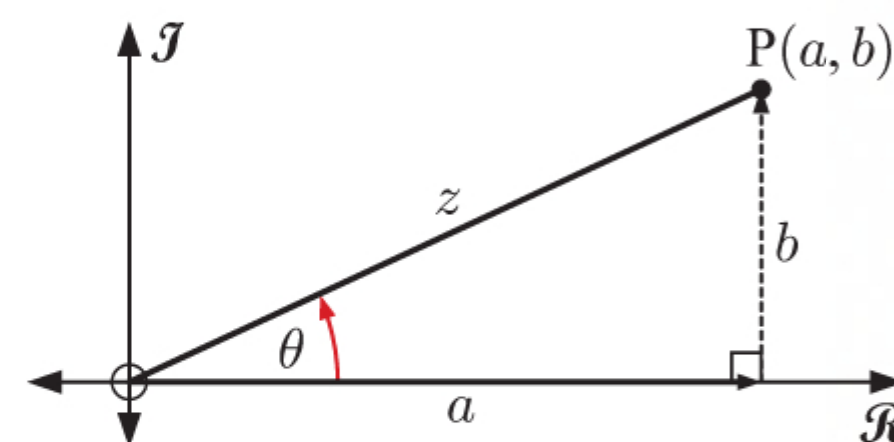
Properties of complex conjugates:

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 - z_2)^* = z_1^* - z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$ and $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$, $z_2 \neq 0$
- $(z^n)^* = (z^*)^n$ for $n \in \mathbb{Z}^+$
- $z + z^*$ and zz^* are real.

The complex plane (Argand plane)

On the complex plane, the x -axis is called the **real axis** and the y -axis is called the **imaginary axis**.

The complex number $z = a + bi$ is represented by the vector $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$.



Modulus and argument

- The **modulus** of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is $|z| = \sqrt{a^2 + b^2}$.
- If z is represented by $P(a, b)$ on the Cartesian plane, the **argument** of z is the angle θ , where $-\pi < \theta \leq \pi$ is measured anticlockwise between the positive real axis and \overrightarrow{OP} .

Properties of modulus and argument:

- $|wz| = |w||z|$ and $\arg(wz) = \arg w + \arg z$
- $\left| \frac{w}{z} \right| = \frac{|w|}{|z|}$ and $\arg\left(\frac{w}{z}\right) = \arg w - \arg z$ provided $z \neq 0$
- $|z^*| = |z|$ and $\arg(z^*) = -\arg z$
- $zz^* = |z|^2$

For points P_1 and P_2 on the complex plane defined by $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$, the distance between P_1 and P_2 is $|z_1 - z_2|$.

Polar and Euler form

The complex number z can be written in **polar form** $z = |z| \operatorname{cis} \theta = |z|(\cos \theta + i \sin \theta)$ or **Euler form** $z = |z|e^{i\theta}$, where θ is the argument of z .

$$\operatorname{cis} \theta \times \operatorname{cis} \phi = \operatorname{cis}(\theta + \phi)$$

$$\frac{\operatorname{cis} \theta}{\operatorname{cis} \phi} = \operatorname{cis}(\theta - \phi)$$

$$\operatorname{cis}(\theta + k2\pi) = \operatorname{cis} \theta \text{ for all } k \in \mathbb{Z}.$$

De Moivre's theorem

$$(|z| \operatorname{cis} \theta)^n = |z|^n \operatorname{cis} n\theta$$

The **n th roots of a complex number c** are the solutions to $z^n = c$.

There are *exactly* n n th roots of c , and the sum of the roots is 0.

The **n th roots of unity** are the solutions of $z^n = 1$. If w is the root with the smallest positive argument, the roots are $1, w, w^2, w^3, \dots, w^{n-1}$, and $1 + w + w^2 + w^3 + \dots + w^{n-1} = 0$.

SYSTEMS OF LINEAR EQUATIONS

You should be able to solve systems of linear equations, up to three equations in three unknowns, using algebra and technology.

Using algebra, we write the system in **augmented matrix form**, then use **elementary row operations** to **reduce** the system to **row echelon form**. The three legitimate row operations are to:

- interchange rows
- replace any row by a non-zero multiple of itself
- replace any row by itself, plus a multiple of another row.

We can then determine whether there are **no solutions** (the system is inconsistent), a **unique solution**, or **infinitely many solutions**.

SKILL BUILDER QUESTIONS

- 1 The fluoride concentration of lakes in a particular region was found to be 3×10^{-4} g per litre.
 - a One lake has 5.6×10^8 litres of water. Find the amount of fluoride in the lake, giving your answer in scientific notation.
 - b Another lake contains 4.13×10^7 g of fluoride. Find the volume of the lake.
- 2 Find k given the consecutive arithmetic terms:
 - a $-2, k+4, k^2+11$
 - b $k-5, 2k, 2k^2$
- 3 Twins Pierre and Francesca were each given \$100 on their 10th birthday. They immediately put their money into their individual money boxes. Each week throughout the next year they added a portion of their weekly pocket money. Pierre added \$10 each week. Francesca added 50 cents the first week, \$1 the next, \$1.50 the next, and so on, adding an extra 50 cents each subsequent week.
 - a How much did Francesca add to her money box in the last week before her 11th birthday?
 - b Find the total amount that each child had added to his or her money box after 8 weeks.
 - c Who had more money in their money box after one year? Explain your answer.
- 4 Find the general term u_n of the geometric sequence which has:
 - a $u_5 = 324$ and $u_{10} = 78\,732$
 - b $u_8 = -10$ and $u_{12} = -160$
- 5 Consider the sequence $2, 2\sqrt{3}, 6, 6\sqrt{3}, \dots$
 - a Show that the sequence is geometric.
 - b Find the formula for its general term.
 - c Find the 10th term.
 - d Find the first term which exceeds 1000.
- 6 An endangered species of bird has population 217. However, with a successful breeding program it is expected to increase by 42% each year.
 - a Find the expected population size after:
 - i 5 years
 - ii 10 years.
 - b How long will it take for the population to reach 30 000?
- 7 Paige invests €500 in an account that pays 7.2% p.a. compounded monthly.
 The amount of money in Paige's account at the end of each month follows a geometric sequence with common ratio r .
 - a Find the value of r .
 - b Find the value of the account after 3 years.
 - c Given that inflation averages 2% p.a. over the 3 years, find the real value of the investment after 3 years.
- 8 Find the sum of:
 - a $11 + 15 + 19 + 23 + \dots$ to 20 terms
 - b $7 + 12.5 + 18 + 23.5 + \dots + 106$
 - c $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$ to 100 terms
 - d the integers from 1 to 200 not divisible by 3.
- 9 An arithmetic sequence has terms $u_7 = 1$ and $u_{15} = -23$.
 - a Find the first term u_1 and common difference d .
 - b Find the 27th term u_{27} .
 - c Find the sum of the first 27 terms of the series.
- 10 The first term of a finite arithmetic series is 18 and the sum of the series is -210 . The common difference is -3 . Suppose there are n terms in the series.
 - a Show that $\frac{n}{2}(39 - 3n) = -210$.
 - b Hence find n .
- 11 The first four terms of a geometric sequence are 0.125, 0.25, 0.5, and 1.
 - a Write down the common ratio r .
 - b Find the 20th term u_{20} .
 - c Find the sum of the first 10 terms.
- 12 Kapil invested 2000 rupees in a bank account on January 1st 2012. The account pays 8.25% per annum compounded annually.
 - a Find the total value of Kapil's investment on January 1st 2019.
 - b Would it have been a better option for Kapil to invest his money in an account paying 8% per annum interest compounded monthly? Justify your answer.

- 13 a** An infinite geometric series is defined by $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k$.
- i** Find the first term u_1 and common ratio r . **ii** Find the sum of the series.
- b** A finite arithmetic series is defined by $\sum_{k=1}^n (k - 4)$.
- i** Find the first term u_1 and common difference d . **ii** Find the sum of the series, in terms of n .
- c** Find n such that the sums of the series in **a** and **b** are equal.
- 14 a** Find the sum to infinity of the infinite geometric series $1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots$
- b** When a ball is dropped from a height of 1 m, on each bounce it returns to 60% of the height it reached previously. Find the total distance travelled by the ball until it stops bouncing.
- 15** Consider the series $\sum_{k=1}^{\infty} 12(x - 2)^{k-1}$.
- a** For what values of x will the series converge? **b** Evaluate the sum of the series when $x = \sqrt{5}$.
- 16** Find x if $\sum_{k=1}^{\infty} \left(\frac{4x}{3}\right)^{k-1} = \frac{5}{2}$.
- 17** Without using a calculator, write in simplest rational form:
- a** $4^{\frac{5}{2}}$ **b** $49^{-\frac{3}{2}}$ **c** $27^{\frac{5}{3}}$
- 18** Expand and simplify:
- a** $x^{\frac{1}{2}}(x^{-\frac{1}{2}} + 2x - x^{\frac{1}{2}})$ **b** $5^x(5^{-x} + 5^{3x})$ **c** $2^{-2x}(2^{2x+3} - 2^{-4x} + 3)$
- 19** Solve for x :
- a** $5 \times 2^x = 160$ **b** $8^{2x-3} = 16^{2-x}$ **c** $\left(\frac{1}{3}\right)^{2x-5} = 27$ **d** $25^x + 2(5^x) = 35$
- 20** Find:
- a** $\log_4 8$ **b** $\log_9\left(\frac{1}{27}\right)$ **c** $\log_9\left(\frac{1}{3\sqrt{3}}\right)$
- 21** Solve for x :
- a** $\log_3 x = 2$ **b** $\log_x 27 = 3$ **c** $\log_5(2x - 1) = -1$
- 22** Find x in terms of a if $a > 1$ and $\log_a(x + 2) = \log_a x + 2$.
- 23** Simplify by writing as a single logarithm or as a rational number:
- a** $\frac{1}{4} \ln 81 + \ln 12 - \ln 4$ **b** $3 \log_9 2 - \log_9 24$ **c** $5 + \log_2 3 - \frac{1}{2} \log_2 49$
- 24** If $x = \log_a 5$, write in terms of x :
- a** $\log_a(5a)$ **b** $\log_a\left(\frac{125}{a^2}\right)$ **c** $\log_{25a} 5$
- 25** Simplify without using a calculator:
- a** $\frac{\log_2 9}{\log_2 3}$ **b** $\frac{\log_5 8}{\log_5 4}$ **c** $\frac{\log_3(0.25)}{\log_3 64}$
- 26** Solve for x :
- a** $3 \log_5 x = \log_5 24 + \log_5\left(\frac{1}{3}\right)$ **b** $\log_2 x = \log_2 12 - \log_2(7 - x)$
- c** $\ln(x^2 - 3) - \ln(2x) = 0$ **d** $\log_3 x + \log_3(x - 2) = 1$
- 27** Solve for x :
- a** $\log_{\frac{1}{9}} x = \log_9 5$ **b** $\log_2 x - \log_8 x = 3$ **c** $\log_{27}(x^4) = \log_9 x - \log_3(\sqrt[5]{9})$
- 28** Write $\frac{8}{\log_5 9}$ in the form $a \log_3 b$ where $a, b \in \mathbb{Z}$.
- 29** Solve for x exactly:
- a** $3^x = 15$ **b** $e^{2x} - 20 = e^x$ **c** $3 \times 4^x - 2^x = 0$
- 30** Solve for x :
- a** $9^x - 6(3^x) + 8 = 0$ **b** $25^x - 5^{x+1} + 6 = 0$ **c** $2 \times 3^{2x} + 3^{x+1} = 5$

31 The population of kangaroos on an island is given by $K(t) = 3200 \times (0.85)^t$, where t is the time in years.

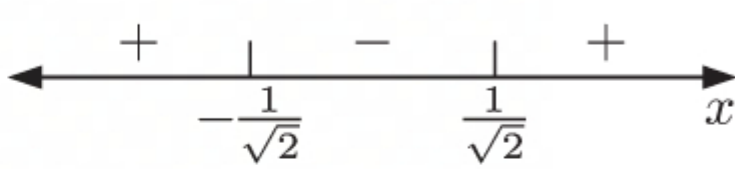
- a** What was the initial kangaroo population?
- b** How many kangaroos were on the island after 5 years?
- c** Use technology to estimate how many years it will take for the kangaroo population to fall to 1000.
- d** Check your answer to **c** using logarithms.

32 State, with justification, whether each statement is true or false:

- a** If $x > 1$ then $\frac{1}{x} < 1$.
- b** If $\frac{1}{x} < 1$ then $x > 1$.
- c** $x > 1$ if and only if $\frac{1}{x} < 1$.

33 Prove that the sum of three consecutive odd integers is divisible by 3.

34 The following “proof” by deduction is incorrect. Identify the incorrect step and write the correct solution to the inequality.

$$\begin{aligned}
 & 2x^3 \geq x \\
 \therefore & 2x^2 \geq 1 \\
 \therefore & x^2 \geq \frac{1}{2} \\
 \therefore & x^2 - \frac{1}{2} \geq 0 \\
 \therefore & \left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right) \geq 0 \\
 \therefore & x \geq \frac{1}{\sqrt{2}} \quad \text{or} \quad x \leq -\frac{1}{\sqrt{2}}
 \end{aligned}$$


35 a Show that $(a + b)^3 - (a - b)^3 = 2b(3a^2 + b^2)$.

b Check this result for $a = 2$, $b = 1$.

36 Suppose $x, y, z > 0$. Show that $xyz = 1 \Leftrightarrow \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = x^2z + y^2x + z^2y$.

37 Prove that $5n^3 - 3n^2 - 2n$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

38 Prove that 5 never divides $n^2 + 2$ for all $n \in \mathbb{Z}$.

39 Find a counter example which disproves that the difference between consecutive cubes is prime.

40 Find a counter example which disproves that if n is a positive, odd integer, then $n^2 + 2^n$ is prime.

41 Let $m, n \in \mathbb{Z}$ and p be prime. Prove, by constructing the contrapositive, that if p divides mn , then p divides at least one of m or n .

42 Let a be a real number. Show that if a^3 is irrational then a is irrational.

43 Prove by contradiction that $\log_5 9$ is irrational.

44 Suppose $p, q \in \mathbb{Q}$, $p < q$.

a Show that there is a rational number r such that $p < r < q$.

b Hence prove that there are infinitely many rational numbers between p and q .

45 Prove that $1 \times 2 + 2 \times 5 + 3 \times 8 + \dots + n(3n - 1) = n^2(n + 1)$ for all $n \in \mathbb{Z}^+$.

46 Prove that $\sum_{i=1}^n i \times 2^{i-1} = (n - 1)2^n + 1$ for all $n \in \mathbb{Z}^+$.

47 Use the principle of mathematical induction to prove that $3^n > n^2 + n$ for all $n \in \mathbb{Z}^+$.

48 Prove by mathematical induction that, for all $n \in \mathbb{Z}^+$, $\sin x + \sin 3x + \sin 5x + \dots + \sin[(2n - 1)x] = \frac{\sin^2(nx)}{\sin x}$, $0 < x < \pi$.

49 a i Using mathematical induction, prove that $\sum_{r=1}^n r^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$.

ii Hence find $1^3 + 2^3 + 3^3 + \dots + 100^3$.

b Prove that $\sum_{r=1}^n r^3 = \left(\sum_{r=1}^n r\right)^2$.

50 Prove, using mathematical induction, that $\sum_{r=1}^n (r^2 - r) = \frac{n(n^2 - 1)}{3}$ for all $n \in \mathbb{Z}^+$.

51 Use mathematical induction to prove that, for all $n \in \mathbb{Z}^+$, $\frac{d^n}{dx^n}(xe^x) = (x + n)e^x$.

- 52** Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$, $\frac{d^n}{dx^n}(\ln x) = (-1)^{n-1} \frac{(n-1)!}{x^n}$.
- 53** **a** Prove that $\text{cis}(\theta_1 + \theta_2) = \text{cis } \theta_1 \text{cis } \theta_2$.
b Use the principle of mathematical induction to prove that $\text{cis}(\theta_1 + \theta_2 + \dots + \theta_n) = \text{cis } \theta_1 \text{cis } \theta_2 \text{cis } \theta_3 \dots \text{cis } \theta_n$ for all $n \in \mathbb{Z}^+$.
- 54** Prove by induction that $2n^3 - 3n^2 + n + 31 \geq 0$ for all $n \in \mathbb{Z}$, $n \geq -2$.
- 55** Look at the menu alongside.

How many different combinations are there for a person to order an entrée, a main course, and a dessert?



- 56** A password must contain 4-6 characters, where each character is either an English letter or a digit from 0 to 9. Given that the password is not case sensitive and must contain at least one digit, how many valid passwords are there?
- 57** Consider all 4-digit integers where all the digits are different and the first digit is non-zero.
- a** How many of these numbers are there?
b How many of these numbers have a 7 as one of the four digits?
- 58** 20 individuals apply for 3 positions at a local supermarket. The positions are floor manager, cleaner, and cashier. How many different ways can these positions be filled if:
- a** there are no restrictions
b only 5 of the applicants are eligible for the floor manager position
c 9 of the applicants apply only for the cleaner position?
- 59** At a reunion between 6 men and 5 women, each person shakes hands once with every other person. Find:
- a** the total number of handshakes
b the number of handshakes between a man and a woman.
- 60** Solve for n : $\binom{n}{3} = 3\binom{n-1}{2} - \binom{n-1}{1}$.
- 61** **a** A squad of 22 students contains 11 students each from schools A and B. A combined team of 11 must be chosen to be sent away interstate.
 In how many ways can the team be selected and a captain be chosen if the captain *must* come from school A?
b Show that the answer you found in **a** can be expressed as $\sum_{k=1}^{11} k \binom{11}{k}^2$.
c Generalise the results from **a** and **b** to show that $1\binom{n}{1}^2 + 2\binom{n}{2}^2 + 3\binom{n}{3}^2 + \dots + n\binom{n}{n}^2 = n\binom{2n-1}{n-1}$.
- 62** **a** Write down the first 5 rows of Pascal's triangle.
b Hence expand and simplify: **i** $\left(x + \frac{1}{x}\right)^5$ **ii** $(1 - \sqrt{2})^5$
- 63** If $(a + bx)^n = 1 - 12x + 54x^2 - \dots$, $a > 0$, $n \in \mathbb{Z}^+$, find the values of a , b , and n .
- 64** Consider the binomial expansion of $(a + b)^6$.
- a** Write down the general term in the expansion. **b** Given that $\binom{6}{4} = 15$, find the coefficient of a^4b^2 .
- 65** Find the coefficient of x^5 in the expansion of $(x + 2)(1 - x)^{10}$.
- 66** **a** Use technology to find 7C_r for $r = 0, 1, \dots, 7$. **b** Hence find r such that ${}^7C_r = 35$.
c In the expansion of $(2x + k)^7$, $k > 0$, the coefficient of x^3 is 10 times the coefficient of x . Find the value of k .
- 67** Find the coefficient of x^3 in $(3x^2 - 7)(x - 2)^3$.

- 68** **a** Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate $\binom{6}{2}$. **b** Hence state the value of $\binom{6}{4}$.
c Given that $\binom{6}{3} = 20$, write down the expansion of $(x-2)^6$. Simplify your answer.
- 69** Consider the binomial expansion of $(3x+5)^{\frac{2}{5}}$.
a Write down the first 4 terms.
b State the interval of convergence for the complete expansion.
c Use **a** to estimate $36^{\frac{1}{5}}$. Check your answer by direct calculation.
- 70** **a** Find the binomial expansion of $\frac{1}{(2+3x)^3}$ up to the term in x^3 .
b The coefficient of x^2 in $\frac{(ax+1)^4}{(2+3x)^3}$ is $\frac{243}{16}$. Find the possible values of a .
- 71** Write as the sum of partial fractions:
a $\frac{3}{x^2+x-2}$ **b** $\frac{7x+5}{2x^2+3x-2}$ **c** $\frac{1-9x}{6x^2-13x+6}$
- 72** Write $\frac{3x^2+4x+12}{(x-3)(x^2+2x+2)}$ as the sum of partial fractions in the form $\frac{A}{x-3} + \frac{Bx+C}{x^2+2x+2}$.
- 73** Solve for x :
a $3x^2 = -27$ **b** $x^2 + x + 1 = 0$ **c** $2x^2 + x + 5 = 0$
- 74** If $z = 3 - 5i$ and $w = 7 + 2i$, find in simplest form:
a $3z - 2w$ **b** zw^2 **c** $\frac{3i}{zw}$
- 75** If $z = 2 + i$ and $w = 3 + 5i$, find:
a $\operatorname{Re}(z - 3w)$ **b** $\operatorname{Im}(iw^2)$ **c** $\operatorname{Re}\left(\frac{z}{w}\right)$
- 76** Write in the form $a + bi$ where $a, b \in \mathbb{Q}$:
a $\frac{3+4i}{1-3i}$ **b** $\frac{3}{i}\left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}\right)^2$
- 77** Suppose $\frac{z+2}{z-2} = i$. Find z in the form $a + bi$ where $a, b \in \mathbb{R}$.
- 78** Solve for z : $z^2 - z + 1 + i = 0$
- 79** Find the exact values of $x, y \in \mathbb{R}$ such that:
a $(3-2i)(x-yi) = -i$ **b** $(x+yi)^2 - (x-yi)^2 = x - y + 16i$
- 80** Suppose $z = iz^*$ where $z = x + iy$ and $x, y \in \mathbb{R}$. Deduce that $x = y$.
- 81** Prove that $(zw)^* = z^*w^*$.
- 82** Show that, for any complex number $z \neq 0$, $\frac{z}{z^*} + \frac{z^*}{z}$ is always real.
- 83** Simplify the expression $(w + 3z^*) + (z - w^*)^*$ using the properties of conjugates.
- 84** z and w are complex numbers such that $\frac{w}{z} = 1 + i$ and $w - 2z^* = -1 - 5i$. Find z and w .
- 85** Find $\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$.
- 86** Suppose $z = 2 + i$ and $w = 3 - 2i$. Find:
a $2z + w$ **b** $w^* - z$ **c** $z^* + 2w + 2i$.
 Illustrate your answers on separate Argand diagrams.
- 87** Suppose $z = x + yi$ where $x, y \in \mathbb{R}$. If $|z-3| = |z-1|$, deduce that $x = 2$.
- 88** Given $|z| = 3$, find:
a $|3z|$ **b** $|(2+i)z|$ **c** $\left|\frac{2i}{z^2}\right|$
- 89** Find the complex number z that satisfies the equation $\frac{10}{z} + \frac{15}{z^*} = 5 + 2i$ given $|z| = \sqrt{5}$.
- 90** z is a complex number where $|z| = 1$ and $\arg z \in [0, \frac{\pi}{2}]$.
 Given that $\arg\left(\frac{z}{z+2}\right) = \frac{\pi}{4}$, find $|z+2|$.

91 On an Argand plane, points P, Q, and R represent the complex numbers z_1 , z_2 , and z_3 respectively.

If $i(z_3 - z_2) = z_1 - z_2$, what can be deduced about triangle PQR?

92 Illustrate on an Argand diagram the complex numbers z satisfying:

a $|z + 3 - 2i| = 2$ **b** $|z - i| > 1$ **c** $\arg(z + 1) = -\frac{\pi}{4}$ **d** $\frac{\pi}{4} \leq \arg(z - i) < \frac{\pi}{2}$

93 If $z = r \operatorname{cis} \theta$, write z^4 , $\frac{1}{z}$, and iz^* in polar form.

94 Use the properties of cis to simplify the following. Convert your answer to exact Cartesian form if possible.

a $2 \operatorname{cis} \frac{\pi}{7} \operatorname{cis} \frac{6\pi}{7}$ **b** $(\operatorname{cis} \frac{5\pi}{12})^2$ **c** $\frac{\sqrt{8} \operatorname{cis} \frac{3\pi}{16}}{\sqrt{2} \operatorname{cis}(-\frac{5\pi}{16})}$ **d** $\operatorname{cis}(\theta + 15\pi)$

95 Suppose $z = \sqrt{3} + i$ and $w = 2 - 2i$.

- a** Write z and w in polar form. **b** Hence find zw in polar form.
c Describe the transformation to z when it is multiplied by w .

96 Let $z = \frac{-1 + i\sqrt{3}}{4}$ and $w = \frac{\sqrt{2} + i\sqrt{2}}{4}$.

a Write z and w in the form $r \operatorname{cis} \theta$ where $r > 0$ and $-\pi < \theta \leq \pi$.

b Show that $zw = \frac{1}{4}(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12})$.

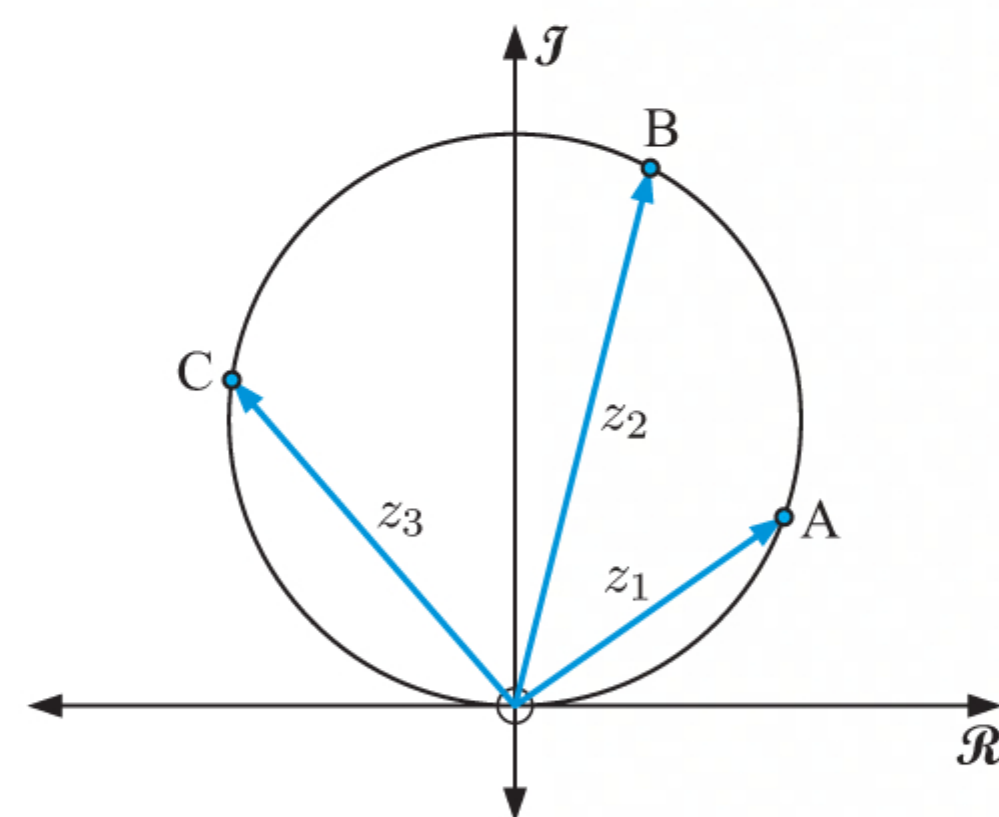
c Hence find the exact values of $\cos \frac{11\pi}{12}$ and $\sin \frac{11\pi}{12}$.

97 If $z = \cos \theta + i \sin \theta$ where $0 < \theta < \frac{\pi}{4}$, find the modulus and argument of $1 - z^2$.

98 Points O, A, B, and C lie on a circle. Suppose z_1 represents \overrightarrow{OA} , z_2 represents \overrightarrow{OB} , and z_3 represents \overrightarrow{OC} .

a What vectors are represented by $z_1 - z_2$ and $z_3 - z_2$?

b Hence find the value of $\arg\left(\frac{z_3}{z_1}\right) + \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$.



99 Using the sum and product of roots, find the real quadratic equations with roots $3 \operatorname{cis} \frac{5\pi}{6}$ and $3 \operatorname{cis} \frac{7\pi}{6}$.

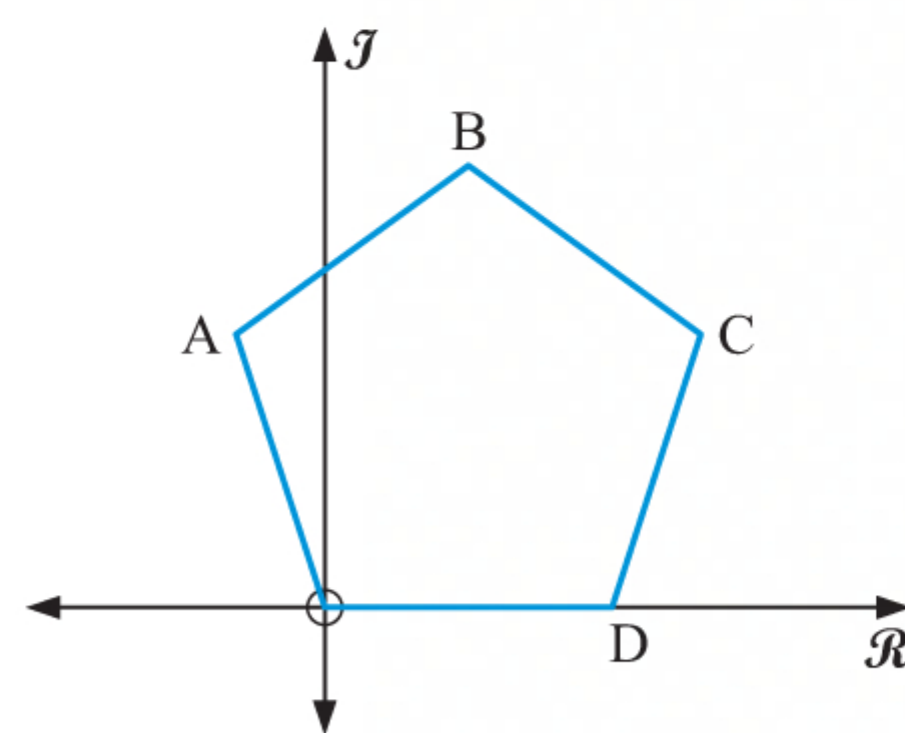
100 OABCD is a regular pentagon with side length 1. Let $z_1 \equiv \overrightarrow{OA}$, $z_2 \equiv \overrightarrow{OB}$, $z_3 \equiv \overrightarrow{OC}$, and $z_4 \equiv \overrightarrow{OD}$.

a Write in polar form:

i z_1 **ii** $z_2 - z_1$ **iii** z_3

b Find the smallest positive integer n such that z_2^n is a real number.

c Show that $z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$.



101 Write:

a $\sqrt{3} + i$ in polar form and Euler form **b** $2 \operatorname{cis} \frac{5\pi}{6}$ in Cartesian form and Euler form

c $5e^{-i\frac{\pi}{4}}$ in Cartesian form and polar form.

102 a Express $1 + i$ and $\sqrt{3} - i$ in the form $re^{i\theta}$.

b Hence write $z = \frac{-1 - i}{\sqrt{3} - i}$ in the form $re^{i\theta}$.

c Find the smallest positive integer n such that z^n is a real number.

103 a Write $z = \frac{1 + i\sqrt{3}}{1 + i}$ in the form $r \operatorname{cis} \theta$, $r > 0$.

b Hence find the smallest positive value of n such that z^n is:

- i** real **ii** purely imaginary.

104 Write $z = \frac{-1 + 5i}{2 + 3i}$ in polar form. Hence show that $z^{12} = -64$.

105 Use De Moivre's theorem to find the exact value of:

a $(\sqrt{5} \operatorname{cis} \frac{\pi}{8})^6$ **b** $(\sqrt{3} - i)^5$ **c** $(\sqrt{2} + i\sqrt{6})^{\frac{1}{2}}$

106 a Find the cube roots of $-27i$ and display them on an Argand diagram, labelling them z_1 , z_2 , and z_3 .

b Show that $z_2 z_3 = z_1^2$, where z_1 is any of the cube roots found in **a**.

c What is the value of $z_1 z_2 z_3$?

107 a Find the cube roots of $-2 - 2i$, and display them on an Argand diagram.

b By considering the sum of the cube roots, show that $\cos \frac{\pi}{4} + \cos \frac{5\pi}{12} + \cos \frac{13\pi}{12} = 0$.

108 Decide whether each system is consistent, giving reasons for your answers:

a
$$\begin{cases} x - y = 3 \\ 3x - 3y = 5 \end{cases}$$

b
$$\begin{cases} 2x + y + z = 4 \\ x - y - z = 2 \end{cases}$$

109 Use row operations to solve:

a
$$\begin{cases} x + 3y = 6 \\ 2x + 7y = 13 \end{cases}$$

b
$$\begin{cases} x - 4y = 2 \\ 3x + 5y = -11 \end{cases}$$

c
$$\begin{cases} 5x + y = 2 \\ 2x - 3y = 7 \end{cases}$$

110 Consider the system
$$\begin{cases} 2x - y = 4 \\ 6x + ky = 12 \end{cases}$$

a Write the system as an augmented matrix.

b Find the value of k such that the system has infinitely many solutions. Find the solutions in this case.

c For all other values of k , find the unique solution to the system.

111 Solve each system of linear equations:

a
$$\begin{cases} x + 3y - 4z = -5 \\ 2x + y + z = 7 \\ x - 4y + 2z = -1 \end{cases}$$

b
$$\begin{cases} 2x - 3y - z = -8 \\ 3x + y - 2z = 1 \\ 5x - 2y - 3z = -7 \end{cases}$$

112 Consider the system of equations
$$\begin{cases} x - 2y + 3z = 4 \\ 2x - 3y + 2z = 1 \\ 3x - 4y + kz = -2 \end{cases}$$
 where k is a constant.

a There is a unique solution provided that $k \neq k_1$. Find the value of k_1 , and find the unique solution when $k \neq k_1$.

b Discuss the case $k = k_1$.

113 Consider the system of equations
$$\begin{cases} 3x - ay + 2z = 4 \\ x + 2y - 3z = 1 \\ -x - y + z = 12 \end{cases}$$
 where $a \in \mathbb{R}$.

a Show that for one value of a , the system has no solutions.

b Show that there is a unique solution for all other values of a . Find the solution in terms of a .

TOPIC 2: FUNCTIONS

PROPERTIES OF LINES

The **gradient** of the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ is $m = \frac{y\text{-step}}{x\text{-step}} = \frac{y_2 - y_1}{x_2 - x_1}$.

The gradient of any horizontal line is zero. The gradient of any vertical line is undefined.

The **y-intercept** of a line is the value of y where the line cuts the y -axis.

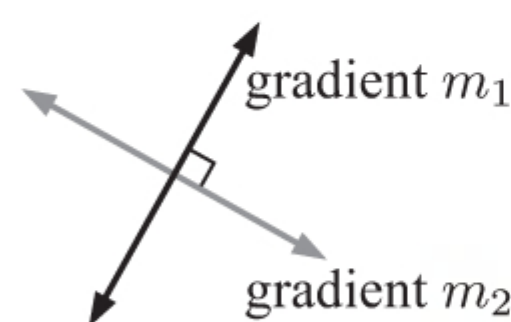
The **x-intercept** of a line is the value of x where the line cuts the x -axis.

PARALLEL AND PERPENDICULAR LINES

The gradients of parallel lines are equal.

The gradients of perpendicular lines are negative reciprocals.

$$m_1 = -\frac{1}{m_2}$$



EQUATION OF A LINE

The equation of a line can be presented in:

- **gradient-intercept form** $y = mx + c$ where m is the gradient and c is the y -intercept.
- **general form** $ax + by = d$
- **point-gradient form** $y - y_1 = m(x - x_1)$

You should be able to find the equation of a line given:

- its gradient and the coordinates of any point on the line
- the coordinates of two distinct points on the line.

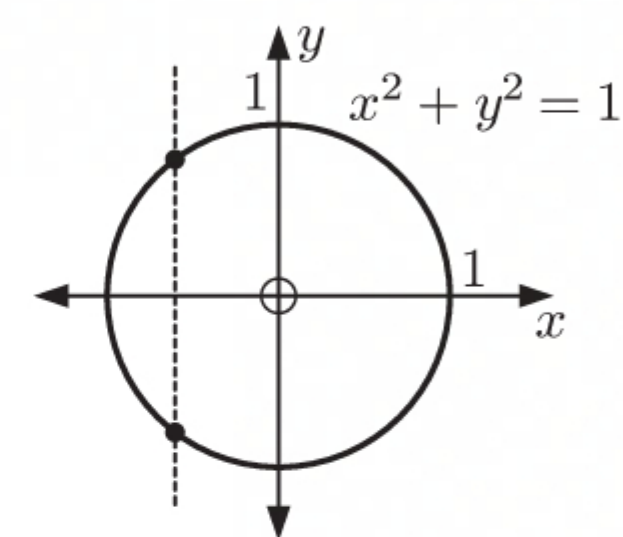
FUNCTIONS $f : x \mapsto f(x)$ OR $y = f(x)$

A **relation** between variables x and y is any set of points in the (x, y) plane.

A **function** is a relation in which no two different ordered pairs have the same x -coordinate or first component. For each value of x there is at most one value of y or $f(x)$. We sometimes refer to y or $f(x)$ as the **image** of x .

We test for functions using the **vertical line test**. A graph is a function if no vertical line intersects the graph more than once.

For example, the graph of the circle $x^2 + y^2 = 1$ shows that this relation is not a function.



The **domain** of a relation is the set of values that x can take.

To find the domain of a function, remember that we cannot:

- divide by zero
- take the square root of a negative number
- take the logarithm of a non-positive number.

The **range** of a relation is the set of values that y or $f(x)$ can take.

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g is $f \circ g : x \mapsto f(g(x))$.

In general, $f(g(x)) \neq g(f(x))$, so $f \circ g \neq g \circ f$.

The **identity function** is $f(x) = x$.

INVERSE FUNCTIONS

A function is:

- **one-to-one** if there is only one value of x for each value of y
- **many-to-one** if there is more than one value of x with the same value of y .

The function $y = f(x)$ has an **inverse function** $y = f^{-1}(x)$ if and only if it is one-to-one.

Many-to-one functions do not have an inverse function.

However, we can often restrict the domain of a many-to-one function to make it a one-to-one function. This restricted function will have an inverse function.

If $y = f(x)$ has an inverse function $y = f^{-1}(x)$, then the inverse function:

- must satisfy the vertical line test
- is a reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- has range equal to the domain of $f(x)$
- has domain equal to the range of $f(x)$.

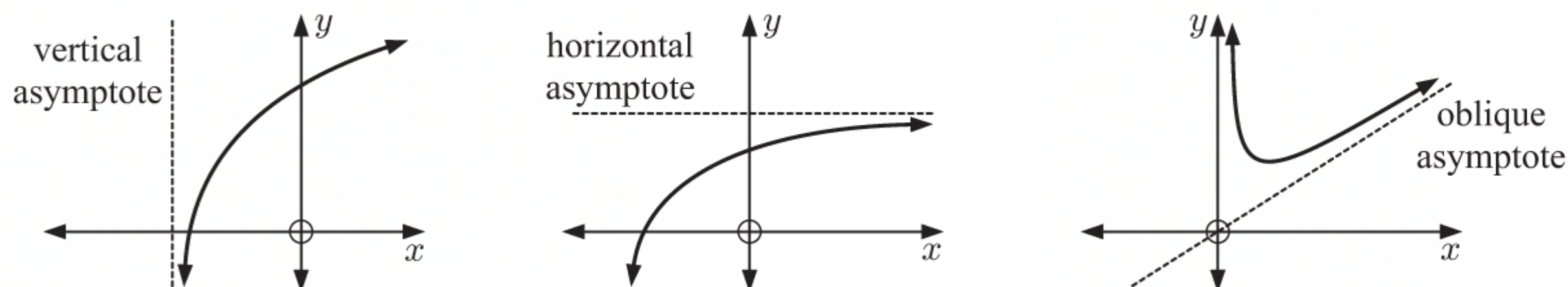
An invertible function f is **self-inverse** if $f^{-1} = f$. The graph of a self-inverse function is symmetrical about the line $y = x$.

GRAPHS OF FUNCTIONS

The **x -intercepts** of a function are the values of x for which $y = 0$. They are the **zeros** of the function.

The **y -intercept** of a function is the value of y when $x = 0$.

An **asymptote** is a line that the graph *approaches* or begins to look like as it tends to infinity in a particular direction.



To find vertical asymptotes, look for values of x for which the function is undefined:

- If $y = \frac{f(x)}{g(x)}$, find where $g(x) = 0$.
- If $y = \log_a(f(x))$, find where $f(x) = 0$.

To find horizontal asymptotes, consider the behaviour as $x \rightarrow \pm\infty$.



You should be able to use technology to solve equations graphically.

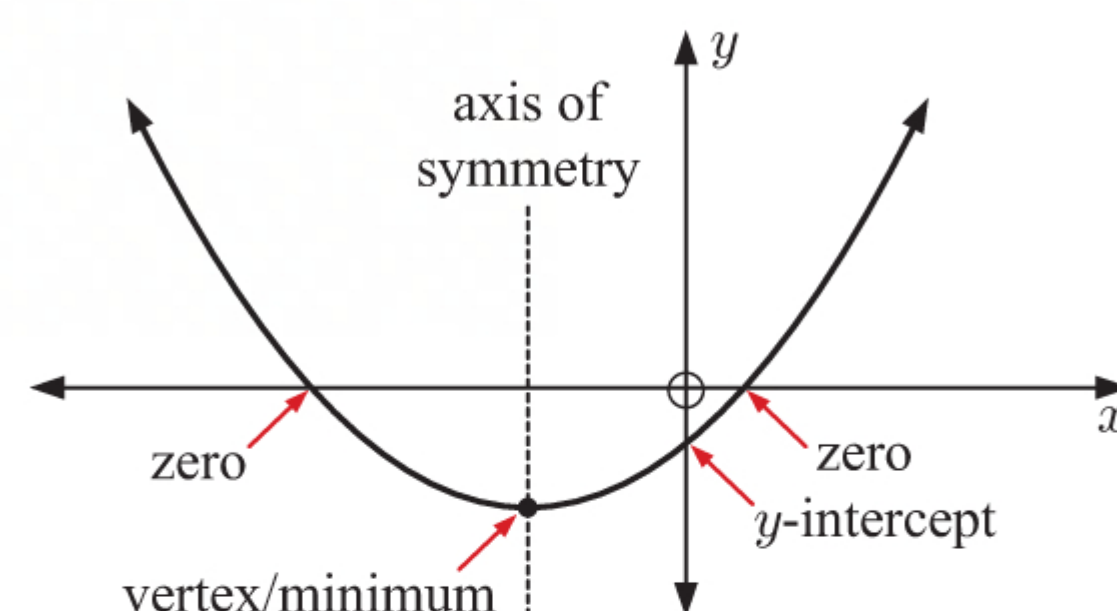
QUADRATICS

Quadratic functions

A **quadratic function** has the form $y = ax^2 + bx + c$, $a \neq 0$.

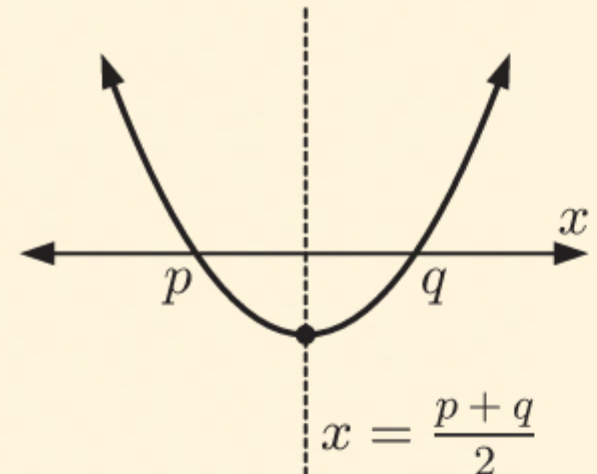
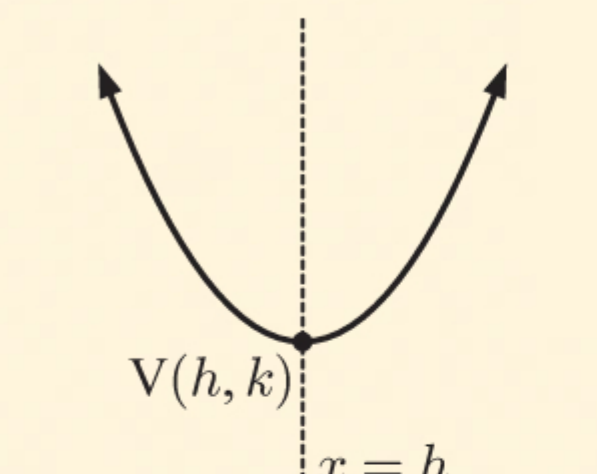
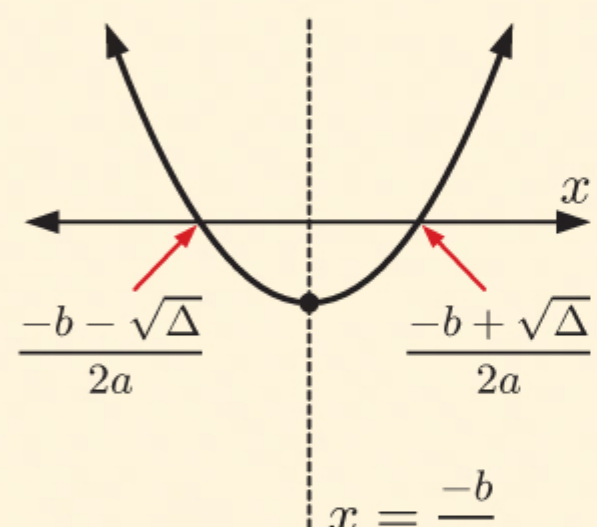
The graph is a parabola with the following properties:

- It is *concave up* if $a > 0$  and *concave down* if $a < 0$. 
- Its axis of symmetry is $x = \frac{-b}{2a}$.
- Its vertex has x -coordinate $\frac{-b}{2a}$. The y -coordinate of its vertex is found by substituting $x = \frac{-b}{2a}$ into the function.
 - If $a > 0$ the vertex is a minimum turning point.
 - If $a < 0$ the vertex is a maximum turning point.



The quadratic function has **discriminant** $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the graph cuts the x -axis twice.
- If $\Delta = 0$, the graph *touches* the x -axis.
- If $\Delta < 0$, the graph does not cut the x -axis.

$y = a(x - p)(x - q)$ x -intercepts p, q axis of symmetry $x = \frac{p + q}{2}$	
$y = a(x - h)^2 + k$ vertex (h, k) axis of symmetry $x = h$	
$y = ax^2 + bx + c$ axis of symmetry $x = \frac{-b}{2a}$ x -intercepts $\frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac \geq 0$	

Quadratic equations

A quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, can be solved by:

- factorisation
- completing the square
- the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant of the quadratic equation is $\Delta = b^2 - 4ac$.

The quadratic equation has:

- *two real solutions* if $\Delta > 0$
- *one real (repeated) solution* if $\Delta = 0$
- *no real solutions* if $\Delta < 0$.

Sum and product of the roots

If $ax^2 + bx + c = 0$ has roots α and β , then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

RATIONAL FUNCTIONS

- A rational function of the form $y = \frac{ax + b}{cx + d}$, $c \neq 0$, has a vertical asymptote $x = -\frac{d}{c}$ and a horizontal asymptote $y = \frac{a}{c}$.
- For a rational function of the form $y = \frac{ax + b}{cx^2 + dx + e}$, $c \neq 0$:
 - ▶ the horizontal asymptote is $y = 0$
 - ▶ the vertical asymptotes are the zeros of the denominator $cx^2 + dx + e$.
- A rational function of the form $y = \frac{ax^2 + bx + c}{dx + e}$, $d \neq 0$, can be written in the form $y = px + q + \frac{r}{dx + e}$ using polynomial division.

The function has a vertical asymptote $x = -\frac{e}{d}$ and an oblique asymptote $y = px + q$.

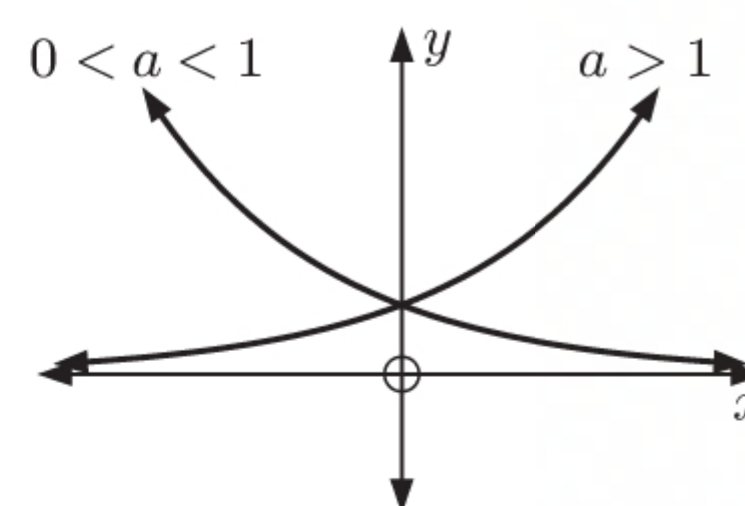
EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The simplest **exponential function** is $f(x) = a^x$, $a > 0$, $a \neq 1$.

If $a > 1$ we have *growth*.

If $0 < a < 1$ we have *decay*.

The graph of $y = a^x$ has the horizontal asymptote $y = 0$.



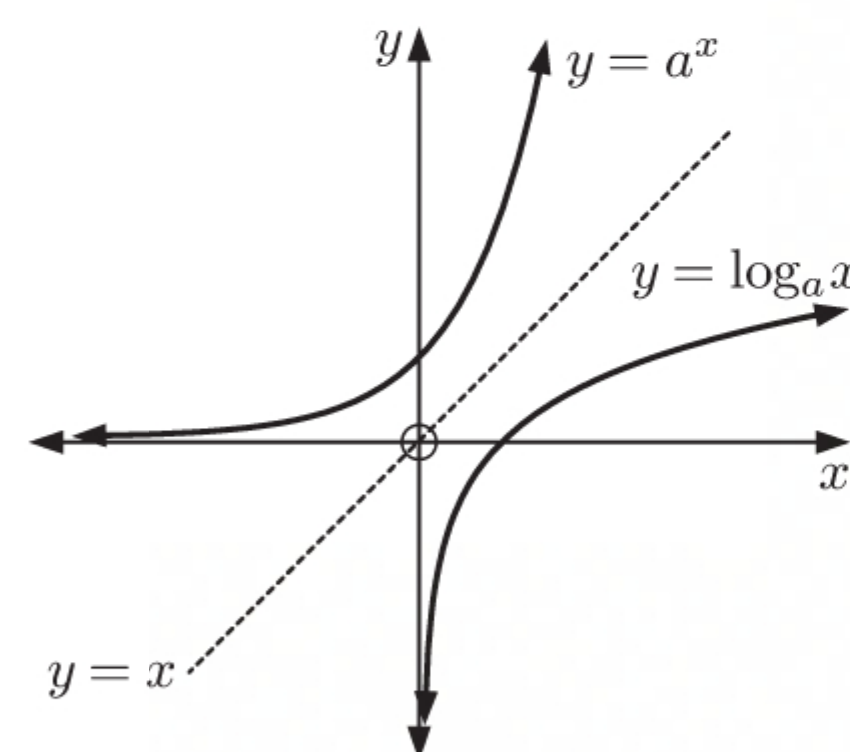
For the general exponential function $y = a^{x-h} + k$ where $a > 0$, $a \neq 1$:

- a controls how steeply the graph increases or decreases.
- h controls horizontal translation.
- k controls vertical translation.
- The equation of the horizontal asymptote is $y = k$.

The **logarithmic function** $y = \log_a x$, $x > 0$ is the inverse function of $y = a^x$.

The graph of $y = \log_a x$ has the vertical asymptote $x = 0$.

The natural logarithmic function $y = \ln x$, $x > 0$ is the inverse function of $y = e^x$.



TRANSFORMATIONS OF FUNCTIONS

- $y = f(x) + b$ **translates** $y = f(x)$ vertically b units.
- $y = f(x - a)$ **translates** $y = f(x)$ horizontally a units.
- $y = f(x - a) + b$ **translates** $y = f(x)$ by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.
- $y = pf(x)$, $p > 0$ is a **vertical stretch** of $y = f(x)$ with scale factor p .
- $y = f(qx)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with scale factor $\frac{1}{q}$.
- $y = -f(x)$ is a **reflection** of $y = f(x)$ in the x -axis.
- $y = f(-x)$ is a **reflection** of $y = f(x)$ in the y -axis.
- If $f^{-1}(x)$ exists, $y = f^{-1}(x)$ is a **reflection** of $y = f(x)$ in the line $y = x$.

EVEN AND ODD FUNCTIONS

A function $f(x)$ is **even** if $f(-x) = f(x)$ for all x in the domain of f .

A function $f(x)$ is **odd** if $f(-x) = -f(x)$ for all x in the domain of f .

MODULUS FUNCTIONS

The **absolute value** or **modulus** function $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

Properties of modulus for all x and y :

- $|x| \geq 0$
- $|-x| = |x|$
- $|x|^2 = x^2$
- $|xy| = |x||y|$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $y \neq 0$
- $|x - y| = |y - x|$

To graph $y = |f(x)|$, we keep the graph for $f(x) \geq 0$, and reflect the graph in the x -axis for $f(x) < 0$.

To graph $y = f(|x|)$, we discard the graph for $x < 0$, and reflect the graph for $x \geq 0$ in the y -axis, keeping what was there.

You should be able to solve modulus equations and inequalities, both algebraically and graphically.

GRAPHS OF RECIPROCAL FUNCTIONS

- If $f(x) > 0$, then $\frac{1}{f(x)} > 0$.
- If $f(x) < 0$, then $\frac{1}{f(x)} < 0$.
- Zeros of $f(x)$ correspond to vertical asymptotes of $\frac{1}{f(x)}$.
- Vertical asymptotes of $f(x)$ correspond to zeros of $\frac{1}{f(x)}$.
- When $f(x)$ is a local minimum which is not a zero, $\frac{1}{f(x)}$ is a local maximum.
- When $f(x)$ is a local maximum which is not a zero, $\frac{1}{f(x)}$ is a local minimum.
- When $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$ and when $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$.

THE GRAPH OF $y = [f(x)]^2$

When $y = [f(x)]^2$ is graphed from $y = f(x)$:

- The points with y -coordinate 0 or 1 are invariant.
- The graph of $y = [f(x)]^2$ *touches* the x -axis at its x -intercepts.
- The graph of $y = [f(x)]^2$ lies above or on the x -axis for all x .
- The vertical asymptotes of $y = f(x)$ are also vertical asymptotes of $y = [f(x)]^2$.

REAL POLYNOMIALS

A **real polynomial** is a function of the form $P(x) = \sum_{r=0}^n a_r x^r$ where $a_r \in \mathbb{R}$ for all $r = 0, 1, 2, \dots, n$.

If $P(x)$ is divided by $ax + b$ until a constant remainder R is obtained:

$$P(x) = Q(x)(ax + b) + R \quad \text{where} \quad \begin{cases} ax + b \text{ is the divisor} \\ Q(x) \text{ is the quotient} \\ R \text{ is the remainder.} \end{cases}$$

α is a **zero** of the polynomial $P(x) \Leftrightarrow P(\alpha) = 0$.

α is a **root** of the polynomial equation $P(x) = 0 \Leftrightarrow P(\alpha) = 0$.

$(x - \alpha)$ is a **factor** of the polynomial $P(x)$ if and only there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

The Remainder theorem

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained then $R = P(k)$.

The Factor theorem

k is a zero of $P(x) \Leftrightarrow (x - k)$ is a factor of $P(x)$.

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree n , then $P(x)$ has exactly n zeros, each of which may be written in the form $a + bi$ where $a, b \in \mathbb{R}$, and some of which may be repeated.

The Fundamental Theorem of Algebra gives the following properties of real polynomials:

- Every real polynomial of degree n can be factorised into n complex linear factors, some of which may be repeated.
- Every real polynomial can be expressed as a product of real linear and real irreducible quadratic factors (where $\Delta < 0$).
- Every real polynomial of degree n has exactly n zeros, some of which may be repeated.
- If $p + qi$ ($q \neq 0$) is a zero of a real polynomial then its complex conjugate $p - qi$ is also a zero.
- Every real polynomial of odd degree has at least one real zero.

Sum and product of roots theorem

For the polynomial equation $\sum_{r=0}^n a_r x^r = 0$, $a_n \neq 0$, the sum of the roots is $-\frac{a_{n-1}}{a_n}$, and the product of the roots is $\frac{(-1)^n a_0}{a_n}$.

Graphs of polynomials

A single zero of a real polynomial indicates its graph *cuts* the x -axis at this point.

A double repeated zero indicates the graph *touches* the x -axis.

A triple repeated zero indicates the graph has a *stationary point of inflection* on the x -axis.

INEQUALITIES IN ONE VARIABLE

You should be able to:

- solve inequalities graphically
- use the absolute value sign in inequalities
- solve $g(x) \geq f(x)$ algebraically where f and g are linear, quadratic, or simple cubic functions
- use sign diagrams.

SKILL BUILDER QUESTIONS

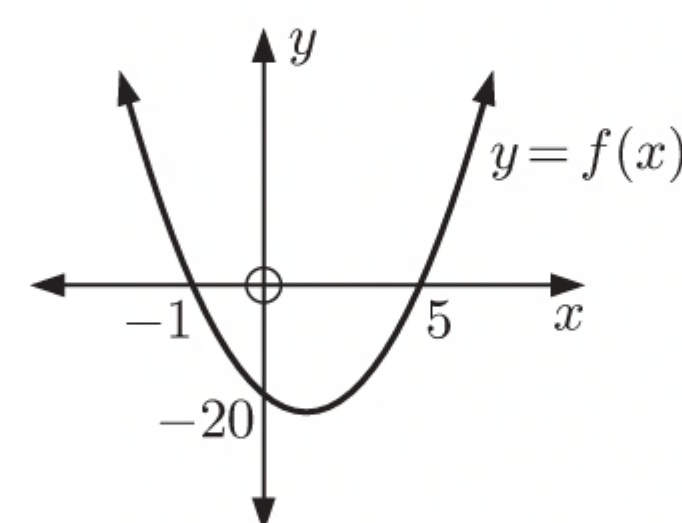
- 1 A line segment has equation $4x - 3y + 2 = 0$. Its midpoint is $(4, 6)$.
 - a State the gradient of:
 - i the line segment
 - ii its perpendicular bisector.
 - b State the equation of the perpendicular bisector. Write your answer in the form $ax + by + d = 0$.
- 2 Find the equation of the line which is:
 - a parallel to $2x - y = -3$ and passes through $(5, 3)$
 - b perpendicular to $y = -4x + 3$ and passes through $(-1, 5)$.
- 3 Line L has equation $y = 3 - 2x$.
 - a If the point $P(3, k)$ lies on line L , determine the value of k .
 - b Write down the gradient of line L .
 - c Find the equation of the line perpendicular to L which passes through point P .
- 4 Tammy buys tickets to a stage show. Tickets cost \$30 for adults, and \$15 for children. She spends a total of \$120 buying tickets for x adults and y children.
 - a Explain why $30x + 15y = 120$.
 - b If Tammy bought tickets for 4 children, how many adult tickets did she buy?
 - c Find the x -intercept of the line $30x + 15y = 120$, and interpret your answer.
 - d Draw the graph of $30x + 15y = 120$. Mark two points on your graph to indicate your answers to **b** and **c**.
- 5 Solve for x :
 - a $2x^2 - 9x = 0$
 - b $x^2 + 8x - 20 = 0$
 - c $4x^2 + 11x = 3$
 - d $(x + 3)(1 - 2x) = -9$
- 6 $x = -2$ is a solution to $x^2 + bx + (b - 2) = 0$.
 - a Find the value of b .
 - b Find the other solution to the equation.
- 7 Find m given that $mx^2 + (m - 2)x + m = 0$ has a repeated root.
- 8 For each of the following quadratic equations:
 - i Find the sum and product of the roots.
 - ii Check your answer by solving the quadratic.
 - a $x^2 - 8x + 15 = 0$
 - b $3x^2 - 4x - 2 = 0$
- 9 Use technology to solve:
 - a $\frac{2}{x} = 5x - 3$
 - b $2^x - x^3 = 0$
- 10 For each of the following functions:
 - i Find the x -intercepts.
 - ii Find the equation of the axis of symmetry.
 - iii Find the coordinates of the vertex.
 - iv State the y -intercept.
 - v Sketch the function.
 - a $y = -4x(x + 3)$
 - b $y = \frac{1}{2}(x + 6)(x - 4)$
 - c $y = -3(x - 2)^2$
 - d $y = 2(x + 5)^2 - 4$
- 11 For each of the following quadratics:
 - i Find the y -intercept.
 - ii Write the function in the form $y = a(x - h)^2 + k$.
 - iii Find the coordinates of the vertex.
 - iv Sketch the graph of the quadratic.
 - a $y = x^2 - 4x + 9$
 - b $y = 4x^2 + 16x + 11$
 - c $y = -3x^2 + 12x - 10$
- 12 For each of the following quadratics:
 - i Find the coordinates of the vertex.
 - ii State whether the vertex is a minimum or a maximum.
 - iii State the range of the function.
 - iv Find the axes intercepts.
 - v Sketch the function.
 - a $y = x^2 - 3x - 4$
 - b $y = -2x^2 - 5x + 7$

13 Find the value(s) of k for which the graph of $y = (k + 3)x^2 - 2kx + (k - 2)$:

- a** cuts the x -axis twice **b** touches the x -axis **c** misses the x -axis.

14 The function f can be written in the form $f(x) = a(x - p)(x - q)$ where $p > q$.

- a** Write down the values of p and q .
b Find a .
c Write down the equation of the axis of symmetry.



15 Consider the graph of $y = -3x^2 - x + 1$.

- a** Describe the shape of the graph.
b Use the discriminant to show that the graph cuts the x -axis twice.
c Find the x -intercepts, rounding your answers to 2 decimal places.
d State the y -intercept.
e Hence sketch the function.

16 For what values of m does the graph of $y = mx^2 + 4x + 6$ lie entirely above the x -axis?

17 The line with equation $y = kx - 2$ is a tangent to the quadratic $y = 3x^2 + x + 1$. Find k .

18 Find the coordinates of the point(s) of intersection of:

- a** $y = x^2 - 4x - 5$ and $y = 3x - 11$ **b** $y = -2x^2 + 5x$ and $y = 5 - 2x$

19 For what values of c does the line $y = 2x + c$ never meet the parabola with equation $y = 3x^2 + 5x + 7$?

20 The graph of the quadratic function $y = f(x)$ has axis of symmetry $x = -3$, y -intercept -3 , and *touches* the x -axis.

- a** Find the quadratic function.
b $y = kx - \frac{9}{4}$ is a tangent to the graph of $y = f(x)$. Find the possible values of k , and the points at which these tangents meet the curve.
c Illustrate your answers to **a** and **b**.

21 A quadratic function cuts the y -axis at 4, and touches the lines $y = x - 5$ and $y = -2x$. Find the quadratic function.

22 A factory manufactures x radios per day.

The production cost of each radio is $\left(26 + \frac{10}{x}\right)$ euros, and the income from selling each radio is $\left(42 - \frac{x}{15}\right)$ euros.

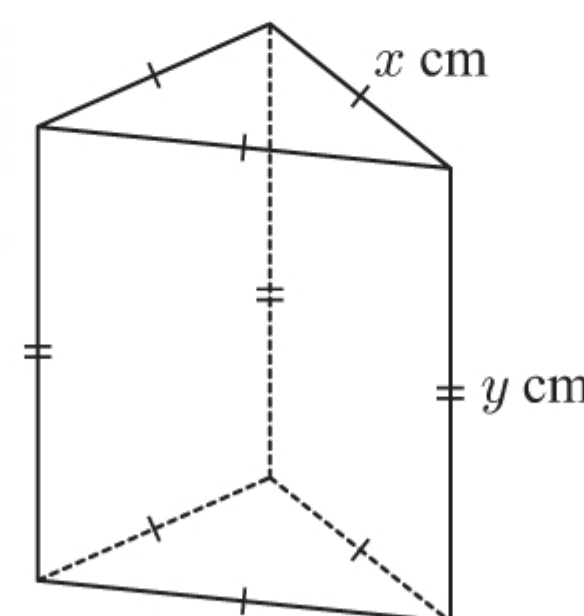
- a** Write down a formula for the *total profit* P from selling all the radios made each day.
b Find the number of radios that should be made per day to maximise profit.
c Calculate the maximum profit per day.

23 Andreas is making an aquarium in the shape of an equilateral triangular prism. The sum of all side lengths of the prism must be 1.8 m.

Let the equilateral triangle ends have sides of length x cm, and the aquarium have height y cm.

- a** Show that the area of the end is $\frac{\sqrt{3}}{4}x^2$ cm².
b Hence show that the total surface area of the aquarium is
 $A = \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x$ cm².

- c** What dimensions should Andreas choose for the aquarium to maximise its surface area?



24 Solve for x :

- a** $(x - 1)(5 - x) \leq 0$ **b** $x^2 + 8x - 20 < 0$ **c** $-9x^2 + 4x + 5 \geq 0$

25 Solve for x :

- a** $x^2 > 9$ **b** $x^2 - 15 \leq 2x$ **c** $3x^2 < 2(5x + 4)$

26 Find the value(s) of k for which the graph of the quadratic function $y = kx^2 - (k - 6)x + (k - 6)$:

- a** cuts the x -axis twice **b** touches the x -axis **c** misses the x -axis.

- 27** Jacob's rainwater tank started leaking. The amount of water in the tank after t hours is given by $W(t) = 1000 - 0.5t$ litres.
- Find $W(0)$, and interpret your answer.
 - Find t when $W(t) = 700$, and explain what this represents.
 - How long will it take for the tank to empty?

- 28** If $f(x) = \frac{x-2}{x-3}$, find in simplest form:

a $f(-x)$

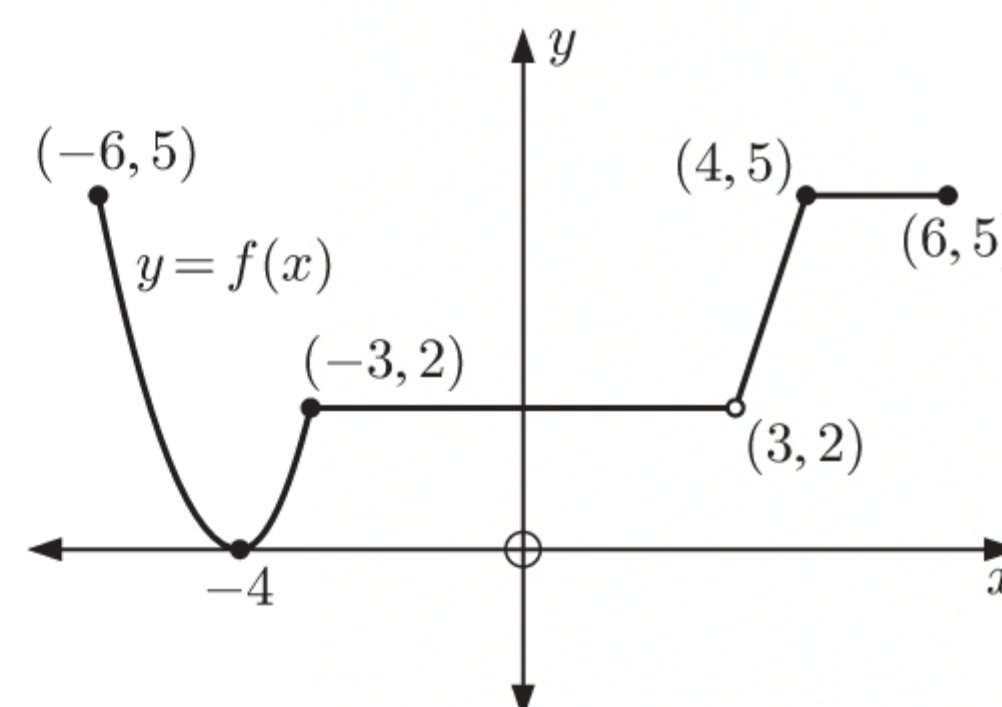
b $f(x+2)$

c $f\left(\frac{1}{x}\right)$

- 29** Consider the graph of $y = f(x)$ alongside.

Decide whether each statement is true or false:

- 0 is in the domain of f .
- 0 is in the range of f .
- 6 is in the range of f .
- 3 is in the domain of f .
- 2 is in the range of f .



- 30** State the domain and range of each function:

a $f(x) = \sqrt{3-2x}$

b $f: x \mapsto \frac{2}{x-3}$

c $f: x \mapsto \frac{1}{\sqrt{x-2}}$

- 31** Consider the function $k(t) = 2t - 4$ for $0 \leq t < 4$, $t \in \mathbb{Z}$.

- List the elements of the domain of $k(t)$.
- List the elements of the range of $k(t)$.
- Sketch the function k on a set of axes, showing all elements in the domain and range.

- 32** The function $f(x) = \sqrt{kx^2 + (2k+1)x + (k+2)}$ has natural domain $x \in \mathbb{R}$.

- Find the possible values of k .
- Find the range of $f(x)$ in terms of k .

- 33** Consider the function $f(x) = \frac{x+2}{x-1}$.

- Find the domain and range of f .
- Write down the equations of the asymptotes of $y = f(x)$.
- Find the axes intercepts.
- Draw a sign diagram for $f(x)$.
- Describe the behaviour of the function near the asymptotes.
- Sketch the function.

- 34** Consider the graph of $y = f(x)$ where $f(x) = 2 + \frac{4}{x+1}$.

- Find the axes intercepts.
- Calculate $f(-2)$.
- Determine the equation of the:
 - horizontal asymptote
 - vertical asymptote.
- Sketch the graph of $y = 2 + \frac{4}{x+1}$. Label the axes intercepts and asymptotes clearly.

- 35** For each of the following functions:

- Find the equations of the asymptotes.
- Find the axes intercepts.
- Draw a sign diagram of the function.
- Hence discuss the behaviour of the function near the asymptotes.
- Sketch the graph of the function.

a $y = \frac{2-x}{x^2+4x-21}$

b $y = \frac{5x-2}{2x^2+9x+9}$

- 36** Consider the function $f(x) = \frac{x+2}{x^2+bx+3}$ where b is a constant.

- Find the axes intercepts.
- Find the possible value(s) of b such that $f(x)$ has:
 - no vertical asymptotes
 - one vertical asymptote
 - two vertical asymptotes.

37 For each of the following functions:

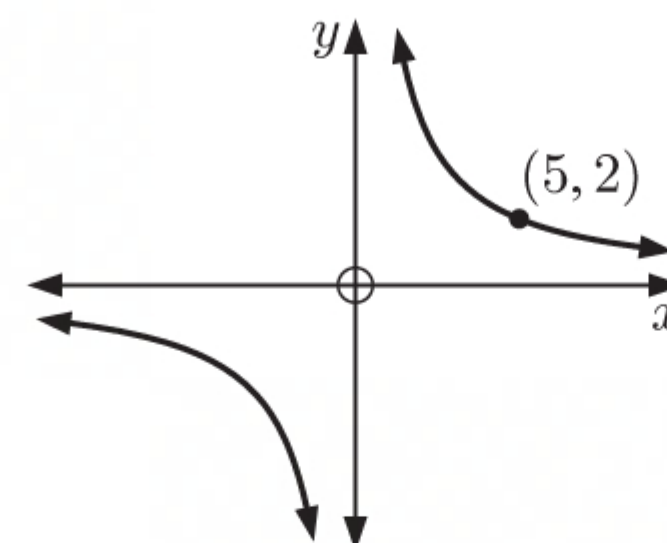
- i** Find the equation of the vertical asymptote.
- ii** Find the axes intercepts.
- iii** Find the oblique asymptote.
- iv** Draw a sign diagram of the function.
- v** Hence discuss the behaviour of the function near its asymptotes.
- vi** Sketch the graph of the function.

a $y = \frac{x^2 - 2x - 8}{x - 3}$

b $y = \frac{6x^2 - 7x - 5}{3x + 2}$

38 Consider the graph of the reciprocal function $f(x) = \frac{k}{x}$.

- a** Find k .
- b** State the domain and range of the function.
- c** Find $f(-\frac{1}{2})$.
- d** Is the function self-inverse? Explain your answer.



39 Consider the functions $f(x) = \frac{1}{x-1} + \sqrt{x+1}$ and $g(x) = x^2$.

- a** State the domain of f .
- b** Find $(f \circ g)(x)$.
- c** Is the domain of $(f \circ g)$ the same as the domain of either f or g ? Explain your answer.

40 Functions f and g are given by $f: x \mapsto e^{x+1}$ and $g: x \mapsto \ln x - 1$.

- a** Find $(f \circ g)(x)$ and state its domain and range.
- b** Find $(g \circ f)(x)$ and state its domain and range.
- c** Graph $y = f(x)$ and $y = g(x)$ on the same set of axes.
- d** State the relationship between f and g .

41 Suppose $f(x) = \sqrt[3]{x}$. Find $g(x)$ given that:

- a** $(f \circ g)(x) = 2x - 1$
- b** $(g \circ f)(x) = 2x - 1$

42 Let $f(x) = \sqrt{x+4}$ and $g(x) = x^2 - 3$.

- a** Find $(f \circ g)(x)$ and state its domain and range.
- b** Find $(g \circ f)(x)$ and state its domain and range.

43 Functions f and g are defined by $f: x \mapsto 3x + 1$ and $g: x \mapsto 4 - x$. Find:

- a** $f(g(x))$
- b** $(g \circ f)(-4)$
- c** $f^{-1}(\frac{1}{2})$

44 Suppose $f: x \mapsto \ln x$ and $g: x \mapsto 3 + x$. Find:

- a** $f^{-1}(2) \times g^{-1}(2)$
- b** $(f \circ g)^{-1}(2)$
- c** a given that $(g \circ f)^{-1}(a) = \sqrt{e}$.

45 Suppose $f: x \mapsto x + 5$ and $g: x \mapsto 7 - 3x$.

- a** Find: **i** $f^{-1}(x)$ **ii** $g^{-1}(x)$ **iii** $(f \circ g)(x)$
- b** Show that $(g^{-1} \circ f^{-1})(x) = (f \circ g)^{-1}(x)$.

46 Let $f(x) = -x^2 - 4x + 7$, $x \leq k$.

- a** Find the largest value of k such that $f^{-1}(x)$ exists.
- b** For this value of k :
 - i** Find $f^{-1}(x)$.
 - ii** State the domain and range of $f^{-1}(x)$.

47 Consider $y = -1 + 2^{-x}$.

- a** Find the axes intercepts.
- b** Find any asymptotes of the function.
- c** State the domain and range of the function.
- d** Hence sketch the function.

48 The exponential function $y = a \times 2^x + b$ passes through the points alongside:

- a** Write down two linear equations which could be used to determine the values of a and b .
- b** Solve the linear equations simultaneously to find a and b .
- c** Hence find the values of p and q .

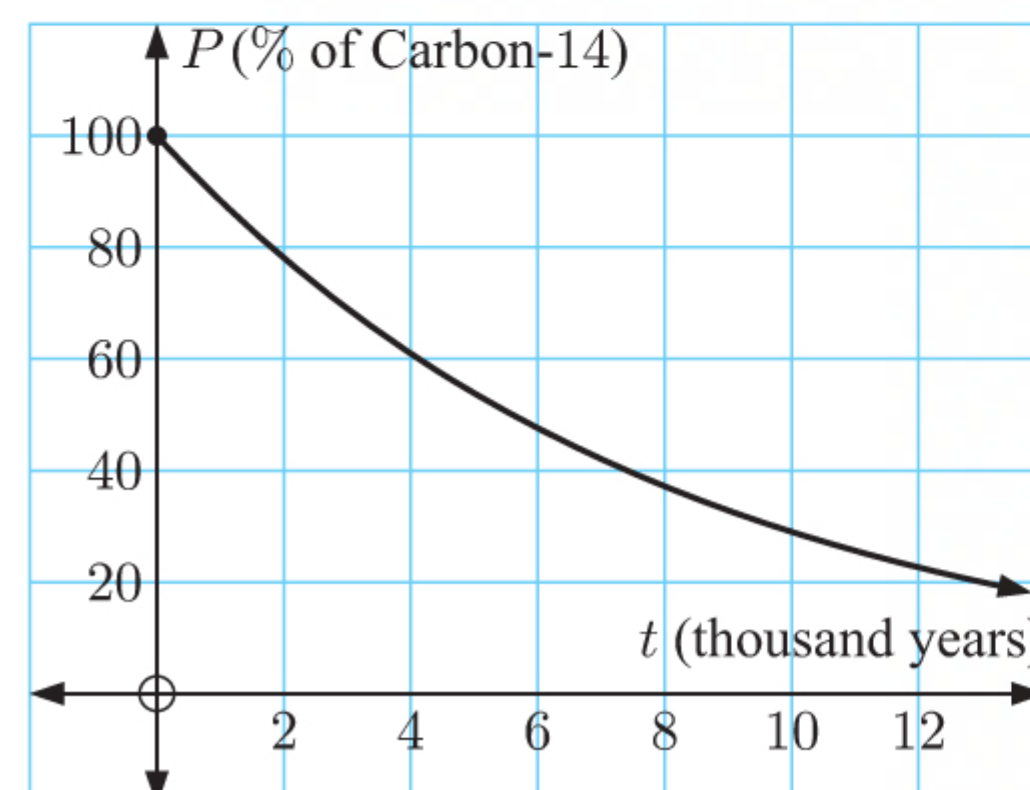
x	0	1	2	3
y	20	p	35	q

49 Consider the exponential function $f(x) = 2 \times \left(\frac{1}{3}\right)^x + 1$.

- a** Find: **i** $f(0)$ **ii** $f(2)$ **iii** $f(-1)$
b State the equation of the horizontal asymptote.
c Sketch the graph of the function.
d State the domain and range of the function.

50 The graph alongside shows the percentage P of radioactive Carbon-14 remaining in an organism t thousands of years after it dies.

- a** Use the graph to estimate:
i the percentage of Carbon-14 remaining after 4000 years
ii the number of years for the percentage of Carbon-14 to fall to 50%.
b The equation of the graph is $P = 100 \times (1.1318)^{-t}$, $t \geq 0$.
i Calculate the percentage of Carbon-14 remaining after 8000 years.
ii How long will it take for the percentage to fall to 15%?



51 The number of people N on a small island t years after settlement, increases according to the formula $N = 120 \times (1.04)^t$.

- a** Find the number of people who started the settlement.
b Find the number of people on the island after 4 years.
c How many years will it take for the number of people to double?

52 Before it is turned on, a refrigerator has an internal temperature of 27°C . Three hours later it has cooled to 6°C .

The internal temperature T (in $^\circ\text{C}$) of the refrigerator t hours after being turned on is given by the function $T(t) = A \times B^{-t} + 3$, where A and B are constants.

- a** Determine the value of: **i** A **ii** B .
b Find the internal temperature of the refrigerator 5 hours after being turned on.
c Write down the minimum temperature that the refrigerator could be expected to reach.

53 Consider $f: x \mapsto e^{x-1}$.

- a** Graph $y = f(x)$. **b** State the domain and range of f .

54 The population P of a species after n months follows the rule $P = 1000 + ae^{kn}$. The initial population was 2000. After 1 year the population was 4000. Find how long it will take for the population to reach 10 000.

55 Show that $\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$.

56 Consider $f(x) = 2^{x-1}$.

- a** Find $f(1)$ and $f(2)$. **b** Graph $y = f(x)$ and its inverse function on the same set of axes.
c Find $f^{-1}(x)$. **d** State the domain and range of $f(x)$ and $f^{-1}(x)$.

57 Consider $f: x \mapsto \ln(x+2) - 5$, $x > -2$.

- a** Find f^{-1} . **b** On the same set of axes, sketch the graphs of f and f^{-1} .
c State the domain and range of f^{-1} .

58 Find the equation of the resulting graph $g(x)$ when:

- a** $f(x) = x^2 - 5x + 6$ is translated 8 units upwards **b** $f(x) = -2x^2 + x + 3$ is translated 1 unit to the right.

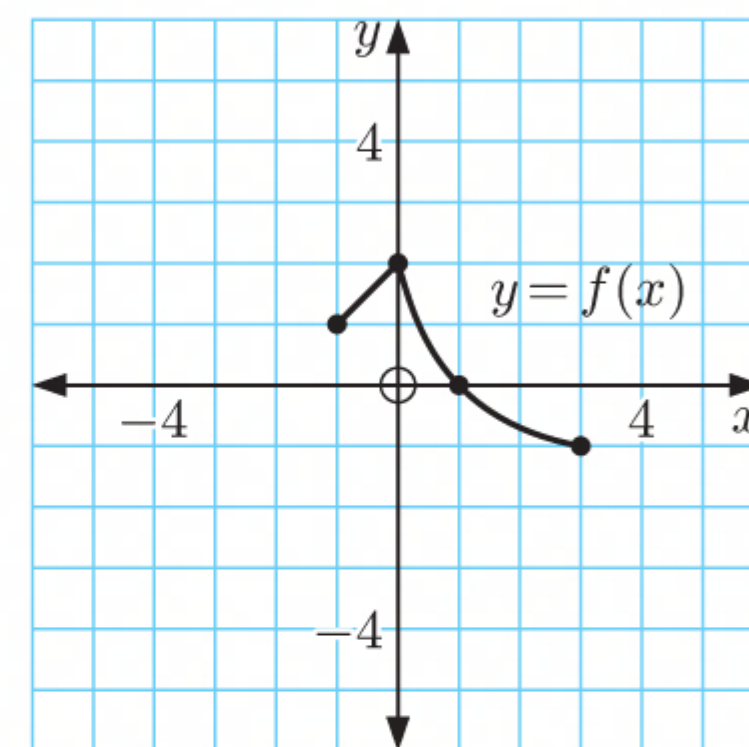
59 The graph of $f(x) = \frac{5}{x+2}$ is translated by $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ to give $g(x)$. Find $g(x)$ in the form $g(x) = \frac{ax+b}{cx+d}$.

60 **a** State the domain and range of $f(x) = \frac{1}{\sqrt{x-4}} + 3$.

- b** What transformation maps $y = \frac{1}{\sqrt{x}}$ onto the function f ?
c Write down the equations of the asymptotes of $y = f(x)$.

61 Copy this graph of $y = f(x)$, and draw the graph of:

- a** $y = f(x + 2)$ **b** $y = 2f(x) - 3$ **c** $y = 4 - f(x)$
d $y = f(-x)$ **e** $y = f(2x)$



62 Consider the function $g : x \mapsto 4 - \ln(x - 2)$.

- a** State the domain and range of g . **b** Write down the equation of the asymptote of $y = g(x)$.
c Write down the function h which is a horizontal stretch of g with scale factor $\frac{1}{2}$.
d Write down the equation of the asymptote of $y = h(x)$.

63 Suppose f and g are functions such that $g(x) = 3f\left(\frac{1}{2}x\right)$.

- a** What transformations are needed to map $y = f(x)$ onto $y = g(x)$?
b Given that $(-6, 3)$ lies on $y = f(x)$, find the coordinates of the corresponding point on $y = g(x)$.
c Given that $(4, -9)$ lies on $y = g(x)$, find the coordinates of the corresponding point on $y = f(x)$.

64 Find the equation of the resulting image when $y = \frac{2}{x}$ is:

- a** reflected in the y -axis **b** translated through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$
c stretched horizontally with scale factor 3.

65 The function $f(x)$ has domain $\{x \mid x < 0, x \geq 2\}$ and range $\{y \mid y \geq 3\}$.

Find the domain and range of: **a** $g(x) = f(x - 3) + 2$ **b** $g(x) = 4 - \frac{1}{2}f(5x)$

66 Let T_A be a horizontal translation 4 units to the right, T_B be a reflection in the x -axis, and T_C be a translation through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Find the resulting function when $y = f(x)$ has the following transformations applied:

- a** T_A then T_B **b** T_B then T_A **c** T_B then T_C **d** T_C then T_B

67 Consider the function $f : x \mapsto \ln(x - 2)$.

- a** State the domain and range of f . **b** Write down the equation of the asymptote of f .
c Find the resulting function when f is stretched vertically with scale factor 3, then reflected in the y -axis.

68 Determine the combination of transformations which transform the function $f(x) = 3x^2 - 12x + 5$ to $g(x) = -3x^2 + 18x - 10$.

69 Consider a function $f(x)$. Find the function which results if $f(x)$ is:

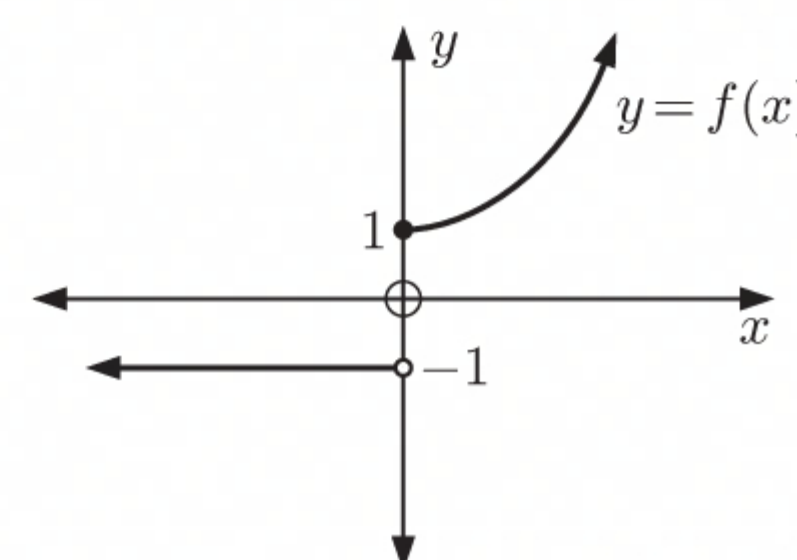
- a** translated through $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ then reflected in the x -axis
b reflected in the y -axis and translated through $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$
c translated through $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ then stretched horizontally with scale factor $\frac{1}{2}$.

70 Fully describe the transformations which map $y = f(x)$ onto:

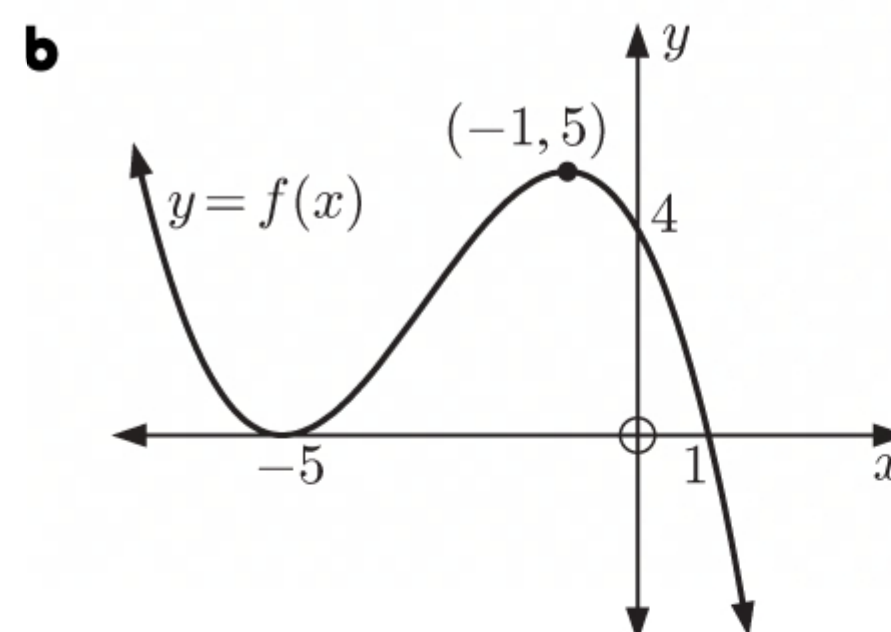
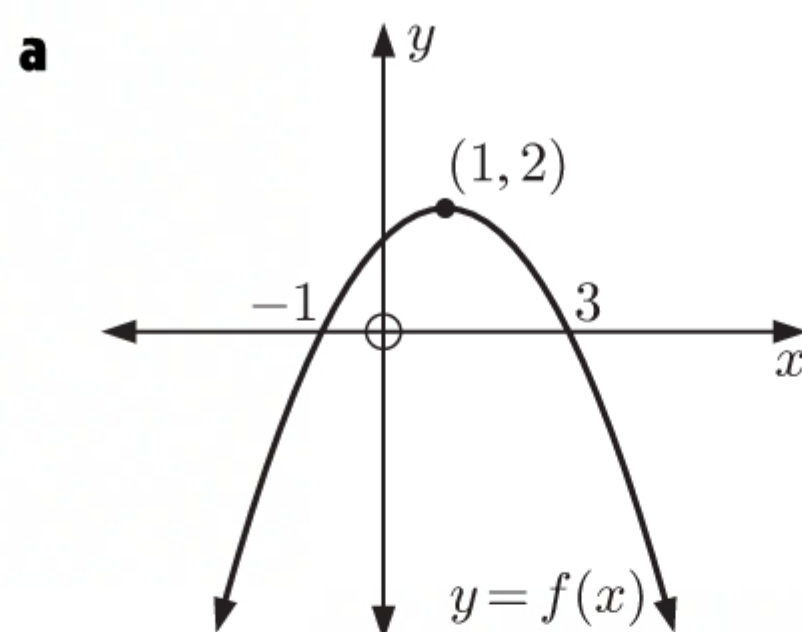
- a** $y = f(2(x - 1)) + 3$ **b** $y = 5 - 2f\left(\frac{1}{4}x\right)$ **c** $y = 6f\left(\frac{1}{3}x - 2\right) + 4$

71 The function $y = f(x)$ is illustrated. Sketch the graphs of:

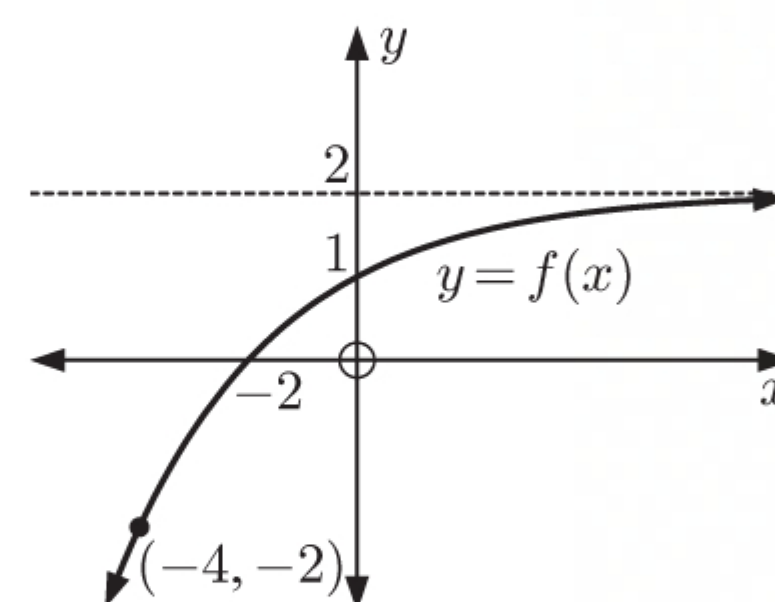
- a** $y = -f(x)$ **b** $y = f(-x)$ **c** $y = f(x - 2)$
d $y = 2f(x)$ **e** $y = \frac{1}{f(x)}$



- 72** Copy the following graphs for $y = f(x)$ and on the same axes graph $y = \frac{1}{f(x)}$:



- 73** Copy the graph alongside, and on the same set of axes, sketch the graph of $y = [f(x)]^2 - 1$.



- 74** Consider the function $f(x) = \frac{6-2x}{x+3}$.

- Find the axes intercepts and asymptotes of the function.
 - Hence find the axes intercepts and asymptotes of $y = [f(x)]^2$.
 - Which points are invariant when $y = f(x)$ is transformed to $y = [f(x)]^2$?
 - Sketch $y = f(x)$ and $y = [f(x)]^2$ on the same set of axes.
- 75**
- Sketch the graph of $f(x) = x^2 - 2x$, $x \in \mathbb{R}$, showing clearly the x -intercepts and vertex.
 - Hence sketch the graph of:
 - $y = f(|x|)$
 - $y = |f(x)|$

- 76** Let $f(x) = \ln(x+3)$. Draw the following graphs, and state the domain and range of each graph.

a $y = f(x)$ **b** $y = |f(x)|$ **c** $y = f(|x|)$

- 77** Solve for x :

a $|3 - 2x| = 5$ **b** $\left| \frac{2x+5}{3-x} \right| = 2$ **c** $|2 - x| = 3|x + 4|$

- 78** Solve graphically:

a $|4x + 3| = 9 - 2x$ **b** $|x^3 - 5x + 1| = \frac{1}{x^2} - 1$ **c** $\ln|x^2 + 2| > \sqrt{x+4}$

- 79** Find exact solutions for:

a $|1 - 4x| > \frac{1}{3}|2x - 1|$ **b** $\frac{x-2}{6-5x-x^2} \leq 0$

- 80** **a** Sketch the graph of $y = |x - 2| + |x|$.

- b** Hence solve for x :

i $|x - 2| + |x| = 2$ **ii** $|x - 2| + |x| \geq 3$

- 81** Determine whether the following functions are odd, even, or neither:

a $y = x - \frac{1}{x}$ **b** $y = \sec 2x$ **c** $y = (x^2 + 1)(x^3 + x)$

- 82** Suppose $f(x) = (5x - 2)\left(\frac{a}{x} + 3\right)$, $a \in \mathbb{R}$ is an odd function. Find the value of a .

- 83** **a** Can an even function have an inverse? Explain your answer.

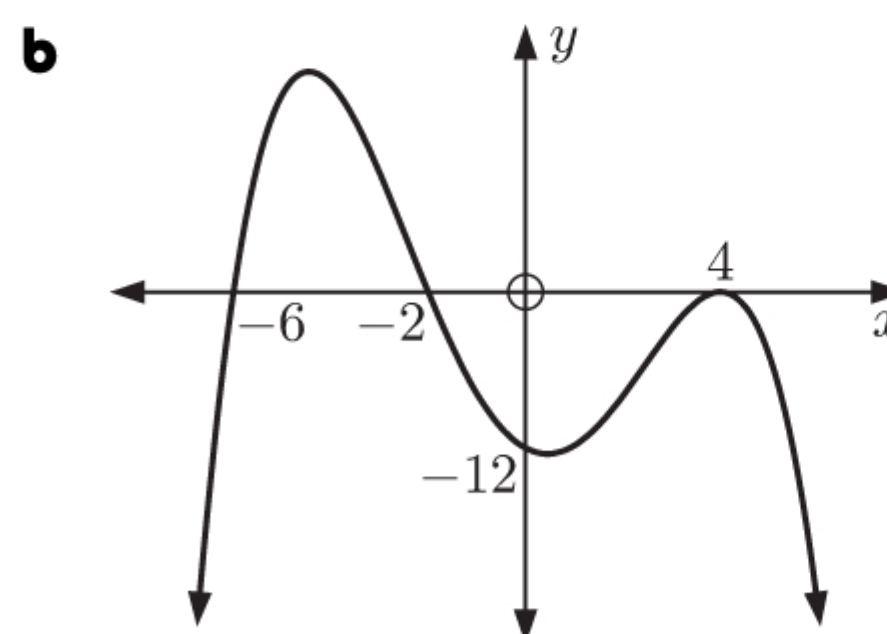
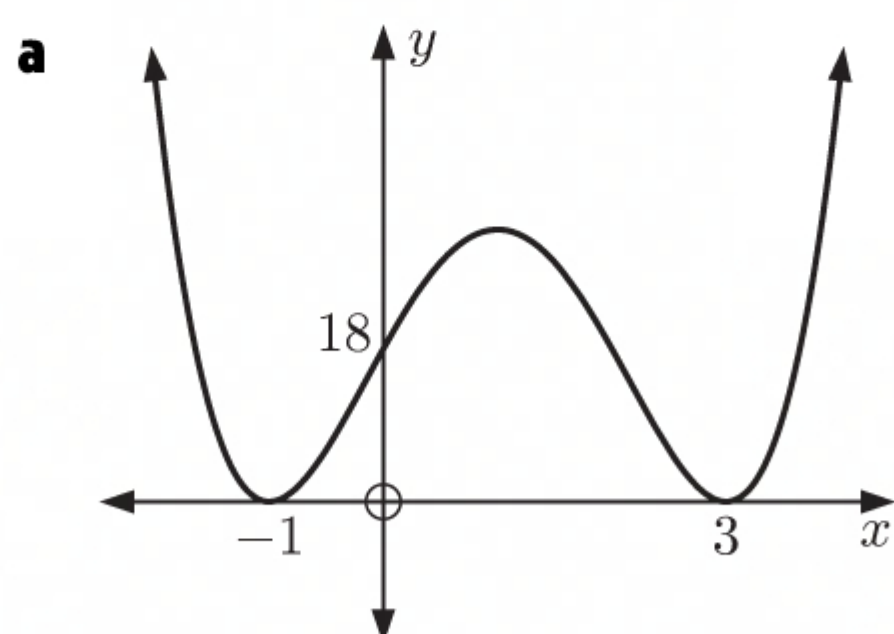
- b** What domain restriction could be placed on $y = x^4 + x^2$ so that the new function obtained has an inverse?

- 84** Find all quartic polynomials with zeros 1, -3 , and $2 \pm i\sqrt{3}$.

- 85** Write $9x^4 + 4$ as a product of quadratic factors.

- 86** For what values of k does $\frac{x^2+3}{x^2-1} = k$ have imaginary roots?

- 87 a** Find real numbers a and b such that $x^4 + 7x^3 + 13x^2 - 9x - 40 = (x^2 + ax - 5)(x^2 + bx + 8)$.
- b** Hence find $x \in \mathbb{R}$ such that $x^4 + 7x^3 + 13x^2 = 9x + 40$.
- 88** $x^3 - 3x^2 - 24x + c$, $c \in \mathbb{R}$, has two identical linear factors. Find the possible values of c , and find the zeros of the polynomial in each case.
- 89** Find the quotient and remainder when $x^4 + 3x^2 + x$ is divided by:
- a** $x + 2$ **b** $(x - 1)^2$
- 90** Perform the following divisions:
- a** $\frac{x^2 + 2}{x - 1}$ **b** $\frac{x^2 + 3x + 5}{x + 4}$ **c** $\frac{3x^2 + 2x - 4}{x - 2}$
- 91** Find m given that $x^3 + mx + m$ leaves a remainder of m when divided by $x - m$.
- 92** When the polynomial $P(x)$ is divided by $(x - 1)(x - 2)$, the remainder is $2x + 3$.
Find the remainder when $P(x)$ is divided by $x - 1$.
- 93** Consider $f(x) = 3x^3 + ax^2 - 5x + b$. When $f(x)$ is divided by $x - 1$ or by $x + 2$, the remainder is 1.
- a** Find a and b .
- b** Hence find the remainder when $f(x)$ is divided by $x - 3$.
- 94** Prove that when $P(x)$ is divided by $(x - a)^2$, the remainder is $P'(a)(x - a) + P(a)$ where $P'(x)$ is the derivative of $P(x)$.
- 95** When $6x^2 + ax + b$ is divided by $x + 3$ the remainder is 22, and when divided by $2x + 1$ the remainder is 2.
Find a and b .
- 96** Find constants a and b given that $3x^3 + ax^2 + bx + 6$ has factors $(x + 3)$ and $(x - 2)$.
- 97** $(x - 3)$ is a factor of $P(x) = 2x^3 + ax^2 + bx - 30$. When $P(x)$ is divided by $x + 1$, the remainder is -12 .
- a** Find a and b . **b** Find the remainder when $P(x)$ is divided by $x + 3$.
- c** Write $P(x)$ in the form $P(x) = (x - 3)(px^2 + qx + r)$.
- d** Find the zeros of $P(x)$.
- 98** $a + 2i$ is a root of $z^2 + bz + (a + 6) = 0$. Find a and b given that $a, b \in \mathbb{R}$.
- 99** $3 - 2i$ is a zero of $P(x) = 2x^3 + mx^2 - (m + 1)x + (3 - 4m)$, $m \in \mathbb{R}$. Find m and the other two zeros of $P(x)$.
- 100** $x^2 + mx + 1 = 0$ has roots α and β . Find m given that $\alpha\beta = \frac{1}{\alpha} + \frac{1}{\beta}$.
- 101** Find the sum and product of the roots of:
- a** $x^2 + 3x - 5 = 0$ **b** $2x^3 + x^2 - 4x + 1 = 0$ **c** $-3x^7 + 2x - 5 = 0$
- 102** A real cubic polynomial $P(x)$ has zeros $2 \pm i\sqrt{3}$ and -1 , and its leading coefficient is 2. Find:
- a** the sum and products of its zeros **b** the coefficient of x^2 **c** the constant term.
- 103** Use axes intercepts to sketch the graph of:
- a** $y = (2x - 1)(x + 2)(x - 3)$ **b** $y = 3(x - 1)(x + 3)^2$ **c** $y = -(2x + 1)(x^2 + x + 1)$
- 104** Find the equation of the cubic whose graph:
- a** cuts the x -axis at -1 , $-\frac{1}{2}$, and 2 , and passes through $(1, -12)$
- b** touches the x -axis at 4 , cuts the x -axis at -2 , and passes through $(7, -27)$
- c** touches the x -axis at -3 , and passes through $(1, -48)$ and $(-2, -9)$.
- 105** Find the equation of the following quartics. Give your answers in expanded form:



106 Find exactly all zeros of:

a $2x^3 + 7x^2 - 2$

b $3x^4 - 13x^3 + 17x^2 - 8x + 4$

c $6x^4 + 35x^3 + x^2 - 45x + 18$

107 Solve for x using technology:

a $x^3 + 5x^2 + 5x - 1 = 0$

b $x^4 - 6x^2 + 3x + 3 = 0$

108 $P(z) = z^3 + az^2 + bz + c$ where $a, b, c \in \mathbb{R}$.

Two of the roots of $P(z)$ are -2 and $-3 + 2i$.

a Find a , b , and c .

b Find the values of z for which $P(z) \geq 0$.

109 Solve graphically for x :

a $x^3 - 4x^2 + 1 < 2x - 5$

b $3x - 0.7x^3 \geq 0.2x^2 - 1.6$

TOPIC 3: GEOMETRY AND TRIGONOMETRY

GEOMETRY OF 3-DIMENSIONAL FIGURES

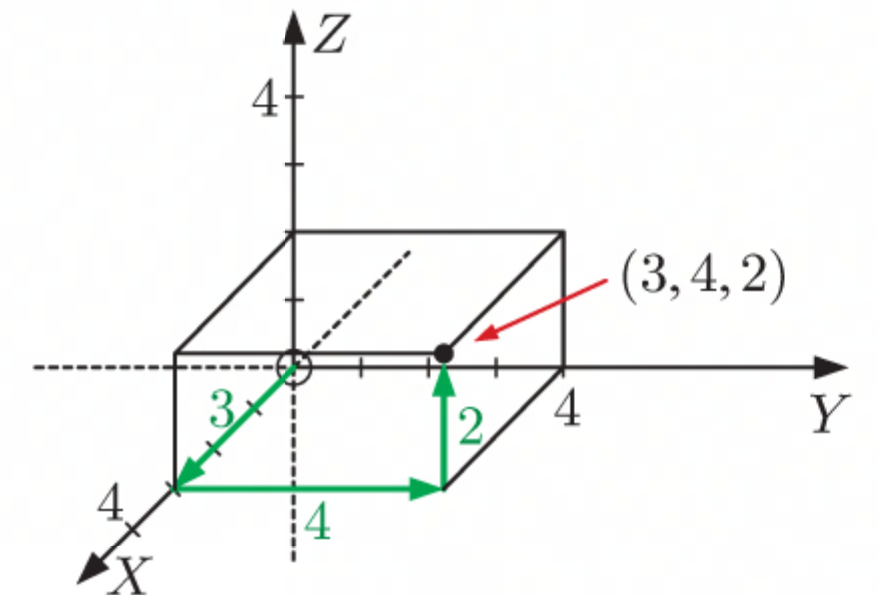
You should be able to calculate the surface area and volume of 3-dimensional figures, including solids of uniform cross-section, pyramids, spheres, and cones.

3-DIMENSIONAL COORDINATE GEOMETRY

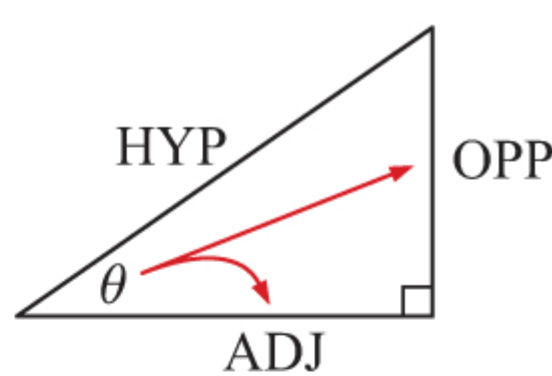
In 3-dimensional coordinate geometry, we specify an origin O, and three mutually perpendicular axes called the X -axis, the Y -axis, and the Z -axis.

For points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the **distance** $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- the **midpoint** of $[AB]$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.



RIGHT ANGLED TRIANGLE TRIGONOMETRY



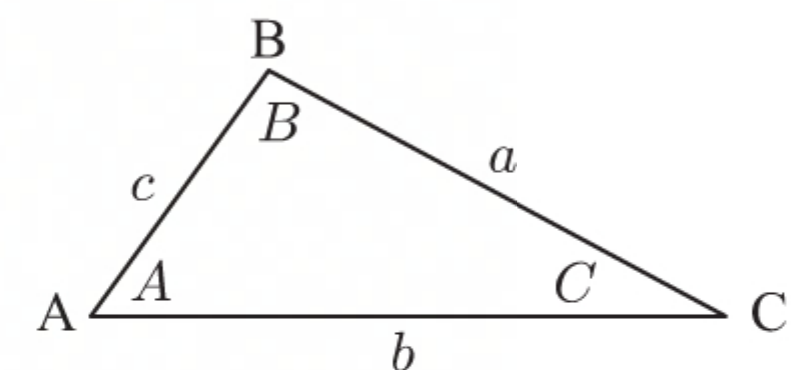
$$\begin{aligned}\cos \theta &= \frac{\text{ADJ}}{\text{HYP}} \\ \sin \theta &= \frac{\text{OPP}}{\text{HYP}} \\ \tan \theta &= \frac{\text{OPP}}{\text{ADJ}}\end{aligned}$$

NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

Area formula: $\text{Area} = \frac{1}{2}ab \sin C$

Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$

Sine rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ or $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



RADIAN MEASURE

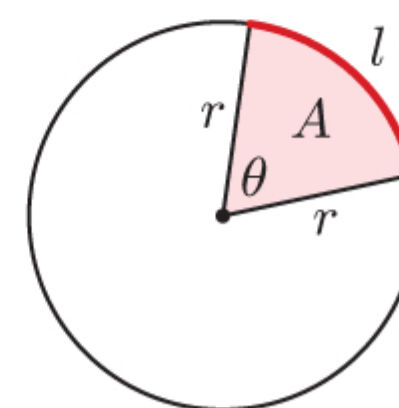
There are $360^\circ \equiv 2\pi$ radians in a circle.

To convert from degrees to radians, multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, multiply by $\frac{180}{\pi}$.

For θ in radians:

- the length of an arc of radius r and angle θ is $l = \theta r$
- the area of a sector of radius r and angle θ is $A = \frac{1}{2}\theta r^2$.



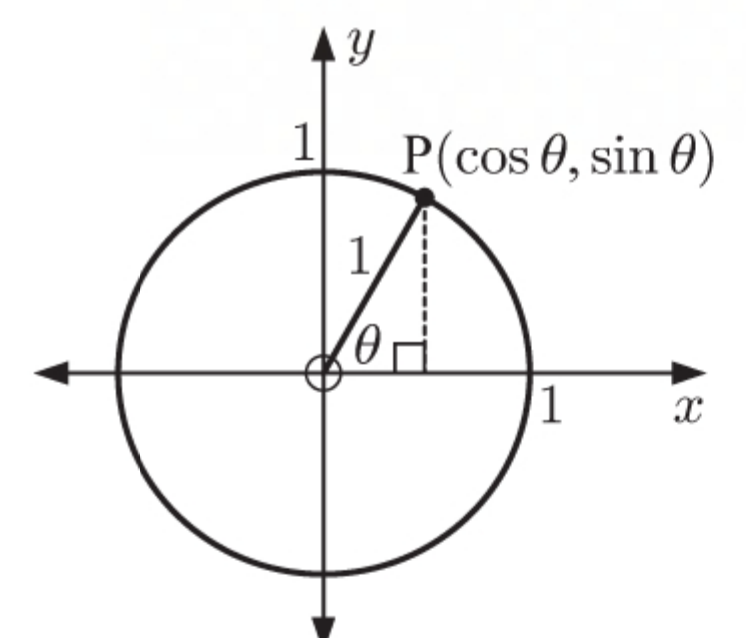
THE UNIT CIRCLE

The **unit circle** is the circle centred at the origin O and with radius 1 unit.

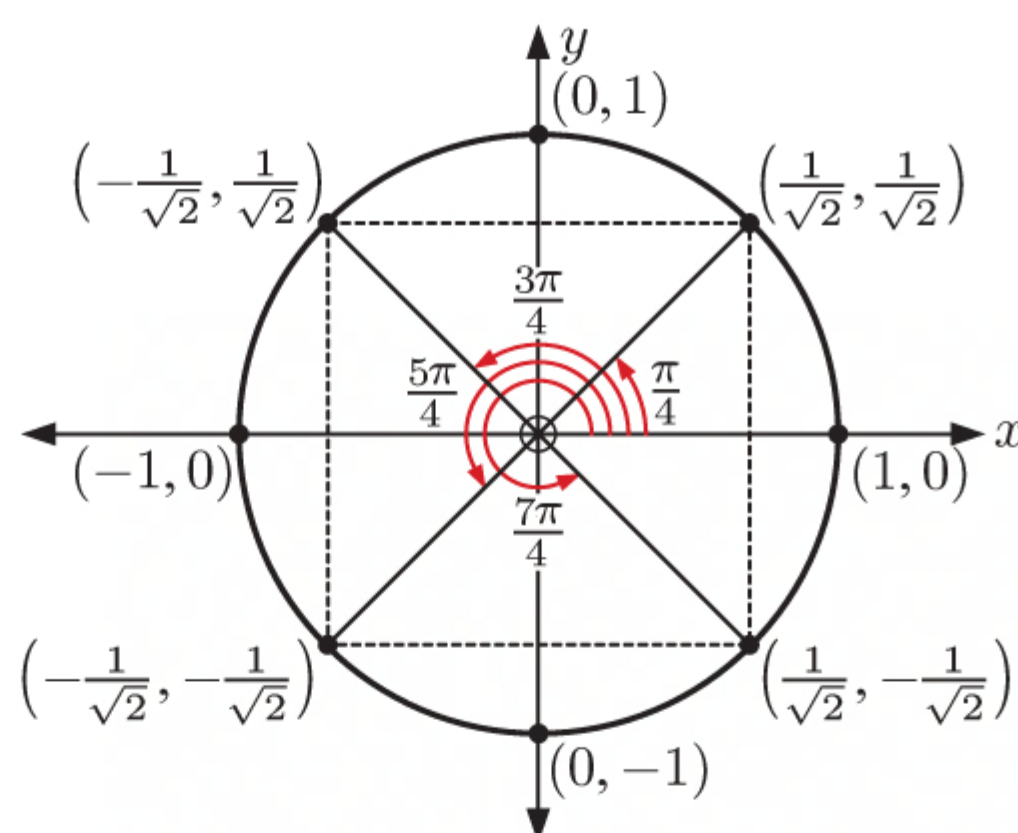
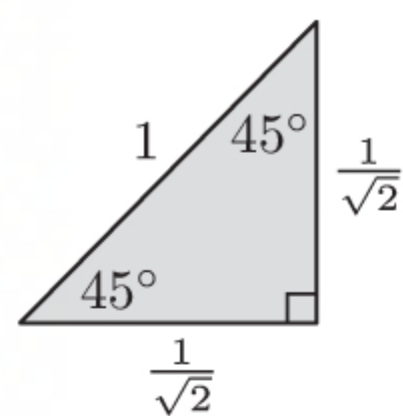
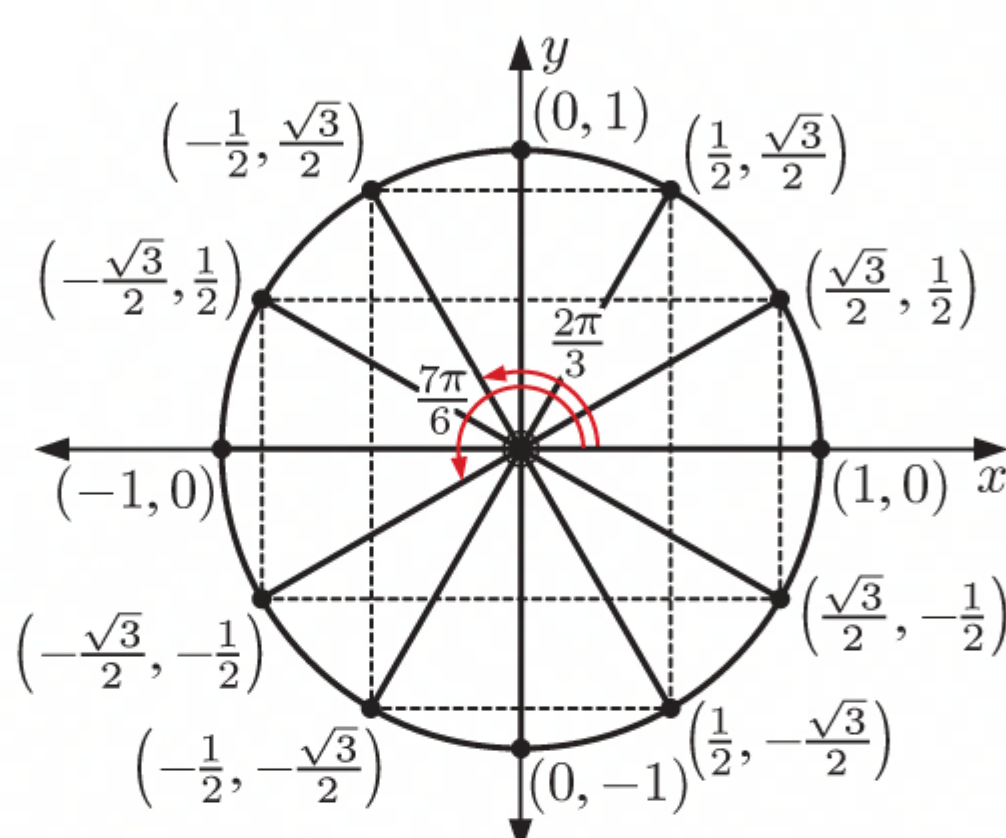
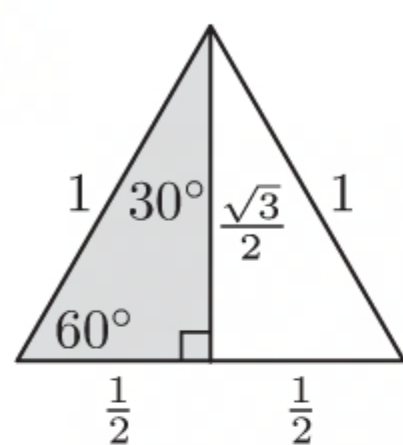
Consider point P on the unit circle where $[OP]$ makes angle θ with the positive x -axis. The coordinates of P are $(\cos \theta, \sin \theta)$.

θ is **positive** when measured in an **anticlockwise** direction from the positive x -axis.

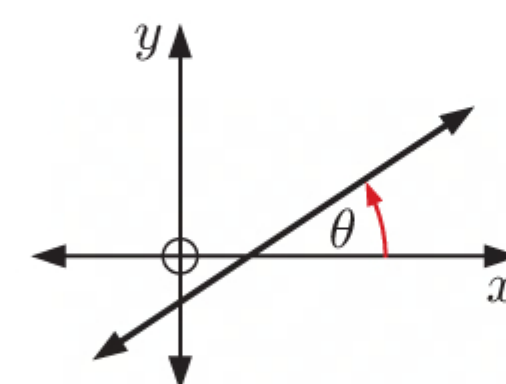
$\tan \theta$ is defined as $\frac{\sin \theta}{\cos \theta}$. $\tan \theta$ is the **gradient** of $[OP]$.



You should memorise or be able to quickly find the values of $\cos \theta$, $\sin \theta$, and $\tan \theta$ that are multiples of $\frac{\pi}{4}$ or $\frac{\pi}{6}$.

Multiples of $\frac{\pi}{4}$ or 45° **Multiples of $\frac{\pi}{6}$ or 30°** 

If a straight line makes an angle of θ with the positive x -axis, then its gradient is $m = \tan \theta$.

**THE SINE FUNCTION**

If we begin with $y = \sin x$, we can perform transformations to produce the **general sine function** $f(x) = a \sin(b(x - c)) + d$, where $b > 0$.

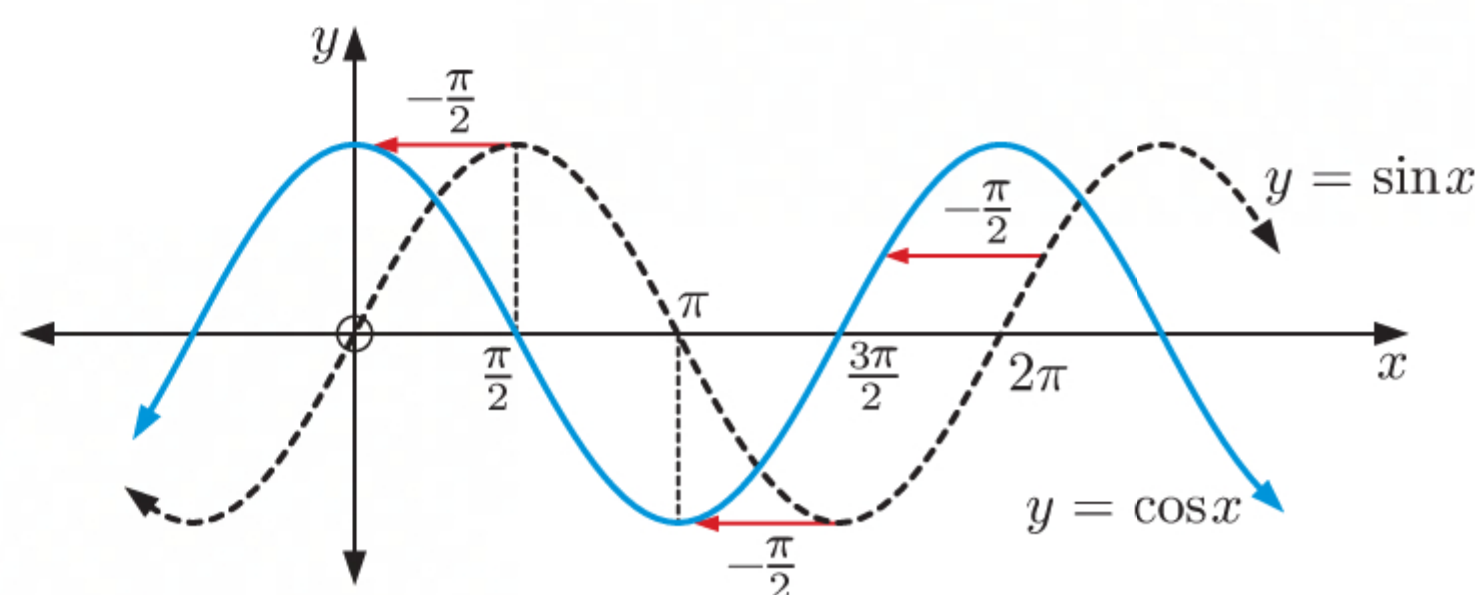
We have a vertical stretch with scale factor $|a|$ and a horizontal stretch with scale factor $\frac{1}{b}$, a reflection in the x -axis is $a < 0$, and a translation through $\begin{pmatrix} c \\ d \end{pmatrix}$.

The general sine function has the following properties:

- the **amplitude** is $|a|$
- the **principal axis** is $y = d$
- the **period** is $\frac{2\pi}{b}$.

THE COSINE FUNCTION

Since $\cos x = \sin(x + \frac{\pi}{2})$, the graph of $y = \cos x$ is a horizontal translation of $y = \sin x$, $\frac{\pi}{2}$ units to the left.

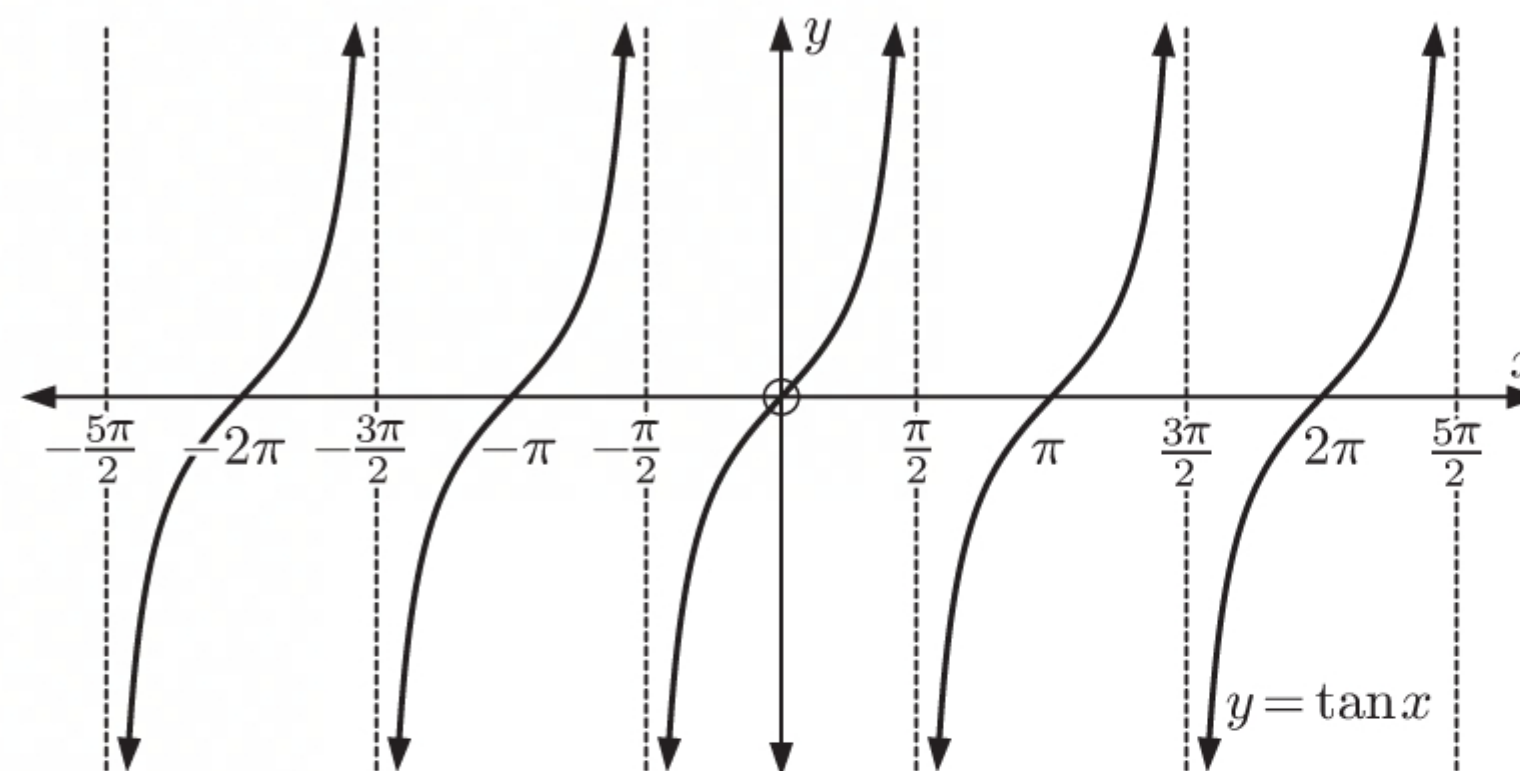


The properties of the **general cosine function** $y = a \cos(b(x - c)) + d$ are the same as those of the general sine function.

THE TANGENT FUNCTION

$y = \tan x = \frac{\sin x}{\cos x}$ is undefined when $\cos x = 0$.

\therefore the graph of $y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.



For the **general tangent function** $y = a \tan(b(x - c)) + d$ where $b > 0$:

- the **principal axis** is $y = d$
- the **period** is $\frac{\pi}{b}$
- the **amplitude** is undefined.

RECIPROCAL TRIGONOMETRIC FUNCTIONS

- $\operatorname{cosec} x = \frac{1}{\sin x}$
- $\sec x = \frac{1}{\cos x}$
- $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

When graphing $\operatorname{cosec} x$, $\sec x$, and $\cot x$, there will be vertical asymptotes corresponding to the zeros of $\sin x$, $\cos x$, and $\tan x$. $\cot x$ will have zeros corresponding to the vertical asymptotes of $\tan x$.

INVERSE TRIGONOMETRIC FUNCTIONS

The **inverse trigonometric functions** are defined as:

Function	Definition	Range
$y = \arcsin x$	$x = \sin y$, $-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$	$x = \cos y$, $-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$	$x = \tan y$, $x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

TRIGONOMETRIC EQUATIONS

Trigonometric equations may be solved:

- graphically (using pre-prepared graphs or technology)
- algebraically

When solving trigonometric equations algebraically, we use the unit circle and the *periodicity* of the trigonometric functions to give us all solutions in the required domain.

An equation of the form $a \sin x = b \cos x$ can be solved as $\tan x = \frac{b}{a}$.

TRIGONOMETRIC IDENTITIES

$\cos(\theta + 2k\pi) = \cos \theta$ and $\sin(\theta + 2k\pi) = \sin \theta$ for all $k \in \mathbb{Z}$.

Negative angles

$\cos(-\theta) = \cos \theta$, $\sin(-\theta) = -\sin \theta$, $\tan(-\theta) = -\tan \theta$

Complementary angles

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$, $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$

Supplementary angles

$\cos(\pi - \theta) = -\cos \theta$, $\sin(\pi - \theta) = \sin \theta$, $\tan(\pi - \theta) = -\tan \theta$

Pythagorean identities

$$\cos^2 \theta + \sin^2 \theta = 1, \quad 1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

Double angle formulae

$$\sin 2\theta = 2 \sin \theta \cos \theta, \quad \cos 2\theta = \begin{cases} \cos^2 \theta - \sin^2 \theta \\ 1 - 2 \sin^2 \theta \\ 2 \cos^2 \theta - 1 \end{cases}, \quad \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Compound angle formulae

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

VECTORS

A **vector** is a quantity with both **magnitude** and **direction**.

Two vectors are **equal** if and only if they have the same magnitude *and* direction.

In examinations:

- scalars are written in italics a
- vectors are written in bold \mathbf{a} .

On paper, you should write vector \mathbf{a} as \underline{a} .

The 3-dimensional base unit vectors are:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The 3-dimensional **zero vector** $\mathbf{0}$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

The general 3-dimensional vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$.

You should understand the following for vectors in both algebraic and geometric forms:

- vector addition
- vector subtraction $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$
- multiplication by a scalar k to produce vector $k\mathbf{v}$ which is parallel to \mathbf{v}
- the magnitude of vector \mathbf{v} , $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- the distance between two points in space is the magnitude of the vector which joins them.

The **position vector** of $A(x, y, z)$ is $\overrightarrow{\mathbf{OA}}$ or $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The **displacement vector** of $B(b_1, b_2, b_3)$ **relative to** $A(a_1, a_2, a_3)$ is $\overrightarrow{\mathbf{AB}} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$.

A, B, and C are **collinear** if $\overrightarrow{\mathbf{AB}} = k\overrightarrow{\mathbf{BC}}$ for some scalar k .

The unit vector in the direction of \mathbf{a} is $\frac{1}{|\mathbf{a}|}\mathbf{a}$.

THE SCALAR OR DOT PRODUCT OF TWO VECTORS

$$\mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Properties of the scalar product:

$$\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$$

$$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$$

$$(k\mathbf{v}) \bullet \mathbf{w} = k(\mathbf{v} \bullet \mathbf{w})$$

$$\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$$

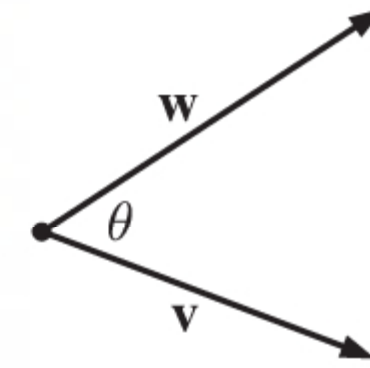
For non-zero vectors \mathbf{v} and \mathbf{w} :

- $\mathbf{v} \bullet \mathbf{w} = 0 \Leftrightarrow \mathbf{v}$ and \mathbf{w} are perpendicular
- $|\mathbf{v} \bullet \mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \Leftrightarrow \mathbf{v}$ and \mathbf{w} are parallel

The angle θ between vectors \mathbf{v} and \mathbf{w} can be found using $\cos \theta = \frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|}$.

If $\mathbf{v} \bullet \mathbf{w} > 0$ then θ is acute.

If $\mathbf{v} \bullet \mathbf{w} < 0$ then θ is obtuse.



THE VECTOR CROSS PRODUCT OF TWO VECTORS

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \\ &= (v_2 w_3 - v_3 w_2) \mathbf{i} - (v_1 w_3 - v_3 w_1) \mathbf{j} + (v_1 w_2 - v_2 w_1) \mathbf{k} \end{aligned}$$

$\mathbf{v} \times \mathbf{w}$ is perpendicular to both \mathbf{v} and \mathbf{w} . Its direction is found using the right hand rule.

Properties of the vector cross product:

If \mathbf{v} and \mathbf{w} are parallel, then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

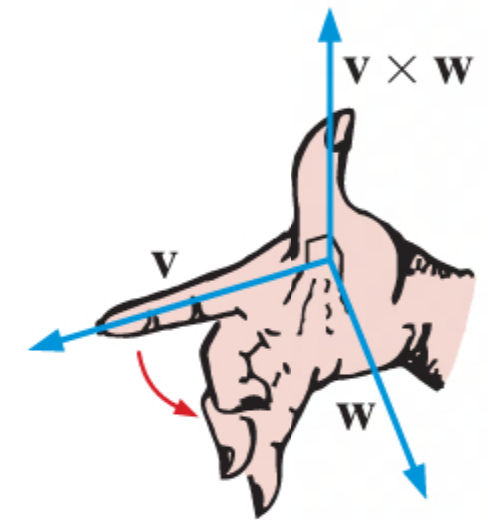
$$(k\mathbf{v}) \times \mathbf{w} = k(\mathbf{v} \times \mathbf{w})$$

Geometric properties:

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \sin \theta \text{ where } \theta \text{ is the angle between the vectors.}$$

$$|\mathbf{v} \times \mathbf{w}| = \text{area of parallelogram formed by vectors } \mathbf{v} \text{ and } \mathbf{w}.$$

$$\frac{1}{2} |\mathbf{v} \times \mathbf{w}| = \text{area of triangle formed by vectors } \mathbf{v} \text{ and } \mathbf{w}.$$

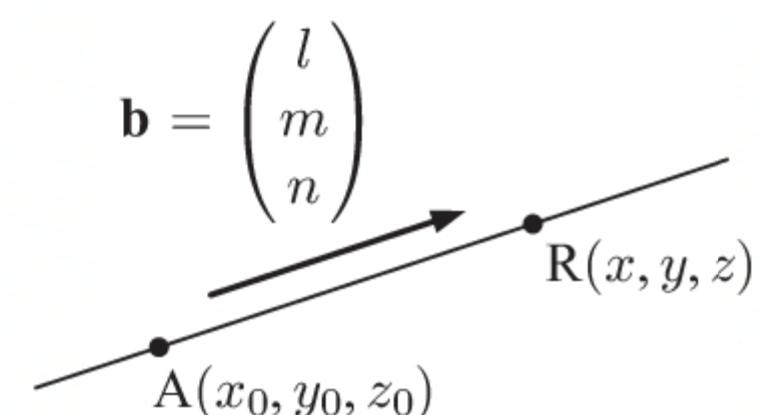


LINES

Suppose $\mathbf{R}(x, y, z)$ is any point on the line,

$\mathbf{A}(x_0, y_0, z_0)$ is a known point on the line,

and $\mathbf{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is the **direction vector** of the line.



Then:

- The **vector equation** of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}, \lambda \in \mathbb{R}$

- The **parametric equations** of the line are:
$$\begin{cases} x = x_0 + \lambda l \\ y = y_0 + \lambda m \\ z = z_0 + \lambda n \end{cases}, \lambda \in \mathbb{R}$$

- The **Cartesian equation** of the line $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}.$

The **acute angle θ between two lines** is given by $\cos \theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|}$ where \mathbf{b}_1 and \mathbf{b}_2 are the direction vectors of the lines.

The shortest distance from point P to a line with direction vector \mathbf{b} occurs at the point R on the line such that \overrightarrow{PR} is perpendicular to \mathbf{b} .

You should be able to determine whether a pair of lines are **parallel**, **coincident**, **intersecting**, or **skew**.

PLANES

The **vector equation of a plane** is $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$

where \mathbf{r} is the position vector of any point on the plane,

\mathbf{a} is the position vector of a known point on the plane,

\mathbf{b} and \mathbf{c} are non-parallel vectors that are parallel to the plane, and

$\lambda, \mu \in \mathbb{R}$.

We say this equation is in **parametric form**.

The vector $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ is a **normal vector** to the plane.

The **normal equation** of a plane is $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$.

If a plane has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and passes through $A(X, Y, Z)$ then it has equation $ax + by + cz = aX + bY + cZ = d$

where d is a constant. This is the **Cartesian equation** of the plane.

If a line has direction vector \mathbf{d} and a plane has normal vector \mathbf{n} , then the acute angle ϕ between the line and the plane is given by

$$\phi = \sin^{-1} \left(\frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \right).$$

If two planes have normal vectors \mathbf{n}_1 and \mathbf{n}_2 , then the acute angle θ between the planes is given by $\theta = \cos^{-1} \left(\frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \right).$

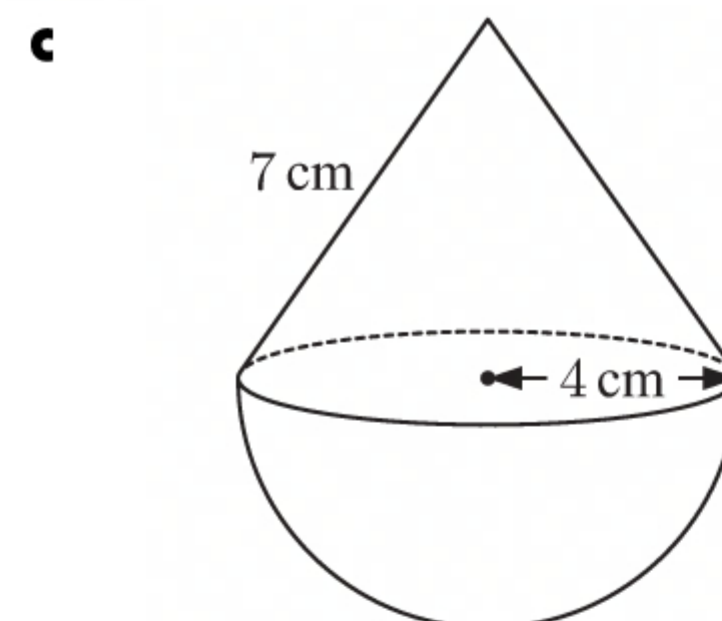
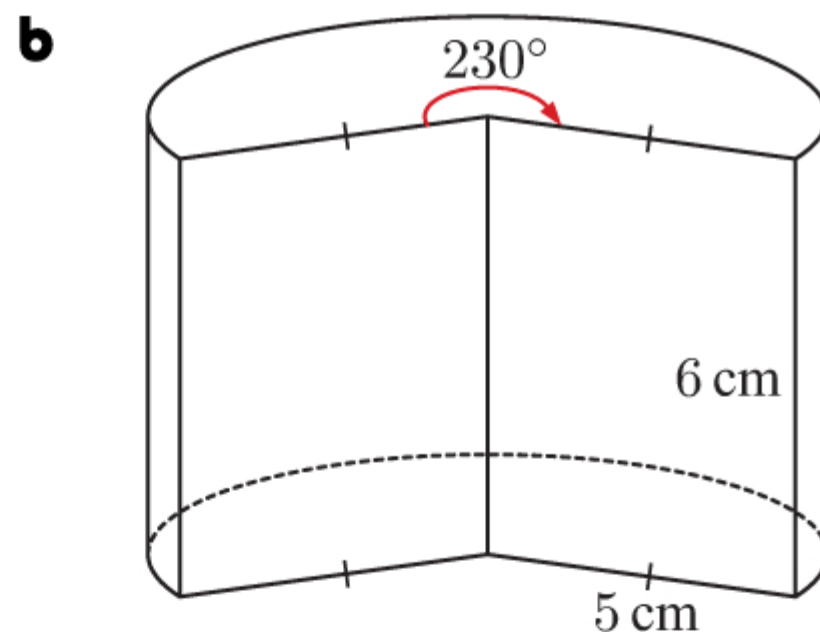
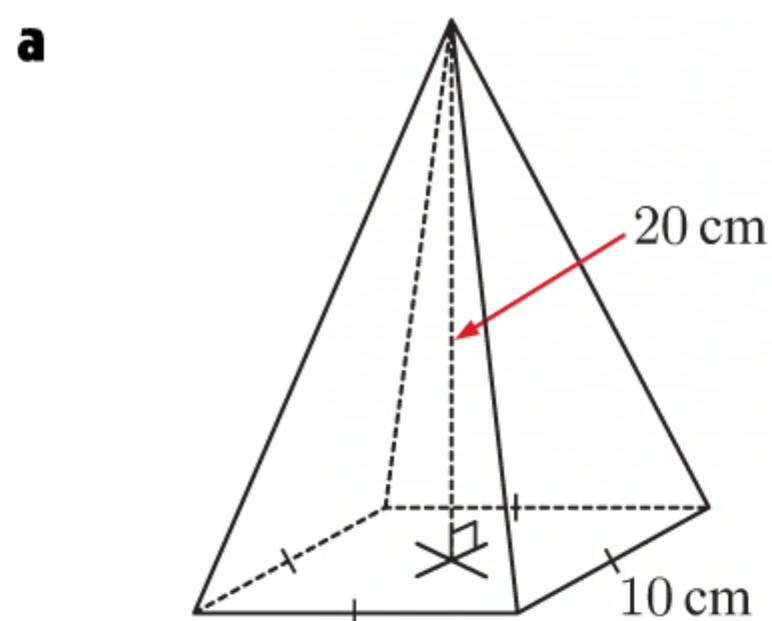
INTERSECTION OF PLANES

To find the intersection of two or three planes, write the system in **augmented matrix form** and use **row reduction** to reduce it to **row echelon form**.

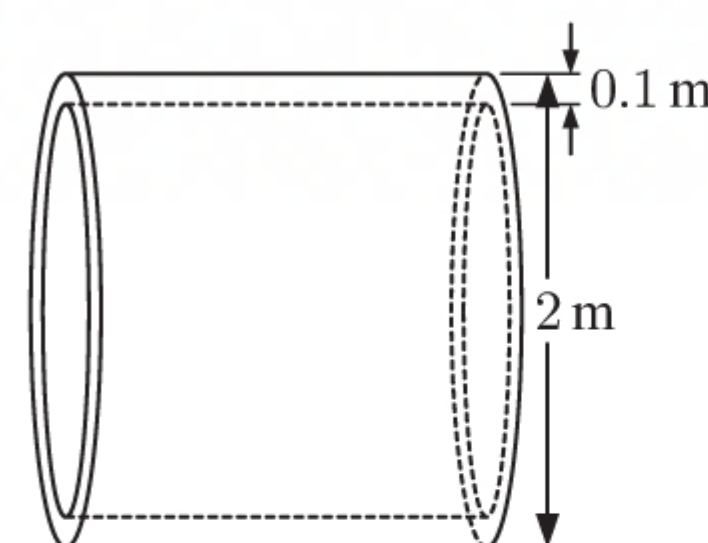
You should be familiar with all possible cases for the intersection of planes.

SKILL BUILDER QUESTIONS

1 Find the surface area of each solid:



2 A pipe used to drain stormwater is made from 3 m^3 of concrete. Find the length of the pipe.



3 A sector of a circle of radius 10 cm has perimeter 40 cm. Find:

a the arc length of the sector

b the area of the sector.

4 A large artificial ice cream for a shop front display is to be made with a hemisphere on top of an inverted cone.

The total height of the structure is 7 m, and the cone is 4 m high.

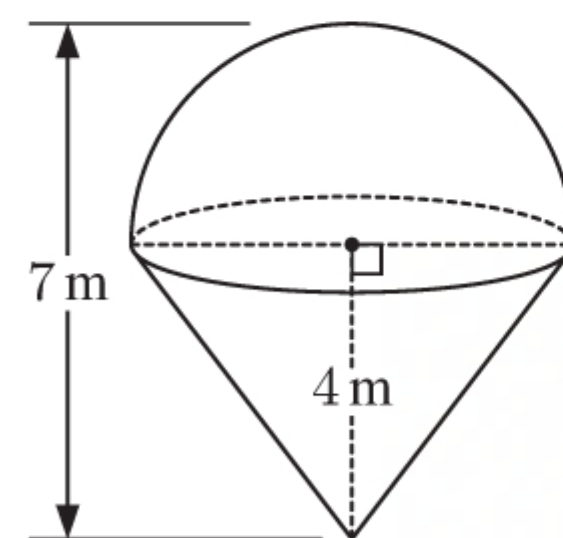
a Show that the radius of the cone is 3 m.

b Calculate the total volume of the ice cream.

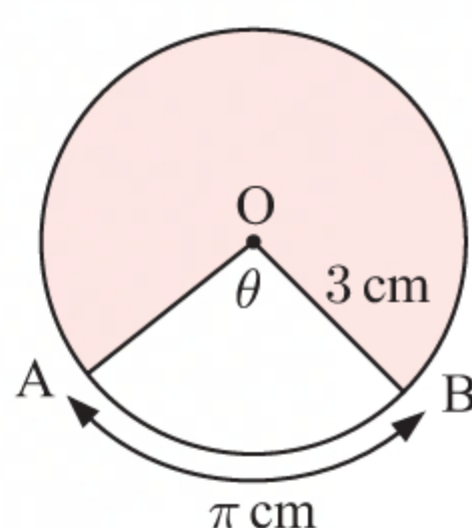
c Find the slant height of the cone.

d Find the total surface area of the ice cream.

e The ice cream is to be made from a lightweight polymer, weighing 1.23 kg per m^2 . Calculate the total weight of the ice cream.



5

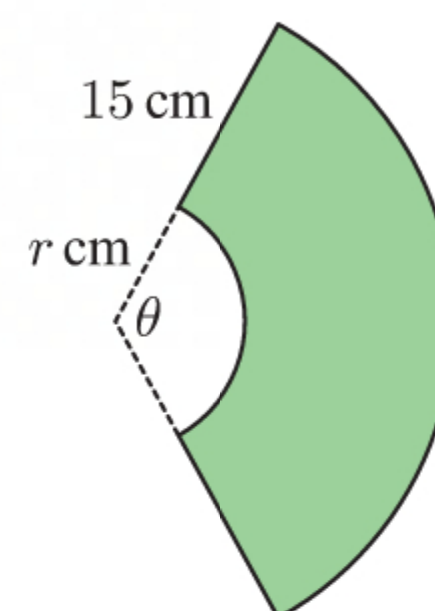
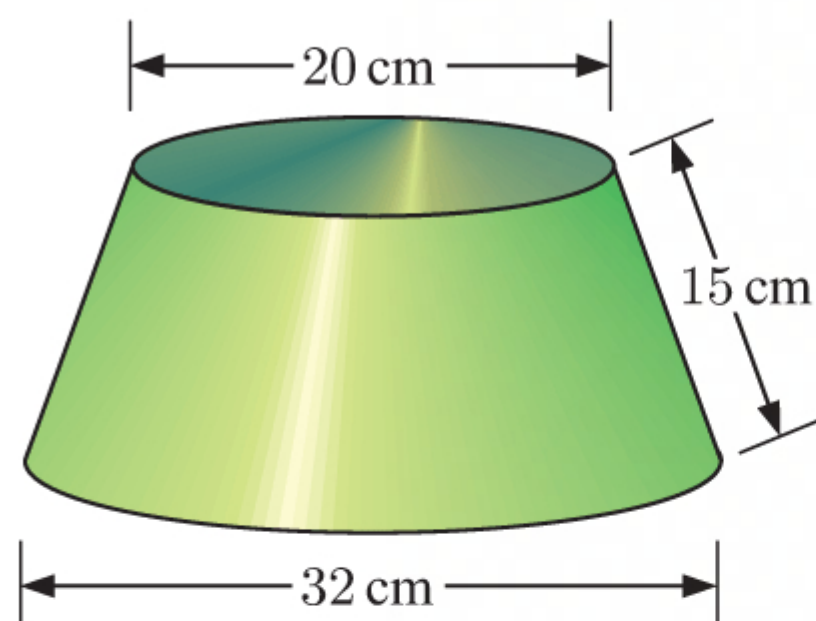


Find:

a θ

b the shaded area.

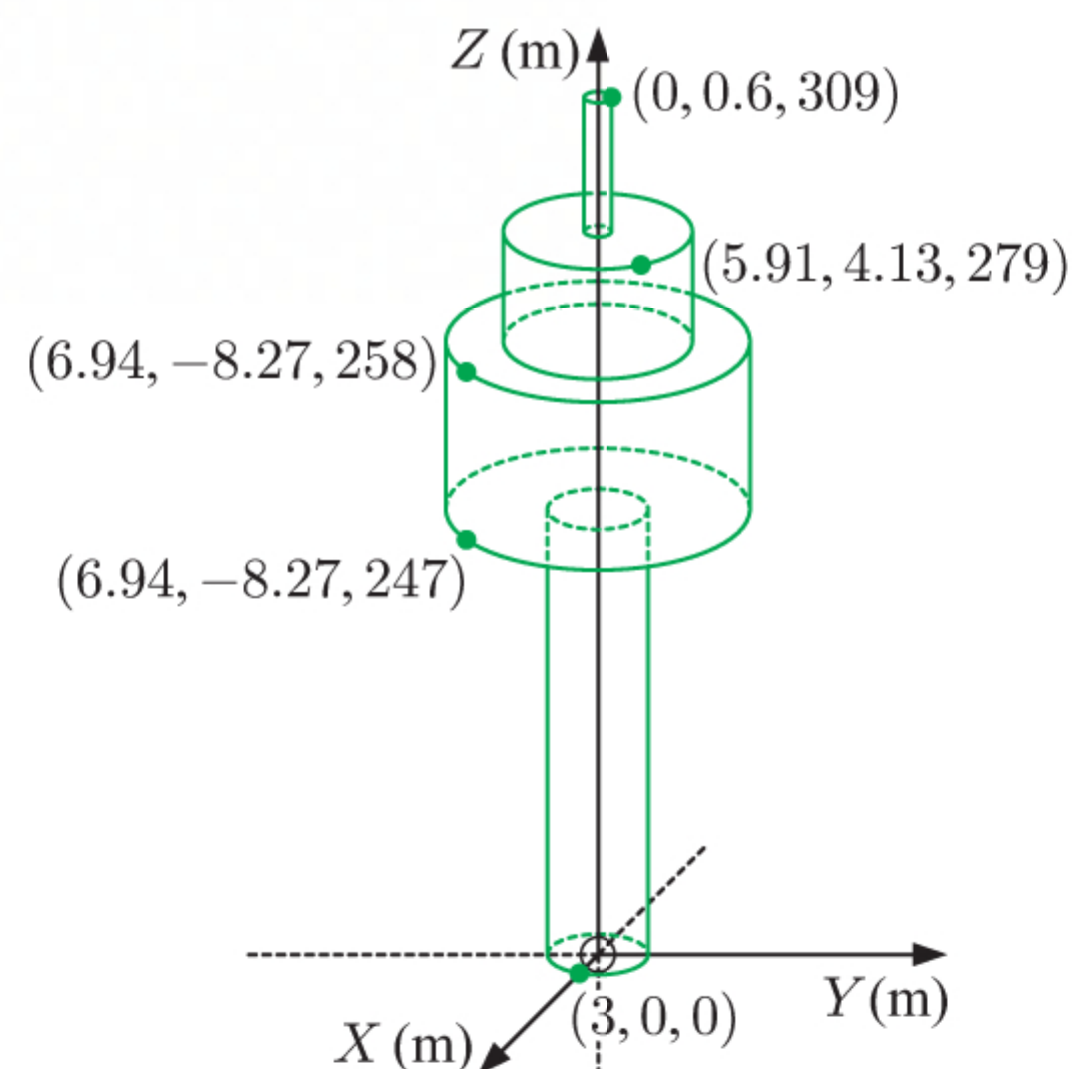
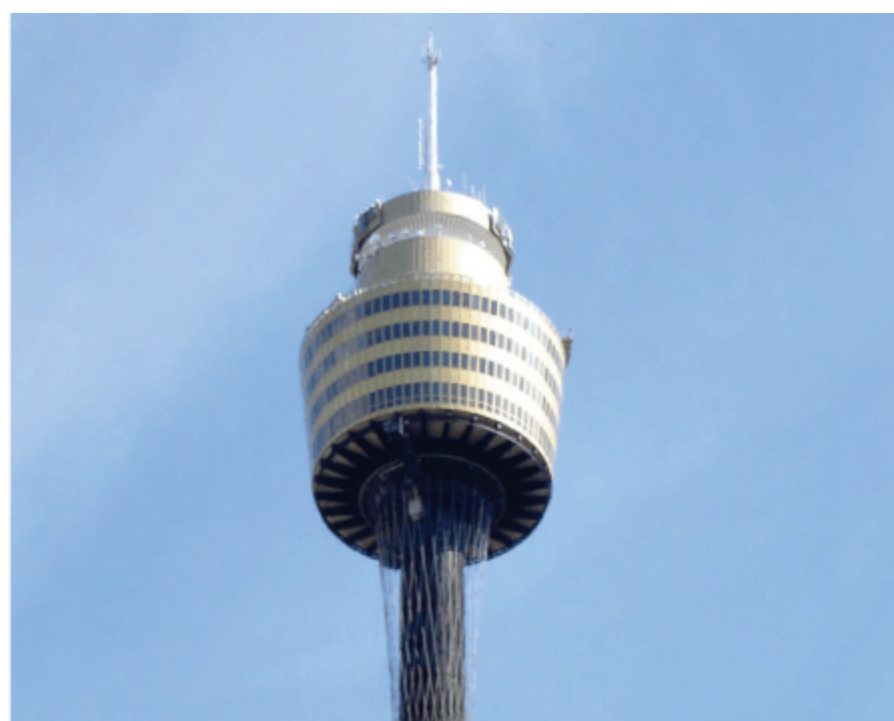
6 The lampshade on the left is made from the sheet of metal on the right. Find r and θ .



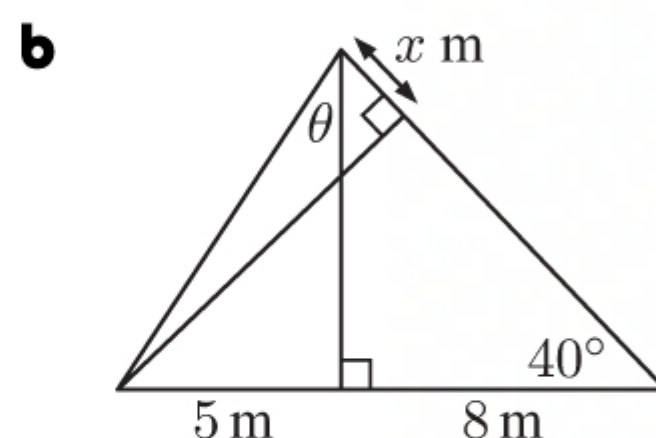
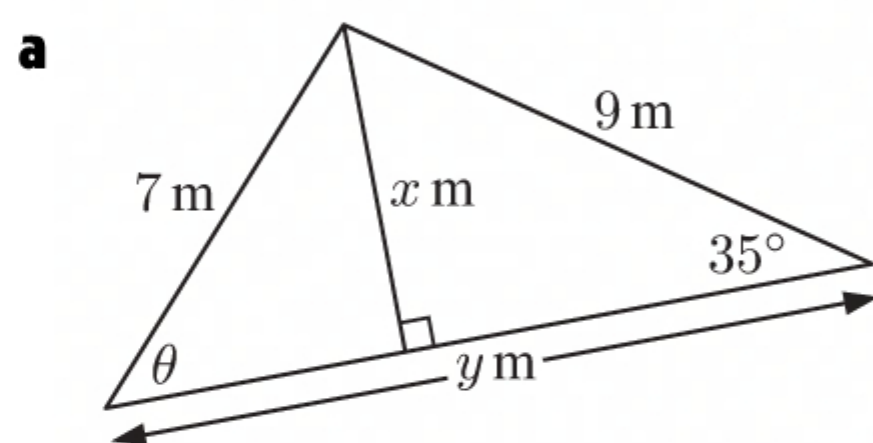
7 The distance from $P(k, 6, -5)$ to $Q(2, -1, -8)$ is 9 units. Find the possible values of k .

- 8** Sydney tower in Australia is the second tallest observation tower in the Southern Hemisphere.

Find the volume of the tower.



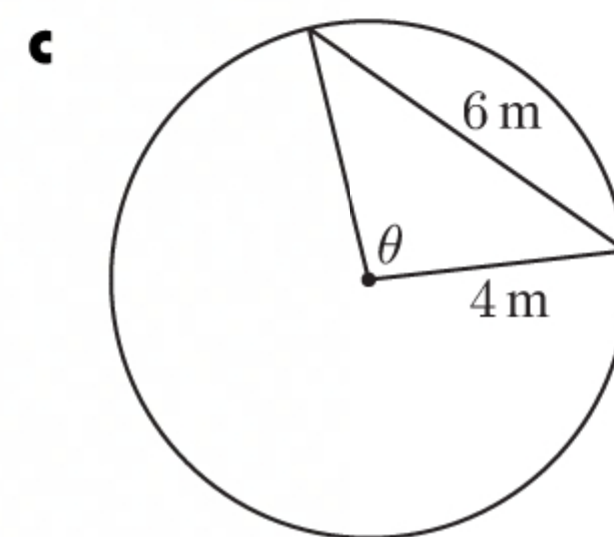
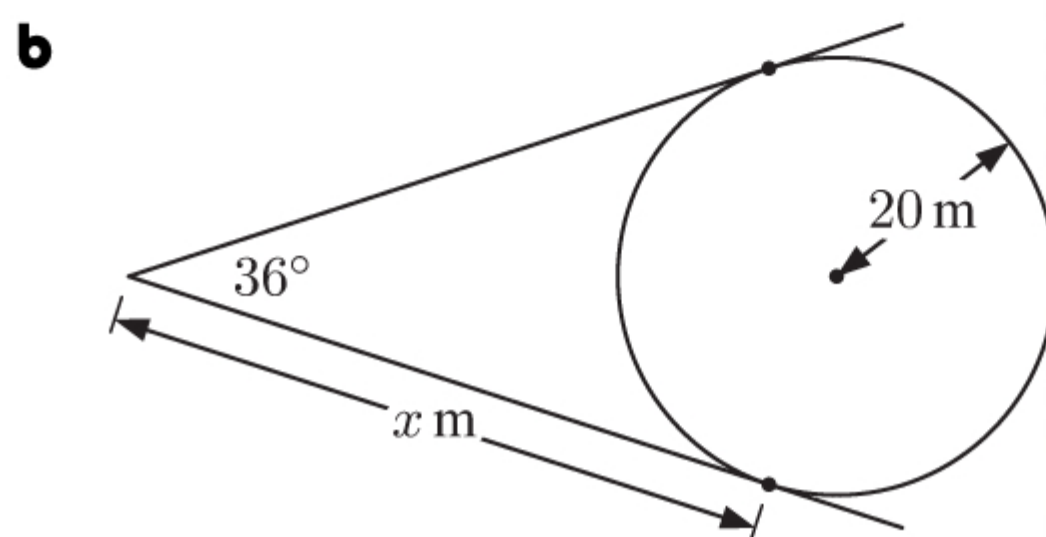
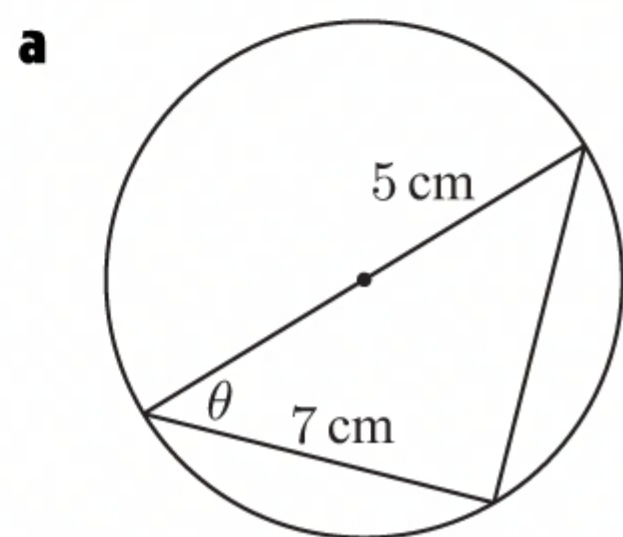
- 9** Find all unknowns in these figures:



- 10** A rhombus has diagonals of length 5 cm and 8 cm.

- Draw a diagram and label it with the given information.
- Find the length of the sides of the rhombus.
- Find the measure of the larger angle in the rhombus.

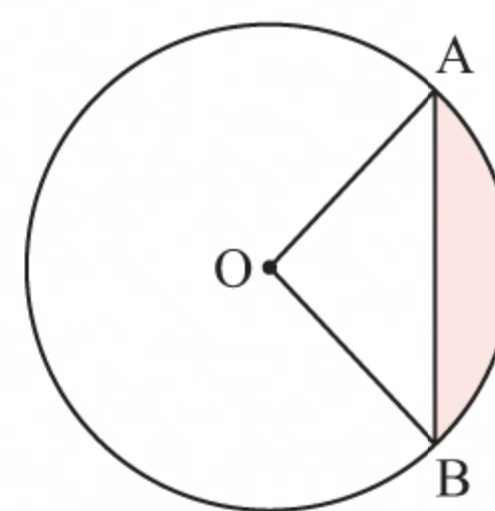
- 11** Find the value of the unknown:



- 12** O is the centre of a circle of radius 32 cm. Chord [AB] is 50 cm long.

Find:

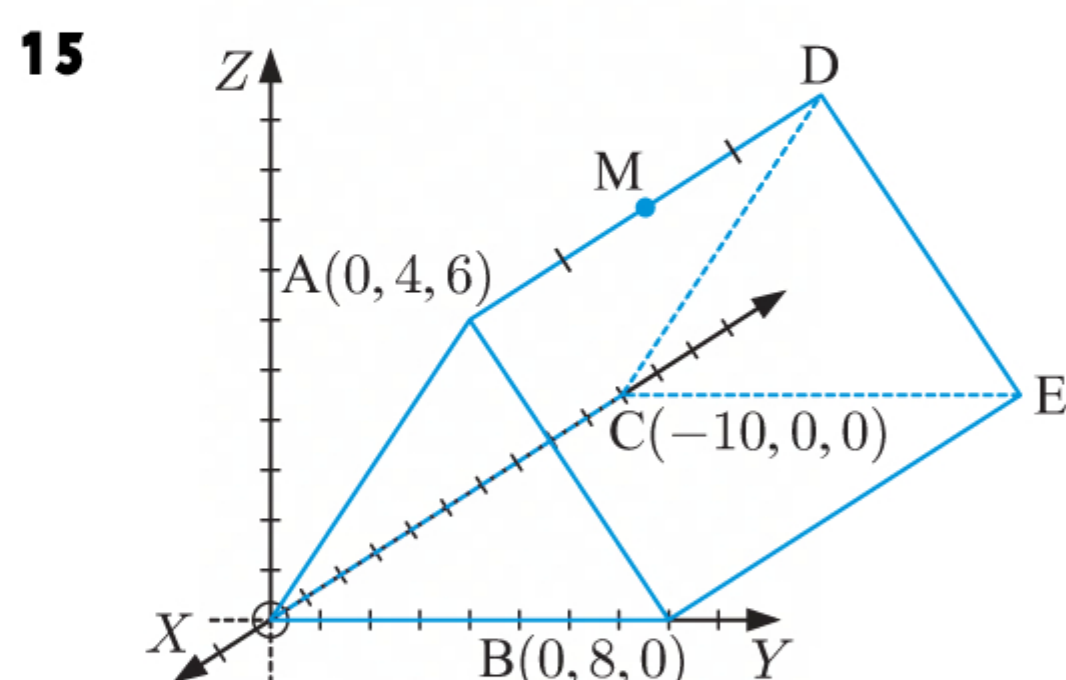
- the measure of \widehat{AOB} , in radians
- the area of the shaded segment.



- 13** At 2:35 pm Fari sees an airplane directly overhead. At 2:38 pm he estimates that the angle of elevation to the plane is 15° . The plane is travelling in a straight line at 110 m s^{-1} . Calculate:

- the height of the plane above the ground
- the angle of elevation to the plane at 2:42 pm.

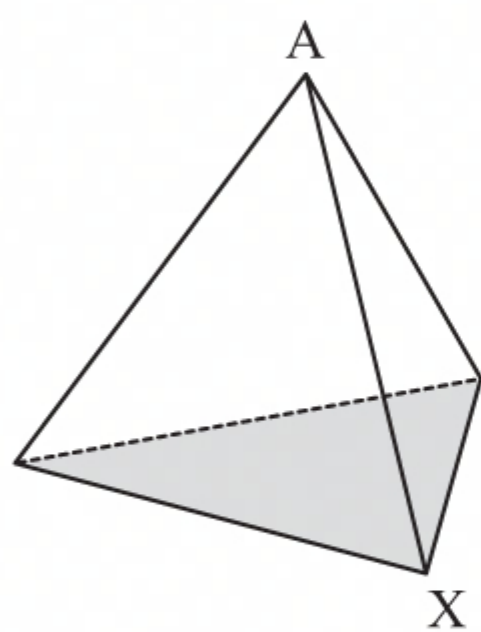
- 14** Pizzerias A and B are 10 km apart. Pizzeria A offers free delivery within 7 km, and pizzeria B offers free delivery within 6 km. Find the area of the region which receives free delivery from *both* pizzerias.



Consider the triangular prism shown.

- State the coordinates of M.
- Find the measure of \widehat{CMD} .
- Find the angle between the following line segments and the base plane BECO:
 - [OD]
 - [EM]

16



All of the faces of this pyramid are equilateral triangles.
Find the angle between $[AX]$ and the shaded base plane.

17 Illustrate and describe these sets:

- a** $\{(x, y, z) \mid z = 2\}$ **b** $\{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3\}$
c $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 16\}$

18 ABC is an equilateral triangle with sides 10 cm long. P is a point within the triangle. P is 7 cm from A and 4 cm from B.

- a** Draw a diagram clearly showing all of the information provided.
b Calculate: **i** \widehat{BAP} **ii** \widehat{CAP}
c Hence find the length of $[CP]$.

19 Triangle ABC has $AB = 8$ cm, $BC = 10$ cm, and $AC = 12$ cm.

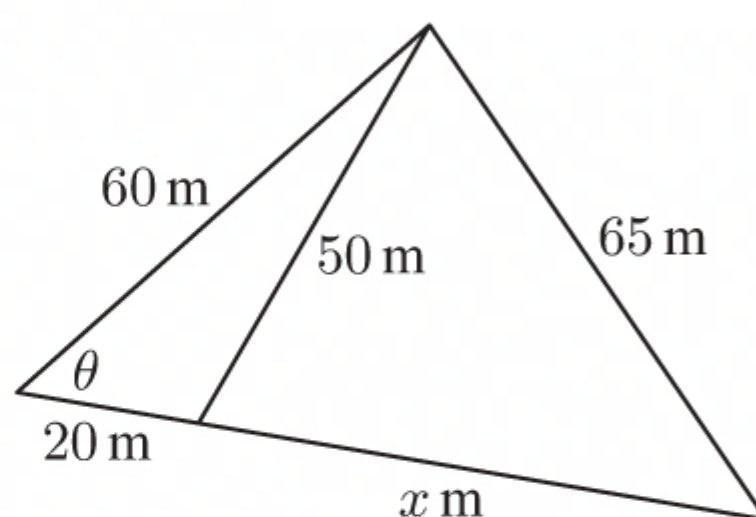
- a** Draw a diagram clearly showing this information.
b Find the smallest angle in triangle ABC.
c Find the area of triangle ABC.

20 Triangle ABC has $AB = 17$ cm, $AC = 19$ cm, and its area is 120 cm^2 .

- a** Draw a diagram to display this information.
b Determine the measure of \widehat{BAC} .
c Determine the length of the remaining side.
d Find the volume of a triangular prism with triangular cross-section ABC, and with length 13.5 cm.

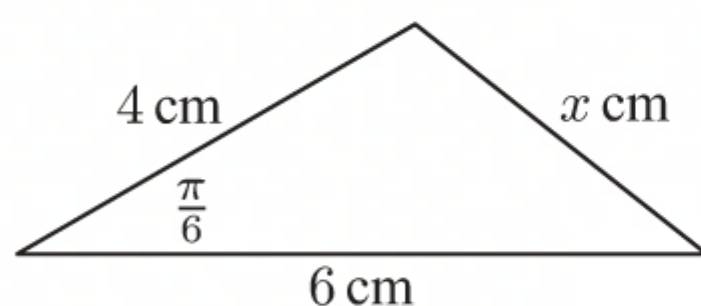
21 **a** Find $\cos \theta$ but not θ .

b Hence find the value of x .

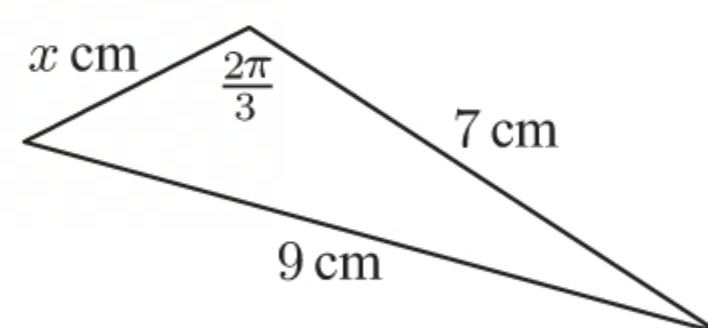


22 Find x :

a

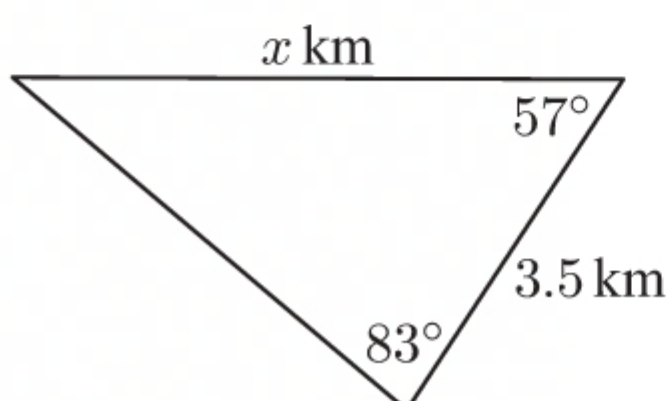


b

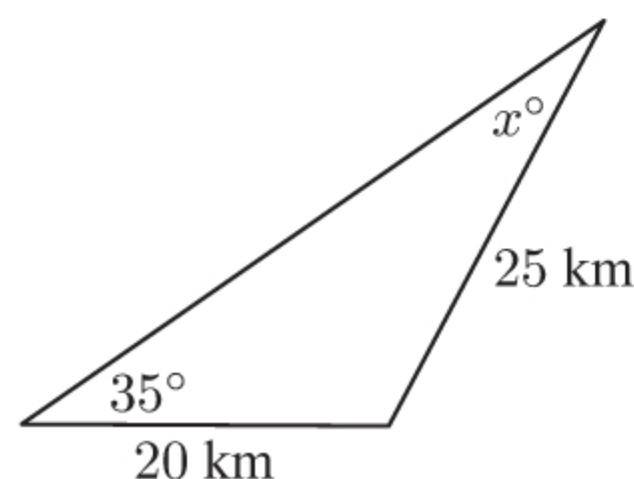


23 Find the value of x :

a



b

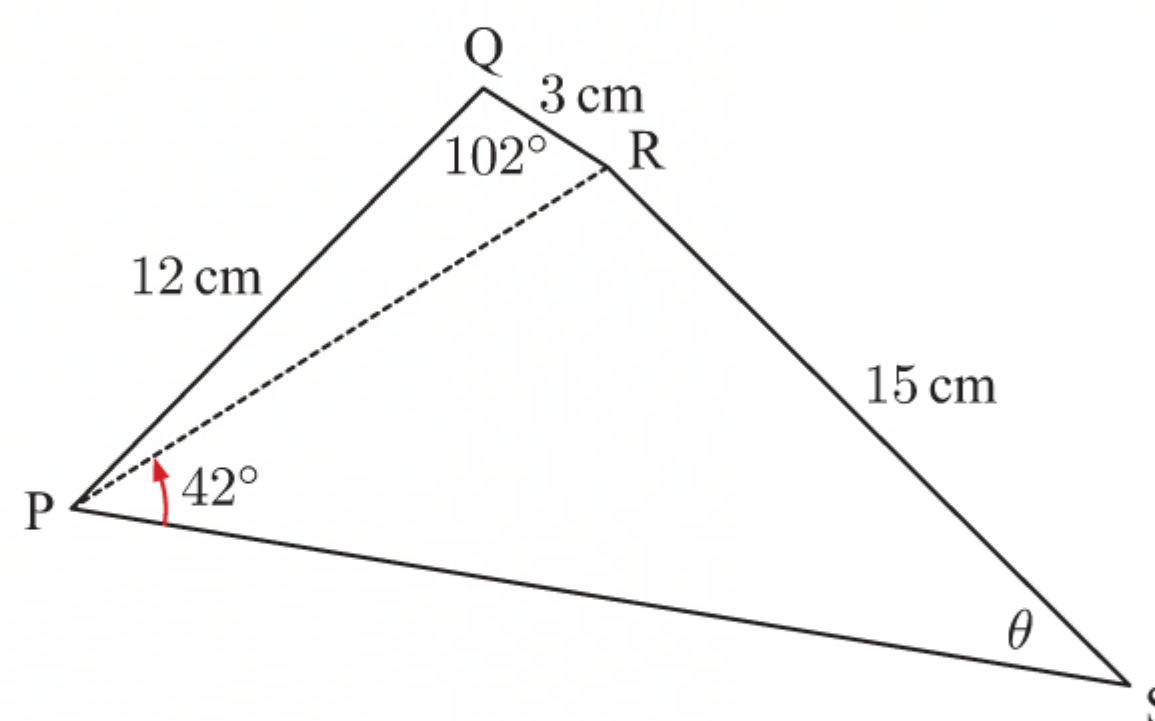


24 In triangle ABC, $AB = 15$ cm, $AC = 12$ cm, and \widehat{ABC} measures 30° .

- a** Find the two possible values of \widehat{ACB} .
b Given that \widehat{BAC} is acute, find its measure.

25 Quadrilateral PQRS has the measurements shown.

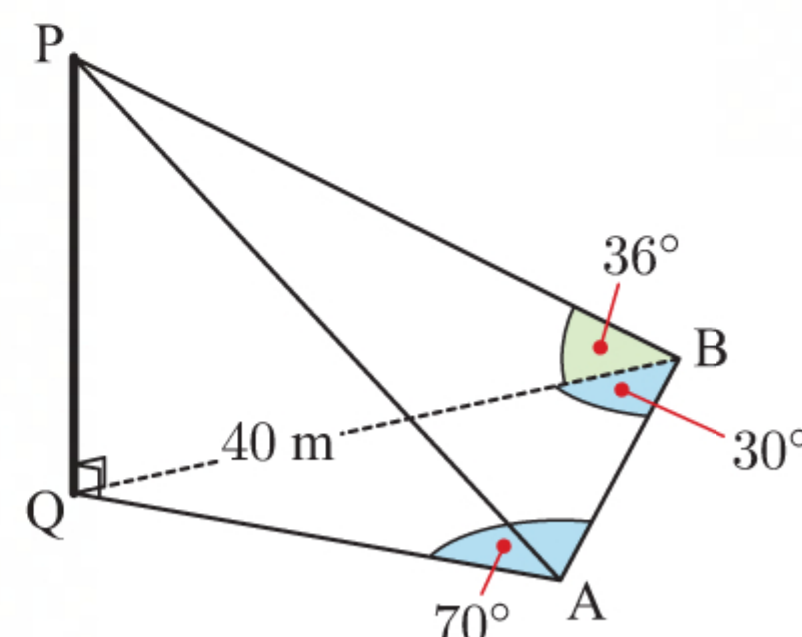
- Find the length of [PR].
- Determine the measure of the angle marked θ .



26 The diagram shows a vertical pole [PQ], which is supported by two wires fixed to the horizontal ground at A and B. $\widehat{PBQ} = 36^\circ$, $\widehat{BAQ} = 70^\circ$, $\widehat{ABQ} = 30^\circ$, and the distance BQ is 40 m.

Find:

- the height of the pole
- the distance between A and B.



27 Trains A and B are 10 km apart, and are approaching the same train station.

Train A is 8 km from the train station on the bearing 071° . Train B is on the bearing 296° from the train station.

- Display this information on a diagram.
- Find the bearing of train B from train A.
- Train B is travelling at an average speed of 7 m s^{-1} . Find, to the nearest second, the time it will take for train B to reach the train station.

28 Given that $\sin \theta = -\frac{1}{2}$ and $\cos \theta = -\frac{\sqrt{3}}{2}$ where $0^\circ < \theta < 360^\circ$, find the exact value of:

- θ
- $\tan \theta$
- $\tan 2\theta$

29 Find the exact value of:

- $\sin \frac{5\pi}{3}$
- $\cos \frac{3\pi}{4}$
- $\tan\left(-\frac{\pi}{3}\right)$

30 Without using a calculator, evaluate:

- $\sin \frac{\pi}{3} \cos \frac{\pi}{4}$
- $2 \tan^2\left(\frac{2\pi}{3}\right) + 1$
- $\frac{\cos \frac{5\pi}{6} \tan^2\left(\frac{3\pi}{4}\right)}{\sin\left(-\frac{\pi}{3}\right)}$

31 Find θ if $0 \leq \theta \leq 2\pi$ and:

- $\cos \theta = -\frac{1}{2}$
- $\sin \theta = \frac{1}{\sqrt{2}}$
- $\tan^2 \theta = \frac{1}{3}$

32 Find the possible exact values of:

- $\cos \theta$ if $\sin \theta = \frac{4}{5}$
- $\sin \theta$ if $\cos \theta = -\frac{2}{7}$.

33 Without using a calculator, find:

- $\sin \theta$ if $\cos \theta = -\frac{1}{4}$ and $\pi < \theta < \frac{3\pi}{2}$
- $\cos \theta$ if $\sin \theta = \frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$
- $\tan \theta$ if $\sin \theta = -\frac{5}{6}$ and $\frac{3\pi}{2} < \theta < 2\pi$
- $\tan \theta$ if $\cos \theta = \frac{1}{3}$ and $0 < \theta < \frac{\pi}{2}$.

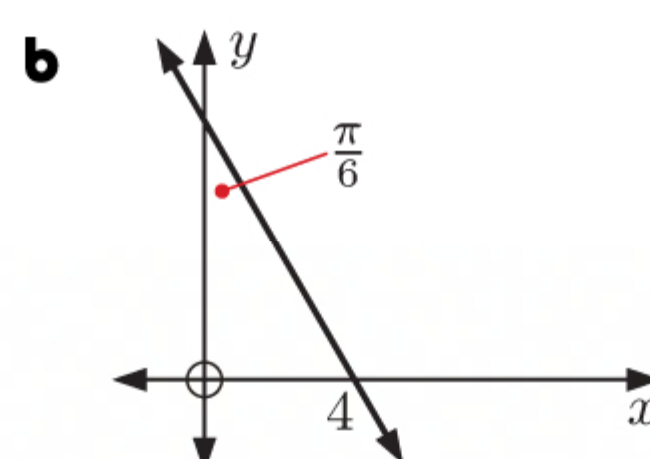
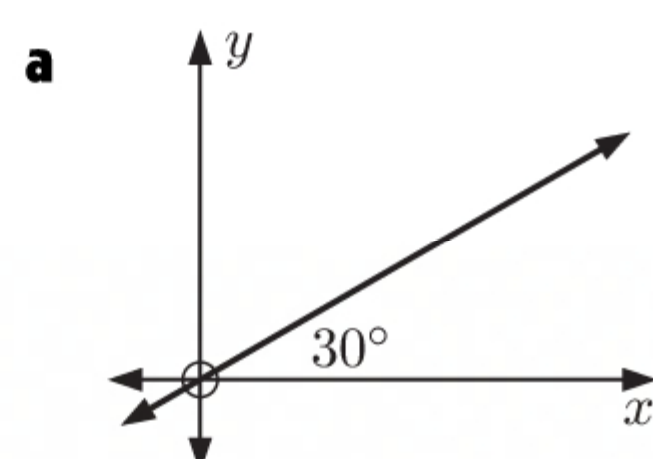
34 Find exact values for $\cos \theta$ and $\sin \theta$ given that:

- $\tan \theta = -\frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$
- $\tan \theta = \frac{1}{\sqrt{2}}$ and $\pi < \theta < \frac{3\pi}{2}$.

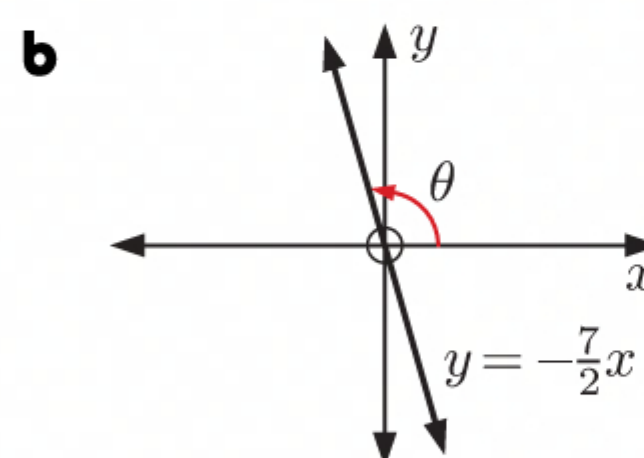
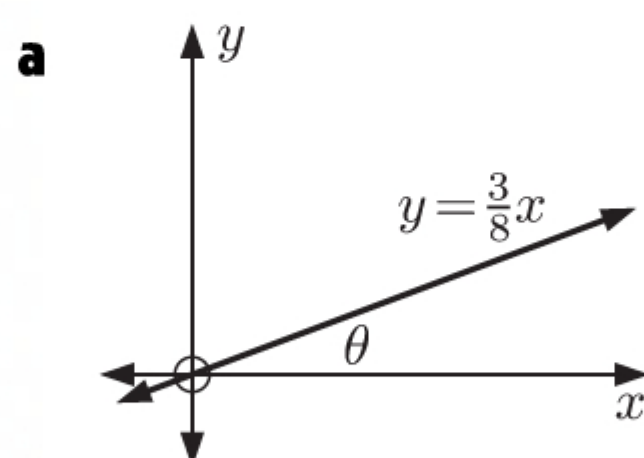
35 Find all θ such that $0^\circ \leq \theta \leq 360^\circ$ and:

- $\cos \theta = -0.3$
- $\sin \theta = -\frac{7}{9}$
- $\tan \theta = -\frac{3}{\sqrt{5}}$

36 Find the equation of the straight line illustrated:



37 Find, in radians, the measure of θ :



38 Find the amplitude, principal axis, and period of:

a $f(x) = \sin 4x$

b $f(x) = -2 \sin \frac{x}{2} - 1$.

39 For each of the following functions:

i State the amplitude.

ii State the principal axis.

iii State the period.

iv Sketch the function.

a $y = 2 \sin(x - \frac{\pi}{3})$ for $0 \leq x \leq 2\pi$

b $y = \sin x + 2$ for $-\pi \leq x \leq \pi$

c $y = 3 \cos 2x$ for $0 \leq x \leq 2\pi$

d $y = \cos \frac{x}{2} - 1$ for $0 \leq x \leq 2\pi$

e $y = \sin(2(x + \frac{\pi}{4}))$ for $0 \leq x \leq 2\pi$

f $y = 10 - 6 \sin 3x$ for $0 \leq x \leq 2\pi$

40 For each of the following functions:

i State the period.

ii Write the equations of the asymptotes.

iii Sketch the function.

a $y = \tan \frac{x}{2}$ for $-2\pi \leq x \leq 2\pi$

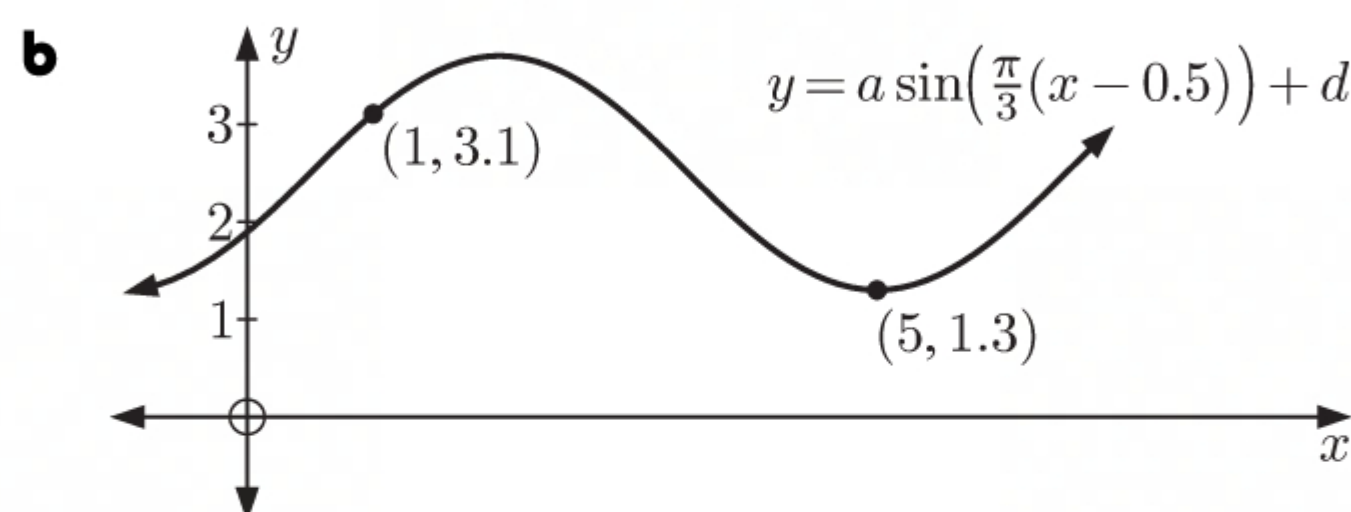
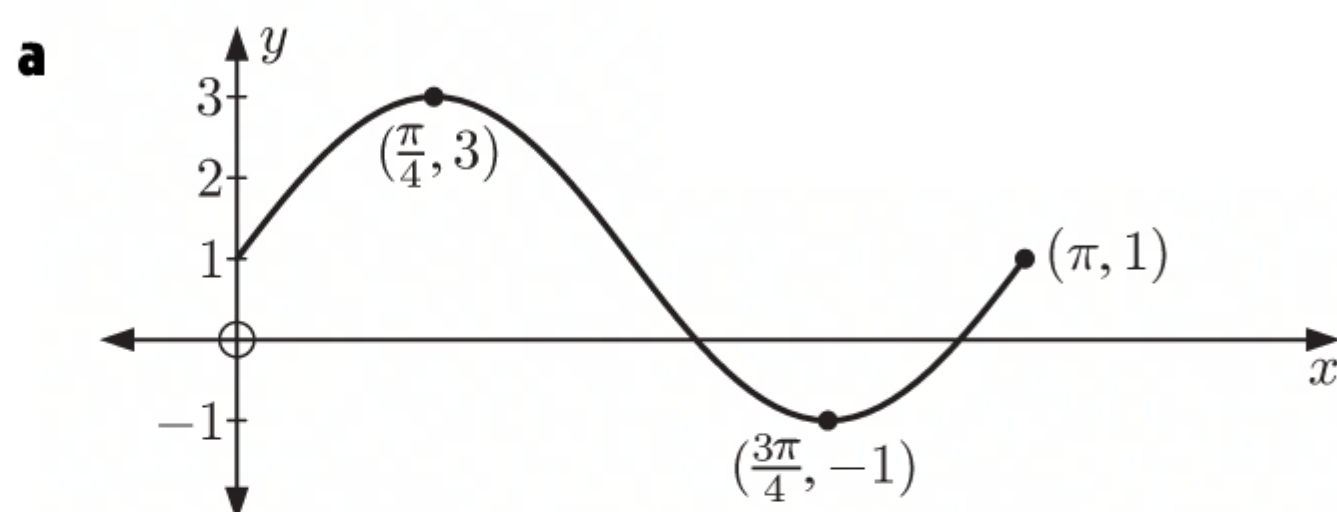
b $y = 5 \tan 3x$ for $0 \leq x \leq \pi$

41 State the transformations which map $y = \sin x$ onto:

a $y = 2 \sin \frac{x}{3}$

b $y = \sin(x + \frac{\pi}{3}) - 4$

42 Find the equation of each sine function.



43 On the same set of axes, sketch the graphs of $f(x) = \sin x$ and $g(x) = -1 + 2f(2x + \frac{\pi}{2})$ for $-\pi \leq x \leq \pi$.

44 a This graph shows $y = a \cos(b(x - c)) + d$ for $1 \leq x \leq 5$.

Use the graph to find the value of:

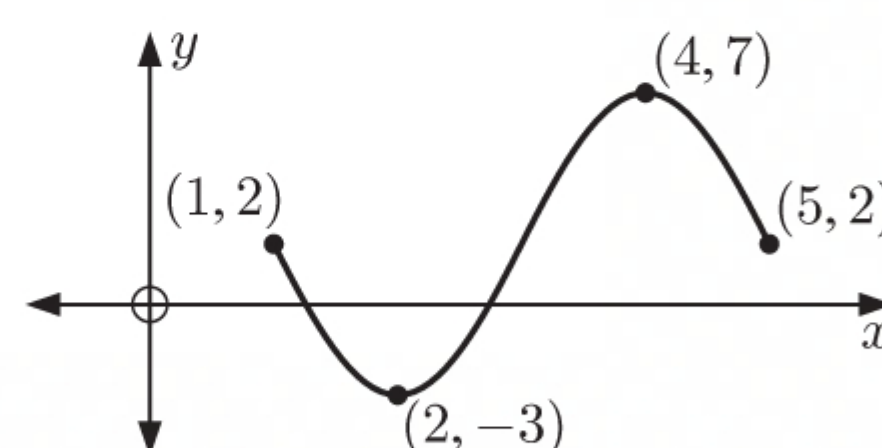
i a

ii b

iii d

iv c

b Write the function in **a** as a sine function.



45 The data in the table below shows the mean daily petrol price in a city for the past 28 days.

Day (t)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Price (P cents per L)	135	135	138	144	147	149	151	150	149	143	140	135	134	130

Day (t)	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Price (P cents per L)	129	128	131	135	136	143	144	148	148	150	151	145	142	137

We want to model the data with a trigonometric function of the form $P = a \sin(b(t - c)) + d$.

a Draw a scatter diagram of the data.

b Without using technology, estimate:

i b

ii a

iii d

iv c

c Check your answers to **b** using technology.

- 46** Suppose $f(x) = 2 \tan(3(x - 1)) + 4$, $-1 \leq x \leq 1$. Find:
- the period of $y = f(x)$
 - the equations of any asymptotes
 - the transformations that transform $y = \tan x$ into $y = f(x)$
 - the domain and range of $y = f(x)$.
- 47** Find the equations of the vertical asymptotes on $[-2\pi, 2\pi]$ for:
- $f(x) = \operatorname{cosec} x$
 - $f : x \mapsto \sec 2x$
 - $g : x \mapsto \cot\left(\frac{x}{2}\right)$
- 48** Solve for x if $0 \leq x \leq 2\pi$:
- $2 \cos x = 5 \sin x$
 - $\tan 3x = 0.9$
 - $4 \sin^2 x = \cos^2 x$
- 49** Solve for x where $-\pi \leq x \leq 3\pi$, giving exact answers:
- $\sqrt{3} \tan \frac{x}{2} = -1$
 - $\sqrt{3} + 2 \sin 2x = 0$
 - $1 - \sqrt{2} \cos 3x = 0$
 - $10 \sin \frac{x}{3} = 5\sqrt{3}$
- 50** Solve exactly:
- $\sin\left(x - \frac{\pi}{3}\right) = 0$, $-2\pi \leq x \leq \pi$
 - $\cos\left(3x + \frac{\pi}{4}\right) = -\frac{1}{2}$, $0 \leq x \leq \pi$
- 51** Find the exact solutions of these equations for $0 \leq x \leq 2\pi$:
- $\sqrt{2} \cos x + 1 = 0$
 - $\sin x = -\sqrt{3} \cos x$
 - $\sin^2 x = \frac{3}{4}$
 - $\tan^3 2x - 3 \tan 2x = 0$
 - $4 \cos^2 x - 3 = 4 \cos x$
 - $4 \cos^4 x + 1 = 5 \cos^2 x$
- 52** The population of butterflies after t years is $P(t) = 4500 + 500 \sin\left(\frac{2\pi}{7}(t - 3)\right)$ for $0 \leq t \leq 10$.
- Find:
 - the initial population
 - the population after 3 years.
 - When is the population:
 - 4200
 - 4900?
 - During what time interval(s) does the population drop below 4300?
- 53** Simplify:
- $2 \sin^2 \theta + 3 \sin^2 \theta$
 - $\cos x \tan x - 2 \sin x$
 - $\frac{-\cos(\frac{\pi}{2} - \theta) \sin(-\theta)}{\cos(-\theta) \sin(\pi - \theta)}$
- 54** Simplify:
- $-3 \sin^2 \theta - 3 \cos^2 \theta$
 - $\sin \theta \cos^2 \theta + \sin^3 \theta$
 - $\frac{\sin^2 \theta - 1}{\cos \theta}$
- 55** Expand and simplify, if possible:
- $(\sin^2 x + 3)^2$
 - $(\tan \alpha + 1)^2$
 - $(\sin x - 1)(\sin x + 1)$
- 56** Solve for x on $0 \leq x \leq 2\pi$, giving your answers as exact values:
- $2 \sin^2 x + 3 \cos x = 3$
 - $\sin 2x + \sin x = 0$
 - $\sin^2 x - \cos^2 x = 0$
- 57** Factorise:
- $1 - \cos^2 x$
 - $2 \cos^2 \alpha - 7 \cos \alpha \sin \alpha - 4 \sin^2 \alpha$
 - $2 \cos^2 \theta - 3 \sin \theta$
- 58** Solve exactly for $0 \leq x \leq 2\pi$:
- $\operatorname{cosec} x = \sqrt{2}$
 - $\sec(\pi - 2x) + 2 = 0$
 - $\cos^2 x + 5 \sin^2 x = \sec^2 x$
- 59** Simplify:
- $\cot x \sec x$
 - $\frac{\operatorname{cosec}(\pi - \theta) \cos(\frac{\pi}{2} - \theta)}{\cot(\pi - \theta)}$
 - $(2 \tan A + 3)^2 + (3 \tan A + 2)^2$
 - $\frac{\cot^2 \theta (1 - \cos^2 \theta)}{1 + \cot^2 \theta}$
- 60** Show that:
- $\frac{1}{\tan \theta - \sec \theta} = -(\tan \theta + \sec \theta)$
 - $(1 - \sin \theta)(1 + \operatorname{cosec} \theta) = \cot \theta \cos \theta$
 - $\frac{\sec x}{\sec x - 1} + \frac{\sec x}{\sec x + 1} = 2 \operatorname{cosec}^2 x$
- 61** Factorise:
- $\cot^2 \beta - \operatorname{cosec}^2 \beta$
 - $3 \sec^2 \alpha + 7 \sec \alpha - 6$
- 62** Solve $\cot \theta + \tan \theta = 2$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- 63** Find the exact value of $\arcsin\left(-\frac{1}{2}\right) + \arctan 1 + \arccos\left(-\frac{1}{2}\right)$.

64 Find, where possible, the exact solutions of:

a $\arcsin(2x - 3) = -\frac{\pi}{6}$

b $\arccos x = -\frac{\pi}{3}$

c $\arctan(2 - x) = \frac{\pi}{4}$

65 Suppose α is an obtuse angle and $\sin \alpha = \frac{2}{3}$. Find the value of:

a $\cos \alpha$

b $\cos 2\alpha$

66 Suppose α is acute and $\cos 2\alpha = \frac{5}{13}$. Find the value of:

a $\sin \alpha$

b $\cos \alpha$

c $\tan \alpha$

67 If $\cos 2x = \frac{5}{8}$, find the exact value of $\sin x$.

68 Solve for x where $-\pi \leq x \leq \pi$:

a $\sin 2x = \sin x$

b $-3 \cos 2x - 14 \sin x + 11 = 0$

c $\sin x + \cos x = 1$

69 Given that A is an acute angle and $\tan 2A = \frac{3}{2}$, find the exact value of $\tan A$.

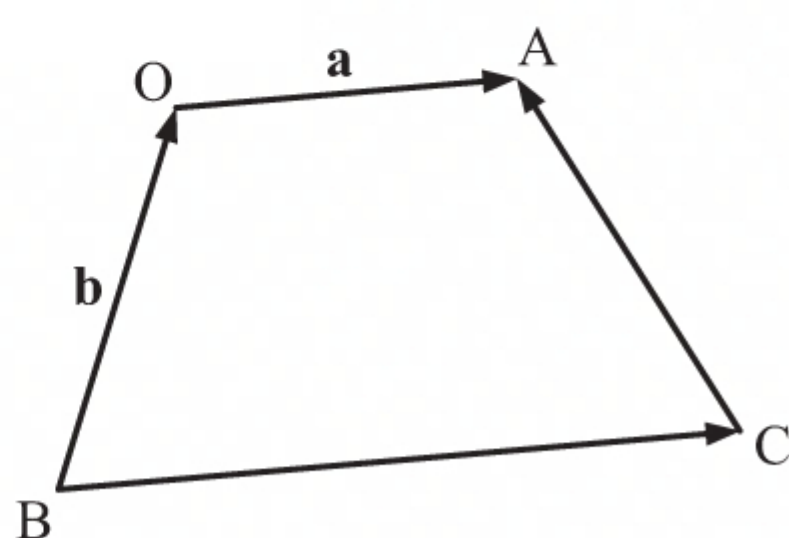
70 Simplify $\cos(\frac{3\pi}{2} - \phi) \tan(\phi + \pi)$.

71 If $\tan \theta = 2$, find the exact value of: **a** $\tan 2\theta$ **b** $\tan 3\theta$

72 If $\sin x = 2 \sin(x - \frac{\pi}{6})$, find the exact value of $\tan x$.

73 Show that $\arctan \frac{1}{4} + \arctan \frac{3}{5} = \frac{\pi}{4}$.

74



In the given figure, $[BC]$ is parallel to $[OA]$ and twice its length.

Write, in terms of \mathbf{a} and \mathbf{b} , vector expressions for:

a \overrightarrow{BC}

b \overrightarrow{CB}

c \overrightarrow{BA}

d \overrightarrow{OC}

e \overrightarrow{AC}

f \overrightarrow{CA}

75 Find k such that:

a $-\frac{1}{3}\mathbf{i} + k\mathbf{j}$ is a unit vector

b $\begin{pmatrix} 3 \\ k \\ k+2 \end{pmatrix}$ has magnitude $\sqrt{61}$ units.

76 On grid paper, illustrate how to find the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$ where $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Check your answer algebraically.

77 If $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, find:

a $\mathbf{c} - \mathbf{a}$

b $\frac{1}{2}\mathbf{c} + 3\mathbf{a}$

c $\mathbf{b} - 2\mathbf{c} - \mathbf{a}$

d $|\mathbf{c} - 3\mathbf{a} + 2\mathbf{b}|$

78 Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$, and $\mathbf{c} = \mathbf{i} + \mathbf{k}$. Find:

a $\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

b $|-5\mathbf{c}|$

c $\frac{1}{|\mathbf{b}|}\mathbf{b}$

d $|2\mathbf{a} - 3\mathbf{b} - \mathbf{c}|$

79 Suppose $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$. Find \mathbf{x} such that:

a $2\mathbf{a} - \mathbf{x} = 4\mathbf{b}$

b $3\mathbf{a} + 2\mathbf{x} = \mathbf{b}$

80 Consider vectors $\mathbf{a} = 3\mathbf{i} - 6\mathbf{j}$ and $\mathbf{b} = 7\mathbf{i} + 2\mathbf{j}$.

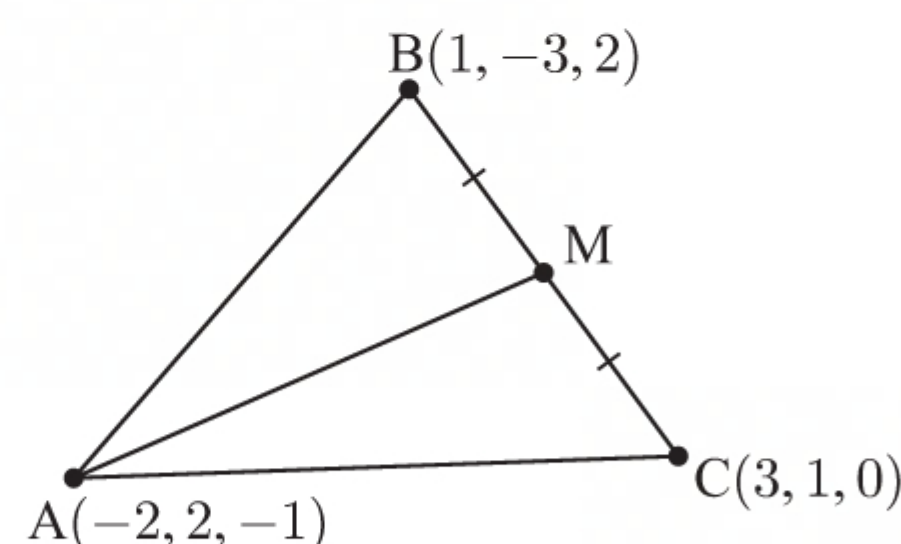
The point $C(5, 22)$ can be located using the vector $\overrightarrow{OC} = r\mathbf{a} + s\mathbf{b}$. Find the values of r and s .

81 Consider the diagram alongside.

a Find the coordinates of M .

b Find vectors \overrightarrow{AB} , \overrightarrow{AM} , and \overrightarrow{AC} .

c Verify that $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$.

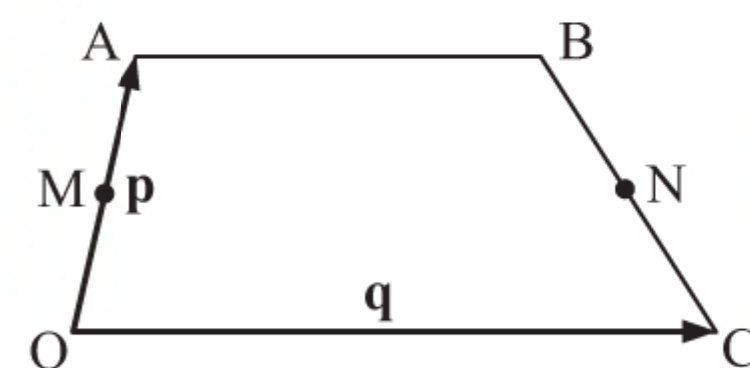


82 Find the vector \mathbf{v} which has:

a the same direction as $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and length 4 units

b the opposite direction to $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$ and length 3 units.

- 83** Find the coordinates of the point which is 6 units from $(2, -1, 3)$ in the direction $\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$.
- 84** For $\mathbf{p} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, find:
- a** $\mathbf{p} \cdot \mathbf{r}$ **b** $\mathbf{q} \cdot (\mathbf{r} + \mathbf{p})$ **c** $(2\mathbf{p} + \mathbf{q}) \cdot \mathbf{r}$ **d** $|\mathbf{q}|^2$
- e** k such that $k\mathbf{p} + \mathbf{q}$ is perpendicular to \mathbf{r} .
- 85** **a** For vectors \mathbf{a} and \mathbf{b} , show that $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2$.
- b** If $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 7$, find $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2$.
- 86** Find the acute angle between two diagonals of the rectangular prism formed by the vectors $2\mathbf{i}$, $3\mathbf{j}$, and $5\mathbf{k}$.
- 87** Consider two vectors \mathbf{x} and \mathbf{y} . Find the value of $|\mathbf{x} + 2\mathbf{y}|$ if:
- a** $\mathbf{y} = -2\mathbf{x}$ and $|\mathbf{x}| = 2$ **b** \mathbf{x} and \mathbf{y} are perpendicular and $|\mathbf{y}| = 3|\mathbf{x}| = 6$.
- 88** Suppose \mathbf{a} and \mathbf{b} are non-zero vectors such that $|\mathbf{a} - \mathbf{b}| = |\mathbf{a} + \mathbf{b}|$. Deduce that \mathbf{a} and \mathbf{b} are perpendicular.
- 89** **a** Given $\mathbf{a} \cdot \mathbf{b} < 0$, what conclusion can you draw about the angle between \mathbf{a} and \mathbf{b} ?
- b** For the vectors $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, find:
- i** $\mathbf{a} \cdot \mathbf{b}$ **ii** the angle between \mathbf{a} and \mathbf{b} , in degrees correct to one decimal place.
- 90** Suppose $|\mathbf{a}| = \sqrt{5}$, and $|\mathbf{b}| = \sqrt{3}$. What can be deduced about \mathbf{a} and \mathbf{b} if:
- a** $\mathbf{a} \cdot \mathbf{b} = 0$ **b** $\mathbf{a} \cdot \mathbf{b} = \sqrt{15}$ **c** $\mathbf{a} \cdot \mathbf{b} = -\sqrt{15}$?
- 91** Use vector methods to prove that joining the midpoints of the sides of a rhombus produces a rectangle.
- 92** In trapezium OABC, $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OC} = \mathbf{q}$, and $\overrightarrow{AB} = k\mathbf{q}$ for some number k .
- a** Show that $\overrightarrow{BC} = (1 - k)\mathbf{q} - \mathbf{p}$.
- b** M and N are the midpoints of [OA] and [BC] respectively. Show that [MN] is parallel to [OC] and $\frac{k+1}{2}$ times its length.
- 93** Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$. Find:
- a** $\mathbf{a} \times \mathbf{b}$ **b** $(\mathbf{b} \times \mathbf{c}) \cdot 2\mathbf{a}$.
- 94** Suppose $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + 2\mathbf{k}$.
- a** Find $\mathbf{a} \times \mathbf{b}$.
- b** Find a vector of length 5 units which is perpendicular to both \mathbf{a} and \mathbf{b} .
- 95** Suppose $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{a} \times \mathbf{b} = \mathbf{j} - 2\mathbf{k}$, and the angle between \mathbf{a} and \mathbf{b} is $\frac{\pi}{4}$. Find $|\mathbf{b}|$.
- 96** Vectors \mathbf{a} and \mathbf{b} are such that $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$. Find the angle between \mathbf{a} and \mathbf{b} , given that it is acute.
- 97** ABCD is a parallelogram with $A(-1, 2, 3)$, $B(0, 2, 4)$, and $C(1, 5, -1)$.
- a** Find the coordinates of D. **b** Find the area of ABCD.
- 98** Let $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{s} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and $\mathbf{t} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ be the position vectors of the points R, S, and T respectively. Find the area of triangle RST.
- 99** $P(2, 4, -1)$, $Q(6, 2, -3)$, and $R(6, k, 0)$ are three points in space. The area of triangle PQR is $\sqrt{44}$ units². Find k .
- 100** Describe each of the following lines using:
- i** a vector equation **ii** parametric equations **iii** a Cartesian equation.
- a** a line parallel to $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ which passes through $(2, -1, 3)$
- b** a line perpendicular to the YZ -plane which passes through $(0, 1, 2)$.



101 Write a vector equation for the line:

- a** parallel to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and passing through the point $(5, 0, -2)$
- b** parallel to $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and passing through the point $(-2, 5, 4)$
- c** perpendicular to the XZ -plane and passing through the point $(2, -4, 1)$.

102 Consider the points $A(1, 3)$ and $B(5, -1)$. Suppose $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, and let M be the midpoint of \overrightarrow{AB} .

- a** Write \overrightarrow{OM} in terms of \mathbf{a} and \mathbf{b} .
- b** Hence find the coordinates of M .
- c** Write a vector equation for the line passing through O and M .
- d** Find the two points on (OM) which are $2\sqrt{10}$ units from M .

103 Suppose A is $(-1, 2, 1)$ and B is $(0, 1, 3)$.

- a** Find the equation of the line (AB) in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$.
- b** Find the angle between (AB) and the line L defined by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, $\mu \in \mathbb{R}$.

104 **a** Find the vector equation of the line through $(-2, -1)$ which is parallel to the vector $3\mathbf{i} + \mathbf{j}$.

b The line in **a** makes an angle of $\frac{\pi}{3}$ with the line $x = 1 + kt$, $y = 2 - t$, $t \in \mathbb{R}$.

Show that k satisfies the equation $13k^2 - 12k - 3 = 0$. Hence find the possible values of k .

105 Trisha rides an escalator which leaves from $(3, 1, 0)$. She moves in the direction $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at 0.5 m s^{-1} .

- a** Find Trisha's velocity vector.
- b** Find Trisha's position after 10 seconds.
- c** The end of the escalator has X -coordinate 0. Find the length of the escalator.
- d** At what angle to the horizontal does the escalator travel?

106 Two ships A and B have paths defined by the equations

$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

respectively, where distances are in kilometres and t is the time in hours.

- a** Find the initial position of each ship.
- b** Find the speed of each ship.
- c** Show that the two ships will pass through the same location, but not at the same time.

107 Consider the point $P(2, -1, 3)$ and the line with Cartesian equation $\frac{x-1}{2} = \frac{3+y}{3} = z$.

- a** Find the coordinates of the foot of the perpendicular from P to the line.
- b** Find the shortest distance from P to the line.

108 **a** Line L_1 passes through $A(1, -1, 2)$ and $B(5, -1, -1)$.

- i** Write a vector equation for the line.
- ii** Find a point on L_1 which is 20 units from A .
- iii** At what point does L_1 meet the YZ -plane?

b Line L_2 passes through $C(4, 1, -\frac{13}{2})$ and is parallel to $-3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

- i** Write a vector equation for the line.
- ii** Show that L_1 is perpendicular to L_2 .
- c** L_1 meets L_2 at the point P .
- i** State the y -coordinate of P .
- ii** Find the coordinates of P .
- d** Find the shortest distance from C to the line L_1 .

109 Line 1 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $s \in \mathbb{R}$.

Line 2 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, $t \in \mathbb{R}$.

Find the point where the two lines meet.

110 Line L_1 passes through the point $(-5, -2)$ and is parallel to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

a Write a vector equation for the line.

b Find the point on the line with x -coordinate -1 .

c The line L_2 is perpendicular to L_1 , and passes through $(4, 5)$.

i Write a vector equation for the line L_2 .

ii Find the point of intersection of L_1 and L_2 .

iii Hence find the shortest distance from $(4, 5)$ to the line L_1 .

111 Consider the system $\begin{cases} mx + 5y = 7 \\ 5x + my = 7 \end{cases}$ where $m \in \mathbb{R}$.

a Find the value(s) of m for which the system has a unique solution, and state the form of the unique solution. Interpret this solution geometrically.

b Discuss the solutions in the other *two* cases.

112 The line passing through the points $A(0, 5, 6)$ and $B(4, 1, -2)$, and the line $\mathbf{r} = \begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $s \in \mathbb{R}$ are intersecting. Find a , and the coordinates of the intersection point.

113 Lines L_1 , L_2 , and L_3 are defined by:

$$L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}, t \in \mathbb{R}$$

$$L_2: x = 5 - r, y = -4 + 2r, z = 1 + r, r \in \mathbb{R}$$

$$L_3: x = 5 - 3s, y = 5 + 4s, z = 1 + 2s, s \in \mathbb{R}$$

a Show that L_1 is parallel to L_2 . **b** Show that L_1 and L_3 intersect, and find the angle between them.

114 Classify the following line pairs as either parallel, coincident, or skew, and find the shortest distance between them.

a $x = 3 + 2t$, $y = 1 - t$, $z = 4 + t$, $t \in \mathbb{R}$ and $x = 4s$, $y = 3 - 2s$, $z = -5 + 2s$, $s \in \mathbb{R}$

b $x = 2t - 1$, $y = 3 - 4t$, $z = 4 - 3t$, $t \in \mathbb{R}$ and $x = -s$, $y = 2s - 1$, $z = 7 - s$, $s \in \mathbb{R}$

c $x = 1 + t$, $y = 2 - t$, $z = 3t - 1$, $t \in \mathbb{R}$ and $x = -2s + 4$, $y = 2s - 1$, $z = 8 - 6s$, $s \in \mathbb{R}$.

115 Find the equation of the plane through A, B, and C, giving your answer in:

i vector form

ii Cartesian form.

a $A(1, 2, 4)$, $B(-1, 0, 3)$, and $C(2, -3, 1)$

b $A(0, -1, 2)$, $B(4, 2, -1)$, and $C(1, -1, 0)$

116 The lines $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} a \\ -1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$, and $\frac{x-4}{2} = 1 - y = \frac{z+2}{3}$ intersect at point P.

a Find the value of a and hence find the coordinates of P.

b Find the acute angle between the two lines.

c Find the equation of the plane which contains the two lines.

117 Suppose line L_1 is defined by $\mathbf{r} = \begin{pmatrix} 8 \\ -13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and line L_2 is defined by $\frac{x+10}{6} = \frac{y-7}{-5} = \frac{z-11}{-5}$.

a Find the coordinates of A, the point of intersection of lines L_1 and L_2 .

b Find the coordinates of B, the point where L_1 meets the plane $3x + 2y - z = -2$.

c The point $C(p, 0, q)$ lies on the plane in **b**.

Find the possible values of p and q if the area of triangle ABC is $\frac{\sqrt{3}}{2}$ units².

- 118** The line $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$ is reflected in the plane $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

Calculate the acute angle between the line and its reflection. Give your answer in radians.

- 119** **a** Find an equation of the plane passing through the points $A(-1, 2, 1)$, $B(2, 1, 3)$, and $C(4, -3, 5)$.
b Find the acute angle between the plane ABC and the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$, $\lambda \in \mathbb{R}$.

- 120** Find the acute angle between the planes with equations:

$$\begin{array}{ll} \mathbf{a} & \begin{array}{l} 2x + y - 3z = 4 \\ x - 2y + z = 3 \end{array} \\ \mathbf{b} & \begin{array}{l} \mathbf{r}_1 = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + 3\mathbf{k}), \quad \lambda, \mu \in \mathbb{R} \\ \mathbf{r}_2 = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(7\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}), \quad \lambda, \mu \in \mathbb{R} \end{array} \end{array}$$

- 121** Solve the system of equations
$$\begin{cases} x + 2y + z = 4 \\ 2x - y + 2z = 3 \\ 3x + 3y + kz = 1 \end{cases}$$
 where k is a real number.

State the geometrical meaning of the different solutions.

- 122** Consider two planes with equations $2x + 4y + z = 1$ and $3x + 5y = 1$. Find:
a the acute angle between the two planes
b any solutions to the system of equations, interpreting your answer geometrically
c all points that lie on the two planes and also on the plane with equation $5x + 13y + 7z = 4$.
- 123** Planes P_1 and P_2 have equations $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ and $\mathbf{r} \cdot (\mathbf{i} - \mathbf{k}) = 5$ respectively.
a Find, to the nearest degree, the size of the acute angle between P_1 and P_2 .
b Find, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, $t \in \mathbb{R}$, the equation of the line in which P_1 and P_2 intersect.

TOPIC 4: STATISTICS AND PROBABILITY

SAMPLING

We obtain data from a **sample** of a population when it is impractical to obtain data from the entire population.

You should know the four main categories of **error** that can arise from sampling:

- **Sampling errors** occur when a characteristic of a sample differs from that of the population.
- **Measurement errors** are inaccuracies in measurement during data collection.
- **Coverage errors** occur when a sample does not truly reflect the population.
- **Non-response errors** occur when a large number of people selected for a survey choose not to respond.

SAMPLING METHODS

- In **simple random sampling**:
 - ▶ Each member of the population has the same chance of being selected in the sample.
 - ▶ Each set of n members of the population has the same chance of being selected as any other set of n members.
- In **systematic sampling**, the sample is created by selecting members of the population at regular intervals.
- In **convenience sampling**, members are chosen for the sample because they are easier to select or more likely to respond.
- In **stratified sampling** or **quota sampling**, the population is divided into subgroups, and the number of members sampled from each subgroup is proportional to the fraction of the population represented by that subgroup. If the members of each subgroup are randomly selected, the sample is a **stratified sample**. If the members are specifically chosen, the sample is a **quota sample**.

TYPES OF DATA AND ITS REPRESENTATION

Categorical data refers to data which describes a particular quality or characteristic.

Discrete data can take any of a set of exact number values $\{x_1, x_2, x_3, \dots\}$. It is normally **counted**.

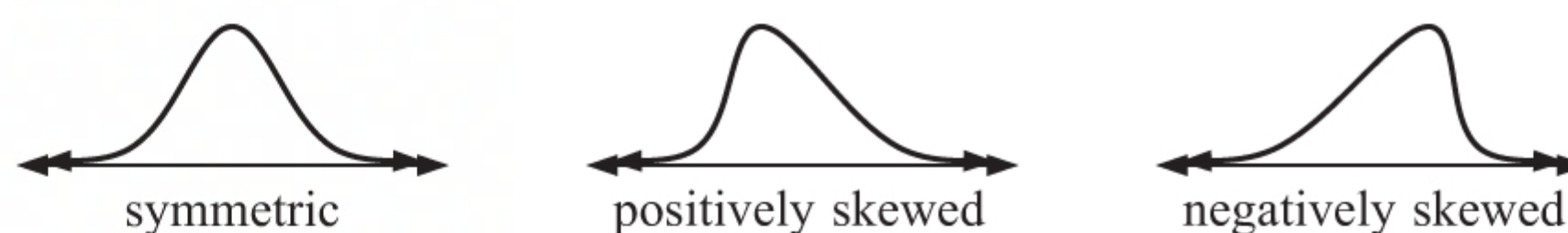
Continuous data can take any numerical value within a certain range. It is normally **measured**.

Grouped data is numerical data which is collected in groups or classes. The **modal class** is the class with the highest frequency.

A **column graph** is used to display discrete data and grouped data. The columns have spaces between them.

A **frequency histogram** is used to display continuous data. The classes are of equal width, and there are no spaces between the columns.

Data may be symmetric, positively skewed, or negatively skewed.



We use a **cumulative frequency graph** to display the cumulative frequency for each data value in a distribution. This enables us to read off the values at each percentile.

MEASURING THE CENTRE OF DATA

The **mean** of a set of scores is their arithmetic average.

For a large population, the **population mean** μ is generally unknown. The **sample mean** \bar{x} is used as an approximation for μ .

For ungrouped data, $\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$

For data in a frequency table, $\bar{x} = \frac{\sum xf}{\sum f}$ where f is the frequency of each value.

For grouped data we can only estimate the mean. We use the **mid-interval value** within each group to represent all scores within that group.

The **median** is the middle value of an ordered data set.

- For an **odd number** of data, the median is one of the original data values.
- For an **even number** of data, the median is the average of the two middle values, and may not be in the original data set.

The **mode** is the most frequently occurring score. If there are two modes we say the data is **bimodal**. For continuous data we refer to a **modal class**.

PERCENTILES

The **k th percentile** is the score a such that $k\%$ of the scores are less than a .

The **lower quartile** (Q_1) is the 25th percentile.

The **median** (Q_2) is the 50th percentile.

The **upper quartile** (Q_3) is the 75th percentile.

You should know how to generate a **cumulative frequency graph** and use it to estimate Q_1 , Q_2 , and Q_3 .

MEASURING THE SPREAD OF DATA

The **range** is the difference between the maximum and the minimum data values.

The **interquartile range** $IQR = Q_3 - Q_1$.

The **variance** σ^2 is the average of the squares of the distances from the mean.

The **standard deviation** σ is the square root of the variance.

You will find formulae for the variance and standard deviation in your formula booklet.

OUTLIERS

Outliers are extraordinary data that are separated from the main body of the data. We test for outliers by calculating upper and lower boundaries:

- upper boundary = $Q_3 + 1.5 \times IQR$
- lower boundary = $Q_1 - 1.5 \times IQR$

Any data outside of these boundaries is considered an outlier.

BOX AND WHISKER DIAGRAMS

A **box and whisker diagram** or **box plot** illustrates the **five-number summary** of a data set:

- minimum value
- Q_1
- median
- Q_3
- maximum value

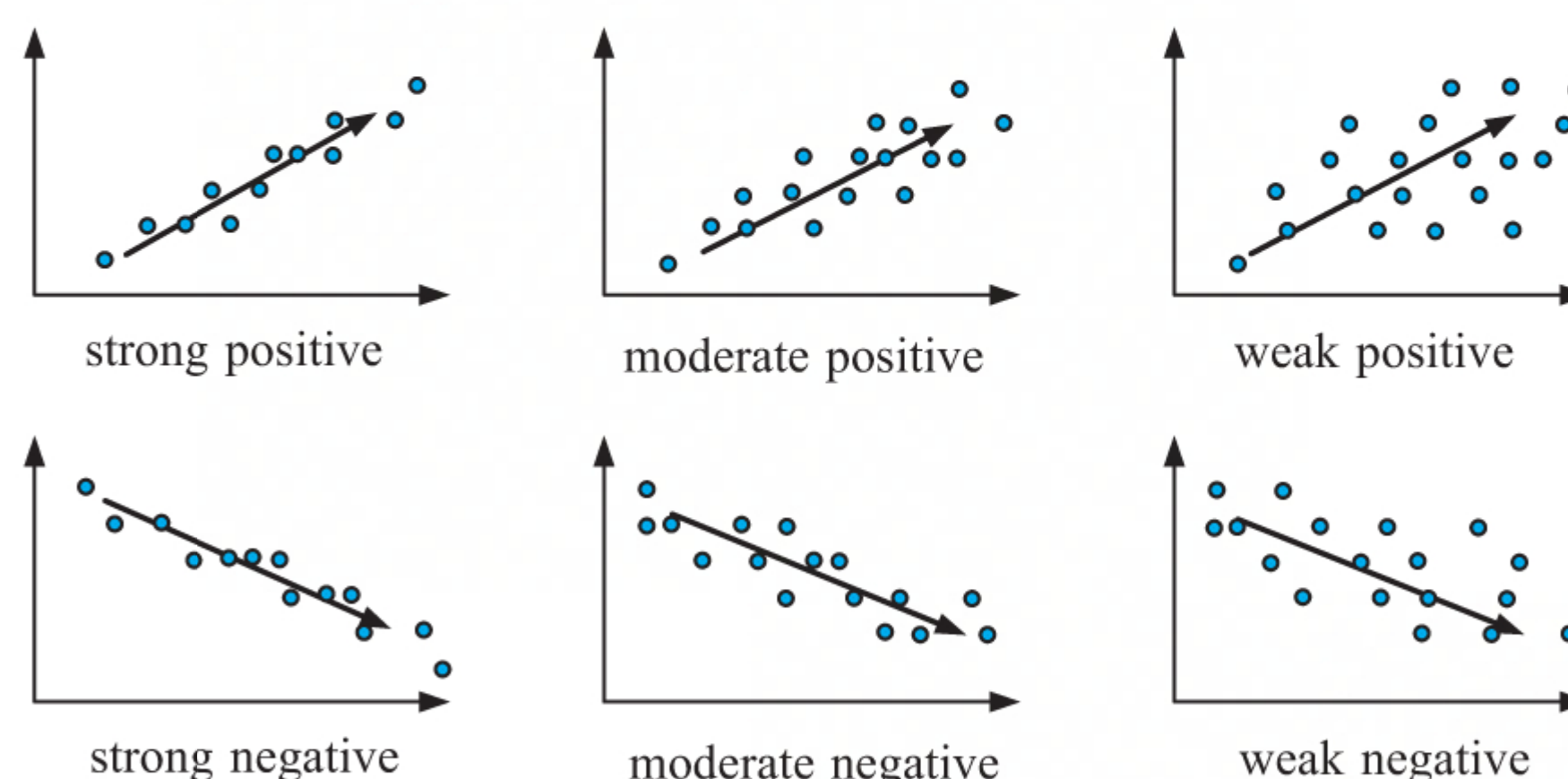


An outlier is indicated by an asterisk *.

BIVARIATE STATISTICS

Correlation refers to the relationship between two numerical variables.

We can use a **scatter diagram** to help identify **outliers** and to describe the correlation between variables. We consider **direction**, **strength**, and **linearity**.



If a change in one variable *causes* a change in the other variable then we say there is a **causal relationship** between them.

To measure the strength of the relationship between two variables, we use **Pearson's product-moment correlation coefficient** r .

The correlation coefficient lies in the range $-1 \leq r \leq 1$.

- The sign of r indicates the direction of correlation.
 - ▶ A positive value for r indicates the variables are positively correlated.
 - ▶ A negative value for r indicates the variables are negatively correlated.
- The size of r indicates the strength of correlation.
 - ▶ A value of r close to $+1$ or -1 indicates strong correlation between the variables.
 - ▶ A value of r close to zero indicates weak correlation between the variables.

Line of best fit

If two variables are linearly correlated, we can draw a line of best fit to illustrate their relationship.

We can draw a **line of best fit by eye**, which passes through the **mean point** (\bar{x}, \bar{y}) , and which fits the trend of the data.

To get a more accurate line of best fit, we use a method called **linear regression**. The line obtained is called the **least squares regression line**. You should be able to find this line using your calculator. In certain situations, it is more sensible to consider the regression line of x against y , rather than the regression line of y against x .

When using a line of best fit to estimate values, **interpolation** is usually reliable, whereas **extrapolation** may not be.

PROBABILITY

A **trial** occurs each time we perform an experiment.

The possible results from each trial of an experiment are called its **outcomes**.

The **sample space** U is the set of all possible outcomes of an experiment.

Experimental probability

In many situations, we can only measure the probability of an event by experimentation.

experimental probability = relative frequency of event

Theoretical probability

If all outcomes are equally likely, the probability of event A is $P(A) = \frac{n(A)}{n(U)}$.

For any event A , $0 \leq P(A) \leq 1$.

For any event A , A' is the event that A does not occur. A and A' are **complementary events**, and $P(A) + P(A') = 1$.

The event that both A **and** B occur is written $A \cap B$.

The event that A **or** B **or both** occur is written $A \cup B$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For **disjoint** or **mutually exclusive** events, $P(A \cap B) = 0$.

Making predictions using probability

If there are n trials of an experiment, and an event has probability p of occurring in each of the trials, then the number of times we *expect* the event to occur is np .

Independent events

Two events are **independent** if the occurrence of each of them does not affect the probability that the other occurs. An example of this is sampling **with replacement**.

For independent events A and B , $P(A \cap B) = P(A)P(B)$.

Dependent events

Two events are **dependent** if the occurrence of one of them *does* affect the probability that the other occurs. An example of this is sampling **without replacement**.

For dependent events A and B , $P(A \cap B) = P(A) \times P(B \text{ given that } A \text{ has occurred})$.

Conditional probability

For any two events A and B , “ $A | B$ ” represents the event “ A given that B has occurred”, and $P(A | B) = \frac{P(A \cap B)}{P(B)}$.

For independent events, $P(A) = P(A | B) = P(A | B')$.

Bayes' theorem

$$\begin{aligned} P(A | B) &= \frac{P(B | A)P(A)}{P(B)} \\ &= \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | A')P(A')} \end{aligned}$$

DISCRETE RANDOM VARIABLES

A **random variable** represents the possible numerical outcomes of an experiment.

A **discrete random variable** can take any of a set of distinct values.

If X is a discrete random variable with possible values $\{x_1, x_2, \dots, x_n\}$ and corresponding probabilities $\{p_1, p_2, \dots, p_n\}$, then:

- $0 \leq p_i \leq 1$ for all $i = 1, \dots, n$
- $\sum_{i=1}^n p_i = p_1 + p_2 + \dots + p_n = 1$
- $\{p_1, p_2, \dots, p_n\}$ describes the **probability distribution** of X .

We can also describe the probability distribution of X using a **probability mass function** $P(x) = P(X = x)$.

The **expectation** of a discrete random variable X is $E(X) = \mu = \sum_{i=1}^n x_i p_i$.

A game where X is the gain to the player is said to be **fair** if $E(X) = 0$.

The **mode** is the data value x_i whose probability p_i is the highest.

$$\begin{aligned} \text{The variance is } \text{Var}(X) &= \sigma^2 \\ &= E[(X - \mu)^2] \\ &= \sum (x_i - \mu)^2 p_i \\ &= \sum x_i^2 p_i - \mu^2 \\ &= E(X^2) - \mu^2 \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

The **standard deviation** is $\sigma(X) = \sqrt{\text{Var}(X)}$.

$$E(aX + b) = aE(X) + b \text{ and } \text{Var}(aX + b) = a^2 \text{Var}(X).$$

THE BINOMIAL DISTRIBUTION

In a **binomial experiment** there are two possible results: success and failure.

Suppose there are n independent trials of the same experiment with the probability of success being a constant p for each trial. If X represents the number of successes in the n trials, then X has a **binomial distribution**, and we write $X \sim B(n, p)$.

The **binomial probability mass function** is $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$ where $x = 0, 1, 2, \dots, n$.

You should be able to use your calculator to find:

- $P(X = x)$ using the binomial probability distribution function
- $P(X \leq x)$ or $P(X \geq x)$ using the binomial cumulative distribution function.

If $X \sim B(n, p)$, then:

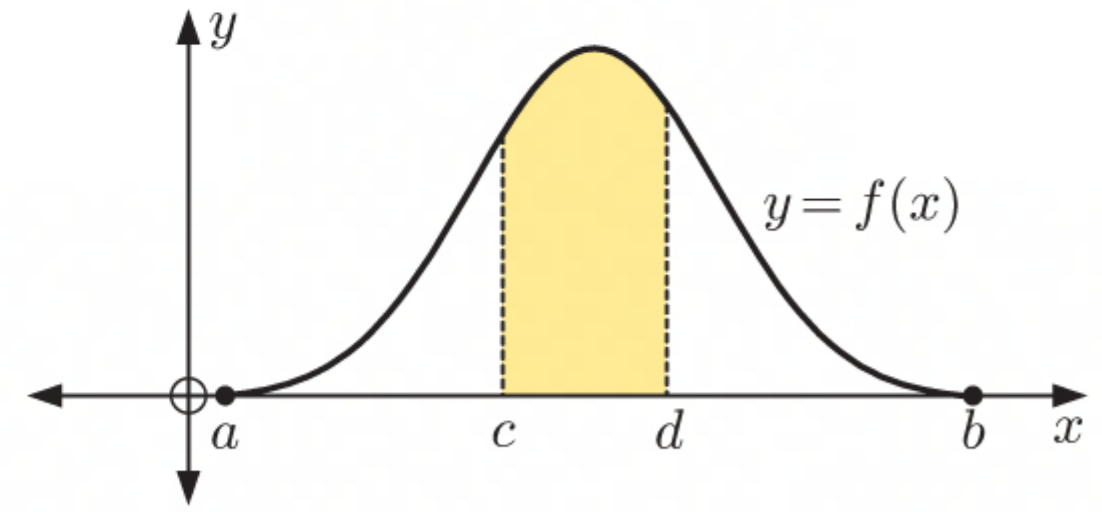
- $E(X) = \mu = np$
- $\text{Var}(X) = np(1 - p)$
- $\sigma = \sqrt{\text{Var}(X)} = \sqrt{np(1 - p)}$

CONTINUOUS RANDOM VARIABLES

A **continuous random variable** may have any possible *measured* value. We consider measurements that are in a particular *interval* rather than an exact value.

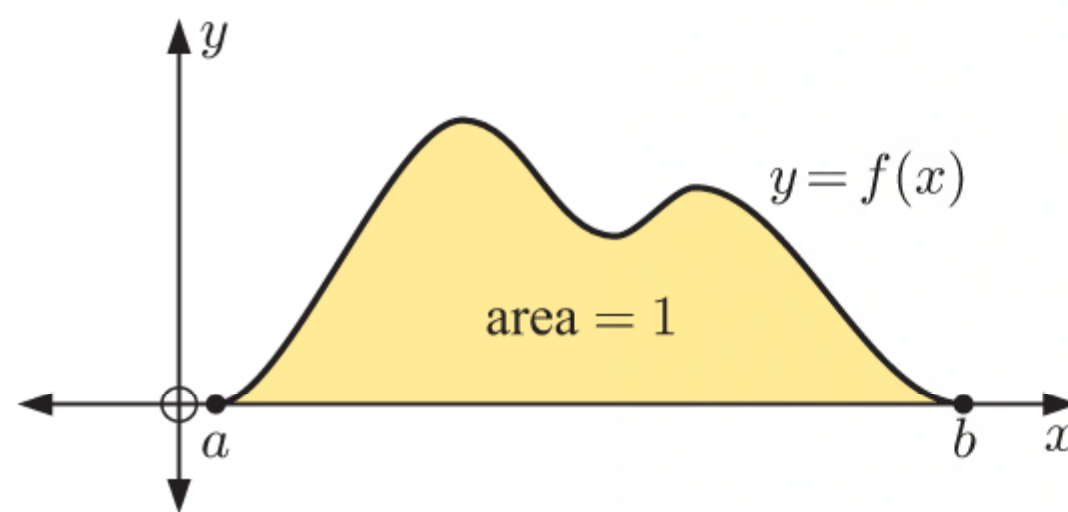
If X is a continuous random variable on the domain $a \leq x \leq b$, the **probability density function** is a function $f(x)$ such that

$$\begin{aligned} P(c \leq X \leq d) &= \int_c^d f(x) dx \\ &= \text{the area under } y = f(x) \text{ between } x = c \text{ and } x = d. \end{aligned}$$



For $f(x)$ to be a valid probability density function for X on the domain $a \leq x \leq b$, it must satisfy:

- $f(x) \geq 0$ for all $a \leq x \leq b$
- $\int_a^b f(x) dx = 1$



The **mean** $\mu = E(X) = \int_a^b x f(x) dx$.

The **mode** is the value of x on $a \leq x \leq b$ which maximises $f(x)$.

The **median** is the value of m such that $\int_a^m f(x) dx = \frac{1}{2}$.

The **variance** $\sigma^2 = \text{Var}(X) = \int_a^b (x - \mu)^2 f(x) dx$
 $= \int_a^b x^2 f(x) dx - \mu^2$

The standard deviation $\sigma(X) = \sqrt{\text{Var}(X)}$.

THE NORMAL DISTRIBUTION

If the random variable X has a normal distribution with mean μ and variance σ^2 , we write $X \sim N(\mu, \sigma^2)$.

The probability density function is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $x \in \mathbb{R}$.

$f(x)$ is a bell-shaped curve which is symmetric about $x = \mu$.

It has the property that:

- $\approx 68\%$ of all scores lie between $\mu - \sigma$ and $\mu + \sigma$
- $\approx 95\%$ of all scores lie between $\mu - 2\sigma$ and $\mu + 2\sigma$
- $\approx 99.7\%$ of all scores lie between $\mu - 3\sigma$ and $\mu + 3\sigma$.

You should be able to use your calculator to find normal probabilities for the situations:

- $P(X \leq a)$
- $P(X \geq a)$
- $P(a \leq X \leq b)$

You should also be able to use your calculator to find the scores corresponding to particular probabilities. These scores are known as **quantiles**.

The standard normal distribution

Every normal X -distribution can be transformed into the **standard normal distribution** or **Z-distribution** using the transformation $Z = \frac{X - \mu}{\sigma}$.

The standard normal distribution has mean 0 and standard deviation 1, so $Z \sim N(0, 1^2)$.

We use Z -distributions when:

- we are looking for an unknown mean μ or variance σ^2
- we are comparing scores from two different normal distributions.

SKILL BUILDER QUESTIONS

- 1 Gerard wants to estimate the average height of the 500 students at his school. He randomly selects a sample of 10 students, and uses a tape measure to find the height of each student.

Explain why this approach may produce a:

- a** coverage error **b** measurement error.

- 2** The students at Hoylebury Middle School are to be surveyed on their attitudes on wearing school uniform. The numbers of students in each year level are shown.

	<i>Boys</i>	<i>Girls</i>
Year 8	135	140
Year 9	130	145
Year 10	125	130

- a**
 - i** What are the advantages of surveying 50 students?
 - ii** What are the disadvantages of surveying all students?

- b** A stratified sample system is used to select 50 students.

- i** How many Year 8 boys will be selected?

- ii** How many girls will be selected in total?

- c** Explain why a stratified sample is better than a random sample in this case.

- 3** Marie is organising a staff lunch in a large office building.

She asks the first 10 people to visit her office for their preferences, and then makes a decision.

- a** Explain why this is a convenience sample.

- b** In what ways will Marie's sample be biased?

- c** Suggest a more appropriate sampling method that Marie should use.

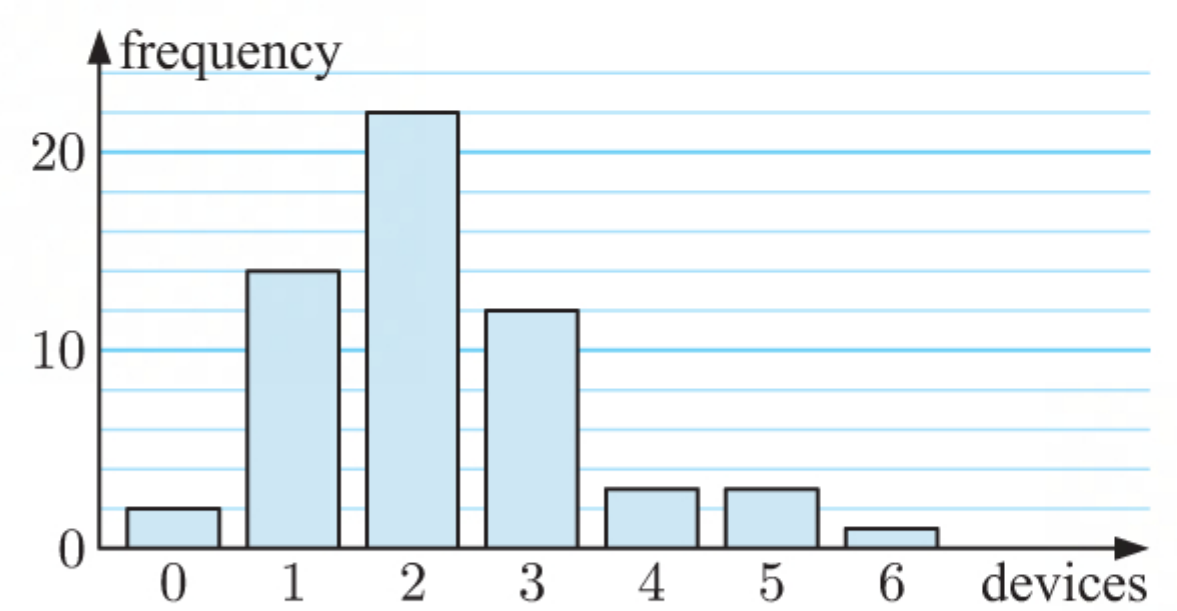
- 4** A ticket inspector checks the tickets of every 20th passenger leaving a train terminal, starting from the 8th passenger.

- a** Identify the sampling method used. Explain your answer.

- b** List the next six passengers to be checked.

- c** Given that 5000 passengers left the terminal that day, find the number of passengers checked.

- 5** A random sample of people were asked “How many devices have you used to browse the internet in the last month?”. The results are displayed in the column graph.



- a** How many people were surveyed?

- b** Find the mode of the data.

- What percentage of people browsed the internet using 1 or 2 devices?

- d** Describe the distribution of the data.

- 6 a** The mean of 7 integers is 14. In ascending order, the integers are 9, 10, a , 13, b , 16, 21.

Find the values of a and b .

- b** In ascending order, a set of six numbers are: 1, 5, 9, 11, 16, p . The mean of the six numbers is the same as their median. Find p .

- 7** Miguel uses an application on his phone to find the amount of sleep he gets each night. The duration of his sleep, in hours, for the past 30 nights are:

7.5	6.8	7.8	6.3	8.6	9.1	7.1	5.8	7.7	7.3	7.7	7.4	11.5	7.1	7.4
8.0	7.6	7.1	9.1	8.0	7.5	7.4	7.5	8.1	8.6	8.7	6.8	7.4	7.7	8.5

- a** Calculate the mean and the median of the data.

- b** Identify the outlier in this data set.

- c** The outlier was the result of a recording error.

- i** Calculate the mean and the median of the data with the outlier removed.

- ii** Which measure of centre is most affected if the outlier is removed?

- 8 The frequency table alongside shows the number of cars owned by different families.
- a Add a column to the table showing the *cumulative frequency* values.
 - b For this data set, calculate the:
 - i mean
 - ii median
 - iii mode.

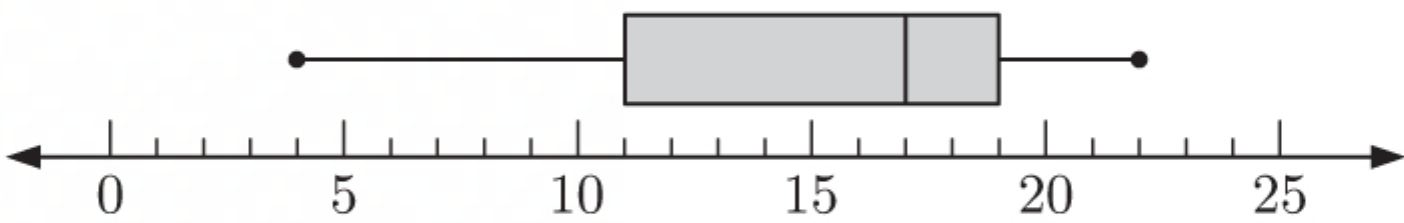
Number of cars	Frequency
0	78
1	117
2	69
3	18
4	2
Total	284

- 9 This table shows the weekly rent for a sample of studio apartments in Italy.
- a Estimate the mean weekly rent.
 - b Find the probability that the weekly rent for a randomly chosen studio apartment will be €140 or greater.

Weekly rent (€r)	Frequency
$80 \leq r < 100$	3
$100 \leq r < 120$	15
$120 \leq r < 140$	26
$140 \leq r < 160$	30
$160 \leq r < 180$	14
$180 \leq r < 200$	1

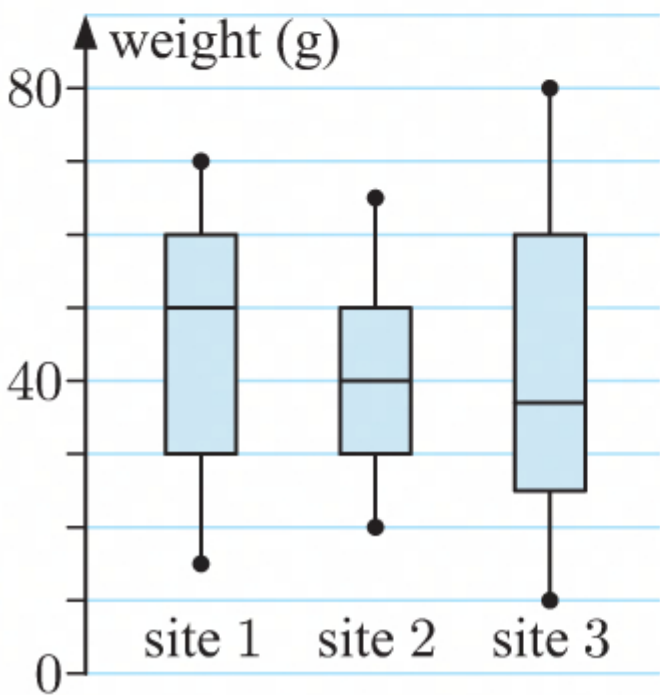
- 10 Consider this data set: 16, 20, 10, 16, 4, 12, 23, 18, 17, 9, 18, 16, 31, 26, 18, 14, 12, 14, 15
- a Write the data set in order, and construct a five-number summary.
 - b Calculate the interquartile range.
 - c Calculate the upper and lower boundaries, and hence identify any outliers in the data set.
 - d Draw a box plot to represent the data.

- 11 A box plot has been drawn to show the heights of some petunia seedlings, in centimetres.



State the:

- a minimum value
 - b maximum value
 - c median
 - d upper quartile
 - e lower quartile
 - f range
 - g interquartile range.
- 12 These parallel box plots show the weights of particular species of fungi collected from 3 different sites in a forest.
- a Write down the five-number summary for site 1.
 - b Which site has the greatest range of weights?
 - c At which site do the weights of fungi have the least variation?
 - d Which site has the highest median weight of fungi?
 - e Which site has the highest proportion of weights above 40 grams?



- 13 The heights of a random sample of trees in an apple orchard are summarised in the table alongside.
- a Construct a cumulative frequency graph for the data.
 - b Estimate the median height.
 - c Estimate the interquartile range.
 - d Estimate the 90th percentile. Interpret your answer.

Height (h m)	Frequency
$7 \leq h < 8$	8
$8 \leq h < 9$	59
$9 \leq h < 10$	74
$10 \leq h < 11$	22
$11 \leq h < 12$	1

- 14 Anthony and Katherine are two musicians in an orchestra. They each recorded the number of hours they spent practising in the 10 days before a performance.

Anthony: $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 4, $4\frac{1}{2}$, 3, $3\frac{1}{2}$, 5, 6, 6

Katherine: 3, $3\frac{1}{2}$, 4, 3, 3, $3\frac{1}{2}$, 4, 4, $4\frac{1}{2}$, 4

- a Calculate the mean and standard deviation of each data set.
- b Which person generally practised for longer?
- c Which person practised more consistently?

- 15** This table shows the distribution of marks obtained on a logic test.
Use technology to find the mean and population standard deviation of the test scores.

Mark	3	4	5	6	7	8	9	10
Frequency	1	3	5	8	4	2	0	1

- 16** A journalist compares the scores given to two camera models by 6 online reviewers.

Camera A	8.5	8	9	7	8.5	7.5
Camera B	7	6	7.5	9	7.5	6

- Draw a scatter diagram of the data.
 - Identify the outlier in the data.
 - It was found that the outlier was a recording error, and was removed.
 - Describe the correlation between camera A's scores and camera B's scores.
 - Does an increase in camera A's scores cause an increase in camera B's scores? Explain your answer.
- 17** Ten students were given aptitude tests on language skills and mathematics. The table below shows the results:

Language (x)	12.5	15.0	10.5	12.0	9.5	10.5	15.5	10.0	14.0	12.0
Mathematics (y)	32	45	27	38	18	25	35	22	40	40

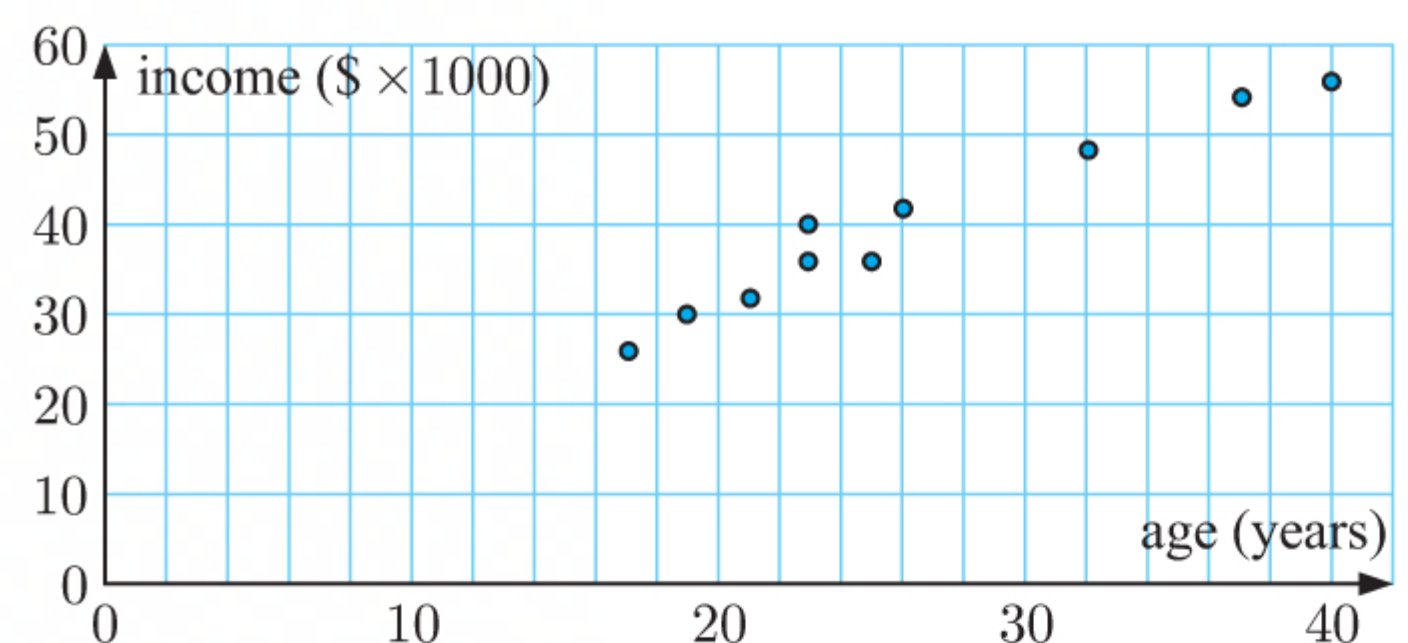
- Plot the data on a scatter diagram.
 - Find the correlation coefficient r .
 - Use your results to comment on the statement: "Those who do well in languages also do well in mathematics."
- 18** This scatter diagram shows the age and annual income of 10 randomly chosen individuals. The mean age is 27 and the mean income is \$40 000.
- Describe the relationship between the age and annual income for these individuals.
 - Do you think there is a causal relationship between the variables? Explain your answer.
 - Draw a line of best fit by eye on the graph.
 - Estimate the annual income for someone who is 30 years old. Comment on the reliability of your estimate.
- 19** A jeweller measured the volume and mass of some samples of silver. He suspects that one of the samples might be fake. The results are listed in the table.

Sample	A	B	C	D	E	F	G	H	I	J	K	L
Volume ($x \text{ cm}^3$)	3	6	4	7	16	8	5	12	9	6	10	11
Mass ($y \text{ g}$)	40	95	50	160	285	130	65	210	155	90	170	190

- Draw a scatter diagram for this data.
 - Calculate Pearson's product-moment correlation coefficient r .
 - Describe the relationship which appears to exist between the volume and mass of the samples of silver.
 - Do you agree with the jeweller that there is a fake sample?
 - Remove the suspect value from the data and find the equation of the regression line for the remaining data.
 - Use your equation to find the expected mass of the sample of silver with the same volume as the suspect sample.
- 20** 9 students sat a Mathematics examination. The number of hours that each of them studied and the results they obtained are shown in the table.

Study time ($x \text{ h}$)	7	6	3	16	15	11	18	32	20
Result ($y \%$)	56	42	25	80	65	60	85	96	90

- Write down the equation of the least squares regression line.
- Describe the correlation between the variables.
- Do you think there is a causal relationship between the variables? Explain your answer.
- Tony's score in the examination was 70%. Use the line of best fit to estimate how long he studied for.
- Interpret the y -intercept and the gradient of the equation of the line of best fit.



21 The average height h (in mm) of grass t days after being mowed, is shown in the table below.

Time (t days)	0	1	2	3	4	5	6	7	8	9
Height (h mm)	5	5.7	5.7	6.2	6.8	7.1	8	8.3	9	9.3

- a Calculate Pearson’s product-moment correlation coefficient r .

b Explain the significance of the size and sign of r .

c The regression line for h against t is $h \approx 0.4879t + 4.9145$. Use this equation to estimate the:

i height of the grass after 14 days

ii time required for the grass height to reach 20 mm.
- 22 A group of friends spend their holiday at a beach. Each day, the friends head out to sea for some fishing. Their distance from the shore and the number of fish caught each day is shown in the table.

Distance from shore (x km)	3.7	1.3	4.3	2.8	0.9
Fish caught (y)	5	4	9	5	2

- a Which regression line should be used to model the relationship between the variables? Explain your answer.

b Use an appropriate regression line to estimate the number of fish caught if the distance from shore is 7 km.

c Comment on the reliability of your estimate.

23 An annual squash tournament groups players into 5 divisions according to their skill level.

The table shows the number of players at the tournament over 3 years.

Find the probability that a player:

- a in the 2017 tournament played in division 1

b in any of the past tournaments played in division 3

c in the 2019 tournament did *not* play in division 2 or 4.

Division	2017	2018	2019
1	4	5	5
2	6	7	8
3	13	12	14
4	18	10	14
5	20	17	16
Total	61	51	57

24 A hospital recorded the age and gender of its 1020 melanoma patients over one year. The data is shown alongside.

- a Complete the table.

b Find the probability that a randomly selected melanoma patient was:

i male

ii female and younger than 40

iii 60 or older, given they were female

iv male, given they were 40 or older.

	< 40	40 - 59	≥ 60	Total
Male	56	127		
Female	75	113	230	
Total				1020

25 A die is rolled, and a square spinner with sectors 1, 2, 3, and 4 is spun.

- a Draw a grid to illustrate the sample space of possible outcomes.

b Use your grid to find the probability of getting:

i two 1s

ii two 5s

iii a sum of 6

iv a 2 and a 3

v a 2 or a 3 (or both)

vi exactly one 4.

26 Suppose $P(A) = 0.37$, $P(B) = 0.41$, and $P(A \cup B) = 0.78$.

- a Find $P(A \cap B)$.

b What can you say about A and B ?
- 27 Given that $P(A) = \frac{23}{50}$, $P(B) = \frac{5}{7}$, and $P((A \cup B)') = \frac{1}{12}$, find $P(A \cap B)$.

28 One ball is drawn from each of the boxes shown.

- a Draw a tree diagram to illustrate the situation.

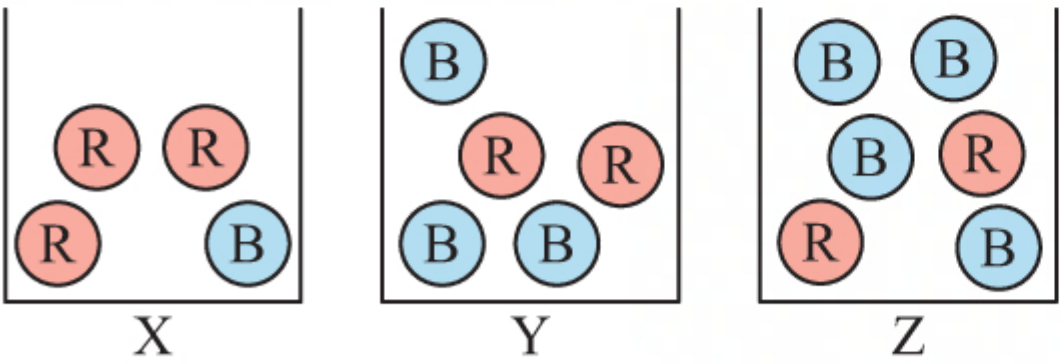
b Find the probability that:

i exactly two red balls are drawn

ii blue balls are drawn from boxes X and Z

iii at most one blue ball is drawn.

c Suppose an extra red ball is added to box Y. Which of the probabilities in **b** will be affected?

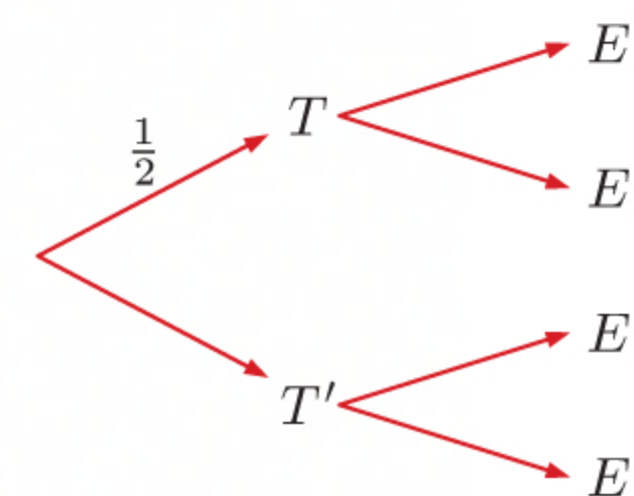


- 29** Suppose you toss a coin and roll a die simultaneously.

Let T represent a tail with the coin and E represent a 2 or a 5 with the die.

- a** Complete the tree diagram showing the probabilities of the different outcomes.

- b** Find: **i** $P(T \cap E')$ **ii** $P(T \cup E')$



- 30** Events A and B are independent. Given that $P(A \cup B) = 0.63$ and $P(B) = 0.36$, find $P(A)$.
- 31** A box of chocolates contains 6 dark brown, 4 light brown, and 2 white truffles. Two truffles are selected from the box without replacement.
Find the probability of selecting:
a 2 white truffles **b** different coloured truffles.
- 32** A box contains 4 blue balls and n red balls. When two balls are drawn from the box without replacement, the probability that both are red is $\frac{1}{3}$. Find n .
- 33** 40% of students in a class own an orange highlighter, 20% own a blue highlighter, and 50% do not own either coloured highlighter.
a Draw a Venn diagram to describe the situation.
b Find the probability that a randomly selected student:
i owns a blue highlighter, given they own an orange highlighter
ii owns an orange highlighter, given they do not own a blue highlighter.
- 34** Suppose $P(X) = \frac{3}{7}$, $P(Y) = \frac{2}{9}$, and $P(X \cup Y) = \frac{3}{5}$.
a Find:
i $P(X \cap Y)$ **ii** $P(X | Y)$ **iii** $P(Y | X)$
b Are X and Y independent events? Explain your answer.
- 35** Suppose $P(A \cap B) = 0.2$ and $P(A' \cap B) = 0.3$. Given that A' and B are independent, find $P(A' \cup B)$.
- 36** **a** If 3 coins are tossed, find the probability that two fall heads and the other falls tails.
b Suppose 3 coins are tossed 400 times. On how many occasions would you expect to see exactly one tail?
- 37** **a** Tickets in a raffle are numbered 1 to 100. A ticket is drawn at random. A is the event that a ticket with a number *less than* 45 is drawn, and B is the event that a ticket with a number *between* 40 and 55 is drawn.
i Are A and B mutually exclusive events? Explain your answer.
ii Find $P(A \cup B)$.
b Suppose $P(A)$ and $P(B)$ are both non-zero. Explain why events A and B cannot be both independent and mutually exclusive at the same time.
- 38** A national cricket team plays 30% of their matches in their home country.
When playing at home, the team wins 40%, draws 35%, and loses 25% of their matches. When playing away from home, the team wins 15% and loses 45% of their matches.
Given that the team did not win their last match, find the probability that the match was played in their home country.
- 39** Two fair dice are rolled. Let X be the difference between the numbers rolled.
a Explain why X is a discrete random variable. **b** State the possible values of X .
c Find $P(X = 3)$.
- 40** Find k for the following probability mass functions:
a $P(x) = k(x + 3)$ for $x = 0, 1, 2, 3, 4$ **b** $P(x) = \frac{k^{x-3}}{x-1}$ for $x = 3, 4, 5$
- 41** A discrete random variable X has probability mass function $P(x) = \frac{a}{(x-3)^2}$ for $x = 0, 1, 2$.
a Find a . **b** Find $P(X = 2)$. **c** Find the mode and median of the distribution.

- 42** A bag contains 3 red tickets and 2 blue tickets. Tickets are selected from the bag, without replacement, until at least one ticket of each colour is selected. Let X be the total number of tickets selected.

- a** State the possible values of X . **b** Find the probability distribution of X .
c Find the mode of X . **d** Find the expected value of X .

- 43** For the probability distribution alongside, find the:

- a** mean μ **b** mode
c variance σ^2 **d** standard deviation σ .

x	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

- 44** The table alongside shows the probability distribution for X .

If $E(X) = 1.55$, find m and n .

x	0	1	2	3
$P(X = x)$	0.3	0.2	m	n

- 45** A random variable X has the probability distribution alongside:

- a** Find k .
b Find $E(X)$, $\text{Var}(X)$, and the standard deviation σ .
c Find the mode and median of X .

x	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	k	$\frac{1}{12}$

- 46** A random variable X has the probability mass function $P(x) = \frac{x^2 + kx}{50}$ for $x = 1, 2, 3, 4$.

- a** Find k . **b** Find the mean of the distribution of X . **c** Find $P(X \geq 2)$.

- 47** A random variable X has probability mass function $P(x) = \frac{x^2}{29}$ for $x = 2, 3, 4$. For this distribution, find the:

- a** mode **b** median **c** mean
d variance **e** standard deviation.

- 48** A bag contains 1 blue ticket, 3 red tickets, and 8 yellow tickets. A player randomly selects a ticket from the bag, and receives \$40 for a blue ticket, \$20 for a red ticket, and \$5 for a yellow ticket.

- a** Calculate the expected return for one trial of this game.
b Given that the game costs \$15 to play, explain why it would not be advisable to play this game.
c Find the number of extra red tickets that should be added to the bag to make the game fair.

- 49** X has probability distribution:

x	1	2	3	4
$P(X = x)$	0.25	0.38	0.17	0.2

Find:

- a** $E(X)$ **b** $\text{Var}(X)$ **c** $\sigma(X)$
d $E(X + 2)$ **e** $\text{Var}(2 - 3X)$ **f** $\sigma(2X - 10)$

- 50** Six questions are asked in a weekly quiz show.

From past shows, the number of correctly answered questions X follows the probability distribution shown.

x	0	1	2	3	4	5	6
$P(X = x)$	0.02	0.02	0.08	0.38	0.35	0.12	0.03

- a** Find:

- i** $E(X)$ **ii** $\text{Var}(X)$ **iii** $\sigma(X)$

- b** Sara is a contestant on the quiz show. She begins the quiz with 20 points, and scores 3 points for each correct answer. Let Y be Sara's score after the quiz. Find:

- i** $E(Y)$ **ii** $\text{Var}(Y)$ **iii** $\sigma(Y)$

- 51** X is a random variable with mean 7 and standard deviation 2. Find $E(Y)$ and $\text{Var}(Y)$ for:

- a** $Y = 4X + 3$ **b** $Y = \frac{1}{2}(5 - X)$ **c** $Y = \frac{2X - 1}{3}$

- 52** 80% of residents in a particular suburb oppose the construction of traffic lights at a particular intersection. A survey of 20 randomly selected residents is conducted.

Find the probability that:

- a** exactly 16 residents oppose the construction **b** 16 or more residents oppose the construction
c between 10 and 15 residents oppose the construction **d** more than 8 residents support the construction.
- 53** 5% of all items coming off a production line are defective. The manufacturer packages the items in boxes of six, and guarantees a refund if more than two items in a box are defective.

- a** On what percentage of boxes will the manufacturer have to pay a refund?
b Patrick purchases 10 boxes. Find the probability that he will get a refund for exactly 1 box.

- 54** A hundred seeds are planted in ten rows of ten seeds per row. Assuming that each seed independently germinates with probability $\frac{1}{2}$, find the probability that the row with the maximum number of germinations contains at least 8 seedlings.

- 55** In a game, a player rolls a biased four-sided die. The probability of obtaining each possible score is shown in the table.

Score	1	2	3	4
Probability	$\frac{1}{12}$	k	$\frac{1}{4}$	$\frac{1}{3}$

- a** Find the value of k .
b Let the random variable X denote the number of 2s that occur when the die is rolled 2400 times. Calculate the exact mean and standard deviation of X .
- 56** A multiple choice test consists of 30 questions with 5 answers to choose from. For each question, only one choice is correct. Let Y be the number of correct answers chosen if each answer is randomly guessed.
- a** Find the mean μ and standard deviation σ of Y . **b** Find $P(Y = 20)$.
c Find $P(Y \geq \mu + 2\sigma)$.

- 57** The continuous random variable X has the probability density function $f(x) = ax^3 + x$, $0 \leq x \leq k$.
 Given that $P(X \leq \frac{1}{2}) = \frac{5}{32}$, find a and k .

- 58** A continuous random variable X has the probability density function $f(x) = \ln x$, $1 \leq x \leq k$.

- a** Find k .
b Find:
 i $E(X)$ **ii** $\text{Var}(X)$ **iii** $\sigma(X)$
c Suppose $Y = 3X - 2$. Find:
 i $E(Y)$ **ii** $\text{Var}(Y)$ **iii** $\sigma(Y)$

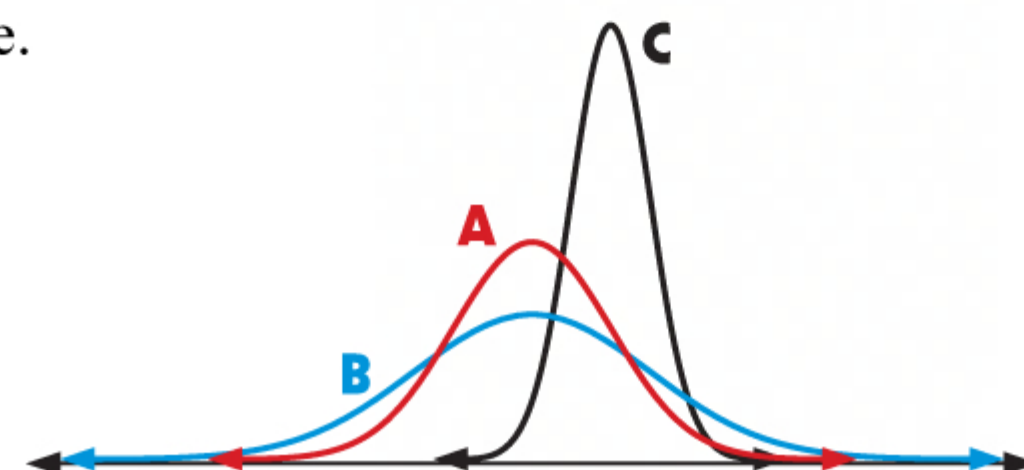
- 59** The continuous random variable X has probability density function $f(x) = \frac{1}{x}$, $a \leq x \leq b$, and mean 4.
a Find a and b . **b** Find the exact median of X .

- 60** A continuous random variable X has probability density function $f(x) = \begin{cases} a(x^2 + 2), & 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$

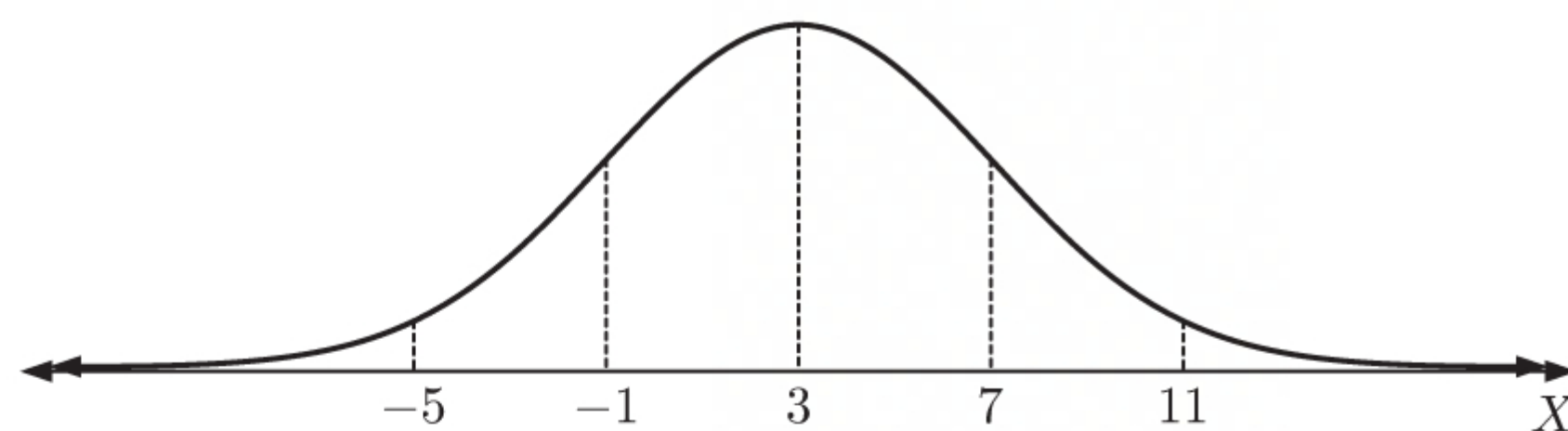
- a** Find the constant a .
b Determine:
 i $P(0.5 \leq X \leq 1.4)$ **ii** $P(X \geq 1)$.
c Find the:
 i median **ii** mean **iii** variance of X .

- 61** Suppose $X \sim N(\mu, \sigma^2)$. Match each pair of parameters with the correct curve.

- a** $\mu = 4, \sigma = 1$
b $\mu = 2, \sigma = 2$
c $\mu = 2, \sigma = 3$



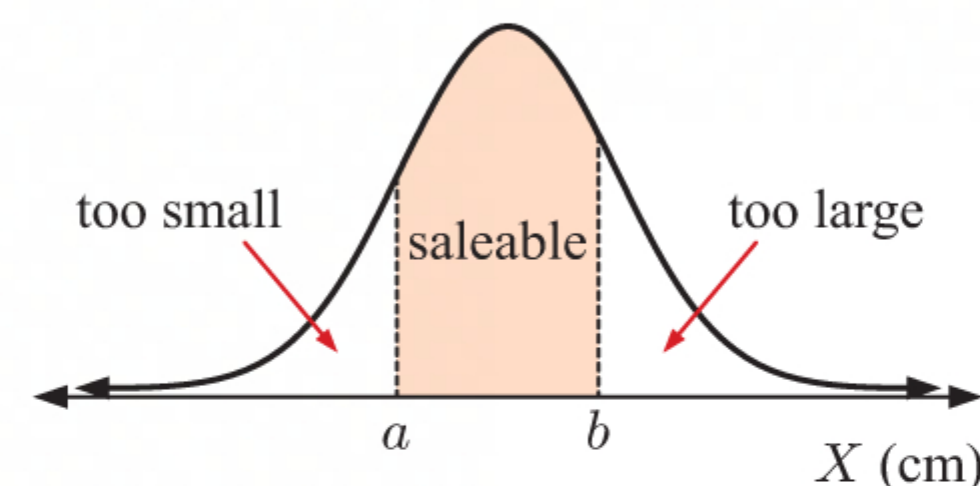
- 62** Consider the distribution curve of $X \sim N(3, 4^2)$ shown:



Copy the above graph, and on the same set of axes sketch the distribution curve for:

- a** $N(1, 4^2)$ **b** $N(3, 2^2)$ **c** $N(2, 64)$
- 63** Suppose a population is normally distributed with mean $\mu = 30$ and standard deviation $\sigma = 5$. Copy and complete:
- a** Approximately 68% of the population lies between and 35.
b Approximately 95% of the population lies between 20 and
c Approximately of the population lies between 15 and 45.
- 64** Containers of a particular brand of ice cream have a capacity of 1050 mL. They are advertised as containing 1 litre of ice cream. The quantity of ice cream added to each container is normally distributed with mean 1020 mL and standard deviation 17 mL.
- a** Find the probability that the container has less than the advertised capacity.
b Find the percentage of containers that overflow.
c A sample of 75 containers are taken. Find the probability that at most three of the containers overflow.
- 65** The volume of drink dispensed by a coffee machine is normally distributed with mean 254 mL and standard deviation 2.3 mL.
- a** Find the probability that a randomly selected drink from the machine will have volume less than 254 mL.
b Find the percentage of drinks dispensed by the machine which have volume between 252 mL and 256 mL.
c A sample of 80 drinks is taken from the machine. Determine the number of drinks which will be expected to have volume at least two standard deviations above the mean.
d The machine operator guarantees that at least 95% of drinks will have volume at least 250 mL.
i Is the guarantee valid?
ii A technician adjusts the machine so the standard deviation is now 2.5 mL. What effect does this have on the operator's guarantee?
- 66** A machine fills bottles with tomato sauce. Each bottle is filled independently of all other bottles. The volume of sauce in each bottle is normally distributed with mean 500 mL and standard deviation 2.5 mL. Bottles are deemed to require extra sauce if the machine delivers less than 495 mL.
- a** Calculate the probability that a randomly selected bottle requires extra sauce.
b From a sample of 200 bottles, calculate the probability that at least 8 bottles require extra sauce.
- 67** The time taken for a skier to complete a particular downhill run is normally distributed with mean 45 seconds and standard deviation 4 seconds.
- a** Find the probability that the skier completes:
i one downhill run in under 40 seconds **ii** two consecutive downhill runs in under 40 seconds each.
b The skier completes a total of 60 independent runs. How many times would you expect the run to take between 44 seconds and 47 seconds?
- 68** The mean birth weight of babies in a population is normally distributed with mean 3.4 kg and standard deviation 300 grams.
- a** What proportion of babies in this population have birth weights:
i in excess of 4 kg **ii** between 3 kg and 4 kg?
b A *low birth weight* corresponds to any newborn weighing in the lowest 10% of birth weights. State the weight below which a baby is classified as having a *low birth weight*.

- 69** The length of a zucchini is normally distributed with mean 24.3 cm and standard deviation 6.83 cm. A supermarket buying zucchinis in bulk finds that 15% of them are too small and 20% of them are too large for sale. The remainder, with lengths between a cm and b cm, are able to be sold.



- a** Find a and b .
 - b** A zucchini is chosen at random. Find the probability that:
 - i** it is of saleable length
 - ii** its length lies between 20 cm and 26 cm
 - iii** its length is less than 24.3 cm.
- 70** The lengths of adult fish of a certain species are normally distributed with mean 40 cm and standard deviation 5 cm.
- a** Find the probability that a randomly chosen adult fish of this species is:
 - i** longer than 45 cm
 - ii** between 35 cm and 50 cm long.
 - b** Determine the minimum length of the longest 10% of this species of fish.
 - c** A randomly selected fish is shorter than 48 cm. Find the probability that it is between 40 cm and 44 cm long.
- 71** The continuous random variable X is normally distributed with $P(X < 56) = 0.8$.
- a** How many standard deviations from the mean is a score of 56?
 - b** If the standard deviation of X is 4, find the mean of the distribution. Give your answer correct to one decimal place.
- 72** The lengths of steel rods cut by a machine are normally distributed with mean 13.8 cm. It is found that 1.5% of all rods are less than 13.2 cm long.
- a** Find the probability that a randomly selected rod has length between 13.2 cm and 13.8 cm.
 - b** Find the standard deviation of rod lengths produced by this machine.
- 73** Suppose X is normally distributed with $P(X \leq 24) = 0.035$ and $P(X \geq 33) = 0.262$. Find the mean and standard deviation of X correct to 3 significant figures.

TOPIC 5: CALCULUS

LIMITS

If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to a , we say that $f(x)$ has a **limit** of A as x approaches a , and we write $\lim_{x \rightarrow a} f(x) = A$.

The limit $\lim_{x \rightarrow a} f(x)$ **exists** and is equal to A if $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = A$.

We say that $f(x)$ **converges** to A as x approaches a .

RATES OF CHANGE

The **instantaneous rate of change** of a variable at a particular instant is given by the **gradient of the tangent** to the graph at that point.

$\frac{dy}{dx}$ gives the rate of change in y with respect to x .

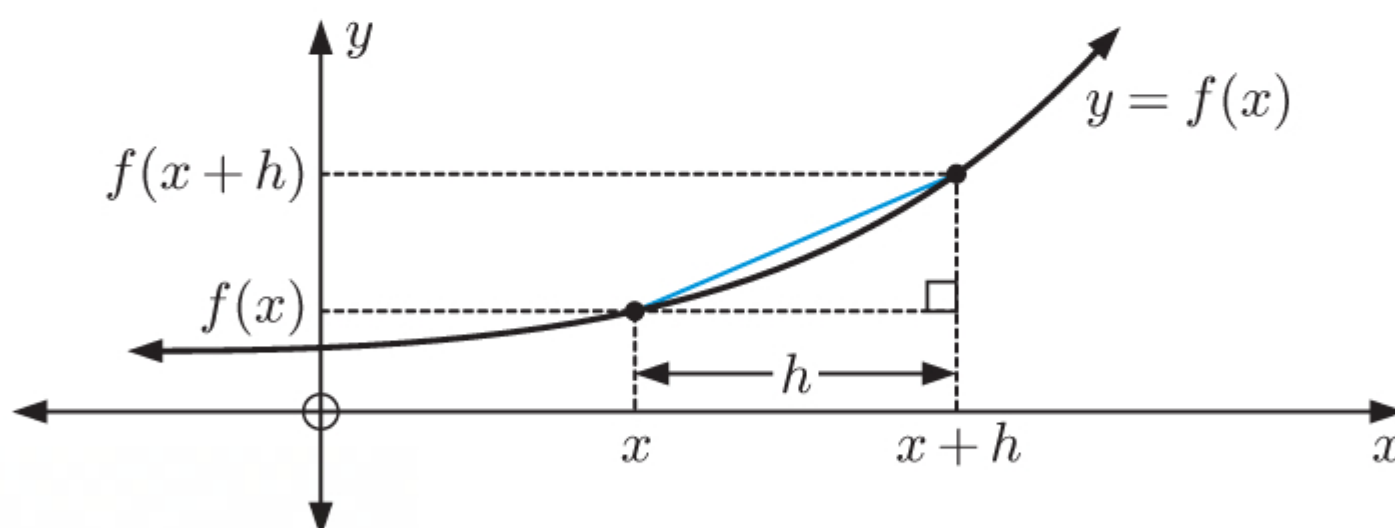
If $\frac{dy}{dx}$ is positive, then as x increases, y also increases.

If $\frac{dy}{dx}$ is negative, then as x increases, y decreases.

DIFFERENTIATION

The **gradient function** or **derivative function** $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ provides:

- the rate of change of f with respect to x
- the gradient of the tangent to $y = f(x)$ for any value of x .



When we use the limit definition to find a derivative, we call this the **method of first principles**.

CONTINUITY AND DIFFERENTIABILITY

- A function f is **continuous at** $x = a$ if $f(a)$ and $\lim_{x \rightarrow a} f(x)$ exist and are equal.
 f is also continuous at $x = a$ if:
 - ▶ a is the left endpoint of an interval on which f is defined, and $f(a)$ and $\lim_{x \rightarrow a^+} f(x)$ exist and are equal, or
 - ▶ a is the right endpoint of an interval on which f is defined, and $f(a)$ and $\lim_{x \rightarrow a^-} f(x)$ exist and are equal.
- A function f is **differentiable at** $x = a$ if:
 - ▶ f is continuous at $x = a$, and
 - ▶ $f'_-(a) = \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h}$ and $f'_+(a) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$ both exist and are equal.

RULES OF DIFFERENTIATION

$f(x)$	$f'(x)$	Name of rule
c	0	exponentials logarithms
x^n	nx^{n-1}	
$e^{f(x)}$	$e^{f(x)} f'(x)$	
$\ln f(x)$	$\frac{f'(x)}{f(x)}$	
$\sin x$	$\cos x$	trigonometric functions
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	
$\sec x$	$\sec x \tan x$	reciprocal trigonometric functions
$\cot x$	$-\operatorname{cosec}^2 x$	
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$	inverse trigonometric functions
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$	
$\arctan x$	$\frac{1}{1+x^2}$	

$f(x)$	$f'(x)$	Name of rule
$cu(x)$	$cu'(x)$	addition rule product rule quotient rule
$u(x) + v(x)$	$u'(x) + v'(x)$	
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$	

Chain rule

If $y = f(u)$ where $u = u(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

SECOND AND HIGHER DERIVATIVES

The second derivative of $y = f(x)$ is written $f''(x)$ or $\frac{d^2y}{dx^2}$.

The n th derivative of $y = f(x)$ is written $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$.

L'HÔPITAL'S RULE

Suppose $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ on an interval that contains the point $x = a$.

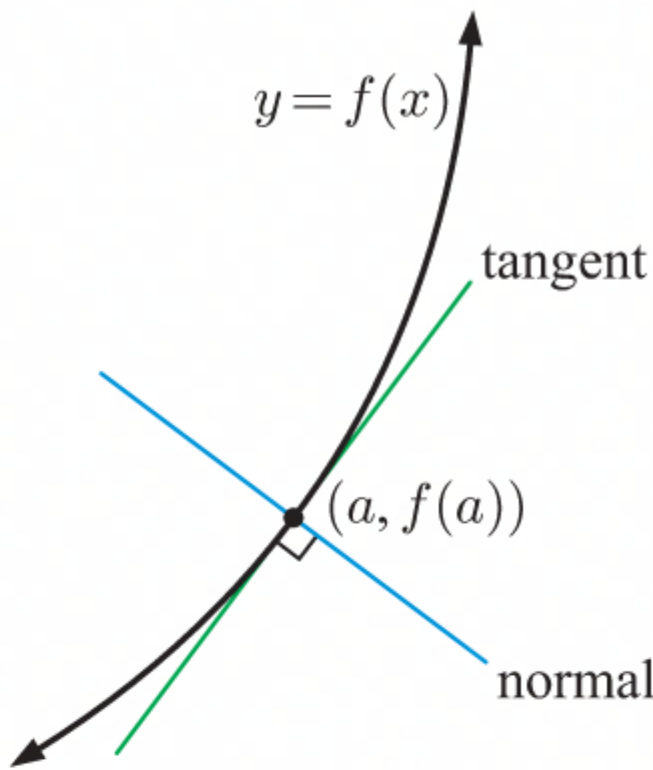
If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, or, if as $x \rightarrow a$, $f(x) \rightarrow \pm\infty$ and $g(x) \rightarrow \pm\infty$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ provided the limit on the right exists.

PROPERTIES OF CURVES

Tangents and normals

For the curve $y = f(x)$:

- The gradient of the tangent at $x = a$ is $f'(a)$.
- The equation of the tangent at $x = a$ is $y = f'(a)(x - a) + f(a)$.
- The gradient of the normal at $x = a$ is $-\frac{1}{f'(a)}$.
- The equation of the normal at $x = a$ is $y = -\frac{1}{f'(a)}(x - a) + f(a)$.



Increasing and decreasing functions

$f(x)$ is **increasing** on an interval $S \Leftrightarrow f(a) \leq f(b)$ for all $a, b \in S$ such that $a < b$.

$f(x)$ is **decreasing** on $S \Leftrightarrow f(a) \geq f(b)$ for all $a, b \in S$ such that $a < b$.

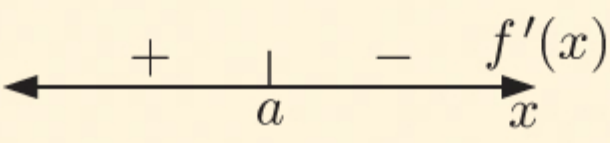
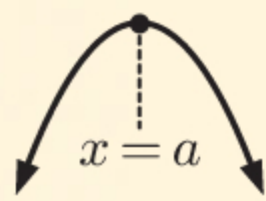
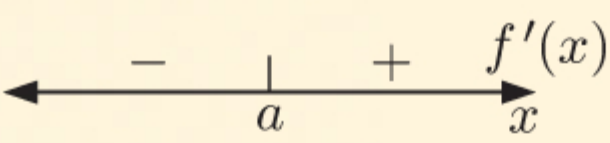
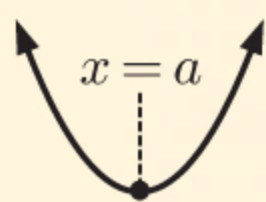
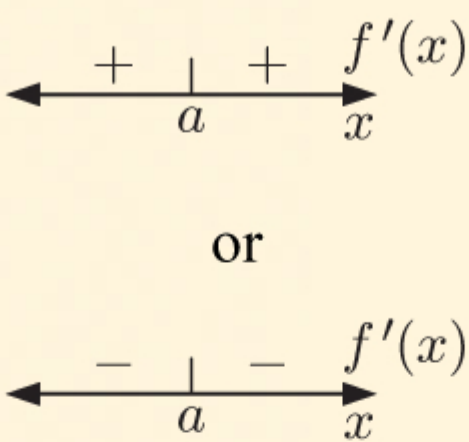
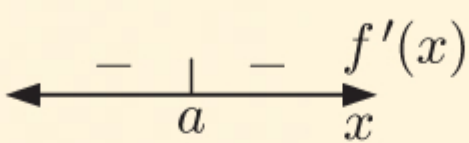
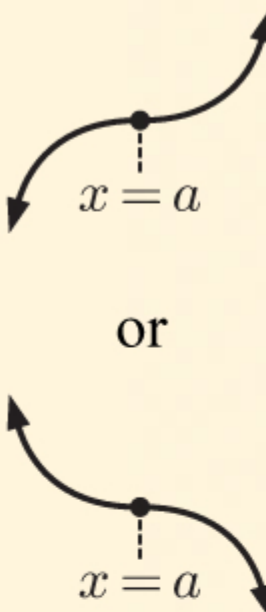
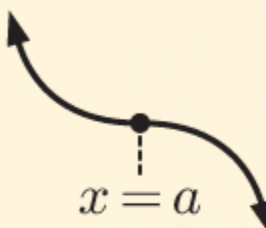
For most functions:

- $f(x)$ is increasing on $S \Leftrightarrow f'(x) \geq 0$ for all x in S .
- $f(x)$ is decreasing on $S \Leftrightarrow f'(x) \leq 0$ for all x in S .

Stationary points

A **stationary point** of a function is a point such that $f'(x) = 0$.

You should be able to identify and explain the significance of local and global maxima and minima, as well as stationary and non-stationary inflections.

Stationary point where $f'(a) = 0$	Sign diagram of $f'(x)$ near $x = a$	Shape of curve near $x = a$
local maximum		
local minimum		
stationary inflection	 or 	 or 

Shape

If $f''(x) \leq 0$ for all $x \in S$, the curve is **concave down** on the interval S .

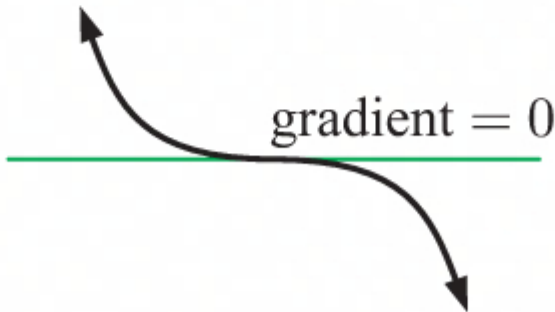


If $f''(x) \geq 0$ for all $x \in S$, the curve is **concave up** on the interval S .

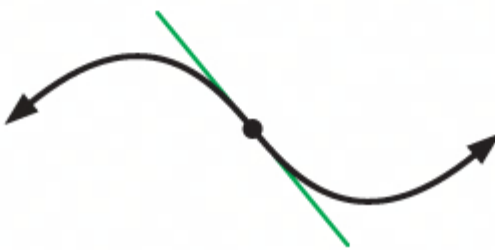
There is a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes on either side of $x = a$. It corresponds to a change in shape of the curve.



If $f'(a) = 0$, the point of inflection is a **stationary inflection**: the tangent at $x = a$ is horizontal.



If $f'(a) \neq 0$, the point of inflection is a **non-stationary inflection**: the tangent at $x = a$ is *not* horizontal.



OPTIMISATION PROBLEMS

It is important to remember that a local minimum or maximum does not always give the minimum or maximum value of a function in a particular domain. You must check for other turning points in the domain, and the values of the function at the end points of the domain.

Optimisation problem solving method

- Step 1:* Draw a large, clear diagram of the situation.
- Step 2:* Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example x . You should write down what domain restrictions there are on x .
- Step 3:* Find the **first derivative** and find the value(s) of x which make the first derivative **zero**.
- Step 4:* For each stationary point, use the **sign diagram test** or **second derivative test** to determine whether you have a local maximum or local minimum.
- Step 5:* Identify the optimal solution, also considering end points where appropriate.
- Step 6:* Write your answer in a sentence, making sure you specifically answer the question.

RELATED RATES

If the variables x and y are related, then $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are **related rates**.

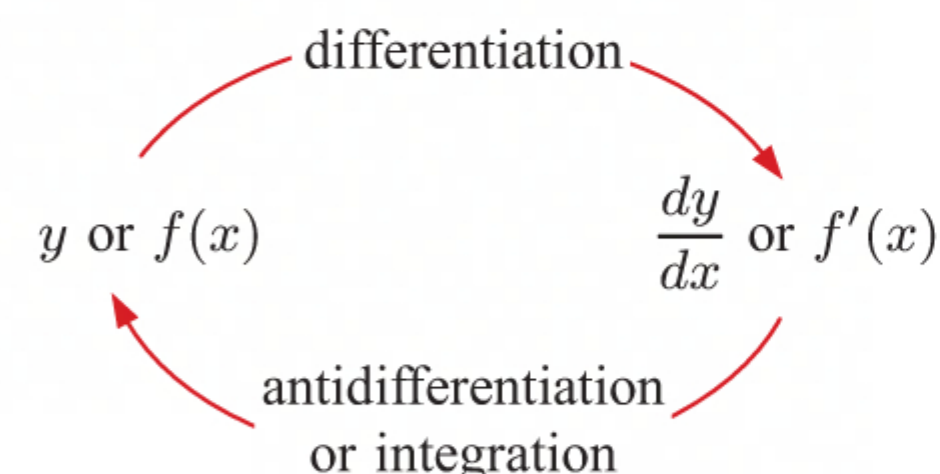
To solve problems involving related rates, we:

- Write an equation connecting the variables.
- Differentiate the equation with respect to time t .
- Substitute the values for the *particular case* corresponding to some instant in time, and solve to find the required unknown.

INTEGRATION

Antidifferentiation or **integration** is the reverse process of differentiation.

The **antiderivative** or **integral** of $f(x)$ is the simplest function $F(x)$ such that $F'(x) = f(x)$.



Techniques for integration

When integrating, we use the rules for differentiation in reverse. Do not forget to include the **constant of integration**.

Function	Integral
k	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c, \quad n \neq -1$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln x + c$
e^{ax+b}	$\frac{1}{a}e^{ax+b} + c, \quad a \neq 0$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c, \quad a \neq 0, \quad n \neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a} \ln ax+b , \quad a \neq 0$
$\cos(ax+b)$	$\frac{1}{a} \sin(ax+b) + c, \quad a \neq 0$
$\sin(ax+b)$	$-\frac{1}{a} \cos(ax+b) + c, \quad a \neq 0$
$\sec^2(ax+b)$	$\frac{1}{a} \tan(ax+b) + c, \quad a \neq 0$
a^x	$\frac{1}{\ln a} a^x + c, \quad a > 0, \quad a \neq 1$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) + c, \quad a \neq 0$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + c, \quad a \neq 0$

Trigonometric identities are often useful for integration, in particular:

- $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$
- $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$
- $\tan^2 x = \sec^2 x - 1$

Integration by substitution

$$\int f(u) \frac{du}{dx} dx = \int f(u) du$$

When using substitution to evaluate a definite integral, make sure you change the limits of integration to correspond to the new variable.

Integration by parts

$$\int uv' dx = uv - \int u'v dx$$

DEFINITE INTEGRALS

Fundamental Theorem of Calculus

For a continuous function $f(x)$ with antiderivative $F(x)$, $\int_a^b f(x) dx = F(b) - F(a)$.

Properties of definite integrals

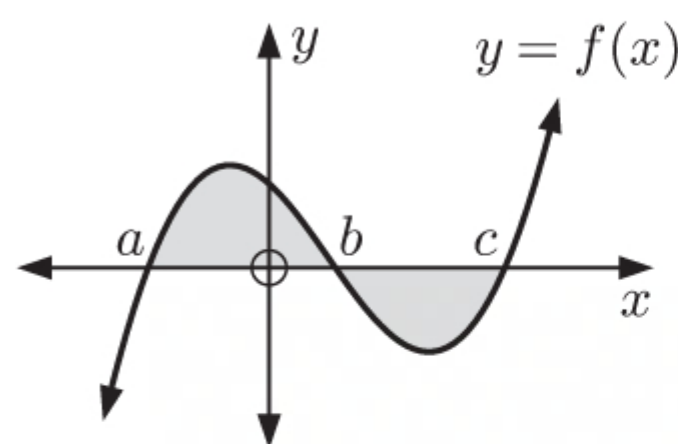
- $\int_a^a f(x) dx = 0$
- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_b^a f(x) dx = -\int_a^b f(x) dx$
- $\int_a^b cf(x) dx = c \int_a^b f(x) dx$

To find the total area enclosed by $y = f(x)$ and the x -axis between $x = a$ and $x = b$, we need to be careful about where $f(x) < 0$.

On an interval $c \leq x \leq d$ where $f(x) < 0$, the area is $-\int_c^d f(x) dx$.

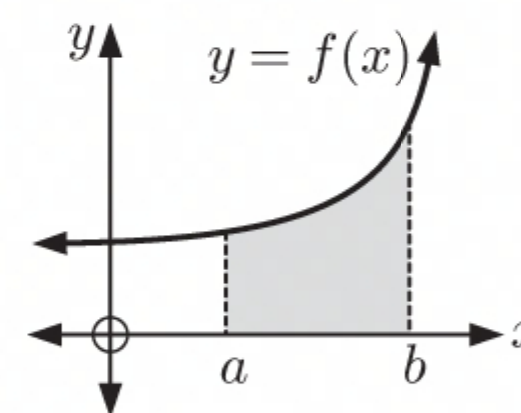
For example:

$$\begin{aligned} \text{The total shaded area} &= \int_a^b f(x) dx - \int_b^c f(x) dx \\ &\neq \int_a^c f(x) dx. \end{aligned}$$



Area under a curve

If $f(x)$ is a continuous *positive* function on the interval $a \leq x \leq b$, then $\int_a^b f(x) dx$ is the area under the curve between $x = a$ and $x = b$.



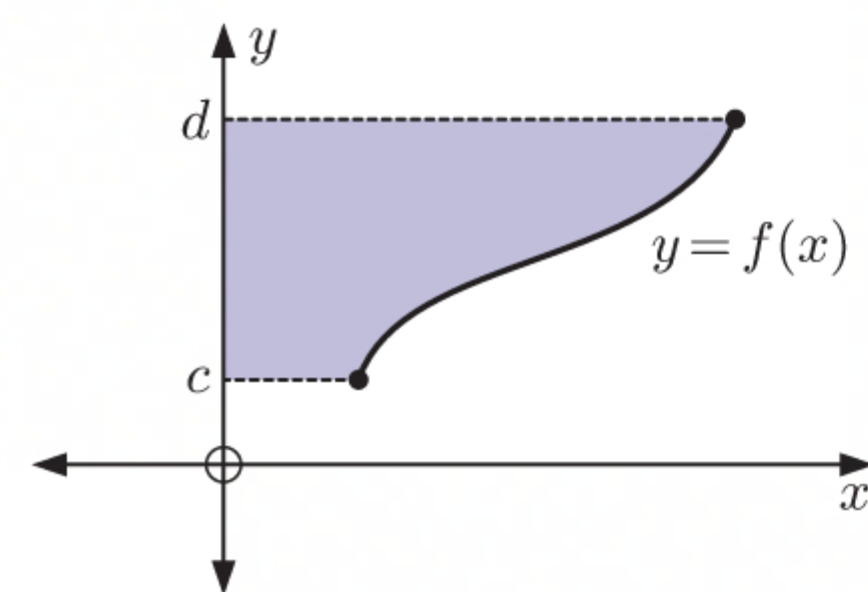
The area between two functions

The area *between* two functions $f(x)$ and $g(x)$ where $f(x) \geq g(x)$ on $a \leq x \leq b$ is given by $A = \int_a^b [f(x) - g(x)] dx$.

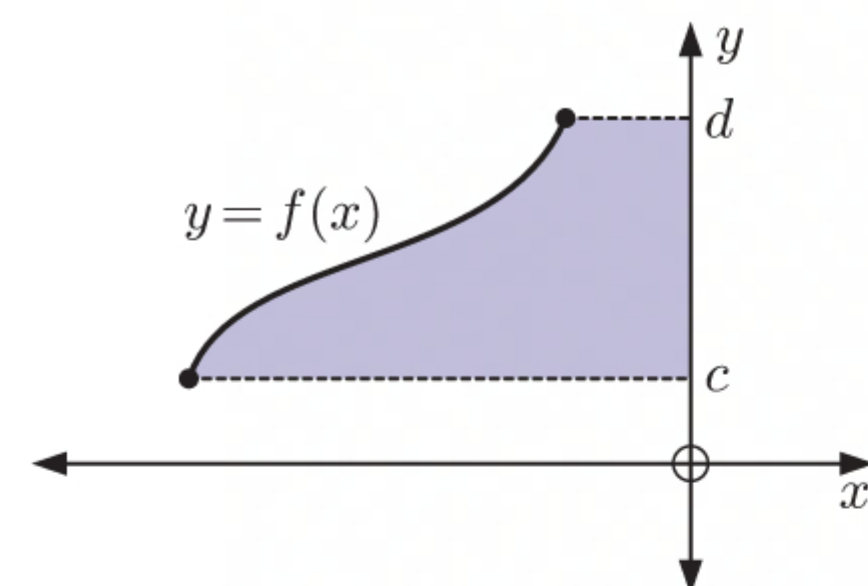
The area between a curve and the y -axis

Consider an invertible function $f(x)$.

- If $x = f^{-1}(y) > 0$ for $c \leq y \leq d$, shaded area = $\int_c^d f^{-1}(y) dy$



- If $x = f^{-1}(y) < 0$ for $c \leq y \leq d$, shaded area = $-\int_c^d f^{-1}(y) dy$

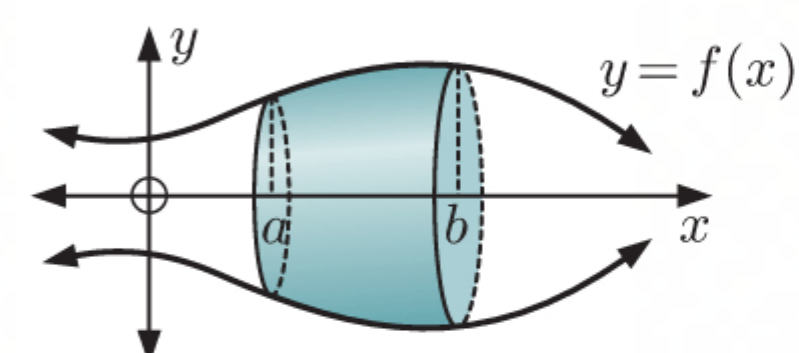


Solids of revolution

- When the region enclosed by $y = f(x)$, the x -axis, and the vertical lines $x = a$ and $x = b$ is revolved through 2π about the x -axis to generate a solid, the volume of the solid is given by

$$V = \pi \int_a^b [f(x)]^2 dx$$

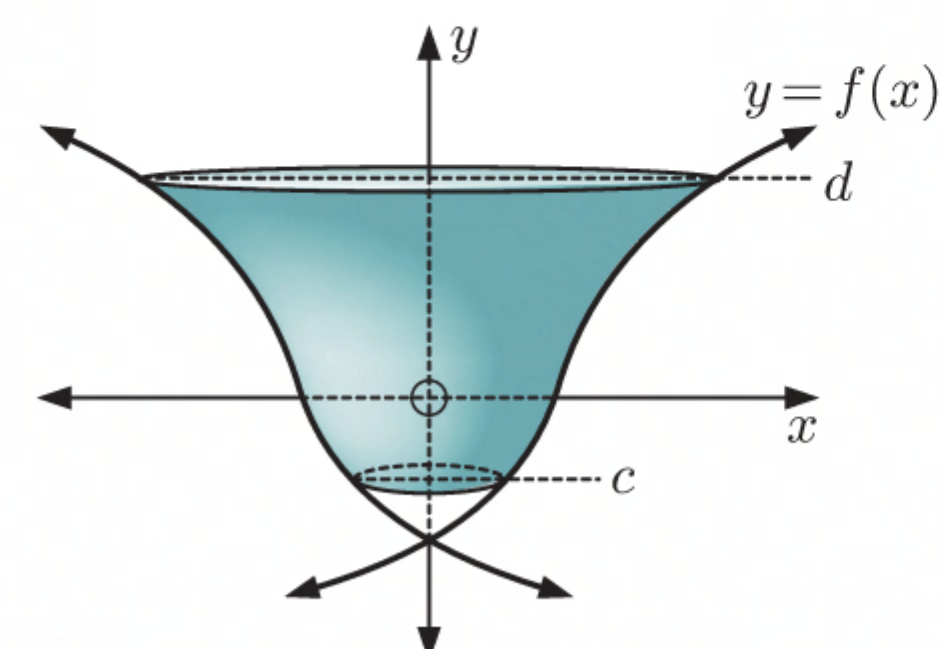
$$\text{or } \pi \int_a^b y^2 dx$$



- When the region enclosed by the invertible function $y = f(x)$, the y -axis, and the horizontal lines $y = c$ and $y = d$ is revolved through 2π about the y -axis to generate a solid, the volume of the solid is given by

$$V = \pi \int_c^d [f^{-1}(y)]^2 dy$$

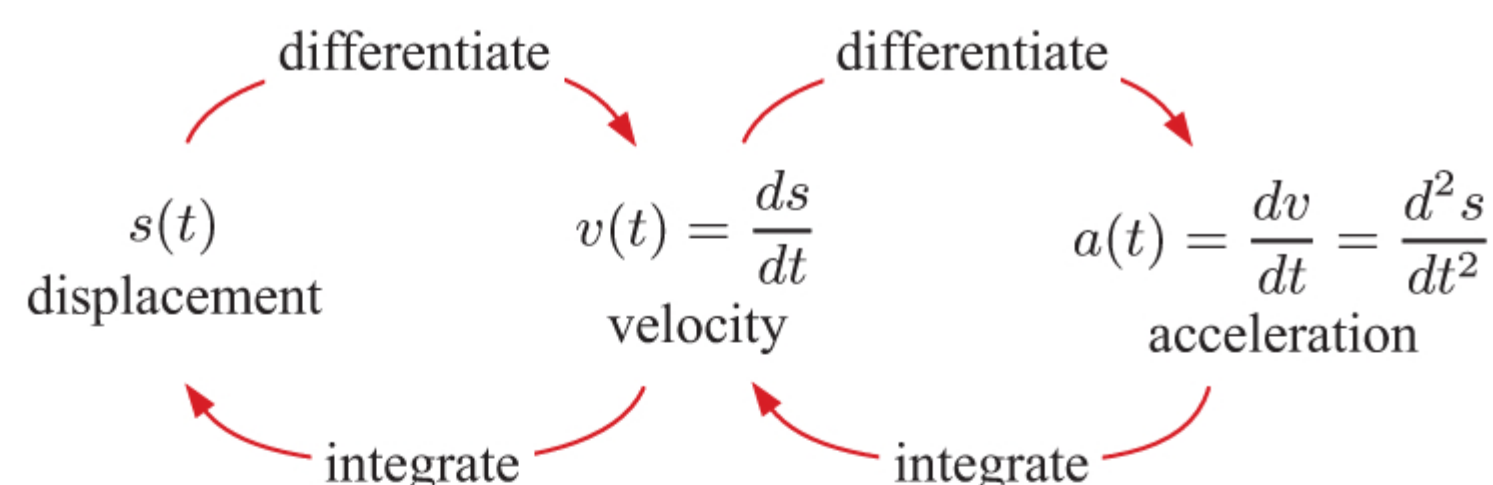
$$\text{or } \pi \int_c^d x^2 dy.$$



KINEMATICS

Suppose an object moves along a straight line.

Its position relative to the origin at time t is given by a displacement function $s(t)$. Its instantaneous velocity is given by $v(t) = s'(t)$, and its instantaneous acceleration by $a(t) = v'(t) = s''(t)$.



You should understand the physical meaning of the different signs of displacement, velocity, and acceleration.

Displacement:

$s(t)$	Interpretation
$= 0$	The object is at O
> 0	The object is to the right of O
< 0	The object is to the left of O

Velocity:

$v(t)$	Interpretation
$= 0$	The object is instantaneously at rest
> 0	The object is moving to the right
< 0	The object is moving to the left

Acceleration:

$a(t)$	Interpretation
> 0	The velocity of the object is increasing
< 0	The velocity of the object is decreasing
$= 0$	The velocity of the object may be at a maximum or a minimum

Speed

The **speed** at any instant is the magnitude of the object’s velocity. If $S(t)$ represents the speed then $S = |v|$.

If the signs of $v(t)$ and $a(t)$ are the same then the speed of the object is increasing.

If the signs of $v(t)$ and $a(t)$ are different then the speed of the object is decreasing.

Displacement and distance travelled

For the time interval $t_1 \leq t \leq t_2$:

- displacement $= s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t) \, dt$
- total distance travelled $= \int_{t_1}^{t_2} |v(t)| \, dt$.

DIFFERENTIAL EQUATIONS

A **differential equation** is an equation involving a derivative of a function.

Euler’s method allows us to approximate the solution curve to the differential equation $\frac{dy}{dx} = f(x, y)$ with particular solution passing through the point (x_0, y_0) .

At each stage we perform the iterative procedure $\begin{cases} x_i = x_{i-1} + h \\ y_i = y_{i-1} + hf(x_{i-1}, y_{i-1}) \end{cases}$ where h is the step size.

To approximate $y(x_n)$ where $x_n = x_0 + nh$, we perform the procedure n times.

Separable differential equations are differential equations of the form $\frac{dy}{dx} = f(x)g(y)$.

To solve these equations, we rearrange the equation and integrate both sides with respect to x to obtain the form $\int \frac{1}{g(y)} \, dy = \int f(x) \, dx$. The variables are now separated, so we can find the two integrals separately and solve the equation.

Homogeneous differential equations are differential equations of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

These equations can be solved using the substitution $y = vx$ where v is a function of x . This substitution will reduce the differential equation to a separable differential equation.

The **integrating factor method** is used to solve differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$. We:

- Calculate the **integrating factor** $I(x) = e^{\int P(x) dx}$, omitting the constant of integration.
- Multiply both sides of the differential equation by $I(x)$.
- Simplify the LHS and hence obtain $I(x)y = \int I(x)Q(x) dx + c$, where c is a constant.
- Integrate to obtain the general solution.

MACLAURIN SERIES

Consider a continuous function f which can be differentiated infinitely many times. The **Maclaurin series expansion** of $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

The **n th degree Maclaurin polynomial** approximation to f is $M_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)x^k}{k!}$.

Some important Maclaurin series are:

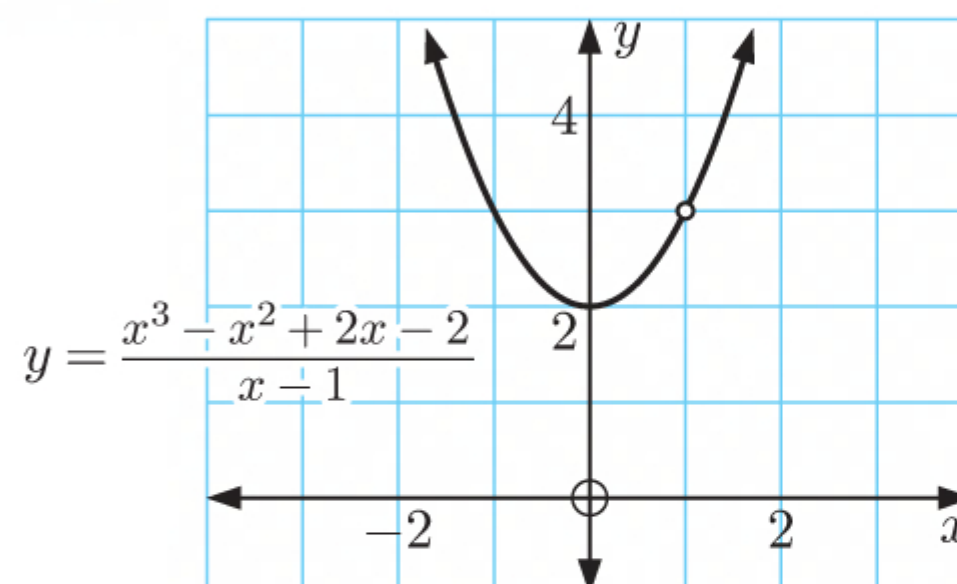
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$
- $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ for $|x| \leq 1$
- $\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ for $-1 < x \leq 1$
- $(1+x)^p = 1 + \sum_{k=1}^{\infty} \frac{p(p-1)\dots(p-k+1)}{k!} x^k$ for $p \in \mathbb{R}$, $|x| < 1$

By performing operations such as addition, subtraction, multiplication, division, composition, differentiation, and integration to these series, we can generate Maclaurin series for other functions.

SKILL BUILDER QUESTIONS

1 The graph of $f(x) = \frac{x^3 - x^2 + 2x - 2}{x - 1}$ is shown alongside.

- a Explain why the function is undefined at $x = 1$.
b Discuss the continuity of $f(x)$.



2 Consider the function $f(x) = \frac{x + 3}{x^2 + 3x}$.

- a For what values of x is $f(x)$ undefined?
b Find, if possible:

i $\lim_{x \rightarrow 0} f(x)$

ii $\lim_{x \rightarrow 1} f(x)$

iii $\lim_{x \rightarrow -3} f(x)$

3 Evaluate:

a $\lim_{x \rightarrow -1} (x^2 - 3x + 1)$

b $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 + 3x - 15}$

4 Evaluate:

a $\lim_{x \rightarrow \infty} \frac{4x - 1}{x + 3}$

b $\lim_{x \rightarrow \infty} \frac{2x + 3}{x^2 - x - 1}$

5 Find:

a $\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta}$

b $\lim_{\theta \rightarrow 0} \frac{\theta^2 - 3\theta}{\sin \theta}$

c $\lim_{x \rightarrow 0} \frac{3x + \sin x}{4x - \sin x}$

6 Let $f(x) = \frac{1}{x^2 + x - 6}$.

- a Sketch the graph of $y = f(x)$.

- b Discuss the continuity of $f(x)$.

7 Suppose $f(x) = \begin{cases} kx^2, & x \leq 1 \\ 2x + 3, & x > 1 \end{cases}$.

Find the value of k such that f is continuous on \mathbb{R} .

8 Find $\frac{dy}{dx}$ from first principles given:

a $y = x^2$

b $y = 3x + 5$

c $y = -3x^2 + x - 1$

9 Consider the function $f(x) = x^3 + 8x^2 + 5x + 3$.

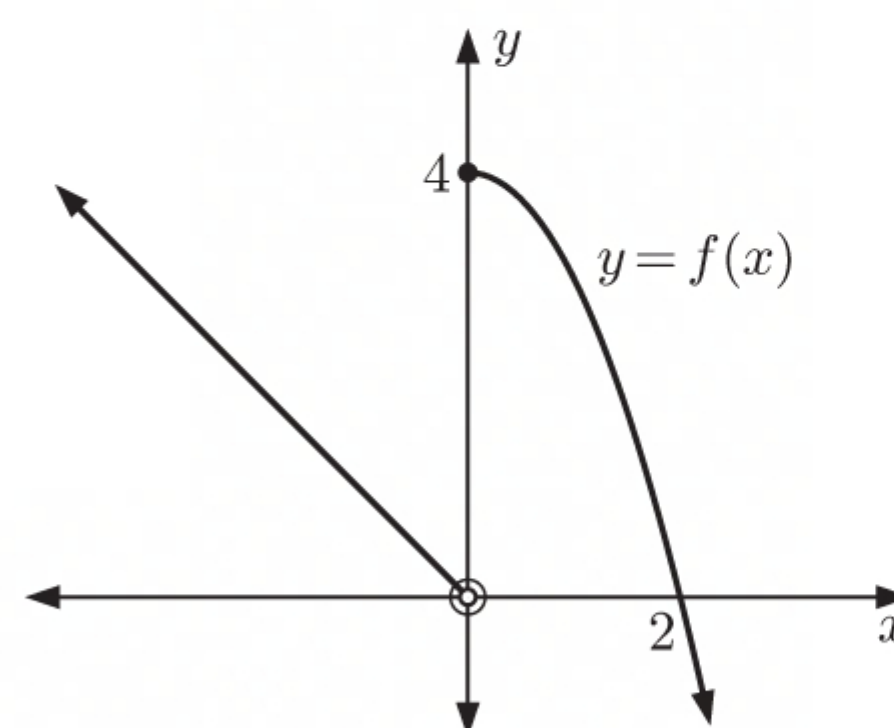
- a Find $f'(x)$ from first principles.

- b Hence find the point(s) on the graph at which the tangent has gradient: i 0 ii 17

10 Consider the graph of $y = f(x)$ alongside.

$f(x)$ is not continuous at $x = k$.

- a Find the value of k .
b Explain why $f(x)$ is not differentiable at $x = k$.



11 Find constants a and b such that:

a $f(x) = ax + \frac{b}{x^2}$, $f(1) = 8$, and $f'(1) = -7$

b $f(x) = ax^b$, $f(2) = \frac{32}{b}$, and $f'(1) = 8$.

12 Using the rules of differentiation, find $\frac{dy}{dx}$ if y is:

a $(x^2 - 3x)^5$

b $\frac{3}{(x^2 + 3)^3}$

c $\sqrt{x^2 - 3x}$

13 The gradient function of $f(x) = (ax + b)^c$ is $f'(x) = 81x^2 + 108x + 36$. Find the constants a , b , and c .

- 14** Find $\frac{dy}{dx}$ for:
a $y = x^2\sqrt{x^2 + 2x}$ **b** $y = \sqrt{x}(2x + 3)^4$ **c** $y = (2x + 1)^3(x - 5)^2$
- 15** Suppose $y = (x - 2)^2(2x - 1)$. For what values of x does $\frac{dy}{dx} = 36$?
- 16** Find the gradient of the tangent to:
a $y = \frac{x^3}{x^2 - 1}$ at $x = 2$ **b** $y = \frac{\sqrt{x}}{2x + 5}$ at $x = 4$
- 17** Find $f'(t)$ if:
a $f(t) = 20te^{-0.1t}$ **b** $f(t) = \frac{100}{1 + 7e^{-\frac{t}{4}}}$ **c** $f(t) = \frac{t + 9}{e^t}$
- 18** Given $f(x) = e^{ax+2} + x^2$ and $f(2) = f'(2)$, find a .
- 19** Find:
a $\frac{d}{dx}\left(\ln\left(\frac{x-4}{x^2+4}\right)\right)$ **b** $\frac{d}{dx}(\ln(x\sqrt{x^2+4}))$ **c** $\frac{d}{dx}\left(\ln\left(\frac{\sqrt{x^2+1}}{(x+3)(x-2)}\right)\right)$
- 20** Differentiate with respect to x :
a $3\sin(x-4)$ **b** $12x - 2\cos\frac{x}{3}$ **c** $x^2\sin 3x$ **d** $(\sin x)e^{\cos x}$
- 21** Find $f'(x)$ for:
a $f(x) = \sqrt{\sin(2x+1)}$ **b** $f(x) = \cos\frac{x}{2}\sin\frac{x}{3}$ **c** $f(x) = \ln\left(\frac{\sin x}{x}\right)$
- 22** Find the gradient of the tangent to:
a $f(x) = \cos^4 x$ at the point where $x = \frac{3\pi}{4}$ **b** $f(x) = \frac{3\sin^2 x}{\cos 2x}$ at the point where $x = -\frac{\pi}{3}$.
- 23** Find $f'(x)$ if:
a $f(x) = 4^x$ **b** $f(x) = x \times 3^{2x}$ **c** $f(x) = 3^{x^2-x-2}$ **d** $f(x) = \frac{4^{\sqrt{x}-x}}{5^{3x}}$
- 24** Suppose $y = k^{k^x}$, $k > 0$.
a Find $\frac{dy}{dx}$. **b** If $\frac{dy}{dx} = 2k^x \times y$, find the value of k .
- 25** Find the derivative of:
a $y = \log_2 x + \log_3(x^3)$ **b** $y = x \log_5\left(\frac{1}{x}\right)$
c $y = \log_5((x+1)^{\log_2 x})$ **d** $y = \frac{x}{\log_2(\sqrt{x})}$
- 26** Find $\frac{dy}{dx}$ for:
a $y = \left[\log_3\left(\frac{x-3}{2+x^2}\right)\right]^5$ **b** $y = \log_2(e^{\log_2 x} \log_3(x+1))$
- 27** Find $\frac{dy}{dx}$ for:
a $y = \tan 2x$ **b** $y = \tan(3x-4)$ **c** $y = \tan(2^x + x^2)$
- 28** Differentiate with respect to x :
a $\sec 5x$ **b** $\sqrt{\cot x}$ **c** $e^{3x} \operatorname{cosec}(x^2)$
- 29** Suppose $f(x) = \cot(2x+c)$ and $f'\left(\frac{\pi}{4}\right) = -\frac{8}{3}$. Find the possible values of c given that $0 < c < 2\pi$.
- 30** **a** Find the derivative of:
i $f(x) = \frac{1}{2} \arcsin x$ **ii** $g(x) = \sin 2x$
b Hence show that $f'(g(x))g'(x) = 1$ for $-\frac{\pi}{4} < x < \frac{\pi}{4}$.
- 31** Find $\frac{dy}{dx}$ for:
a $y = \arccos\left(\frac{1}{2}\sin x\right)$ **b** $y = e^{-2x} \arctan 2x$ **c** $y = (2-x^2) \arcsin(5^x)$
- 32** Find $\frac{d^2y}{dx^2}$ for:
a $y = \frac{3}{x^2}$ **b** $y = 2x^3 + 3x^2 + 2$ **c** $y = \frac{x+3}{6-x}$

33 Given $f(x) = \ln(\cos x)$, find:

a $f\left(\frac{\pi}{4}\right)$ **b** $f'\left(\frac{\pi}{4}\right)$ **c** $f''\left(\frac{\pi}{4}\right)$.

34 Find $\frac{d^2y}{dx^2}$ for:

a $y = x2^x$ **b** $y = \log_3(2x^2)$ **c** $y = \tan(x+1)$

35 Let $f(x) = \arcsin(\sqrt{1-x})$.

a Show that $f'(x) = \frac{-1}{2\sqrt{x-x^2}}$. **b** Hence find $f''(x)$.

36 Find $\frac{dy}{dx}$ if:

a $x^2 - xy^2 + y = 21$ **b** $e^y \sin 2x = 1$

37 Find $\frac{d^2y}{dx^2}$ for:

a $x^3 + y^2 = 7$ **b** $2^x + 2^y = 1$

38 Evaluate, if possible, using l'Hôpital's rule:

a $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2 + x}$ **b** $\lim_{x \rightarrow 0} \frac{e^{x^2} - \ln(x+e)}{x}$ **c** $\lim_{x \rightarrow -\infty} (x-2)3^x$

39 Evaluate, if possible, using l'Hôpital's rule:

a $\lim_{x \rightarrow \infty} \frac{x^2}{2^x}$ **b** $\lim_{x \rightarrow 0} \frac{x^2 - \sin x}{x - 3 \sin x}$ **c** $\lim_{x \rightarrow 0} \frac{5^x - 2^x}{x^2 - 2x}$

40 Evaluate, if possible:

a $\lim_{x \rightarrow 0^+} \frac{\arctan(\sqrt{x})}{x}$ **b** $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\sin^2 x)}{\cos^2 x}$

41 a Show that $\lim_{x \rightarrow \infty} x \ln\left(1 - \frac{2}{x}\right) = -2$.

b By writing $\left(1 - \frac{2}{x}\right)^x$ as $e^{x \ln\left(1 - \frac{2}{x}\right)}$, find $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x$.

42 Evaluate $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$.

43 Let $g(x) = -x \cos x$.

a Find $g'(x)$. **b** Find the equation of the tangent to the graph $y = g(x)$ at the point where $x = \frac{\pi}{3}$.

44 Let $f(x) = -x^2 + 4x$.

a Find $f'(x)$. **b** Find the equation of the tangent to $y = f(x)$ at the point where $x = k$.
c Suppose this tangent has positive gradient and passes through $(4, 9)$. Find the value of k .

45 Consider the curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x-x^2}$.

a Find the point at which these curves meet.
b Show that the tangents to the curves have the same gradient at this intersection point.
c Find the equation of the common tangent.

46 Consider the curve $y = \frac{a}{x} - x^2 + 1$ where $a \in \mathbb{R}$. The gradient of the tangent to the curve is -5 when $x = 2$.

a Find the value of a . **b** Determine the equation of the tangent to the curve at $x = 2$.

47 Let $f(x) = \frac{x+2}{\sqrt{x-1}}$.

a State the domain of $f(x)$. **b** Find the equation of the normal to $y = f(x)$ at $x = 10$.

48 Consider the cubic function $y = x^3 + ax^2 + bx + 3$.

a The tangent to the function at $(1, 8)$ has equation $y = 2x + 6$. Determine the values of a and b .
b Find the equation of the normal to the function at $x = -1$.

49 Consider the function $f(x) = \arcsin(-2\sqrt{x})$.

a Find the domain and range of $f(x)$.
b Find: **i** $f\left(\frac{1}{16}\right)$ **ii** $f'\left(\frac{1}{16}\right)$
c The tangent to $y = f(x)$ at the point P has gradient -4 . Find the coordinates of P.

- 50** Find the equation of the tangent to:
- a** $y = \operatorname{cosec} \frac{x}{2}$ at the point where $x = \frac{\pi}{3}$ **b** $y = \arcsin 2x$ at the point where $y = \frac{\pi}{4}$.
- 51** Find the equation of the normal to:
- a** $y = 5^{2x} + x^2$ at $x = 0$ **b** $y = x \log_3(x+1)$ at $x = 2$.
- 52** Consider the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
- a** Find $\frac{dy}{dx}$. **b** Find the equations of the tangents to the ellipse at $x = -1$.
c Find the point where the tangents in **b** intersect.
- 53** Consider the curve $ax \sec y = 4$, $0 \leq y \leq 2\pi$. The tangent to the curve at the point where $x = 4$, $0 \leq y \leq \pi$, has gradient $\frac{\sqrt{3}}{12}$.
- a** Find a . **b** Find the points on the curve at which the tangent has gradient $-\frac{1}{4}$.
- 54** **a** Show that if $f(x) = \ln\left(\frac{1-2x}{x^2+2}\right)$, then $f'(x) = \frac{2(x-2)(x+1)}{(1-2x)(x^2+2)}$.
b On what intervals is $f(x)$ decreasing?
- 55** Find the exact coordinates and nature of the stationary points of:
- a** $y = xe^{-x}$ **b** $y = \frac{x-3}{x^2-5}$
- 56** Find the greatest and least values of:
- a** $f(x) = x^3 - 2x^2$ for $-1 \leq x \leq 1$ **b** $f(x) = x^2 - \frac{27}{x}$ for $-6 \leq x \leq -1$
c $f(x) = x^3 - 6x^2 + 12x - 10$ for $0 \leq x \leq 5$.
- 57** Consider the function $f(x) = \frac{e^{3x}}{kx}$, $k \neq 0$.
- a** Find the x -coordinate of the stationary point.
b For what values of k is the stationary point:
i a local minimum **ii** a local maximum?
c Given that the stationary point has y -coordinate $-\frac{e}{2}$, find k and determine the nature of the stationary point.
d State the location and nature of the stationary point of $g(x) = -f(2x)$.
- 58** **a** Differentiate $x^{\frac{1}{x}}$ with respect to x .
b Hence find the coordinates of the stationary point of the function $f(x) = x^{\frac{1}{x}}$.
- 59** Find the coordinates and nature of the stationary points of:
- a** $y = \frac{\sin x}{\tan x + 1}$ on $-\pi \leq x \leq \frac{\pi}{2}$ **b** $y = \operatorname{cosec} 2x$ on $0 \leq x \leq \pi$ **c** $y = 4^x - 2^x$
- 60** Consider $g(x) = 3 - 2 \cos 2x$.
- a** Find $g'(x)$. **b** Sketch $y = g'(x)$ for $-\pi \leq x \leq \pi$.
c Write down the number of solutions to $g'(x) = 0$ for $-\pi \leq x \leq \pi$.
d Mark a point M on the sketch in **b** where $g'(x) = 0$ and $g''(x) > 0$.
- 61** For each of the following functions, determine the interval(s) on which the function is:
- i** concave up **ii** concave down.
- a** $y = \frac{x^2 - 5}{x + 2}$ **b** $y = \log_2 x + \frac{x^2}{2}$
- 62** Let $f(x) = xe^{1-2x^2}$.
- a** Find $f'(x)$ and $f''(x)$.
b Find the exact coordinates of the stationary points of the function and determine their nature.
c Find the exact x -coordinates of the inflection points of the function.
d Use technology to help sketch the function, showing the information you have found.
- 63** The function $f(x) = \frac{a \ln bx}{x}$ has an inflection point at $\left(\frac{e\sqrt{e}}{2}, \frac{9}{e\sqrt{e}}\right)$. Find the constants a and b .

64 Find and classify all points of inflection of:

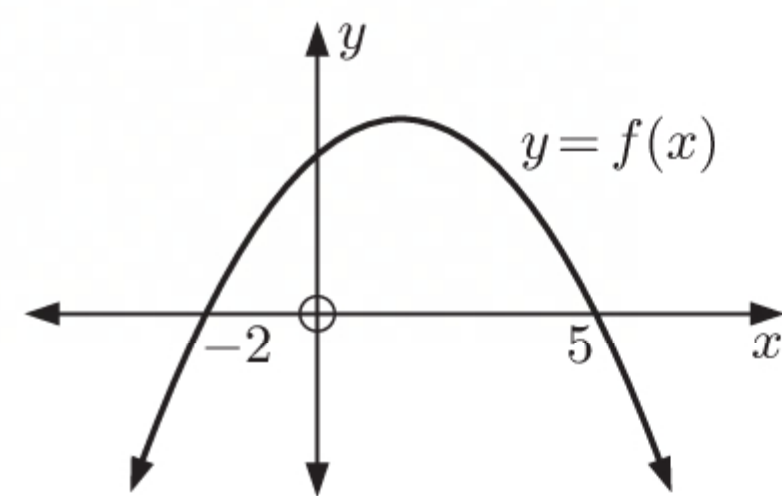
a $f(x) = \cot x$ for $-\pi \leq x \leq \pi$

b $f(x) = \arctan(x^2)$

65 For the function $y = f(x)$ with graph shown, sketch the graphs of:

a $y = f'(x)$

b $y = f''(x)$



66 The graph of $y = f'(x)$ is shown alongside.

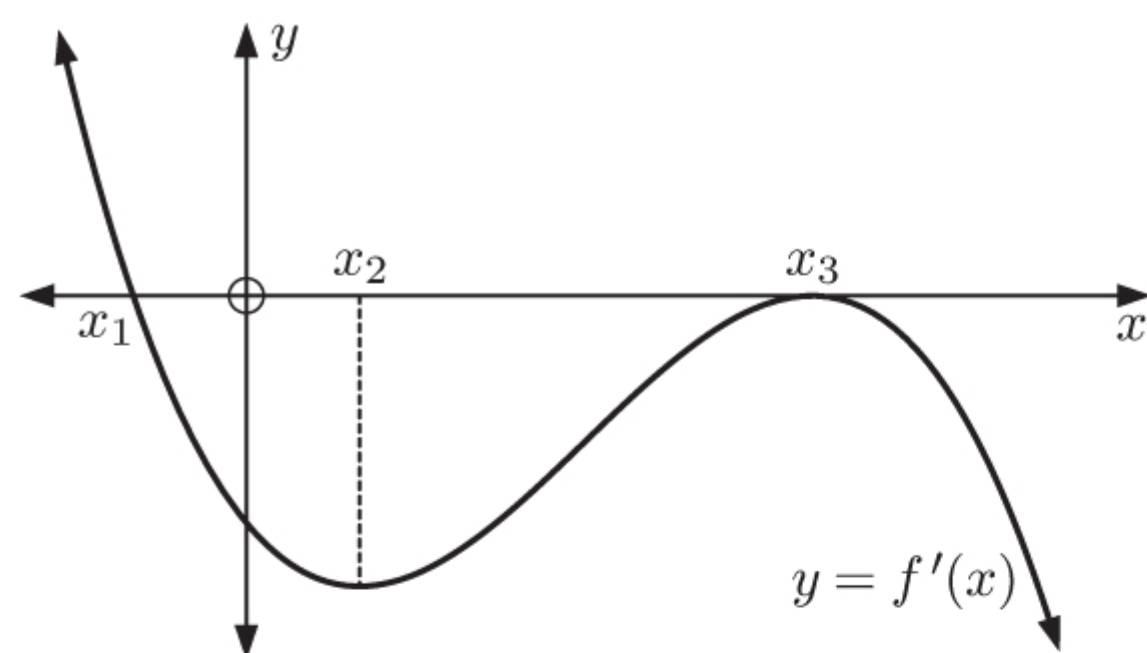
a For what values of x is $f(x)$:

i increasing

ii concave down?

b Given that $y = f(x)$ passes through the origin, sketch the graph of $y = f(x)$.

c Sketch the graph of $y = f''(x)$.



67 The volume of water in a tank is given by $V(t) = 10t^2 - \frac{1}{3}t^3$ litres, where t is the time in minutes and $0 \leq t \leq 30$.

a Find $V(5)$ and explain what this represents.

b Find $V'(t)$ in fully factorised form. Do not forget to include units.

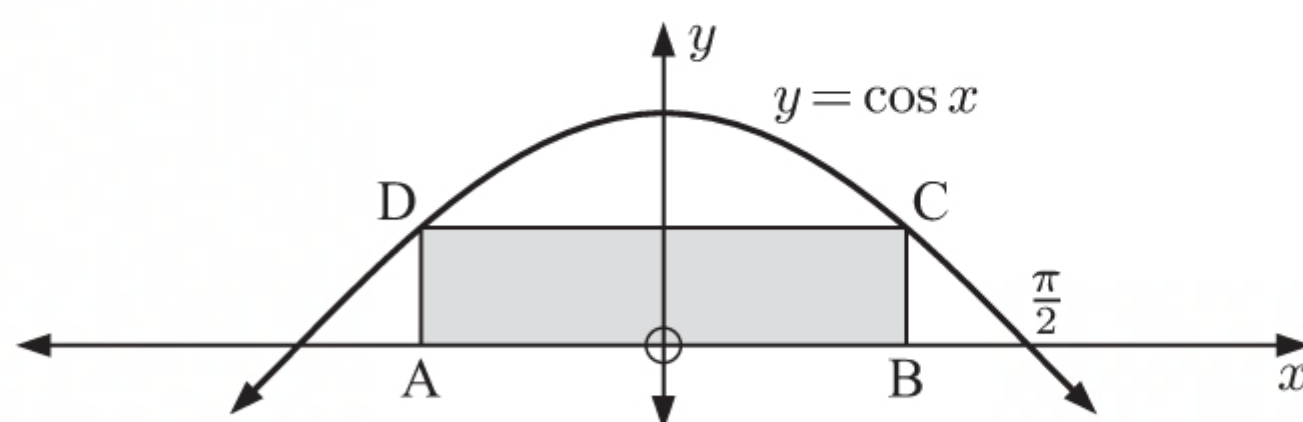
c Find t when $V'(t) = 0$.

d Find $V'(5)$ and $V'(25)$.

e Determine the time(s) at which the volume is increasing by 75 litres per minute.

68 Rectangle ABCD is inscribed under one arch of $y = \cos x$. Suppose the point C has x -coordinate x .

Find the coordinates of C such that ABCD has maximum area.



69 The number of bacteria found in a sample of human tissue t hours after infection occurred, is modelled by the function $N = (8 - t)e^{t-6}$ million, $0 \leq t \leq 8$.

The graph of this function is shown alongside.

a Show that $\frac{dN}{dt} = (7 - t)e^{t-6}$.

b Find the coordinates of the:

i turning point

ii point of inflection

iii t -intercept.

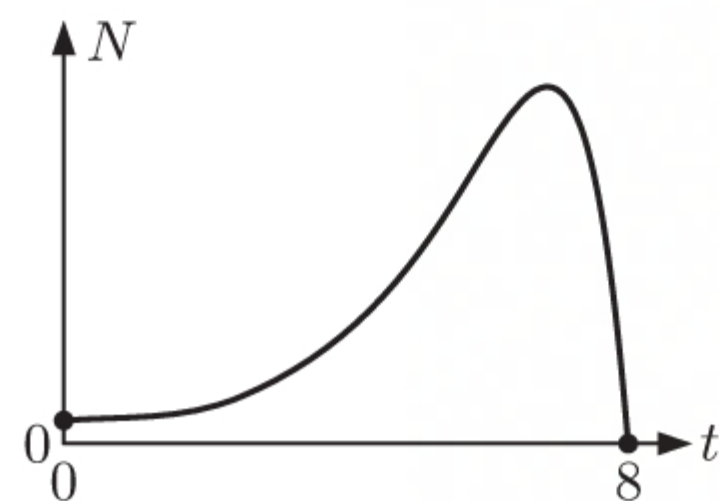
c Use these coordinates to state:

i the time when all the bacteria are dead

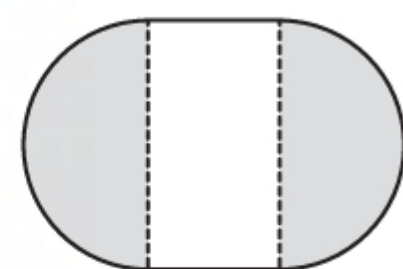
ii the maximum number of bacteria reached in the sample

iii the time at which the rate of increase of the bacteria is a maximum.

d Copy the graph and indicate clearly which points on the graph represent your answers to **c**.



70 An ornamental pond of area A is to be built with straight sides and semi-circular ends as shown. The cost of tiling per unit length is 25% greater along the rounded ends than along the straight walls. Show that the total cost of tiling the walls is minimised when the shaded area is $\frac{2}{3}A$.



71 Terry wants to fence off a rectangular garden plot of area 48 m^2 . Three sides will be fenced with strong wire mesh costing \$18 per metre, and the remaining side will be fenced with corrugated iron costing \$30 per metre.

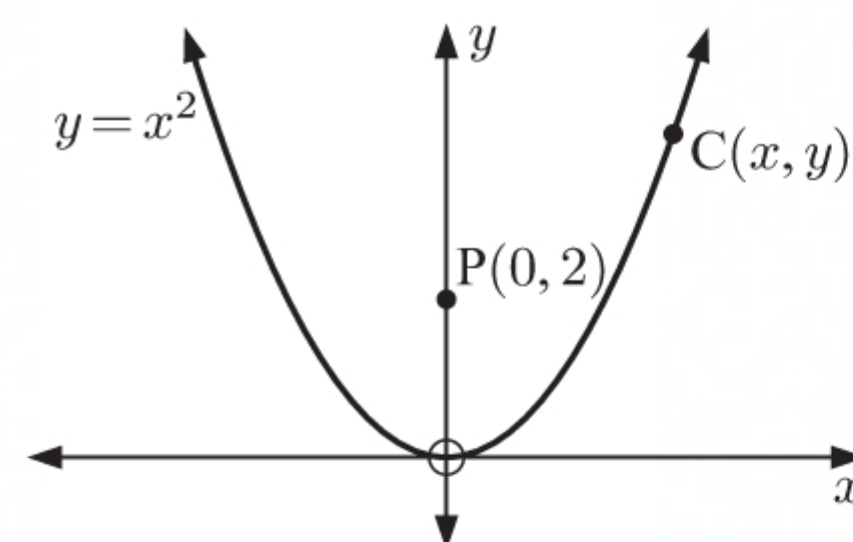
a By letting x be the length in metres of the side fenced with corrugated iron, show that the cost of fencing is

$$C = 48\left(\frac{36}{x} + x\right) \text{ dollars.}$$

b Find the dimensions of the garden plot which will minimise the cost of fencing.

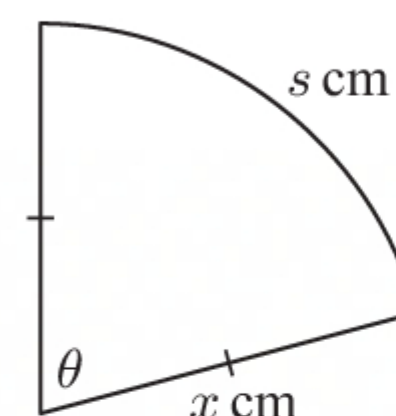
- 72** A comet travels in an orbit which can be described by the equation $y = x^2$ as shown in the diagram.

- a** Show that the distance of the comet at $C(x, y)$ from an observer at the point $P(0, 2)$ is given by $s(x) = \sqrt{x^4 - 3x^2 + 4}$.
- b** Find the shortest and the greatest distance between the comet and the observer for $-2 \leq x \leq 2$.



- 73** A 40 cm piece of wire is bent to form a sector of a circle with radius x cm.

- a** Write θ in terms of x .
- b** Show that the area of the sector is given by $A = 20x - x^2$ cm².
- c** Find x and θ for which A is a maximum.

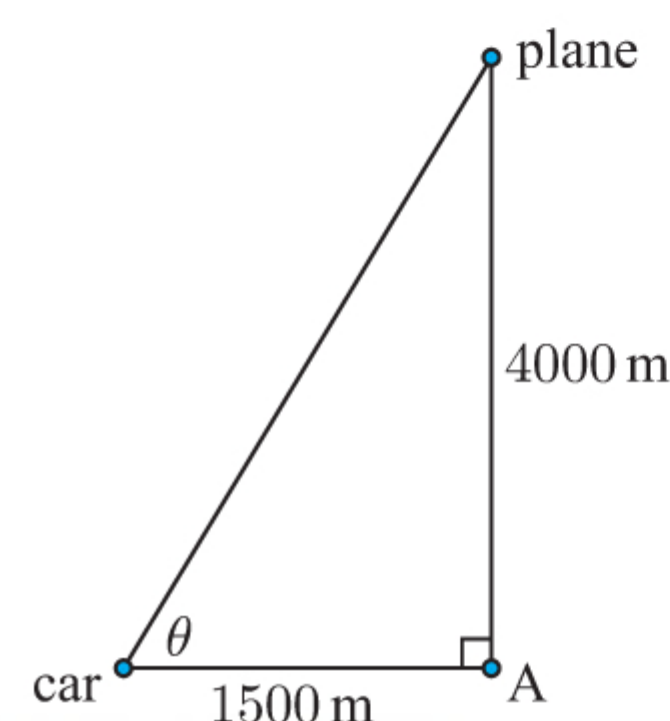


- 74** A ball of ice cream with initial radius 8 cm takes 5 minutes to melt. During this time, its radius decreases at a constant rate.

- a** Find the rate of change of volume of the ice cream ball 2.5 minutes after it begins to melt.
- b** Find the average rate of change of volume for the last 4 minutes of melting time.

- 75** The kinetic energy of a moving object is given by $K = \frac{1}{2}mv^2$ where m is its mass in kg and v is its velocity in km s⁻¹. When a rocket is fired, its initial kinetic energy increases at 50 000 units s⁻¹, while its mass initially decreases at 10 kg s⁻¹. If a rocket initially has mass 4000 kg and velocity 8 km s⁻¹, at what rate is the rocket's velocity changing?

- 76** A skydiver jumps from a plane 4000 m above ground level, and descends towards A at 50 m s⁻¹. At the same time a car which is 1500 m from A drives towards A at 12 m s⁻¹. Find the rate at which the angle of elevation θ from the car to the skydiver is changing after 1 minute.



- 77** Integrate with respect to x :

a $3x^2 + 2x + 1$

b e^{4x}

c $\cos(2x + 1)$

- 78** **a** Given $f(x) = \sqrt{xe^x}$, find $f'(x)$.

b Hence find $\int \frac{2e^{\frac{x}{2}}(1+x)}{\sqrt{x}} dx$.

- 79** By considering $\frac{d}{dx}(x^2 \ln x)$, find $\int x \ln x dx$.

- 80** Suppose $f'(x) = (x^2 + 2)^2$ and that $f(1) = \frac{8}{15}$. Find $f(x)$.

- 81** Suppose $f'(x) = \sqrt{4x + 5}$ and that $f(0) = -\frac{\sqrt{5}}{6}$.

- a** For what values of x is $f'(x)$ defined?

- b** Find $f(x)$.

- 82** Find $f(x)$ given that:

a $f''(x) = e^x + 2x - 1$, $f'(0) = 4$, $f(0) = 1$

b $f''(x) = 2 + \sin x$, $f'(\pi) = 1$, $f(\frac{\pi}{2}) = \frac{\pi^2}{4}$

c $f''(x) = \frac{2}{\sqrt{x}} + 3x$, $f(1) = -\frac{19}{3}$, $f(4) = \frac{64}{3}$

- 83** Integrate with respect to x :

a $2 \sin(x - 3) + e^{3x}$

b $\frac{2}{5x - 1}$

c $\cos(5 - 7x)$

- 84** Find:

a $\int (2 \sin^2 x - 1) dx$

b $\int (\sin 2x - \cos 2x)^2 dx$

c $\int (\cos x + 2)^2 dx$

- 85** Find:

a $\int 6^x dx$

b $\int \left(\frac{2}{x} - \frac{5}{x \ln 3} \right) dx$

c $\int (\cos x - \sec x \tan x) dx$

- 86** Find $f(x)$ given that $f'(x) = \frac{2}{1+x^2}$ and $f(-\sqrt{3}) = \pi$.

- 87** A curve has gradient function $f'(x) = a \times 2^x$ where a is a constant. Find $f(x)$ given that $f(0) = -1$ and $f(1) = 2$.
- 88** Find:
- a** $\int \frac{-10}{\sqrt{36-4x^2}} dx$ **b** $\int \frac{2}{9+4x^2} dx$ **c** $\int \frac{3}{(x+4)^2 + (x-4)^2} dx$
- 89** Find $\int \frac{x^3 + 5x^2 - 3x + 2}{x-3} dx$.
- 90** Find:
- a** $\int 3x^2(5+x^3)^4 dx$ **b** $\int xe^{x^2+2} dx$ **c** $\int \frac{2(\ln x)^2}{x} dx$
- 91** Integrate with respect to x :
- a** $\sqrt{x^2+3x-1}(2x+3)$ **b** $\frac{e^x+2}{e^x+2x}$ **c** $\frac{6-8x}{2x^2-3x+2}$
- 92** Find:
- a** $\int \sin^4 x \cos x dx$ **b** $\int 3x^3 \sin(x^4) dx$ **c** $\int \frac{\sin 2x}{(3-\cos 2x)^3} dx$
- 93** **a** Show that $\cos^4 x = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$. **b** Hence find $\int \cos^4 x dx$.
- 94** Integrate with respect to x :
- a** $e^{-3x} + 2 \sec^2\left(\frac{\pi}{3} - x\right)$ **b** $7^{3x+2} - \sec^2\left(\frac{x}{2} + \frac{\pi}{6}\right)$
- 95** Use the substitution $x = 2 \sin \theta$ to find $\int \sqrt{4-x^2} dx$.
- 96** Find $\int 3x^2 \sqrt{x+4} dx$ using the substitution $u = x+4$.
- 97** Find:
- a** $\int x\sqrt{3-x} dx$ **b** $\int \frac{2x}{\sqrt{x-4}} dx$ **c** $\int x^3 \sqrt{4-x^2} dx$
- 98** Find $\int x^{-\frac{3}{2}} \sqrt{1-x} dx$ using $x = \cos^2 \theta$.
- 99** **a** Write $\frac{4-x}{(x-3)(x-5)}$ as a sum of partial fractions.
b Hence find $\int \frac{4-x}{(x-3)(x-5)} dx$.
- 100** **a** Write $\frac{x^2+x-5}{(x+2)(x+1)^2}$ as a sum of partial fractions in the form $\frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.
b Hence find $\int \frac{x^2+x-5}{(x+2)(x+1)^2} dx$.
- 101** Use integration by parts to find:
- a** $\int x \sin 4x dx$ **b** $\int (2x+1) \ln x dx$ **c** $\int x2^x dx$ **d** $\int \sin 2x \cos 3x dx$
- 102** Use integration by parts to find:
- a** $\int e^x \sin 2x dx$ **b** $\int \arccos x dx$ **c** $\int x \tan^2 x dx$
- 103** Use integration by parts to find $\int x \arctan x dx$.
Check your answer using differentiation.
- 104** **a** Differentiate $\sec x + \tan x$, and hence find $\int \sec x dx$.
b Find $\int \sec^3 x dx$.
- 105** Find a given that $\int_a^{2a} \sqrt{x} dx = 2$.
- 106** If $y = x\sqrt{4-x}$, find $\frac{dy}{dx}$ and simplify your answer. Hence evaluate $\int_0^2 \frac{8-3x}{\sqrt{4-x}} dx$.

107 Find:

a $\int_1^5 \frac{2x^3 + 1}{x^2} dx$

b $\int_{-1}^1 e^x (2 - 3e^{-x})^2 dx$

c $\int_0^2 \frac{3}{5 - 2x} dx$

d $\int_{-2}^{\frac{\pi}{4}-2} \sin^2(x + 2) dx$

e $\int_0^1 (2x + 3)(x^2 + 3x + 4)^3 dx$

f $\int_{-2}^2 8xe^{x^2+1} dx$

108 Find a if $\int_0^a \frac{x}{x^2 + 1} dx = 3$ and $a > 0$.**109** Find:

a $\int_2^3 3^x dx$

b $\int_{-\sqrt{3}}^1 \frac{5}{1 + x^2} dx$

c $\int_2^{2\sqrt{3}} \frac{1}{\sqrt{16 - x^2}} dx$

110 a Simplify $\sin(2 \arcsin x)$, $0 \leq x \leq 1$.**b** Hence find $\int_0^1 \sin(2 \arcsin x) dx$.**111** Find the exact value of:

a $\int_3^5 \frac{x}{x^2 - 8} dx$

b $\int_0^2 x\sqrt{x^2 + 1} dx$

c $\int_1^4 \frac{3e^{\sqrt{x}}}{\sqrt{x}} dx$

112 Find the exact value of:

a $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} dx$

b $\int_0^{\frac{\pi}{3}} \cos^3 x \sin x dx$

c $\int_0^{\arcsin \frac{\pi}{8}} \sin^2(\sin x) \cos x dx$

113 Use the substitution:

a $x = 2 \sin \theta$ to find $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}} dx$

b $u = x + 3$ to find $\int_1^6 x\sqrt{x + 3} dx$.

114 Use integration by parts to evaluate:

a $\int_1^e x \ln x dx$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x^2 \sin x dx$

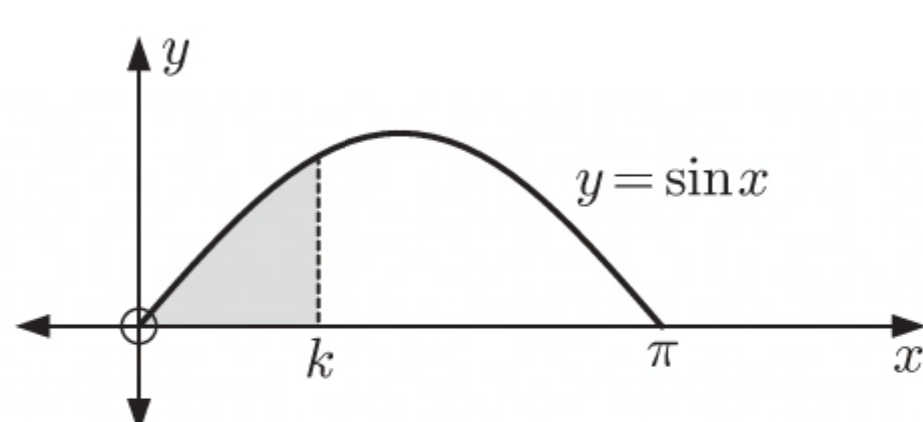
c $\int_0^{\pi} e^{2x} \cos 3x dx$

115 a Find constants A , B , and C such that $\frac{x + 5}{(x^2 + 5)(1 - x)} = \frac{Ax + B}{x^2 + 5} + \frac{C}{x - 1}$.**b** Hence find the exact value of $\int_2^4 \frac{x + 5}{(x^2 + 5)(1 - x)} dx$.**116** Evaluate, if possible: **a** $\int_1^{\infty} \left(\frac{1}{x^3} - \frac{1}{x^4} \right) dx$ **b** $\int_{\frac{2}{\pi}}^{\infty} \frac{1}{x^2} \cos \frac{1}{x} dx$ **117 a** Find $\int e^{-x} \cos x dx$.**b** Hence find the total area between $f(x) = e^{-x}(\cos x + 1)$ and the x -axis for $x \geq 0$.**118** Evaluate using technology:

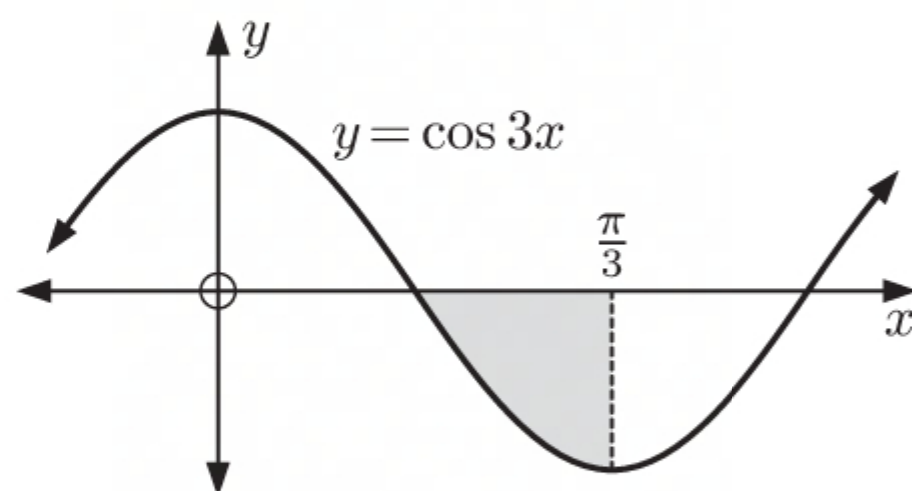
a $\int_0^2 \sqrt{x} e^{x^2} dx$

b $\int_1^3 e^x \sin(x^2) dx$

c $\int_{-1}^1 \sqrt{x^2 + \cos x} dx$

119 For a continuous function $f(x)$ defined on the interval $a \leq x \leq b$, the length of the curve can be found using $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$. Use technology to find the length of:**a** $f(x) = x^2$ on the interval $0 \leq x \leq 1$ **b** $f(x) = \sin x$ on the interval $0 \leq x \leq \pi$.**120**The shaded region has area 0.42 units². Find k , correct to 2 decimal places.**121** Find the area of the region bounded by:**a** $y = x^2 + x$, the x -axis, $x = 2$, and $x = 4$ **b** $y = \sin 2x$, the x -axis, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$ **c** $y = \frac{1}{\sqrt{x+2}}$, the x -axis, $x = 2$, and $x = 7$ **d** $y = e^{-3x}$, the x -axis, $x = 0$, and $x = 1$.

- 122** Find the shaded area:



- 123** The area of the region bounded by $f(x) = -\frac{10}{x+5}$, the x -axis, $x = 0$, and $x = k$, is $10 \ln 3$ units².

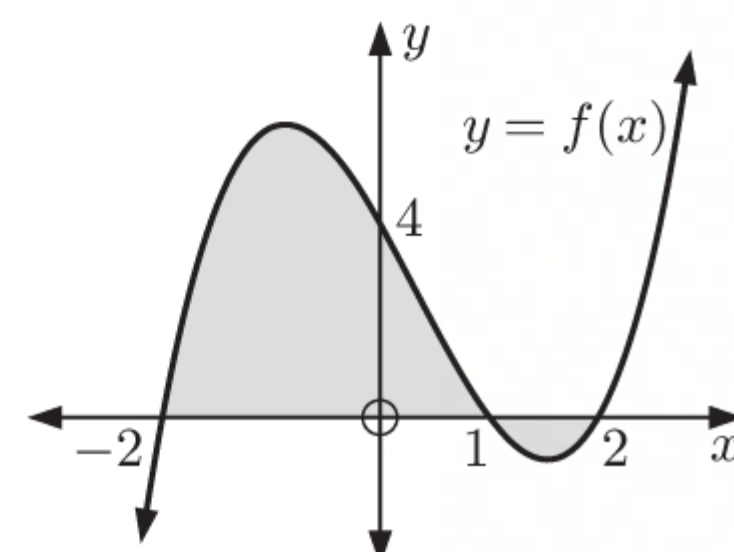
Find the possible values of k .

- 124** Consider the graph of $f(x) = x^3 - x^2 - 4x + 4$.

a Find $\int_{-2}^2 f(x) dx$.

b Explain why the value obtained in **a** does not represent the shaded area.

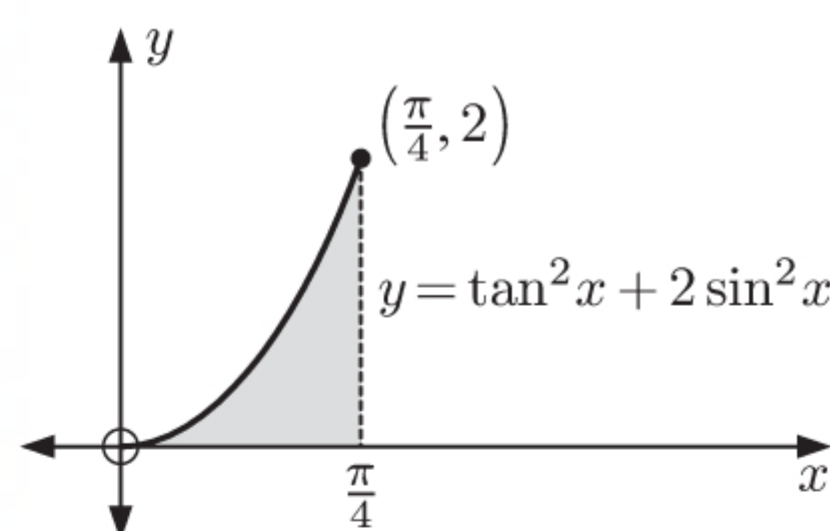
c Find the shaded area.



- 125 a** Sketch the graph of $f(x) = \sin(x + \frac{\pi}{6})$ for $-2\pi \leq x \leq 2\pi$.

b Find the area that lies above the line $y = \frac{1}{2}$ and below the graph of $y = f(x)$ for $-2\pi \leq x \leq 2\pi$.

- 126** Find the shaded area.



- 127** Consider the function $f(x) = \cos 2x \times e^{\cos x + \sin x}$.

a State the period of $f(x)$.

b Sketch the function for the interval $0 \leq x \leq 4\pi$.

c Find:

i $\int f(x) dx$ using the substitution $w = e^{\cos x + \sin x}$

ii the first positive x -intercept of the function.

d Hence find the area enclosed by the function and the x -axis from $x = 0$ until the first positive x -intercept.

- 128** Consider the function $f(x) = (2 - \frac{1}{x})e^{-x}$, $x > 0$.

a Find the zero of $f(x)$.

b Discuss the behaviour of $f(x)$ near $x = 0$ and as $x \rightarrow \infty$.

c Find the position and nature of the stationary point.

d Sketch the graph of $y = f(x)$, showing all the above information.

e Find, correct to 3 decimal places, the area enclosed by $y = f(x)$ and the line $y = x - 1$.

- 129 a** Write down the derivative with respect to x of: **i** $f(x) = \ln x$ **ii** $F(x) = x \ln x - x$.

b What is the relationship between $f(x)$ and $F(x)$?

c Sketch the graph of $y = f(x)$.

d Let O be the origin, and let P be the point on $y = f(x)$ with x -coordinate t , $1 < t \leq e$.

i Show that the area bounded by [OP], $y = f(x)$, and the x -axis, is $(t - \frac{1}{2}t \ln t - 1)$ units².

ii For what value of t will this area be maximised?

e i Write down the equation of the tangent to $y = f(x)$ at the point P(t , $\ln t$).

ii Find t such that this tangent passes through the origin.

- 130** Find the area enclosed by:

a $y = x^3 - 2x^2 - 3x$ and $y = 5x - 4x^2$

b $y = 2x^3 - 5x + 4$ and $y = x^3 + 2x^2 - 2$

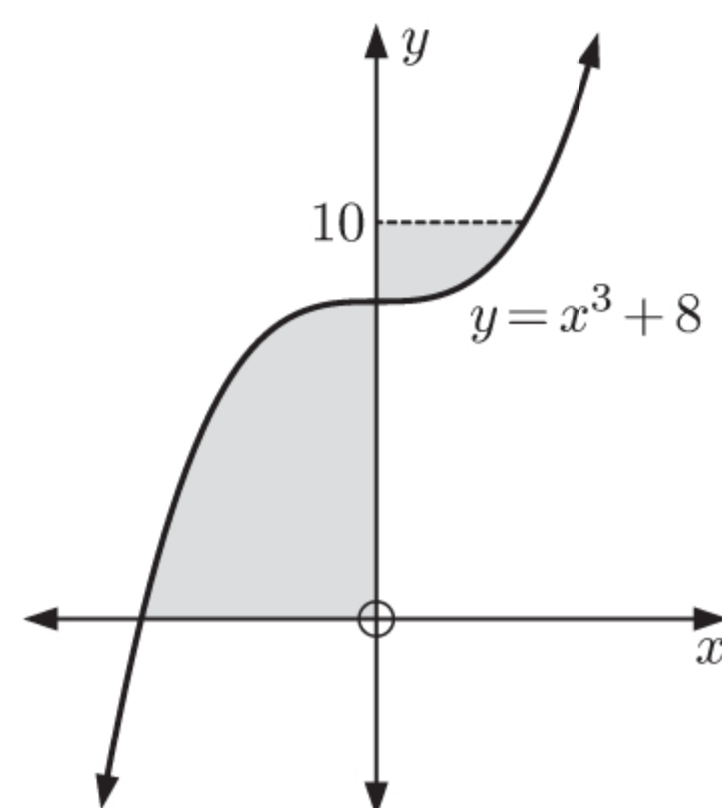
- 131** The rate at which a tree grows t years after planting is given by $G(t) = \frac{2.5}{t+1}$ metres per year.

a Explain why the tree is always growing taller.

b Evaluate the following integrals, and interpret their meaning: **i** $\int_0^5 G(t) dt$ **ii** $\int_5^{10} G(t) dt$

c After 15 years, the tree is struck by lightning and is cut down. How much did the tree grow over its lifetime?

- 132** Find the shaded area:

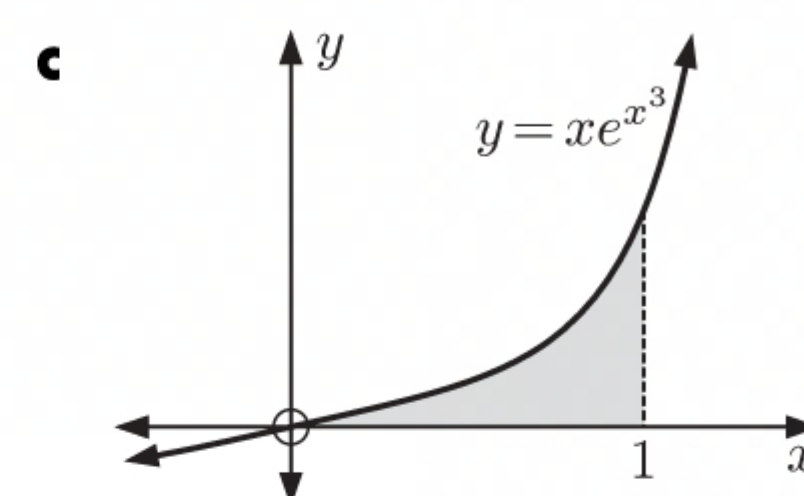
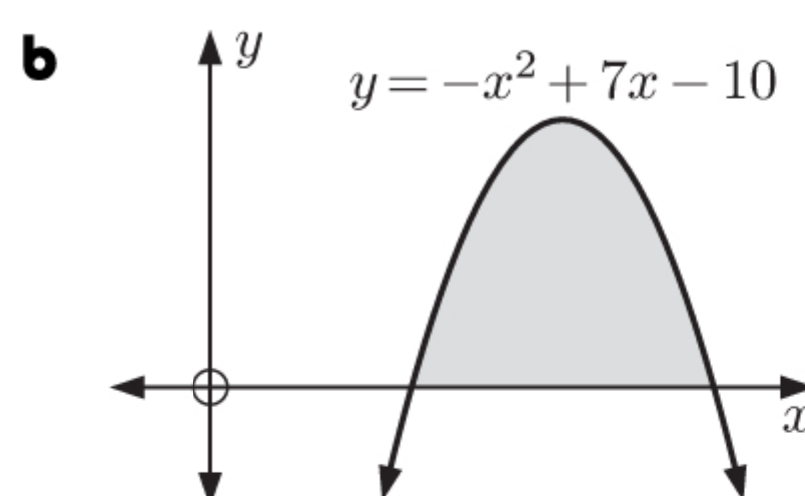
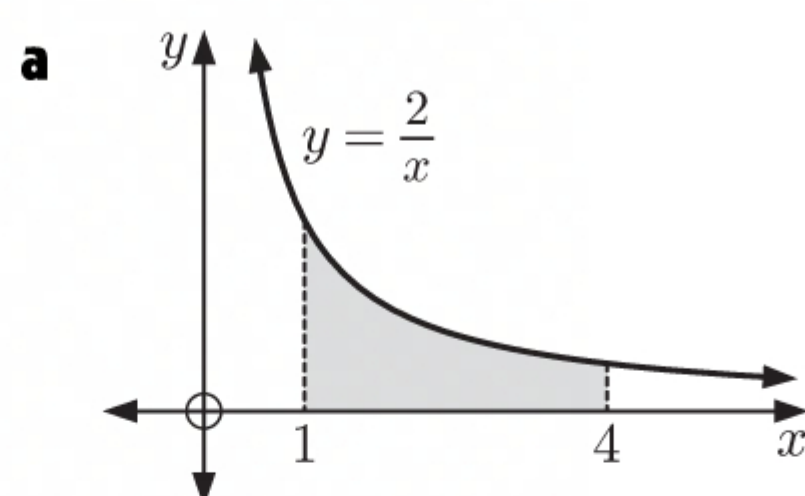


- 133** Find the exact area of the region bounded by:

a $y = 3 - \ln(4 - x)$, the y -axis, and the line $y = 5$

b $y = \arcsin(x - 1) - \frac{\pi}{2}$ and the axes.

- 134** Find the volume of revolution when the shaded region is revolved through 2π about the x -axis.



- 135** Find the volume of revolution when the following regions are revolved through 2π about the x -axis.

a $y = \sin \frac{x}{2}$ for $0 \leq x \leq 2\pi$

b $y = 4 \sin x \cos x$ for $0 \leq x \leq \frac{\pi}{2}$

c $y = 1 + \tan x$ for $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$

- 136** **a** Find the exact area of the region bounded by $y = \frac{1}{\sqrt{x}}$, the x -axis, and the lines $x = 1$ and $x = 9$.

b If the area in **a** is rotated 360° about the x -axis, what is the volume of the resulting solid?

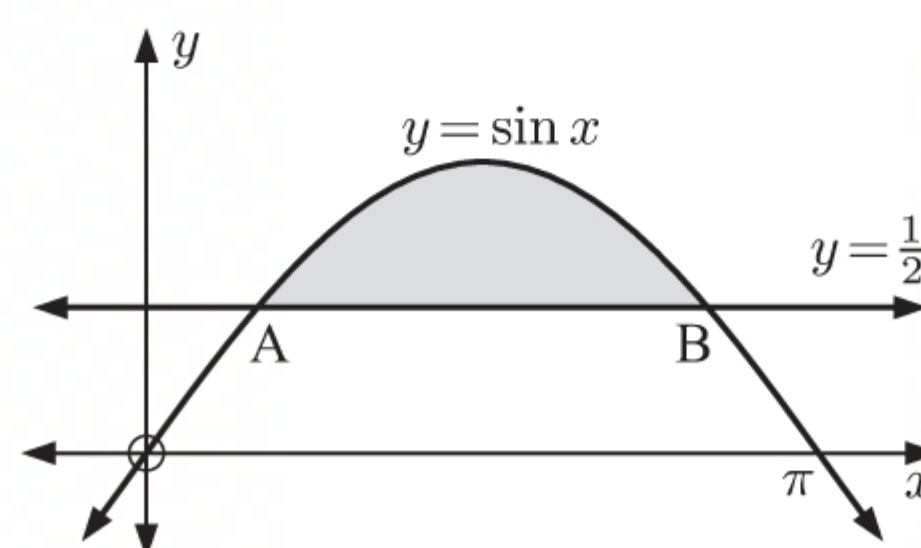
- 137** Find the exact volume of the solid formed when the region enclosed by $y = \ln x$, the axes, and the line $y = \ln 3$ is rotated about the y -axis.

- 138** Find the exact volume of revolution when the relation $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $x \geq 0$ is rotated 2π about the y -axis.

- 139** The shaded region is revolved 2π about the x -axis.

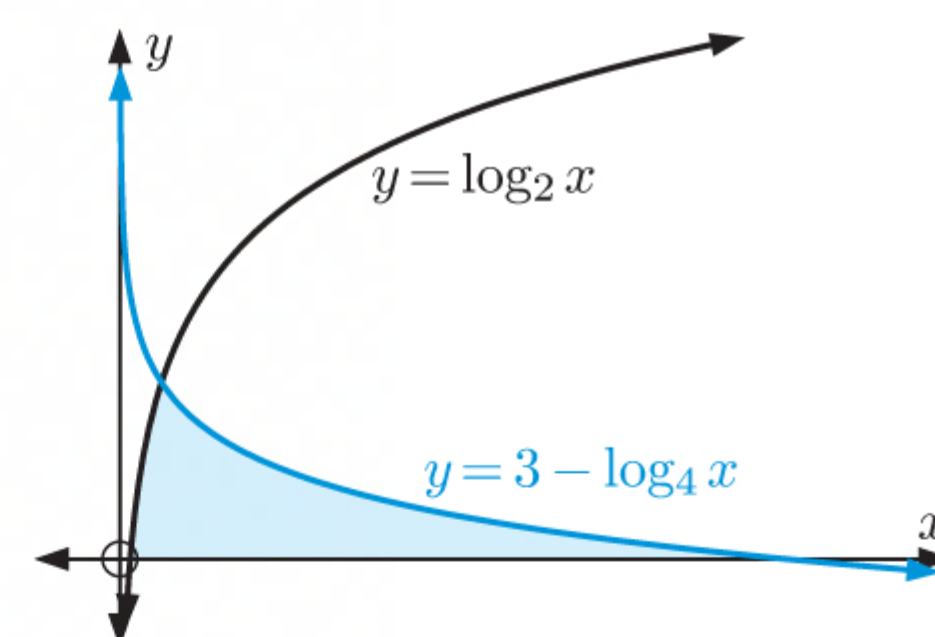
a Find the exact coordinates of A and B.

b Find the volume of revolution.



- 140** Find exactly the volume of revolution formed by rotating the region enclosed by $y = x^2 + 3x + 3$ and $y = 2x + 5$ through 360° about the x -axis.

- 141** Find the volume of the solid formed when the shaded region is revolved 2π about the y -axis. Give your answer in the form $\frac{a\pi}{\ln 4}$ units³, where $a \in \mathbb{Z}$.



142 Consider the function $f(x) = x^{-\frac{2}{3}}$, $x \geq 1$.

- a** Explain why the area between the curve and the x -axis for $x \geq 1$ is infinite.
- b** Find the volume of the solid formed when $y = f(x)$ is revolved about the x -axis.

143 A particle is moving in a straight line with velocity given by $v(t) = t^3 - 3t^2e^{0.05t}$, where $t \geq 0$ is in seconds, and distance units are in metres.

Use technology to find:

- a** the greatest speed reached by the particle in the first 4 seconds of motion
- b** the total distance travelled by the particle in the first 4 seconds of motion.

144 For the first 6 seconds of its motion, a particle moving in a straight line has velocity given by $v = t^3 - 9t^2 + 24t$ m s⁻¹, where t is the time in seconds.

- a** Find the acceleration function for the particle.
- b** Find the greatest velocity of the particle in the first 6 seconds.
- c** At what times in the first 6 seconds is the speed of the particle decreasing?

145 A particle moves on a straight line with acceleration given by $2 - 3t$ m s⁻². Initially the particle has displacement 3 m. When $t = 1$ s, the particle is momentarily at rest.

- a** Find the velocity function of the particle.
- b** At what other time is the particle momentarily at rest?
- c** Find the displacement function of the particle.

146 A particle moves in a straight line with displacement function $s(t) = 12t - 3t^3 + 1$ cm, where $t \geq 0$ is in seconds.

- a** Find the velocity and acceleration functions for the particle's movement.
- b** Find the speed of the particle after:
 - i** 1 second
 - ii** 2 seconds.
- c** When is the particle's:
 - i** velocity decreasing
 - ii** speed decreasing?

147 A particle moves from rest along a straight line. Its velocity is given by $v = 2\sqrt{t} - t$ m s⁻¹, where $t \geq 0$ is the time in seconds.

- a** Find the speed of the particle after 5 seconds.
- b** Find the acceleration function of the particle.
- c** Show that the particle changes direction after 4 seconds.
- d** Find the total distance travelled in the first 9 seconds.

148 An object moves in a resisting medium such that its velocity v , decreases at a rate $\frac{dv}{dt} = -kv$, where k is a positive constant. The initial velocity of 100 m s⁻¹ is reduced to 40 m s⁻¹ in 2 seconds.

- a** Show that $k = \frac{1}{2} \ln\left(\frac{5}{2}\right)$.
- b** Find the distance the object travels in the medium in the first 2 seconds.

149 Derive the Maclaurin series representation $\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^k (k+1)x^k$.

150 **a** Use the Maclaurin series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to find the Maclaurin series expansion for 3^x . Write the series up to the term in x^4 .

b Hence estimate the value of $\sqrt{3}$.

151 **a** Use the rule $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to write the first 8 terms of the Maclaurin series for $e^{i\theta}$.

b By considering the case $\theta = \pi$, show that:

$$\text{i} \quad \frac{\pi^2}{2!} - \frac{\pi^4}{4!} + \frac{\pi^6}{6!} - \frac{\pi^8}{8!} + \dots = 2$$

$$\text{ii} \quad \pi + \frac{\pi^5}{5!} + \dots = \frac{\pi^3}{3!} + \frac{\pi^7}{7!} + \dots$$

- 152** **a** Use a Maclaurin series to prove that for $|r| < 1$, the sum of the infinite geometric series $\sum_{k=0}^{\infty} r^k$ is $\frac{1}{1-r}$.
- b** Hence show that $\frac{1}{2-x} = \sum_{k=0}^{\infty} (x-1)^k$ for a particular domain of x .
- c** Use the definition of a Maclaurin series to show that $\frac{1}{2-x} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$.
State the values of x for which this expansion converges.
- d** Verify that $\sum_{k=0}^{\infty} (x-1)^k = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$ when $x = \frac{3}{4}$.
State the complete domain of values for which this result holds.
- 153** **a** Use the Maclaurin series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, $-1 < x \leq 1$ to find the Maclaurin series for $\ln(1-x^2)$. State the domain for which this series is valid.
- b** For the curve $f(x) = \ln(1-x^2)$, find:
- i** the axes intercepts
 - ii** the asymptotes
 - iii** the turning point.
- c** Hence sketch the curve $y = f(x)$.
- 154** The Maclaurin series expansion for $\cos 2x$ is $\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$.
- a** Use the first few terms of $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ to find the terms of $\cos^2 x$ up to x^6 .
- b** Compare your result with the expansion of $\frac{1}{2} + \frac{1}{2} \cos 2x$.
- 155** **a** Given that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$, find the Maclaurin series for $\sec x = \frac{1}{\cos x}$, up to the term in x^4 .
- b** Hence find the Maclaurin series for $\sec^2 x$, up to the term in x^4 .
- c** Hence find the Maclaurin series for $\tan x$, up to the term in x^5 .
- 156** **a** Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, find the Maclaurin series expansion for $\frac{x^2}{e^x}$, up to the term in x^3 .
- b** Hence estimate $\int_{-0.1}^{0.1} \frac{x^2}{e^x} dx$.
- 157** Suppose f is a continuous even function which can be differentiated infinitely many times. Use Maclaurin series to prove that every even numbered derivative of f is an even function.
- 158** Consider the differential equation $\frac{dy}{dx} = e^x - 2x$ with $y(0) = 1$.
- a** Estimate $y(1)$ by applying Euler's method with:
- i** $h = 0.5$ for two steps
 - ii** $h = 0.25$ for four steps.
- b** Find $y(1)$ exactly using the Fundamental Theorem of Calculus. Comment on your results.
- 159** Use Euler's method with step size 0.2 to estimate $y(2)$ given $\frac{dy}{dx} = x - y$, $y(1) = 2$.
- 160** Find particular solutions for the following differential equations and initial conditions:
- a** $\frac{dy}{dx} = \cos x - 2 \sin x$ where $y(3\pi) = 2$
 - b** $\frac{d^2y}{dx^2} = 6x^2 - 15x^{\frac{1}{2}}$ where $y(1) = 2$ and $y'(1) = 1$
 - c** $\sqrt{3-t^2} \frac{dR}{dt} = 4t$ where $R(1) = 4$
- 161** Find the general solution to the following differential equations:
- a** $\frac{dy}{dx} = e^{2x} - \cos x$
 - b** $\frac{dM}{dt} = t\sqrt{t^2 + 5}$
 - c** $\frac{dP}{dx} - \sin^2 x = \tan x$
- 162** **a** Differentiate $y = x \ln x$, $x > 0$ with respect to x .
- b** A management consultant uses a computer to schedule machines in a factory. The time T taken by the computer increases with the number of machines, N , according to $T' = \frac{1}{20}(1 + \ln N)$.
Given that the program takes 10 seconds to schedule 50 machines, determine, to the nearest second, the time required to schedule 100 machines.
- 163** Solve the following differential equations:
- a** $\frac{dy}{dx} = xy^2$
 - b** $\frac{dy}{dx} = 5\sqrt{y}$
 - c** $\frac{dy}{dx} = \frac{xy}{x^2 + 1}$

164 Find the particular solution to:

a $\frac{dP}{dz} = -3P^2z$ given $P = 1$ when $z = 2$

b $\frac{dy}{dx} = x + \frac{1}{3}xy$ given $y = 2$ when $x = 1$.

165 a Write $\frac{x-5}{x^2+4x+3}$ as a sum of partial fractions.

b Find the general solution to $(x^2+4x+3)\frac{dy}{dx} = \frac{x-5}{y^2}$.

166 Find the particular solution to $\frac{dy}{dx} = \frac{2x}{\sin y}$ given that $y(0) = \frac{\pi}{6}$. Find the values of x for which the solution is defined.

167 Suppose the rate of change of population of a bacterial culture is proportional to the population.

a Show that the population P at time t is given by an equation of the form $P = Ae^{kt}$.

b Given that the initial population is 1 million and the population doubles every 3 hours, find A and k .

c How long does it take for the population to reach 10 million bacteria?

d Suppose the bacteria consume a nutrient at a rate proportional to the population, and that the initial population consumes nutrient at 0.05 grams per hour. Let N be the mass of the nutrient consumed after t hours.

i At what rate will the nutrient be consumed after t hours?

ii How many grams of nutrient will be consumed in the time that it takes the population to grow to 10 million bacteria?

168 A vessel contains 100 litres of liquid industrial waste. A faulty tap at the base of the vessel allows liquid to escape at a rate proportional to the square root of the quantity of liquid remaining in the vessel.

a Show that the volume V remaining after t hours is given by $V = \left(\frac{20-kt}{2}\right)^2$ litres, where k is a positive constant.

b If, after 4 hours, 19 litres have escaped, how long will it take for the vessel to empty?

169 a Show that $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$ is homogeneous, and hence solve the differential equation.

b Find the particular solution if it passes through $(1, 4)$.

170 Show that $\frac{dy}{dx} = \frac{y}{x}(\ln y - \ln x + 1)$ is homogeneous, and hence solve for y in terms of x .

171 Consider the differential equation $\frac{dy}{dx} + 4xy = x$.

a Find the integrating factor.

b Solve the differential equation given $y(0) = 2$.

172 Find a general solution to $\frac{dy}{dx} = x - 2y$ using an integration factor.

173 Solve using the integrating factor method: $\frac{dy}{dx} + y \sin x = e^{\cos x}$, $y(\pi) = \frac{1}{e}$

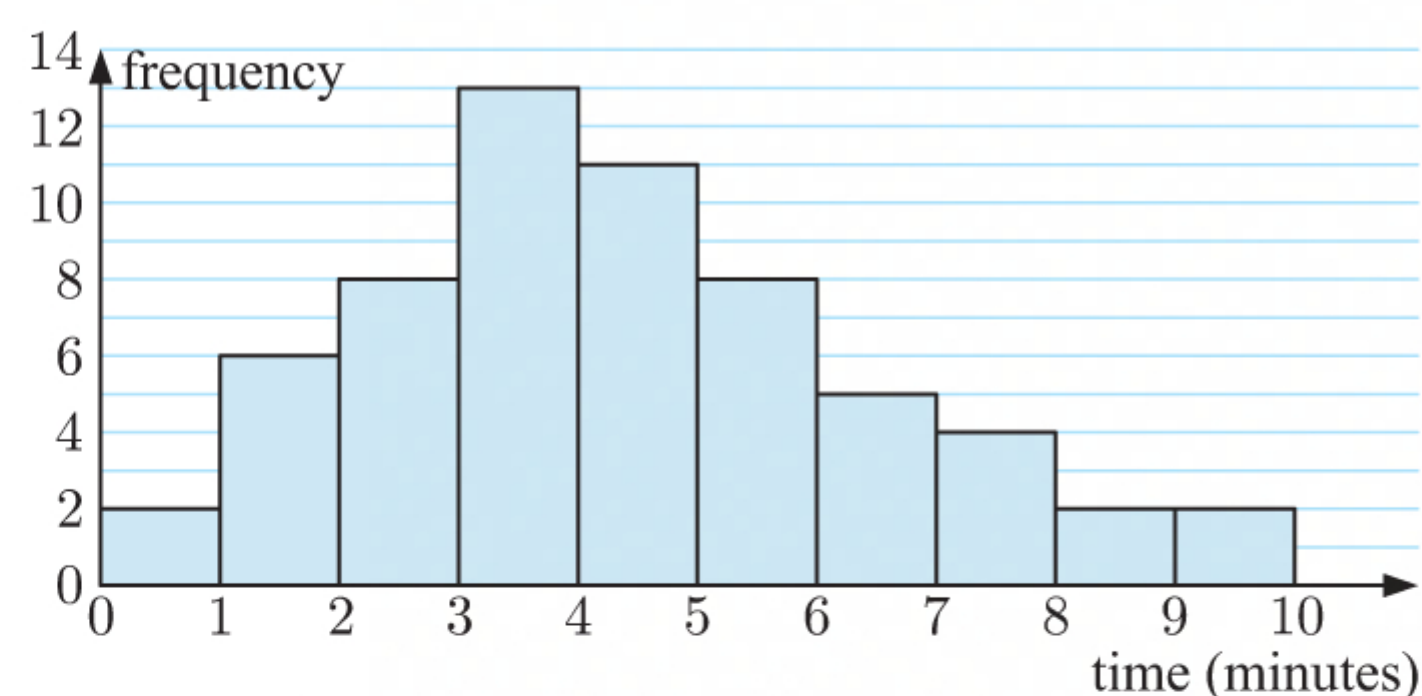
174 The displacement x of a mass on an undamped spring may be modelled by the differential equation $\frac{d^2x}{dt^2} = -k^2x$, where k is a constant.

Use the Maclaurin series expansion for $x = \sum_{m=0}^{\infty} a_m t^m$ to show that $x = a_0 \cos kt + \frac{a_1}{k} \sin kt$.

Mixed questions

MIXED QUESTIONS SET 1

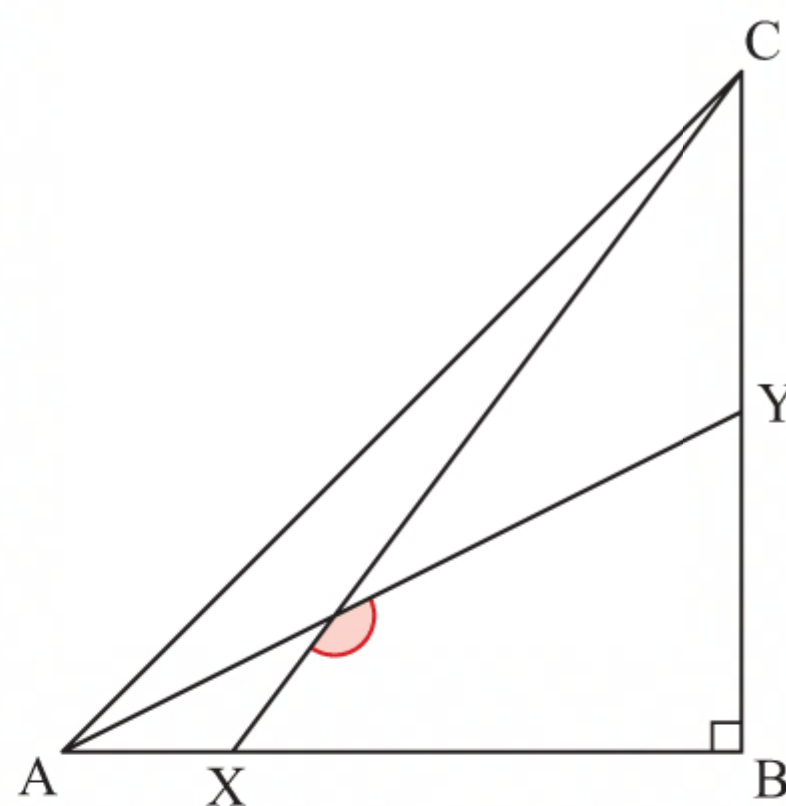
- 1 Consider the quadratic $y = 2x^2 - 9x + 3$.
 - a Find the equation of the axis of symmetry.
 - b Find the coordinates of the vertex.
 - c Find the axes intercepts.
 - d Sketch the function.
- 2 The temperature inside Pam's caravan t hours after 6 am is given by the function $T(t) = 24 + 5 \sin\left(\frac{\pi}{12}(t - 6)\right)^\circ\text{C}$.
 - a Sketch the graph of T against t for $0 \leq t \leq 24$.
 - b Find the temperature inside Pam's caravan at:
 - i 2 pm
 - ii 9 pm.
 - c Find the maximum temperature inside Pam's caravan, and the time at which it occurs.
- 3 Let $f(x) = \ln(x\sqrt{1-2x})$.
 - a State the domain of the function.
 - b Show that $f'(x) = \frac{1-3x}{x(1-2x)}$.
 - c At what point(s) on the graph of $y = f(x)$ does the normal have gradient $-\frac{6}{5}$?
- 4 Consider the functions $f(x) = 5^x$ and $g(x) = 2x + 1$.
 - a Find $(f \circ g)(x)$.
 - b Find $(f \circ g)^{-1}(0.2)$.
 - c When $f(x)$ is horizontally stretched with scale factor k , the resulting graph passes through $(\frac{1}{6}, \sqrt{5})$. Find k .
- 5 The value of a car decreases by 10% each year. After 3 years its value is \$26 244.
 - a Find the original value u_0 of the car.
 - b Write a geometric sequence to describe the value of the car u_n after n years.
 - c In what year will the value of the car fall below \$10 000?
- 6 Before selecting a new mobile phone plan, George reviews the duration of calls he made over the last 3 months. George produced the histogram alongside to illustrate the data he collected.
 - a Write down the modal class.
 - b Organise the data into a frequency table.
 - c Estimate the mean length of a phone call.
 - d Estimate the probability that George's next call will last 6 minutes or longer.
- 7
 - a Simplify $(-1 + i\sqrt{2})^3$.
 - b Write $5 + i\sqrt{2}$ in the form $a^3 \text{cis } \theta$, stating the exact values of a and θ .
 - c Find the exact solutions of $z^3 = 5 + i\sqrt{2}$.
 - d Hence show that $\arctan\left(\frac{\sqrt{2}}{5}\right) + 2\pi = 3 \arccos\left(-\frac{1}{\sqrt{3}}\right)$.
- 8
 - a Given that $\frac{1}{2}$ is a zero of $4x^3 - 8x^2 - 15x + 9$, fully factorise $4x^3 - 8x^2 - 15x + 9$.
 - b Hence solve $4x^3 - 15x > 8x^2 - 9$ exactly.
- 9 Let $f(x) = e^{ax}(x - 3)$, $a \in \mathbb{R}$. Prove by mathematical induction that $f^{(n)}(x) = a^{n-1}e^{ax}[a(x - 3) + n]$, $n \in \mathbb{Z}^+$.



- 10** ABC is a right angled isosceles triangle.

$$AX : XB = 1 : 3 \quad \text{and} \quad BY = YC.$$

Find the measure of the shaded angle.



MIXED QUESTIONS SET 2

- 1 a** Show that $\log_4(x^2 - x + 3) = \log_2 \sqrt{x^2 - x + 3}$.

b Hence solve $\log_2(x + 2) = \log_4(x^2 - x + 3)$

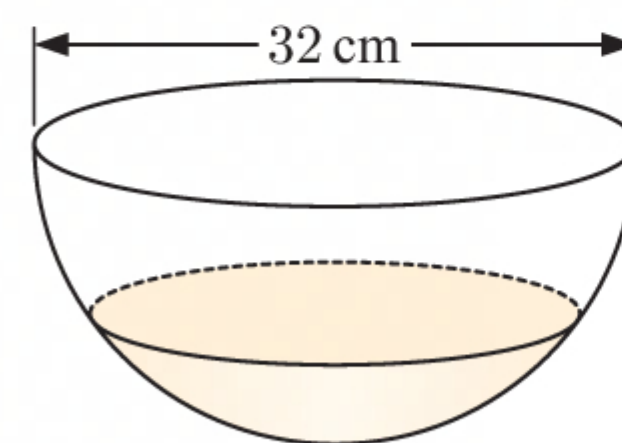
- 2** A hemispherical mixing bowl has dimensions shown.

- a** Find the capacity of the bowl.

- b** Suppose the bowl is 20% full with cake batter.

- i** How many litres of cake batter does it contain?

- ii** The cake batter is poured into a cylindrical cake tin with diameter 25 cm. How high will it reach up the tin?



- 3** Charlie and Charlotte are on a road trip in Australia. They travel 36 km north-west from Wollongong to Picton, then 210 km south-west from Picton to Canberra.

- a** How far is Canberra from Wollongong?

- b** Find the bearing of Wollongong from Canberra.

- 4** The velocity of a boat travelling in a straight line after t seconds is given by $v(t) = 30 - 20e^{-0.2t} \text{ m s}^{-1}$.

- a** Find the boat's:

- i** initial velocity

- ii** velocity after 2 seconds.

- b** How long does it take for the boat's velocity to reach 20 m s^{-1} ? Give your answer correct to two decimal places.

- c** What happens to $v(t)$ as $t \rightarrow \infty$?

- d** Calculate $v'(t)$ and show that the acceleration is always positive.

- e** Graph $v(t)$ against t , showing the information from **a** to **c**.

- f** How far did the boat travel before its velocity reached 20 m s^{-1} ?

- 5** A random variable X has the following distribution table:

x	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	k	$\frac{1}{12}$

- a** Is the random variable X discrete or continuous?

- c** Find the mode and median of X .

- b** Find k .

- d** Find $E(X)$.

- 6 a** Find \overrightarrow{AB} .

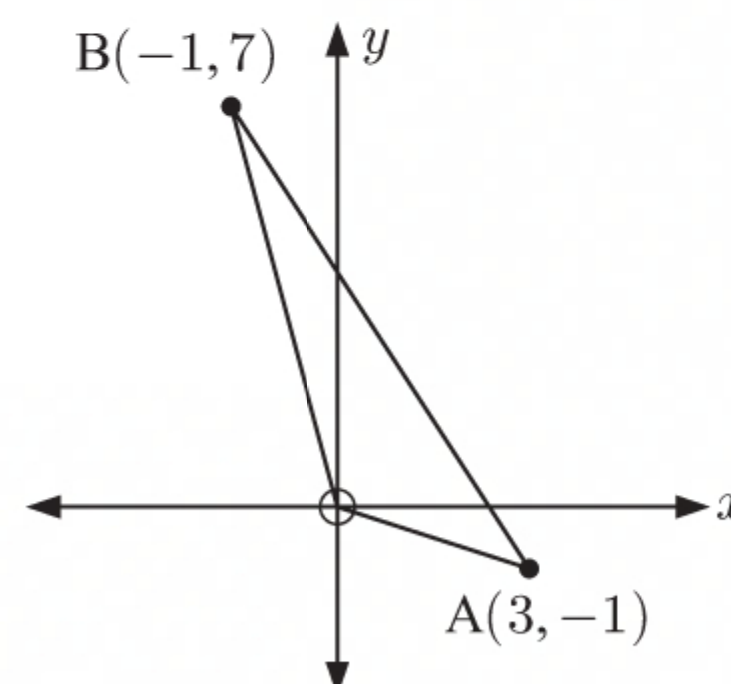
- b** Find the length of:

i \overrightarrow{OA}

ii \overrightarrow{AB} .

- c** Find the measure of \widehat{OAB} in radians.

- d** Hence find the area of triangle OAB.



- 7** $(x - 1)^2$ is a factor of $P(x) = x^4 + ax^3 + 2x^2 + bx - 3$.

- a** Find a and b .

- b** Sketch the graph of $y = P(x)$.

8 Given $f(x) = \begin{cases} x^2 + 1, & x < 0 \\ \sin x, & x \geq 0 \end{cases}$, find if possible:

a $\lim_{x \rightarrow 0^-} f(x)$

b $\lim_{x \rightarrow 0^+} f(x)$

c $\lim_{x \rightarrow 0} f(x)$

9 Let $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$.

a Calculate $I_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx$.

b Calculate $I_1 = \int_0^{\frac{\pi}{2}} x \cos x \, dx$.

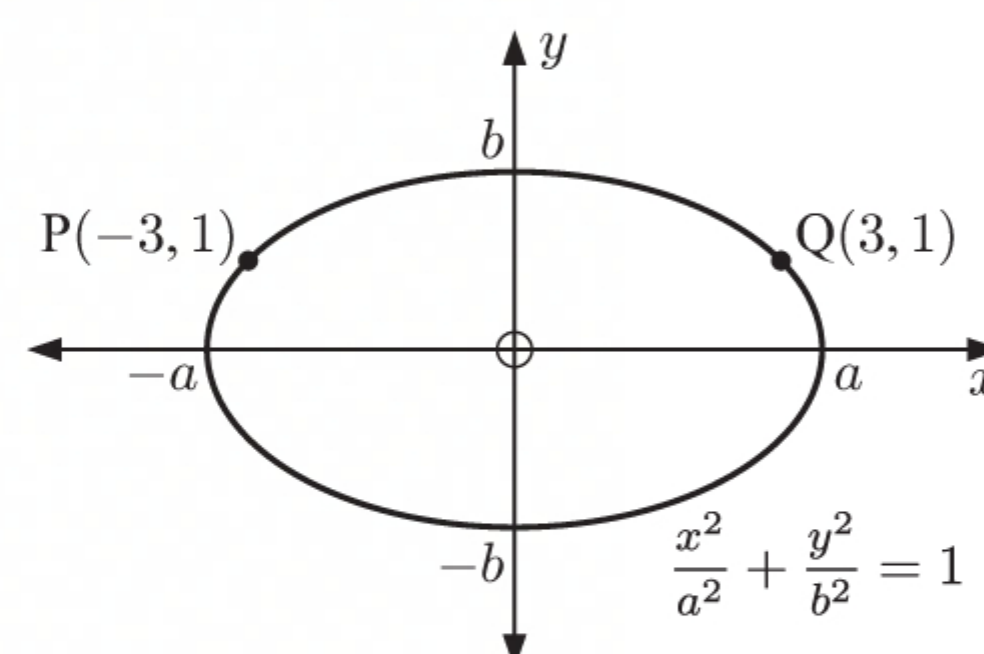
c Show that for $n \geq 2$, $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$.

d Hence find $\int_0^{\frac{\pi}{2}} x^3 \cos x \, dx$.

10 Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > 0$, $b > 0$.

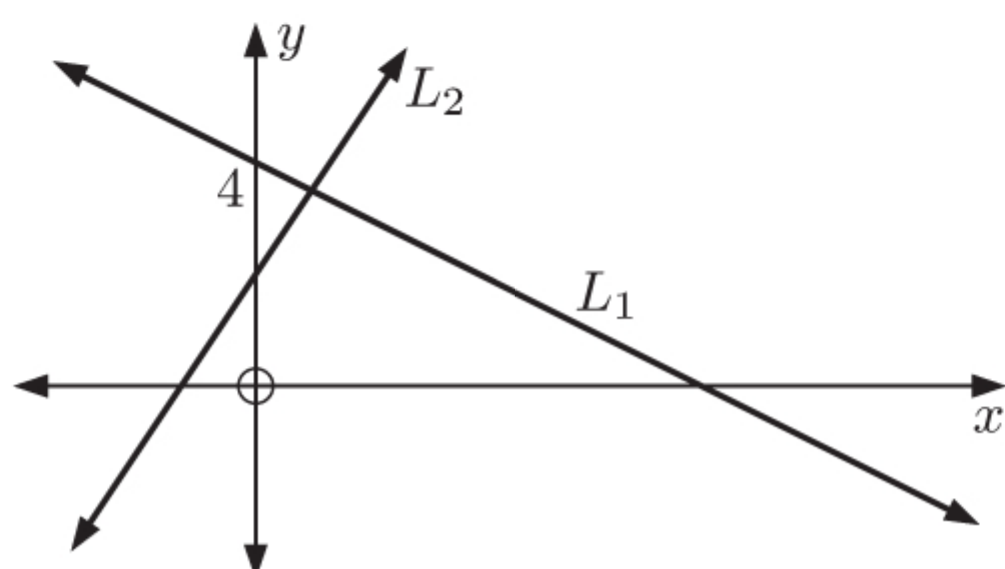
a Find $\frac{dy}{dx}$.

b The tangents at P and Q are perpendicular. Find a and b .



MIXED QUESTIONS SET 3

1



a L_1 has gradient $-\frac{1}{2}$ and passes through $(0, 4)$.

Find the equation of L_1 , giving your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$.

b L_2 passes through $(-2, -1)$ and $(4, 8)$.

Find the point of intersection of L_1 and L_2 .

2 An infinite geometric series has terms $u_1 = 27$ and $u_4 = 8$.

a Find the common ratio r .

b Find the 6th term of the series.

c Using summation notation, write an expression for the sum S of the infinite series.

d Evaluate S .

3 At an athletics competition, Carl ran the 100 m and 200 m events. His times are summarised in the table, along with the event means and standard deviations.

Event	Time (seconds)	μ (seconds)	σ (seconds)
100 m	9.99	10.20	0.113
200 m	17.30	18.50	0.706

a Assuming the times for each event are normally distributed, calculate Carl's z -scores for each event.

b Based on the results of a, in which event did he perform better?

4 Consider the function $f(x) = ax^3 - bx^2$. The line $y = x - 6$ is a tangent to $y = f(x)$ at $x = 3$.

a Find the constants a and b .

b Find the point where the tangent meets $y = f(x)$ again.

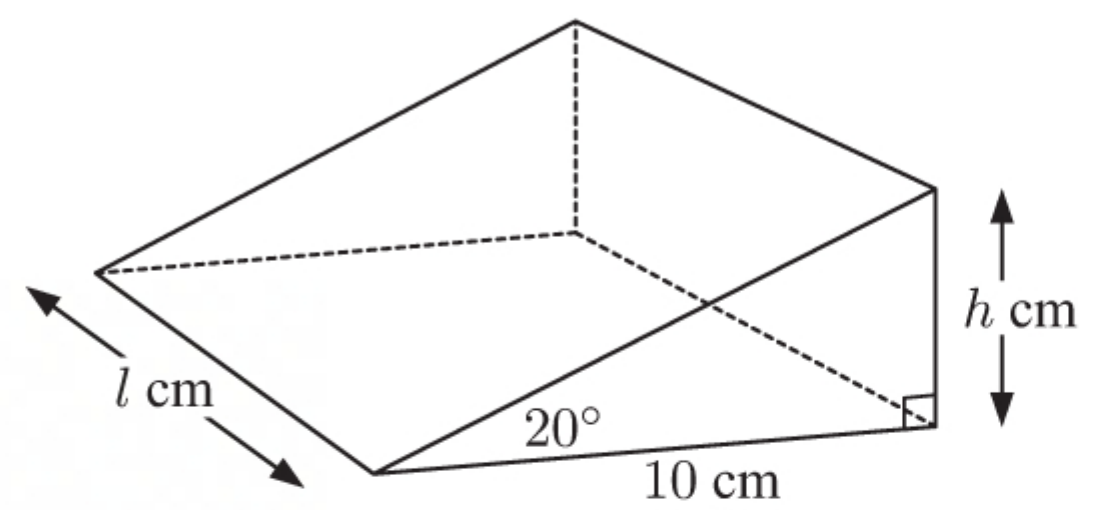
c Graph $y = f(x)$ and $y = x - 6$ on the same set of axes.

5 a Show that $\frac{\sin^2 \theta}{1 + \cos \theta} = 1 - \cos \theta$ for all θ such that $\cos \theta \neq -1$.

b Hence solve $\frac{\sin^2 \theta}{1 + \cos \theta} = \frac{1}{2}$ for $-\pi < \theta < \pi$.

- 6 A manufacturer produces wooden door-stops with the shape of the triangular prism shown.

- Calculate the height h correct to 4 significant figures.
- Determine the area of the triangular end of the prism.
- The volume of the door-stop is 60 cm^3 . Determine its length l .
- Calculate the total surface area of each door-stop. Give your answer correct to 3 significant figures.



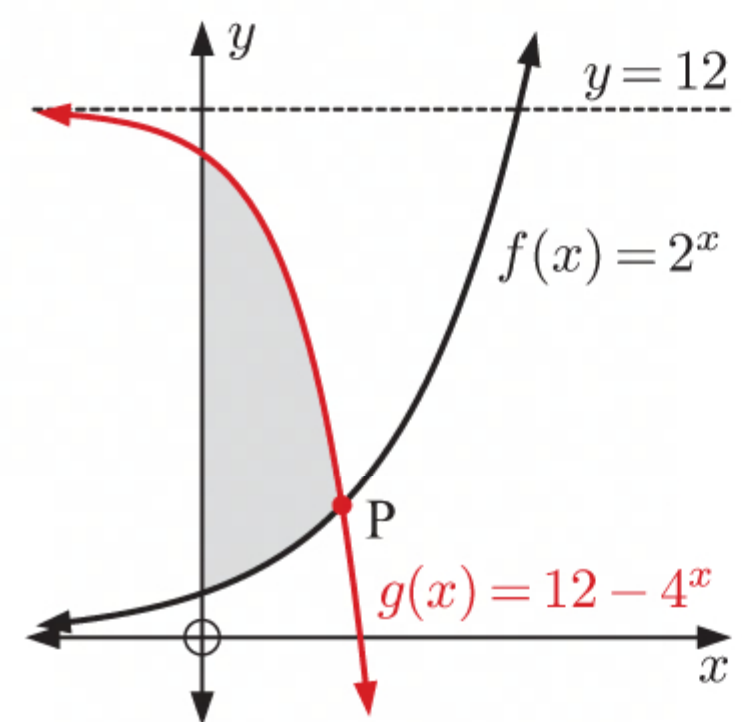
- 7 Given vectors \mathbf{x} and \mathbf{y} , the triangle inequality states that $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Let \mathbf{a} and \mathbf{b} be vectors.

- Using the triangle inequality with $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b} - \mathbf{a}$, show that $|\mathbf{b} - \mathbf{a}| \geq |\mathbf{b}| - |\mathbf{a}|$.
 - Show that $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$.
 - Hence deduce that $|\mathbf{a} - \mathbf{b}| \geq ||\mathbf{a}| - |\mathbf{b}||$.
- 8 The points $(2, 4)$, $(2, -6)$, and $(-1, 3)$ lie on a circle with equation $x^2 + y^2 + ax + by + c = 0$.
- Write three equations in the unknowns a , b , and c .
 - Find the values of a , b , and c , and hence find the coordinates of the circle's centre.

- 9 The functions $f(x) = 2^x$ and $g(x) = 12 - 4^x$ are graphed alongside.

- Find the exact coordinates of P.
- Find the shaded area, giving your answer in the form $\frac{a \ln 3 + b}{\ln 2}$, $a, b \in \mathbb{Z}$.



- 10 Let $P_1(x) = 2x + 3$, and $P_n(x)$ be the antiderivative of $P_{n-1}(x)$ for each $n \in \mathbb{Z}$, $n \geq 2$. Show that the sum of the roots of $P_n(x) = 0$ is $-\frac{3n}{2}$ for all $n \in \mathbb{Z}^+$.

MIXED QUESTIONS SET 4

- 1 The management of a large shopping centre chain sent a survey team to one of its suburban shopping centres. Between 10 am and 3 pm on a very busy Thursday, 100 people in the main mall were asked the following multiple choice question:

“At which type of shopping centre do you prefer to shop?”

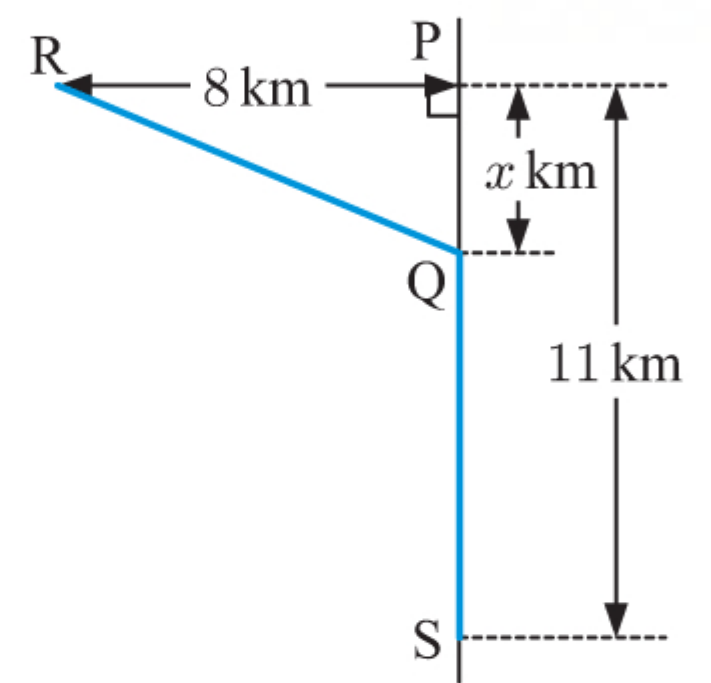
- A** suburban **B** central city **C** equally preferred **D** neither **E** no opinion

- Give *two* reasons why this survey is likely to contain a coverage error.
- The results were: suburban 33%, central city 8%, equally preferred 51%, neither 4%, no opinion 4%

Management concluded that “*more than four times as many people prefer suburban shopping to the central city*”. Explain why this conclusion is unreasonable.

- 2 The current in an electrical circuit t milliseconds after it is switched off is given by $I(t) = 40e^{-0.1t}$ amps.
- What current was flowing in the circuit initially?
 - What current was flowing in the circuit after 100 milliseconds?
 - Sketch $I(t)$ and $I = 1$ on the same set of axes.
 - How long did it take for the current to fall to 1 amp?
- 3 Suppose $a > b > c > 0$.
- Show that: **i** $a^2 - b^2 > 0$ **ii** $b^2 - c^2 > 0$
 - Hence show that $(ab)^2 + (bc)^2 - (ac)^2 > b^4$.

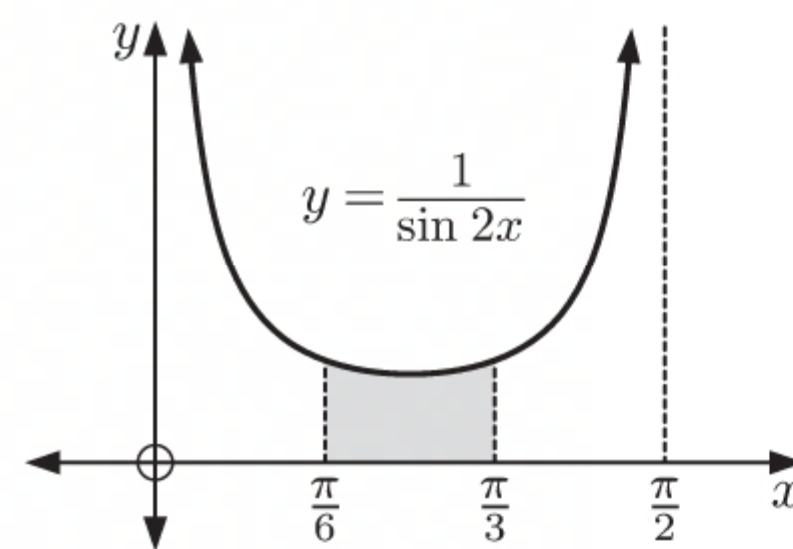
- 4 An offshore oil rig is at point R, 8 km from a straight shore. The point P is on the shore directly opposite the rig. A refinery is on the shore at S which is 11 km from P. A pipeline is to be constructed under the sea from R to reach the shore at the point Q. From Q a pipeline is to be taken overland to S. The cost of the pipeline is \$5 million per km under the sea and \$3 million per km overland.



- a If Q is x km from P, show that the cost to construct the pipeline from R to S is $C(x) = 5\sqrt{x^2 + 64} + 33 - 3x$ million dollars.
- b Find the minimum cost of the pipeline.
- 5 A tinned food company examined a sample of its tins of corn and tins of pineapple for defects. The results are summarised in the table alongside.

	Defective	Not defective
Corn	37	581
Pineapple	24	617

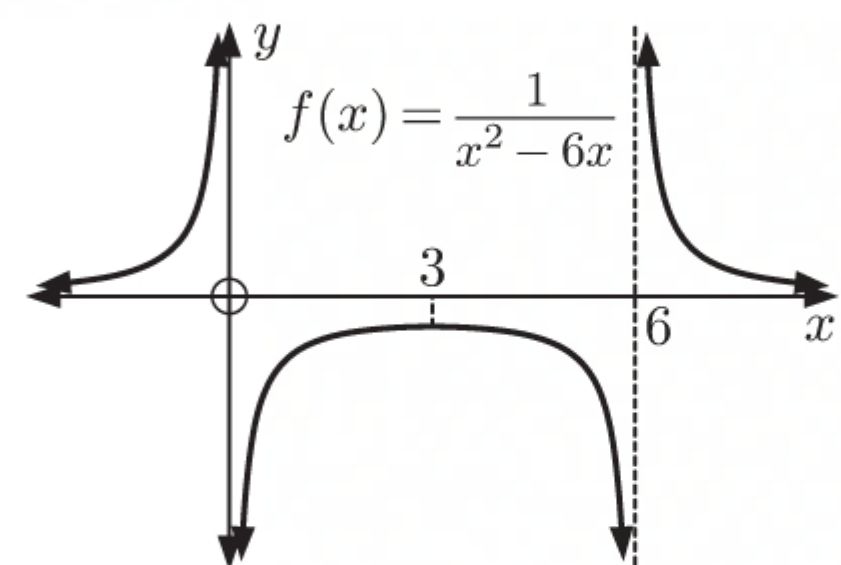
- a How many tins were included in the sample?
- b Estimate the probability that the next randomly selected tin:
- is not defective
 - is a defective tin of pineapple
 - is defective, given it is a tin of corn.
- 6 a If $y = \ln(\tan x)$, $0 < x < \frac{\pi}{2}$, show that $\frac{dy}{dx} = \frac{k}{\sin 2x}$ for some constant k .
- b Alongside is a graph of $y = \frac{1}{\sin 2x}$.
- Show that the shaded area is $\frac{1}{2} \ln 3$ units².



- 7 Find an equation for the plane passing through the point $(3, -1, 2)$, which is parallel to the vectors $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$. Give your answer in:

- a parametric form
- b Cartesian form.

- 8 The graph of $f(x) = \frac{1}{x^2 - 6x}$ is shown alongside.
- a Find the domain and range of $f(x)$.
- b Does $f(x)$ have an inverse function? Explain your answer.
- c Find the inverse function of $g(x) = \frac{1}{x^2 - 6x}$, $x \geq 3$, $x \neq 6$.
- State the domain and range of the inverse function.

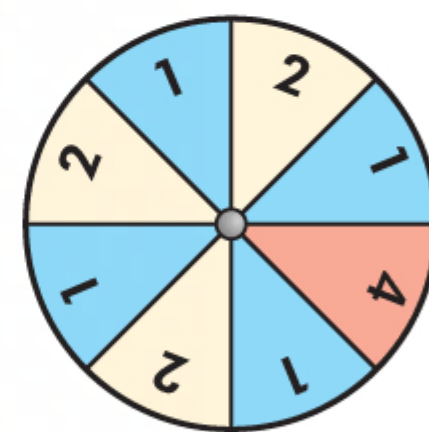


- 9 Solve the differential equation $\frac{dy}{dx} - 3y = 2x^2$, $y(0) = -\frac{4}{27}$ using:
- a an integrating factor
- b a Maclaurin series of the form $y = \sum_{k=0}^{\infty} a_k x^k$.
- 10 One of the five fifth roots of the complex number $a + bi$ is $1 + i$. Without finding a and b , find the other four roots in polar form.

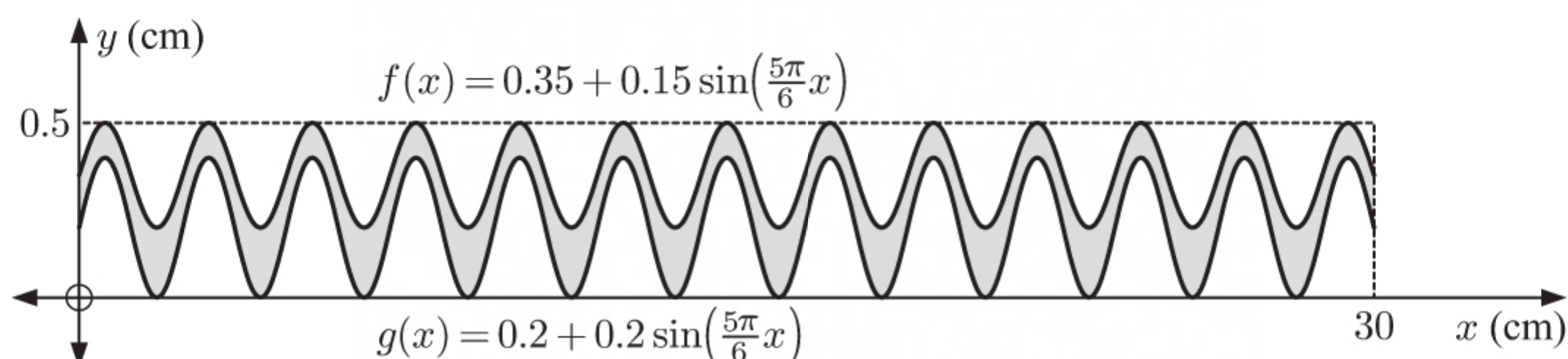
MIXED QUESTIONS SET 5

- 1 a Determine the discriminant of $x^2 + 8x + k = 0$.
- b Hence find the values of k for which the equation has:
- no real roots
 - two distinct real roots.
- 2 Let $f(x) = \frac{x-3}{2-x}$.
- a State the domain and range of f .
- b Write down the equations of the asymptotes of $y = f(x)$.
- c Find the axes intercepts of $y = f(x)$.
- d Sketch $y = f(x)$, showing the features you have found.

- 3 Solve for x exactly: $\sqrt{2} \sin\left(2\left(x - \frac{\pi}{6}\right)\right) = 1$, $-\pi \leq x \leq 2\pi$.
- 4 A game is played in which the wheel shown is first spun by the player, and then by the game operator. The player wins \$ a if their spin is higher than the operator's. It costs \$ k to play the game. Find the relationship between a and k so that the game is fair.



- 5 The cross-section of a 1 m long strip of cardboard is shown below.



Find the volume of the cardboard.

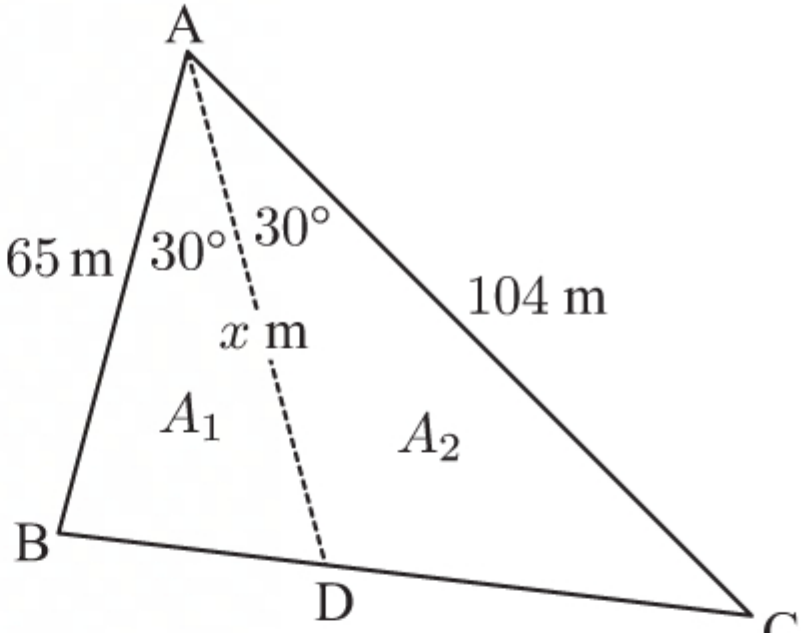
- 6 The population of a hive of bees is given by $P(t) = 120 \times (2.25)^{\frac{t}{3}}$, where t represents time in weeks.
- Sketch $P(t)$ for $0 \leq t \leq 20$.
 - Find the population of bees in the hive after 10 weeks.
 - Write a function for t in terms of P .
 - How long will it take for the population to reach 5000?
- 7 A blood test is developed to detect a rare disease, which affects 0.06% of the population. When the test is administered on an individual with the disease, the disease is detected 98% of the time. When administered on an individual without the disease, a false positive result occurs 1% of the time.
- Find the probability that a randomly selected person tests positive for the disease.
 - If a randomly selected person tests positive for the disease, use Bayes' theorem to find the probability that they actually have the disease.
- 8 The base of a cone lies in the XY -plane, and has centre $(-2, 5, 0)$. The point $(4, 2, 0)$ lies on the edge of the base, and the cone has volume 90π units³.
- Find the height of the cone.
 - Find the surface area of the cone.
 - The point $(k, 7, 2)$ lies on the curved surface of the cone. Find the possible values of k .
- 9 Consider the curve $y = -2x^2 + 3$.
- Differentiate the function from first principles.
 - Find the equation of the normal to the curve at the point where $x = -1$.
 - At what point does the normal in **b** meet the curve again?
- 10 Given the sequence defined by $u_{n+2} = u_n + u_{n+1}$ where $u_1 = u_2 = 1$ and $n \in \mathbb{Z}^+$, use mathematical induction to prove that $u_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$.

MIXED QUESTIONS SET 6

- 1 The table shows the amount of petrol remaining in a motorbike's fuel tank and the number of kilometres travelled. The capacity of the tank is 10 litres.

Remaining fuel (x litres)	10	8	6	4	2	1
Distance (y km)	0	90	190	260	330	370

- Plot this data on a scatter diagram.
- Find the equation of the regression line for y against x .
- Interpret the y -intercept of the regression line.
- The motorbike has travelled 220 km since its tank was refilled.
 - Use your equation to estimate the amount of fuel left in the tank.
 - Find the average distance travelled per litre over the 220 km.

- 2** A particle is initially at rest. It moves in a straight line with acceleration $a(t) = 2 - 6t \text{ m s}^{-2}$, where t is the time in seconds, $t \geq 0$.
- Find the velocity function.
 - Find the change in *displacement* of the particle in the first second.
 - Find the *total distance* travelled by the particle in the first second.
- 3** The probability of Mark waking up early is 0.8. If he wakes up early, he will pack lunch with probability 0.6. If he does not wake up early, he will pack lunch with probability 0.15.
- Display the sample space of possible outcomes on a tree diagram.
 - Hence determine the probability that Mark will pack lunch today.
- 4** A farmer owns a triangular field ABC.
- D is the point on [BC] such that [AD] bisects \widehat{BAC} . The farmer divides the field into two parts A_1 and A_2 by constructing a straight fence [AD] of length x m.
- 
- Use the cosine rule to calculate the length of [BC].
 - Find the total area of the field.
 - Find, in terms of x , the area of:
 - A_1
 - A_2 .
 - Hence find x .
- 5** Suppose that p , q , and r are consecutive odd integers, $p < q < r$. Show that $2q(p + r)$ is a perfect square.
- 6** Consider the curve $y = xe^{2x}$.
- Find the exact value of $k \in \mathbb{R}$ such that $y = k$ is a horizontal tangent to the curve.
 - For which values of $k \in \mathbb{R}$ does the line $y = k$ meet the curve at:
 - exactly one point
 - two distinct points
 - no points?
 - Now consider the family of curves $y = xe^{ax}$, $a \in \mathbb{R}$, $a > 0$.
 - Show that $y = x$ is a tangent to all such curves and find the point of contact.
 - Find the equation of the normal to $y = xe^{ax}$, $a \in \mathbb{R}$, $a > 0$, when $x = 0$, and find the acute angle this normal makes with the x -axis.
- 7** Line L_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$, $t \in \mathbb{R}$. Line L_2 cuts the X -axis at 3 and the Y -axis at -5 .
- Find the point where L_1 meets the XY -plane.
 - Find parametric equations for L_2 .
 - The line L_3 is perpendicular to both L_1 and L_2 , and passes through $(-2, 5, 1)$. Find the equation of L_3 in Cartesian form.
- 8** The probability density function of a continuous random variable X is given by $f(x) = \begin{cases} \frac{1}{3} \sin \frac{x}{2}, & 0 \leq x < \pi \\ -\frac{1}{3} \cos x, & \pi \leq x \leq \frac{3\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$
- Sketch the graph of $f(x)$.
 - Check that $f(x)$ is a valid probability density function.
 - Find $P(\frac{2\pi}{3} < X < \frac{7\pi}{6})$.
 - Let m be the median of X . Show that $m = 2 \arccos \frac{1}{4}$.
 - Find the mean of X .
- 9** Show that $\frac{dy}{dx} = \frac{5x - 2y}{2x - y}$ is homogeneous, and find the particular solution given $y(1) = 3$.
- 10**
 - Use complex number methods to show that $\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$.
 - Hence find the roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

MIXED QUESTIONS SET 7

- Suppose $f(x) = 4x - 3$ and $g(x) = x + 2$. Find the value of x such that $(f \circ g^{-1})(x) = f^{-1}(x)$.
- Suppose $f'(x) = a\sqrt{x} + bx$ where a and b are constants. Find $f(x)$ given that $f(0) = -4$, $f(1) = -1$, $f(2) = 4\sqrt{2}$.

- 3 The data below are the recent sale prices, in thousands of dollars, of houses in two neighbourhoods.

<i>Neighbourhood A:</i>	275	281	320	265	305	258	310	430	285
	290	297	345	195	230	269	300	258	273
<i>Neighbourhood B:</i>	325	300	412	370	297	505	340	333	290
	428	305	520	360	410	275	320	431	410

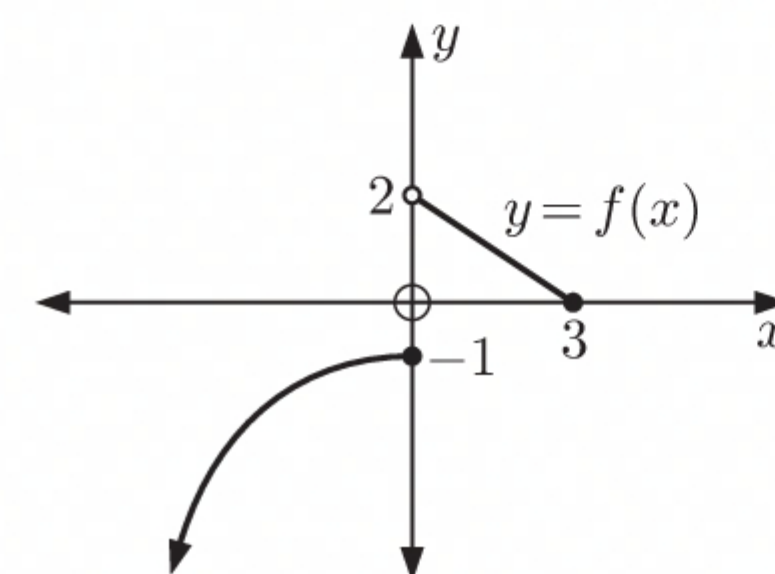
- a Is the data discrete or continuous?
 b Use technology to find the five-number summary for each data set.
 c Display the data in a parallel box plot.
 d Compare and comment on the distributions of each data set.
- 4 In triangle PQR, $\widehat{PRQ} = 40^\circ$, $PR = 12$ cm, and $PQ = 8$ cm.
 a Show that there are two possible measures of \widehat{PQR} .
 b Sketch triangle PQR for each case.
 c In each case, find:
 i the measure of \widehat{QPR}
 ii the perimeter of the triangle.

- 5 Consider the graphs with equations $y = \frac{2}{x}$ and $y = x - 1$.
 a Solve $\frac{2}{x} = x - 1$ using algebra.
 b Use technology to plot the graphs on the same set of axes.
 c Hence solve for x : $\frac{2}{x} < x - 1$

- 6 The coefficients of the first three terms of $(x + a)^3$ form an arithmetic sequence. Find the constant a .

- 7 The function $y = f(x)$ is illustrated.

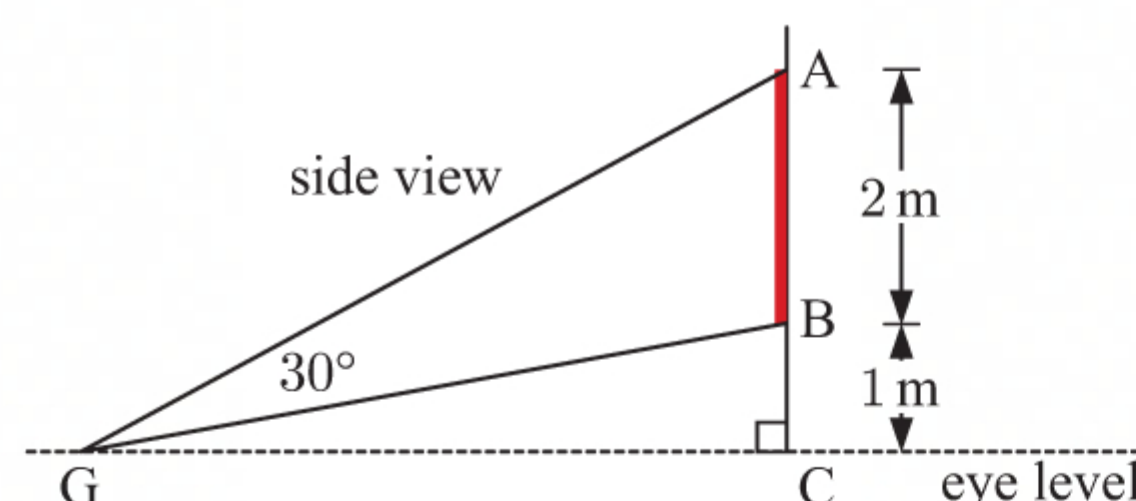
- a State the domain and range of $y = f(x)$.
 b Sketch the graph of:
 i $y = -2f(x)$
 ii $y = |f(x)|$
 c Suppose $g(x) = f(|x|)$.
 i Is $g(x)$ odd or even?
 ii Does $g(x)$ have an inverse?



- 8 [AB] represents a painting on a wall.

The angle of view observed by a girl between the top and bottom of the painting is 30° .

How far is the girl from the wall?



- 9 Let $f(x) = xe^{-2x^2}$ on the interval $0 \leq x \leq 2$.

- a Find the stationary point of $f(x)$ on this interval.
 b Hence determine the maximum and minimum values of $f(x)$ on this interval.
 c The area between $y = f(x)$ and the x -axis, from $x = 0$ to $x = 2$, is revolved through 360° about the x -axis.
 Find the volume of the resulting solid of revolution.

- 10 a Prove that:

i $\frac{\sin a}{\sin \frac{a}{2}} = 2 \cos \frac{a}{2}$ ii $\frac{1}{2} \sin(a + b) + \frac{1}{2} \sin(a - b) = \sin a \cos b$

iii $\frac{1}{2} \cos(a - b) - \frac{1}{2} \cos(a + b) = \sin a \sin b$

- b By letting $a = \frac{P+Q}{2}$ and $b = \frac{P-Q}{2}$ in a iii, show that $\cos P - \cos Q = -2 \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right)$.

- c Show by induction that $\sin \theta + \sin(\theta + a) + \sin(\theta + 2a) + \dots + \sin(\theta + na) = \frac{\sin\left(\frac{(n+1)a}{2}\right) \sin\left(\theta + \frac{na}{2}\right)}{\sin \frac{a}{2}}$ for all $n \in \mathbb{Z}^+$.

MIXED QUESTIONS SET 8

- 1** The heights X of maize plants two months after planting are normally distributed with mean μ cm and standard deviation 6.8 cm. 75% of a crop of maize plants are less than 45 cm high.

a Find:

i μ

ii $P(X < 25)$

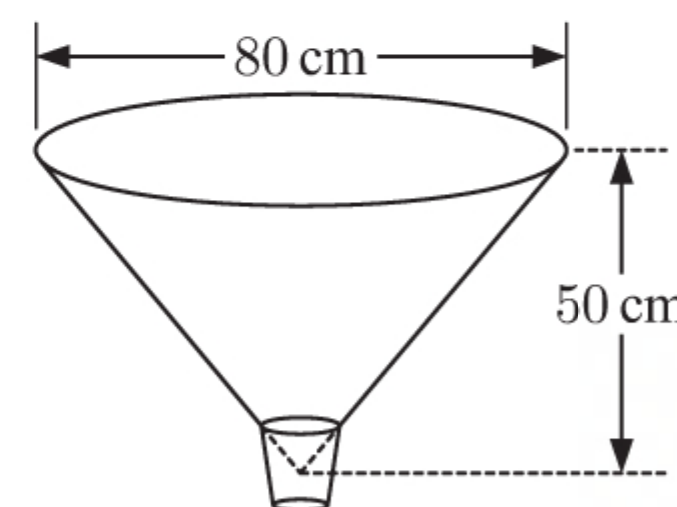
iii a such that $P(X < 25) = P(X > a)$.

b Six maize plants are randomly chosen. Find the probability that exactly four of them are more than 35 cm high.

- 2** A conical funnel is 80 cm wide and 50 cm high.

a Estimate the capacity of the funnel in mL. Write your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

b The funnel is half full with liquid, and its contents are poured into a cylindrical tube 20 cm wide. How high up the tube will the liquid reach?



- 3** At time t seconds, the tip of a pendulum has acceleration $6 \cos 2t \text{ cm s}^{-2}$. At $t = 0$, the pendulum is stationary.

a Find the speed of the tip of the pendulum after 4 seconds.

b Find the total distance travelled by the tip of the pendulum in the first 5 seconds to two decimal places.

- 4** In a busy harbour, the time difference between successive high tides is 12.3 hours. The water level varies by 2.4 metres between high and low tide. The first high tide of the day is 4.7 metres, occurring at 1 am.

a Find a cosine model for the height H of the tide t hours after midnight.

b Sketch a graph of the water level in the harbour for $0 \leq t \leq 24$.

- 5** The probability of rain falling on any day in Dunedin is 0.4. The tree diagram shows the possible outcomes when two consecutive days are considered.

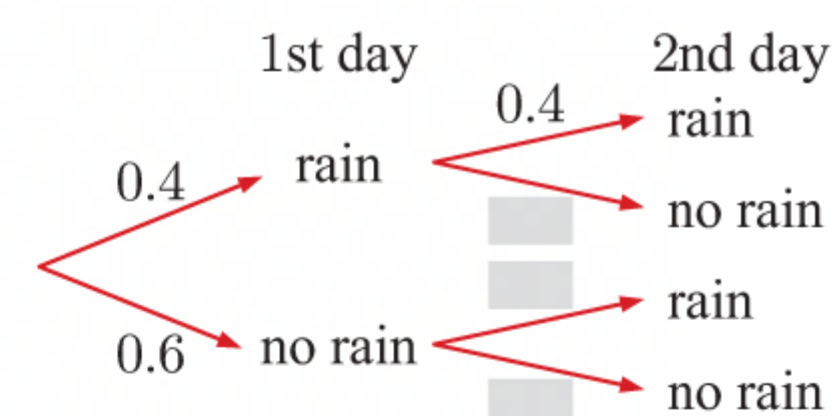
a Complete the tree diagram by filling in the missing probabilities.

b Hence determine the probability of:

i rain on both days

ii no rain on exactly one day.

c Given that rain fell on at least one day, find the probability of rain on the second day.



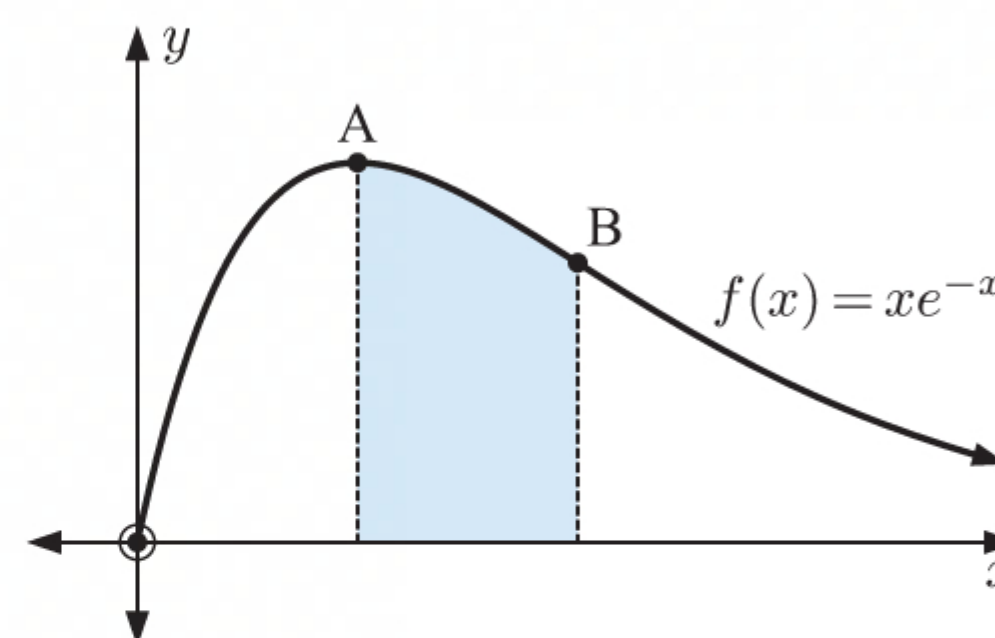
- 6** The graph of $f(x) = xe^{-x}$, $x \geq 0$ is shown.

a Find the y -intercept.

b Find $f'(x)$ and hence find the coordinates of A.

c Find the exact x -coordinate of the point of inflection B.

d Find the area of the shaded region.



- 7** Let $g(x) = \frac{3-x}{x+1}$.

a Sketch $y = g(x)$, clearly showing any axes intercepts and asymptotes.

Hence sketch $y = \frac{1}{g(x)}$ on the same set of axes.

b Explain what happens to $y = \frac{1}{g(x)}$ around $x = -1$.

c State the coordinates of any invariant points.

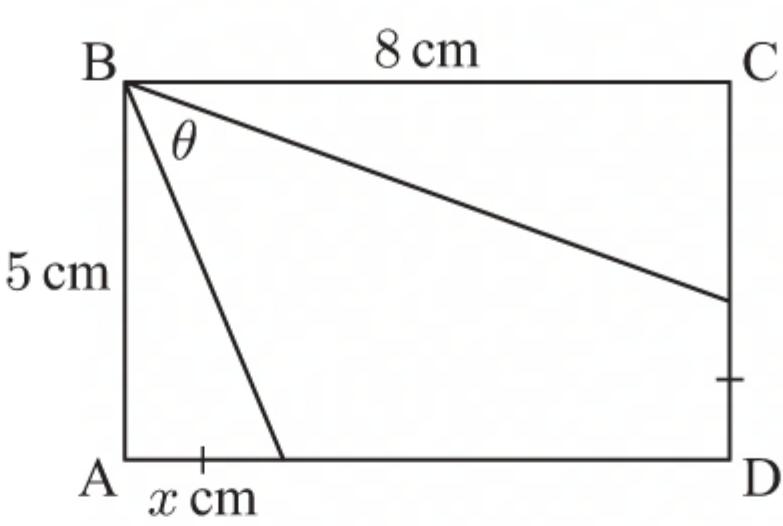
d Explain what transformations can be used to produce $y = g(x)$ from $y = \frac{1}{x}$.

- 8** z is a complex number and z^* is its complex conjugate. Show that if $z^2 = (z^*)^2$ then z is either real or purely imaginary.

- 9** **a** Write $\sqrt{3} \sin x - \cos x$ in the form $A \sin(x + \alpha)$ for $A > 0$ and $0 < \alpha < 2\pi$.

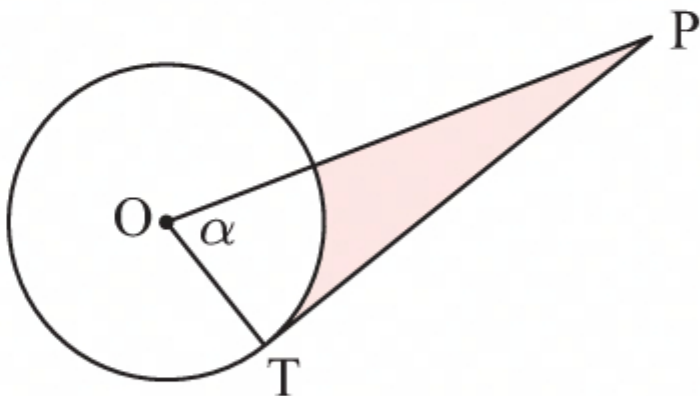
b Hence solve the equation $\sqrt{3} \sin x - \cos x = 1$ for $0 \leq x \leq 2\pi$.

- 10 ABCD is a rectangle.
Find the value of x which minimises θ .



MIXED QUESTIONS SET 9

- 1 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph cuts the x -axis at -4 , passes through $(1, 5)$, and has axis of symmetry $x = -1$.
- 2 [PT] is a tangent to the given circle. The circle has radius 9 cm and $OP = 30$ cm. Find:
a α b the area of the shaded region.



- 3 Solve for $-\pi \leq x \leq \pi$: $2 \sin^2 x = 3 \cos x + 2$
- 4 The distance travelled by two similar toy cars after rolling down a slope was measured 40 times each. The measurements were rounded to the nearest tenth of a metre.

Red car	3.6	4.6	5.6	6.4	4.2	5.3	6.1	4.5
	5.4	4.6	3.9	6.2	5.8	4.5	5.4	6.1
	4.5	5.6	5.7	4.8	3.9	5.6	6.1	5.9
	4.1	5.3	4.2	6.2	7.4	5.4	5.8	4.5
	3.9	5.4	5.7	4.8	5.4	5.7	6.1	6.4

Blue car	Number of rolls	40
	Median distance	4.8 m
	Shortest distance	3.2 m
	Longest distance	6.7 m
	Q_1 Lower quartile	4.1 m
	Q_3 Upper quartile	5.4 m

- a Complete this table of cumulative frequencies for the red car data.
- b Draw the cumulative frequency graph for the distance travelled by the red car.
- c Use the graph to find the following statistics for the red car:
- i median distance
 - ii lower quartile
 - iii upper quartile
- d Draw a parallel box and whisker diagram to display the data for both cars.
- e Compare the statistics for distance travelled by the two toy cars. Is it reasonable to assume that the same machine manufactured these two toys? Explain your answer.

Distance (m)	Cumulative frequency
$3.5 \leq d < 4$	
$4 \leq d < 4.5$	
$4.5 \leq d < 5$	
$5 \leq d < 5.5$	
$5.5 \leq d < 6$	
$6 \leq d < 6.5$	
$6.5 \leq d < 7$	
$7 \leq d < 7.5$	

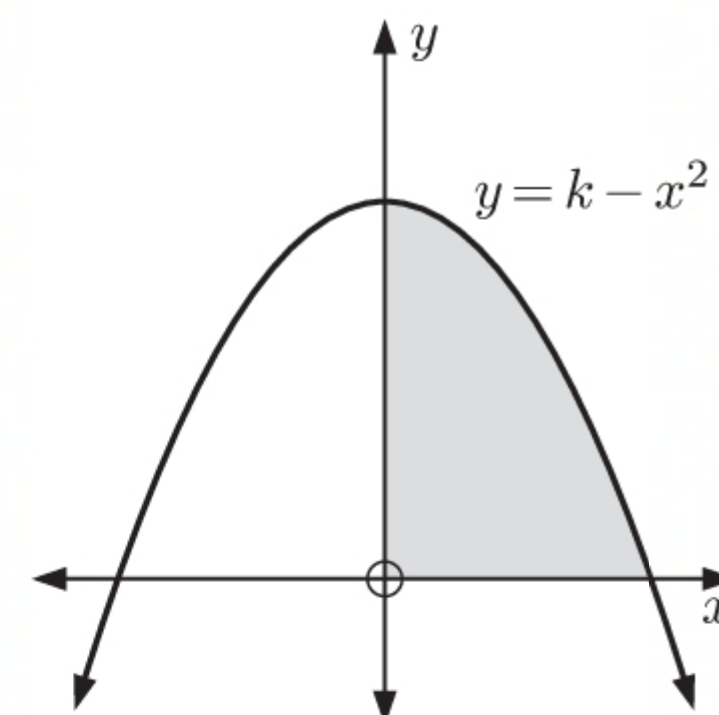
- 5 Solve for x : $4^x + 4 = 17(2^{x-1})$
- 6 An arithmetic sequence has common difference d . The series sums S_3 , S_6 , and S_8 themselves form an arithmetic sequence. Find, in terms of d , the common difference for this sequence.
- 7 There are 12 students in a school’s Hungarian class. Being well-mannered, they line up in a single file to enter the class.
- a How many orders are possible?
- b How many orders are there if:
- i Irena and Eva are among the last four in the line
 - ii Istvan is between Paul and Laszlo and they are all together
 - iii Istvan is between Paul and Laszlo but they are not necessarily together
 - iv there are exactly three students between Annabelle and Holly?
- c Once inside, the class is split into 3 groups of four students each for a vocabulary quiz. In how many ways can this be done if:
- i there are no restrictions
 - ii Ben and Marton must be in the same group?

- 8 $1 + i$ is a zero of $f(x) = x^4 + ax^3 - 2x^2 + 10x + 4a$ where $a \in \mathbb{R}$.
- Find a .
 - Sketch $y = f(x)$, showing all axes intercepts.
 - Verify the sum and product of roots theorem for $f(x)$.
- 9 A plane P has equation $x + y + z = 1$.
- Find the foot of the normal N from $A(1, 1, 1)$ to P .
 - Hence find the shortest distance from A to P .
 - Find the coordinates of the reflection of A in the plane P .

- 10 Consider the graph $f(x) = k - x^2$ alongside, where $k > 0$.

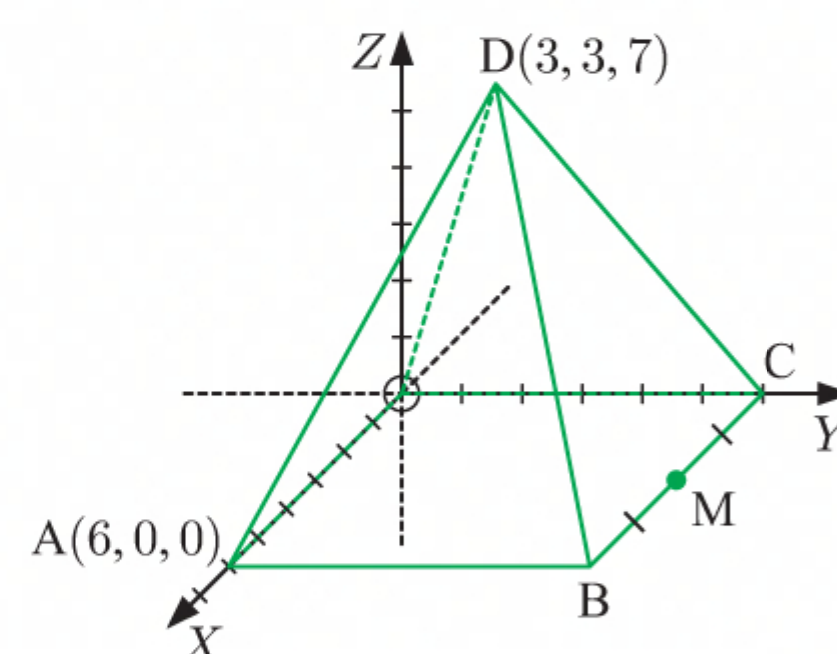
The volume when the shaded area is rotated about the x -axis is equal to the volume when it is rotated about the y -axis.

Find the value of k .

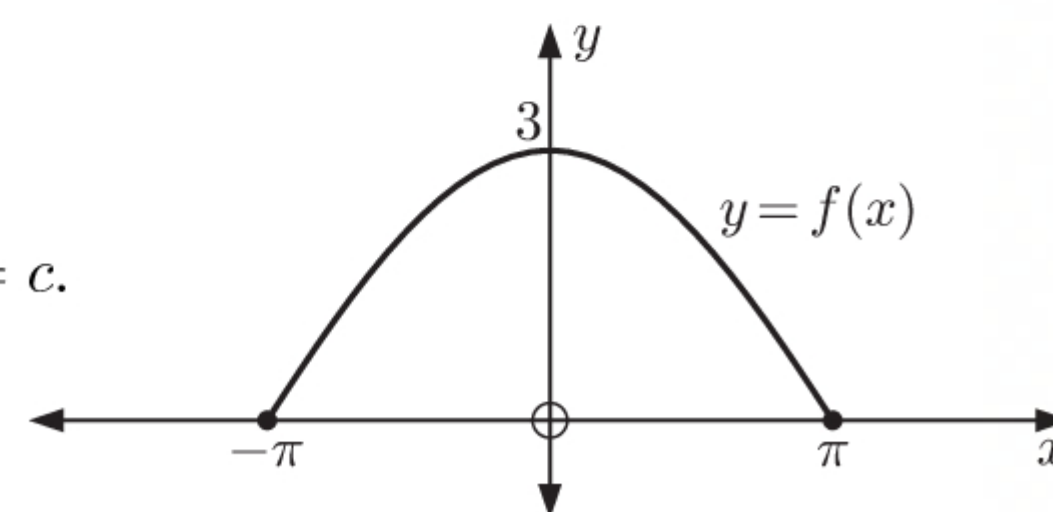


MIXED QUESTIONS SET 10

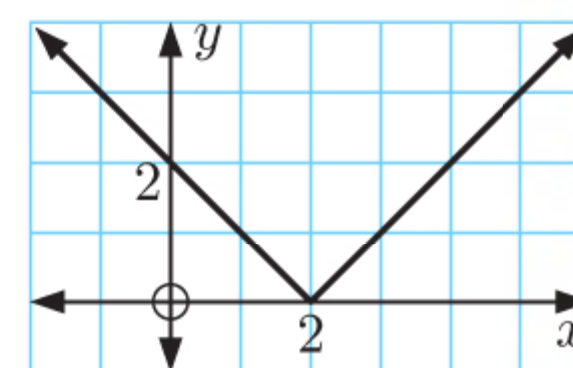
- 1 Consider the square-based pyramid alongside. Find:
- the coordinates of B and C
 - the volume of the pyramid
 - the coordinates of M
 - the surface area of the pyramid.



- 2 The function f has the form $f(x) = a \cos bx$, $-\pi \leq x \leq \pi$.
- State the values of a and b .
 - Find the equation of the normal to $y = f(x)$ at the point where $x = c$.
 - Find the values of c such that the normal passes through the origin.
 - Sketch $y = f(x)$ and the normals found in **b ii** on the same set of axes.

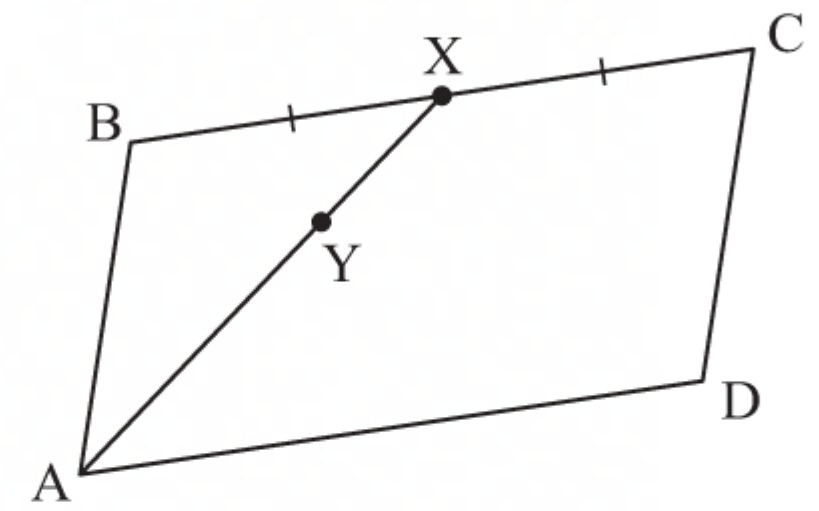


- 3 The graph alongside shows a relation between x and y .
- Is the relation a function? Explain your answer.
 - State the domain and range of the relation.
 - Sketch the result when the graph is translated through $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, then reflected in the x -axis.



- 4 Yiren is filling her swimming pool with water. She suddenly realises it is overflowing and turns the tap off. The water continues to overflow at the rate $R(t) = \frac{12}{\sqrt{t+1}} e^{-\sqrt{t+1}} \text{ L s}^{-1}$ where t is in seconds, $t \geq 0$.
- At what rate is the water still overflowing after 10 seconds?
 - Find $\int R(t) dt$ using an appropriate substitution.
 - Hence find $\int_0^{60} R(t) dt$.
 - Interpret the meaning of the integral found in **c**.
- 5 A university club committee holds weekly meetings. Each committee member has a 70% chance of attending a given meeting. A meeting can only go ahead if at least 10 committee members are present.
- If the club has 15 committee members, what percentage of meetings will go ahead?
 - Find the smallest number of committee members required to ensure that at least 90% of the meetings will go ahead.

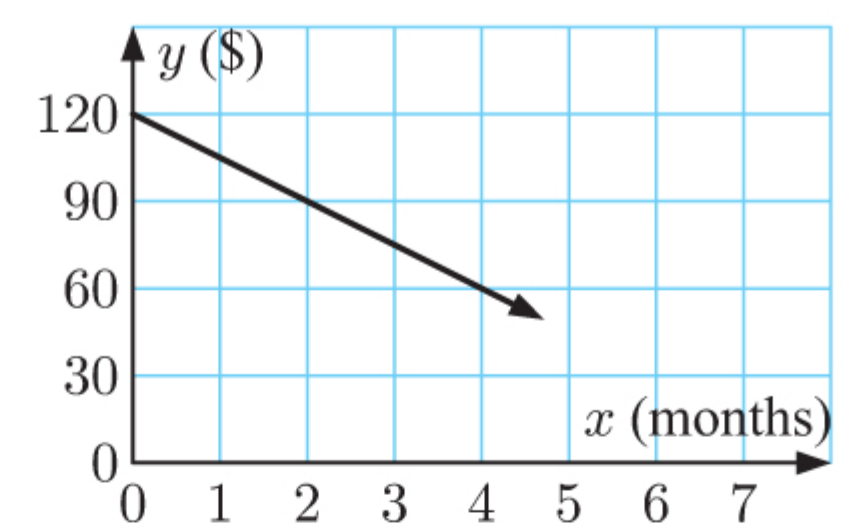
- 6** The displacement of an object after t seconds is $s = \sin\left(\frac{\pi}{(t+1)^2}\right)$ cm, $t \geq 0$.
- Find the displacement of the object after 1 second.
 - Find the first time that the object has displacement 0.5 cm.
 - Show that the velocity of the object is $v = -\frac{2\pi}{(t+1)^3} \cos\left(\frac{\pi}{(t+1)^2}\right)$ cm s⁻¹.
 - Find the first time that the object is stationary.
- 7** The roots of the equation $3x^2 + 3x - 5 = 0$ are α and β . Find a quadratic equation with roots:
- $-\alpha$ and $-\beta$
 - $\frac{\alpha}{2}$ and $\frac{\beta}{2}$
- 8**
- Write $\frac{2x-1}{x^2-x-2}$ in the form $\frac{A}{x+1} + \frac{B}{x-2}$ where A and B are constants.
 - Using binomial expansions, write an expansion for $\frac{2x-1}{x^2-x-2}$ up to the term in x^3 .
 - For what values of x does the complete binomial expansion of $\frac{2x-1}{x^2-x-2}$ converge?
 - Use your expansion to estimate the value of $\frac{2x-1}{x^2-x-2}$ when $x = 0.01$.
- 9** In the given figure, ABCD is a parallelogram. X is the midpoint of [BC], and Y lies on [AX] such that $AY : YX = 2 : 1$.
Prove that B, D, and Y are collinear.



- 10** Consider the differential equation $\frac{dy}{dx} = \frac{1}{1+x^2}$ with $y(1) = \pi$.
- Estimate $y(2)$ using Euler's method with step size 0.2.
 - Use technology to apply Euler's method with step size 0.005 for 200 steps.
 - Find the exact value of $y(2)$ using the Fundamental Theorem of Calculus. Comment on your results.

MIXED QUESTIONS SET 11

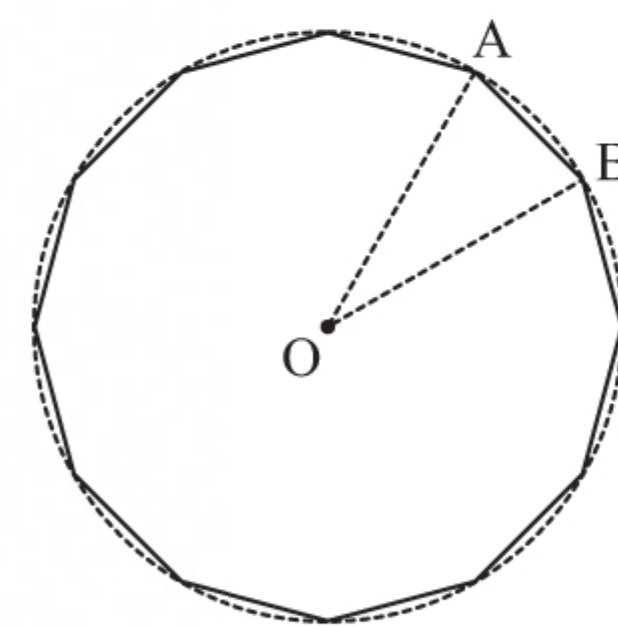
- 1** Let $f(x) = 3 - 4^{-x}$.
- Points $A(2, p)$ and $B(-2, q)$ lie on $y = f(x)$. Determine p and q .
 - For the graph of $y = f(x)$, determine the:
 - y -intercept
 - equation of the horizontal asymptote.
 - Sketch the graph of $y = f(x)$, showing all details from above.
 - Write down the range of $f(x)$.
- 2** Michael purchases a music subscription at the start of the year. The graph shows the amount of money left in the subscription account after x months.
- Find the gradient and y -intercept of the line, and interpret your answers.
 - Find the equation of the line.
 - How long will it take for the account to run out of money?



- 3** A particle moves in a straight line with velocity $v(t) = e^{2t} - 3e^t$ m s⁻¹ at time t seconds, $t \geq 0$.
- Find the initial velocity.
 - Show that the particle is stationary when $t = \ln 3$ seconds.
 - The particle is initially 1 m right of the origin. Show that its position after $\ln 5$ seconds is also 1 m right of the origin.

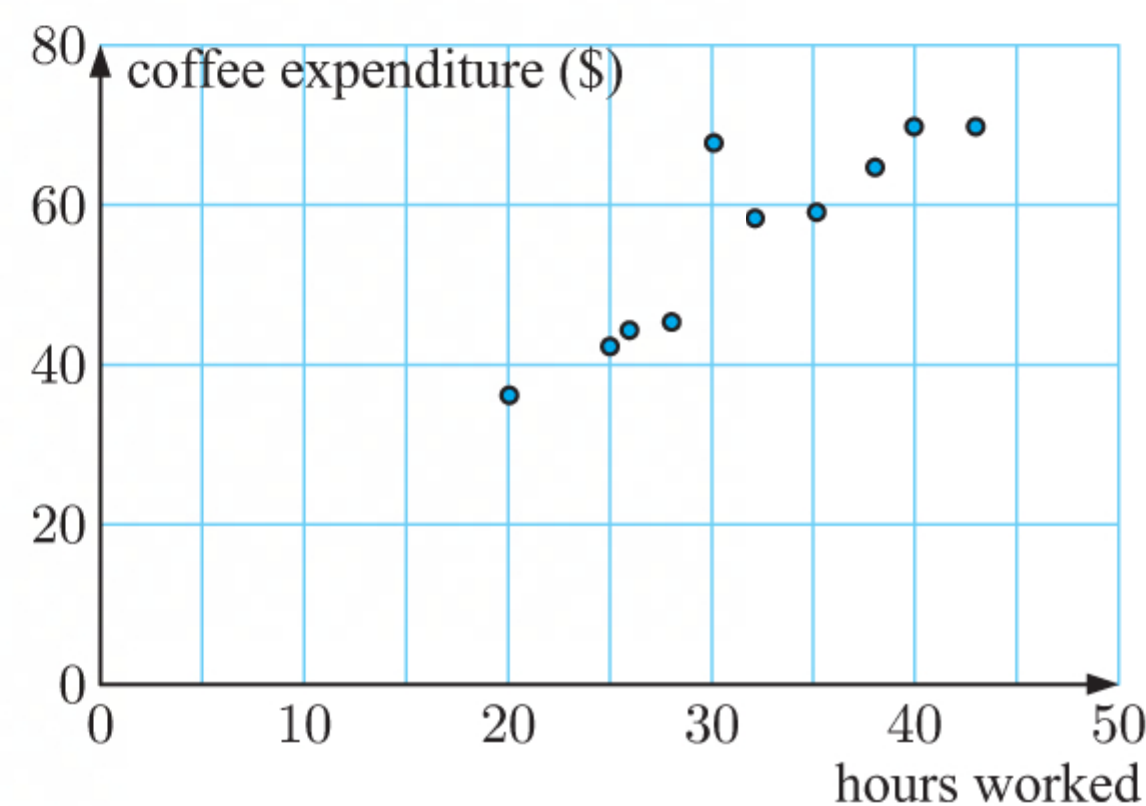
- 4 A regular dodecagon (12-sided polygon) is inscribed in a circle of radius 6 cm. Points A and B are adjacent vertices of the dodecagon, and both lie on the circle.

- Deduce that $\widehat{AOB} = 30^\circ$.
- Show that the area of triangle $AOB = 9 \text{ cm}^2$.
- Hence determine the area of the dodecagon.



- 5 This scatter diagram displays the amount James spends on coffee in the cafeteria against the number of hours he works in the week.

- James worked an average of 32 hours, and his average expenditure was \$56 per week. Plot the mean point $P(32, 56)$ on the graph.
- Draw a line of best fit by eye which passes through P.
- Use this line to predict the amount James will spend on coffee if he works a 35 hour week.
- Describe the nature and strength of the linear relationship between the variables. Comment on whether the prediction in c is reliable.



- 6 Consider the binomial expansion of $\left(kx + \frac{1}{\sqrt{x}}\right)^9$.

- Write down a formula for the general term.
- Given that the constant term is $-10\frac{1}{2}$, find k .

- 7 a Perform row reduction on the system of equations
- $$\begin{cases} x + 3y + kz = 2 \\ kx - 2y + 3z = k \\ 4x - 3y + 10z = 5 \end{cases}$$

- Show that for one value of k , the system of equations has infinitely many solutions. Find the solutions in this case, and give a geometric interpretation of your answer.
- Find the value(s) of k for which the system has no solutions. Give a geometric interpretation of your answer.
- Find the value(s) of k for which the system has a unique solution. Find the unique solution in terms of k . Give a geometric interpretation of your answer.

- 8 Let $h(x) = x^3 - 6tx^2 + 11t^2x - 6t^3$ where $t \in \mathbb{R}$.

- Show that t is a zero of $h(x)$.
- Factorise $h(x)$ as a product of linear factors.
- Hence or otherwise, find the coordinates of the points where the graphs of $y = x^3 + 6x^2$ and $y = -6 - 11x$ meet.

- 9 For $-\pi \leq \theta \leq \pi$, find the exact solution of $4 \operatorname{cosec} 2\theta = \tan 2\theta + 5 \cot 2\theta$.

- 10 a Write $\frac{\ln(1+3x)}{\arctan 2x}$ as the quotient of two Maclaurin series.

- Hence find $\lim_{x \rightarrow 0} \frac{\ln(1+3x)}{\arctan 2x}$.

MIXED QUESTIONS SET 12

- 1 A manufacturer states that the contents of its cereal boxes weigh an average of 320 g. A random sample of 24 boxes was weighed, with the following results recorded in grams:

312	320	326	330	326	322	326	330	331	315	323	316
315	325	311	320	308	325	320	332	316	309	314	324

- Organise the data using a frequency table, with the class intervals $305 \leq w < 310$, $310 \leq w < 315$, and so on.
- Draw a frequency histogram to display the data.
- Describe the distribution of the data.
- Find the modal class of this data.
- Calculate the mean of the data. How does it compare to the manufacturer's claim?

2 Let $y = x(x^2 - 12x + 45)$.

a Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

c Find the inflection point of the function.

e For what values of a does the equation $x^3 - 12x^2 + 45x - a = 0$ have three distinct real roots?

b Find the turning points of the function.

d Sketch the graph of $y = x(x^2 - 12x + 45)$.

3 Suppose $f(x) = 25 - x^2$ and $g(x) = \frac{2}{\sqrt{x}}$.

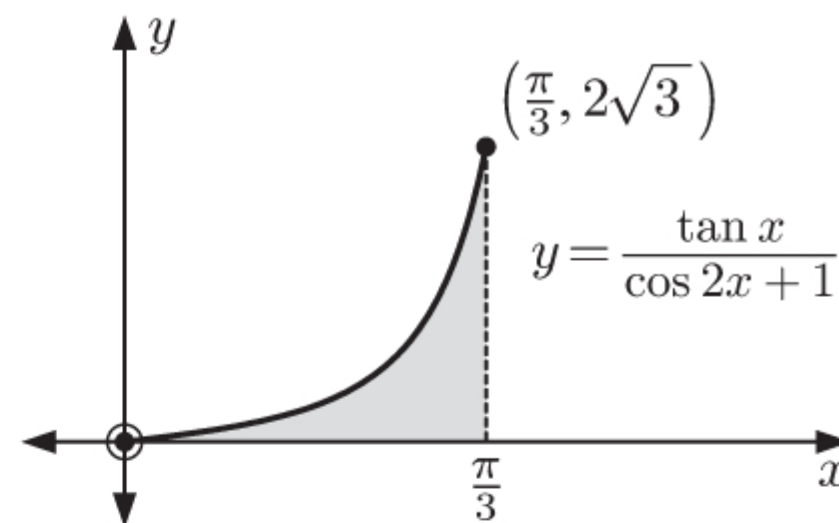
a Find $(g \circ f)(x)$, and state its domain.

b Solve $(g \circ f)(x) = 1$.

c Find the asymptotes of $y = (g \circ f)(x)$.

4 a Show that $\frac{\tan x}{\cos 2x + 1} = \frac{\sin x}{2 \cos^3 x}$.

b Hence find the shaded area.



5 100 diners at a restaurant were given a set three-course meal. After the meal, the diners were asked whether they liked each of the courses. The results are summarised alongside.

a Given that 48 people liked course A , find x and y .

b Which course was the most popular?

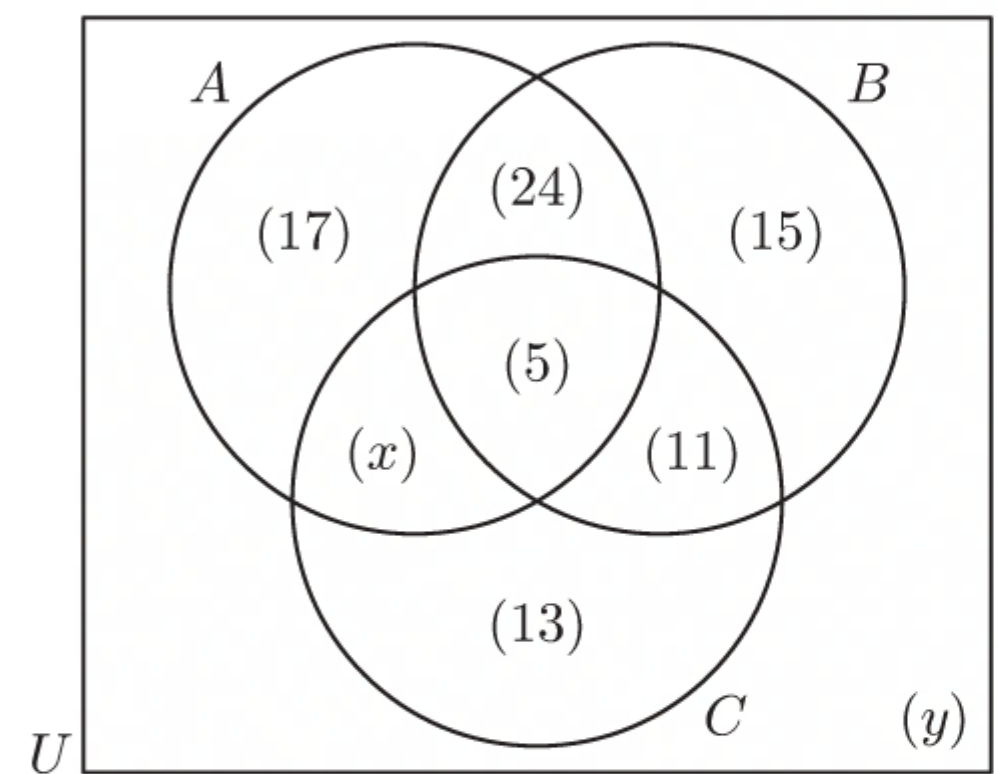
c Find the probability that a randomly selected diner liked:

i all of the courses

ii course B , but not course C

iii exactly two courses, given that the diner liked course C

iv none of the courses, given that the diner disliked course B .



6 a Show that $x = \sin^2 \theta$ and $x = \cos^2 \theta$ satisfy the equation $4x^2 - 4x + \sin^2 2\theta = 0$.

b Hence show that $\sin^2(\frac{\pi}{8}) \times \cos^2(\frac{\pi}{8}) = \frac{1}{8}$.

7 Let $f(x) = \sqrt{1 + 2x}$.

a Find the binomial expansion of $f(x)$ up to the term in x^4 .

b Hence estimate the value of $\sqrt{0.9}$.

c Differentiate the expansion in **a** with respect to x . Give your answer up to the term x^3 .

d Find the product of the expansions in **a** and **c**, up to the term in x^3 . Explain your answer.

8 a Consider the line with equation $\frac{x-1}{2} = \frac{3-y}{3} = z$.

i Find a vector parallel to the line.

ii Find the point on the line with z -coordinate 3.

iii Determine whether the point $(7, -3, 2)$ lies on the line, giving reasons.

b Find an equation of a line perpendicular to the line in **a**, which passes through the point $(5, -3, 2)$. Give your answer in parametric form.

c Determine whether the lines in **a** and **b** intersect. If they do intersect, find the point of intersection. If they do not, state the relationship between the lines.

d Find the acute angle between the line in **a** and the line L with equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

9 Consider the function $f(x) = \arcsin x + a\sqrt{1-x^2}$ where a is a constant.

a State the domain of the function.

b Find the coordinates of the function at the endpoints of its domain.

c Given that f has a stationary point at $x = \frac{1}{2}$, find a and determine the exact coordinates of the stationary point.

d Show that f is concave down over its entire domain.

- 10 a** Prove that $\tan 2x(1 - \tan^2 x) = 2 \tan x$.
- b** Given that $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$, find the Maclaurin series expansion for the following up to the term in x^5 :
- i** $\tan 2x$ **ii** $1 - \tan^2 x$.
- c** Show that the Maclaurin series in **b** are consistent with the result in **a**.

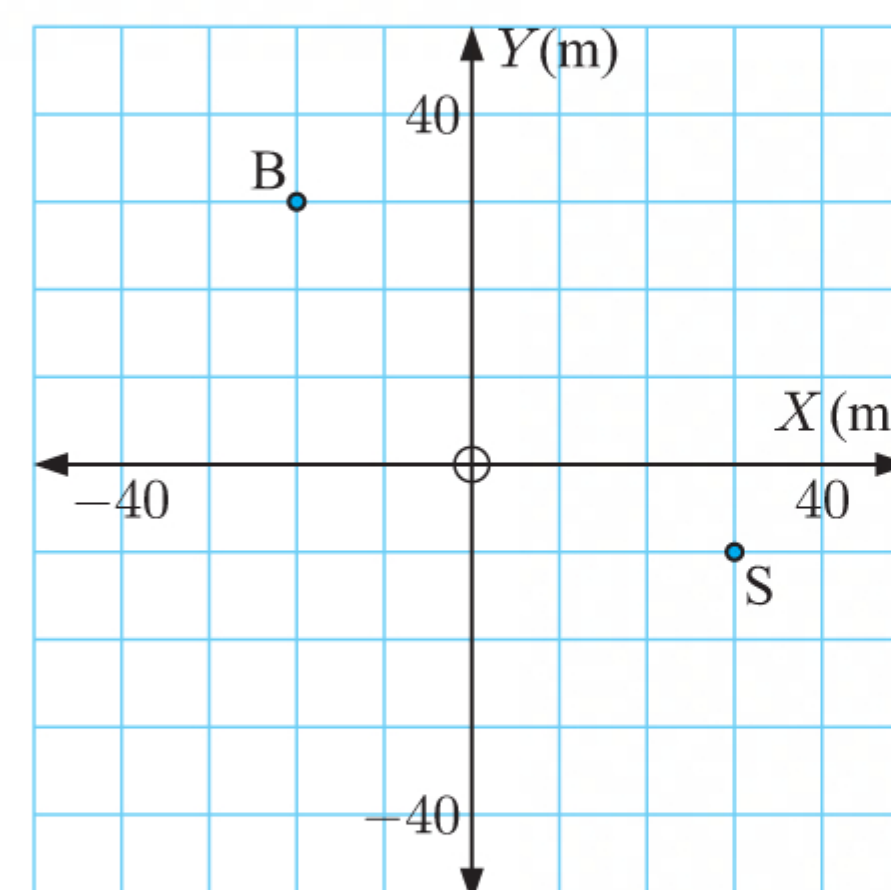
MIXED QUESTIONS SET 13

- 1** Cynthia invested \$2000 in an account that pays 4.4% p.a. interest compounded quarterly for 5 years.
- a** Find the final value of the investment. **b** How much interest did Cynthia earn?
- c** Given that inflation averages 2.5% p.a. over the investment period, find the real value of the investment.

- 2** This grid shows the position of a boat B and a shipwreck S.

The boat's anchor is directly below the boat, 50 m below sea level. The shipwreck is 40 m below sea level. Suppose sea level has Z -coordinate 0.

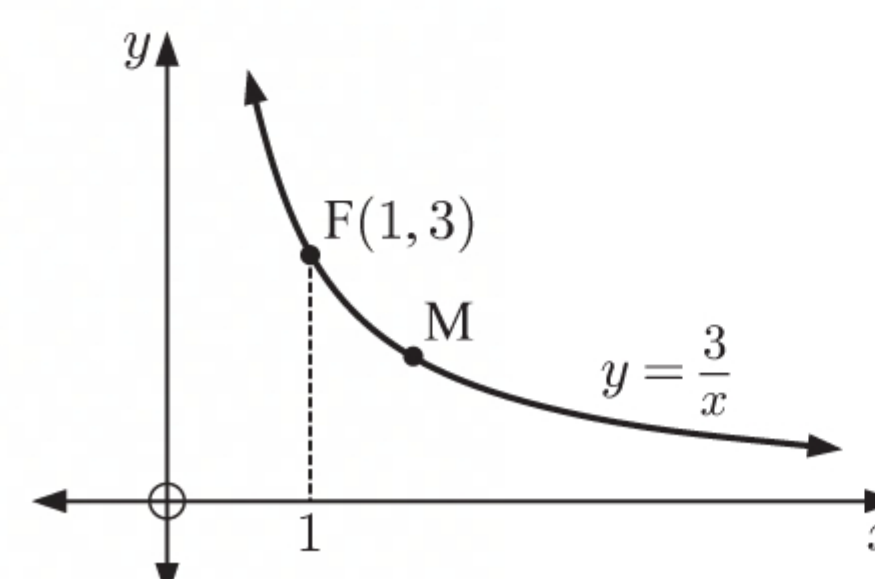
- a** Find the 3-dimensional coordinates of:
- i** the anchor **ii** the shipwreck.
- b** A diver swims from the boat to the shipwreck. How far does the diver swim?
- c** Find:
- i** the angle of depression from the boat to the shipwreck
- ii** the angle of elevation from the anchor to the shipwreck.



- 3** Consider the graph of $y = \frac{3}{x}$.

The point F is on the curve. Let M be a point close to F with x -coordinate $1 + h$.

- a** What is the y -coordinate of M? **b** Show that $3 - \frac{3}{h+1} = \frac{3h}{h+1}$.
- c** Find the gradient of [FM] in terms of h .
- d** Using this expression, state the gradient of the *tangent* at F.



- 4** 160 m of fence is used to enclose a rectangular field.

- a** Given that one side of the field has length x m, find the area of the field in terms of x .
- b** Find the dimensions of the field which would maximise the area.
- c** Suppose the actual area of the field is 1200 m^2 .
- i** Find the dimensions of the field.
- ii** The average production yield for this field is 6.5 kg m^{-2} . Determine the amount of production lost by not using the dimensions which maximise the area.

- 5** The heights of a sample of 80 children from a junior school were measured. The results are shown in the table alongside.

Estimate the: **a** mean **b** standard deviation.

Height (h cm)	Number of students
$80 \leq h < 90$	8
$90 \leq h < 100$	12
$100 \leq h < 110$	17
$110 \leq h < 120$	30
$120 \leq h < 130$	13

- 6** Prove that $2n^3 + 3n^2 + n$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

- 7** Year 12 students at a government school can choose from 16 subjects to study for their Certificate. Seven of these subjects are in Group I, six are in Group II, and the other three are in Group III. Students must study six subjects to qualify for the Certificate.

How many combinations of subjects are possible if:

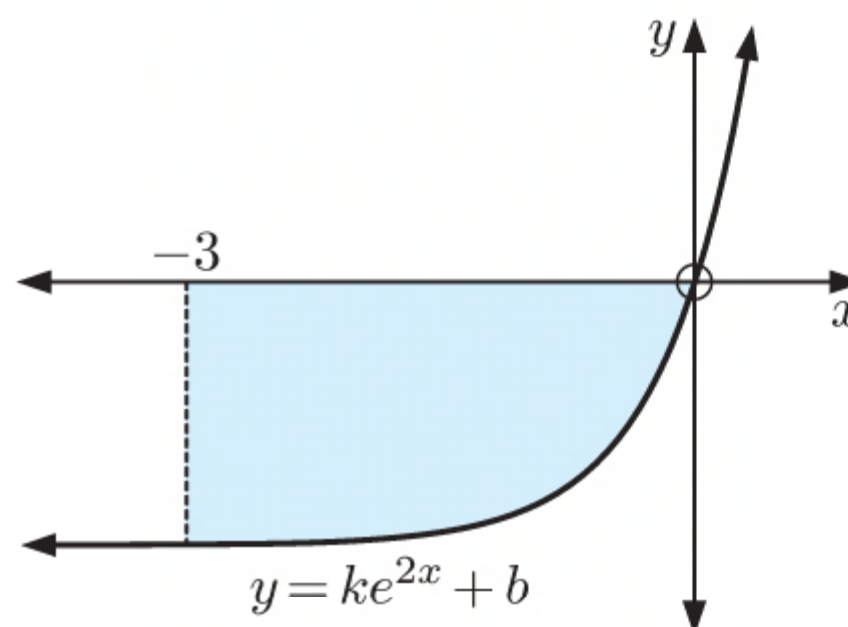
- a** there are no restrictions
- b** students must choose 2 subjects each from Groups I and II, and the remaining subjects can come from any group
- c** French (a Group I subject) is compulsory, and they must choose at least one subject from Group III?

- 8** The real polynomial $P(z)$ of degree 4 has one complex zero of the form $1 - 2i$, and another of the form ai , where $a \neq 0$, $a \in \mathbb{R}$. $P(0) = 10$, and the coefficient of z^4 is 1. Find $P(z)$, leaving your answer in factorised form.
- 9** **a** Find a and k if the line L_1 given by $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ a \\ -1 \end{pmatrix}$ lies on the plane P_1 with equation $3x - ky + z = 3$.
- b** Show that the plane P_2 with equation $2x - y - 4z = 9$ is perpendicular to P_1 .
- c** Find the equation of L_2 , the line of intersection of P_1 and P_2 .
- d** Find the point of intersection of L_1 and L_2 .
- e** Find the acute angle between the lines L_1 and L_2 .
- 10** The growth rate of snakes on Grouse Island is given by $\frac{dP}{dt} = \frac{P}{100} \left(1 - \frac{P}{5160} \right)$, where t is the time in years.
- a** What is the environment carrying capacity of snakes?
- b** Given that there were initially 2280 snakes, write P in terms of t .
- c** When would you expect the population to reach 4000 snakes?

MIXED QUESTIONS SET 14

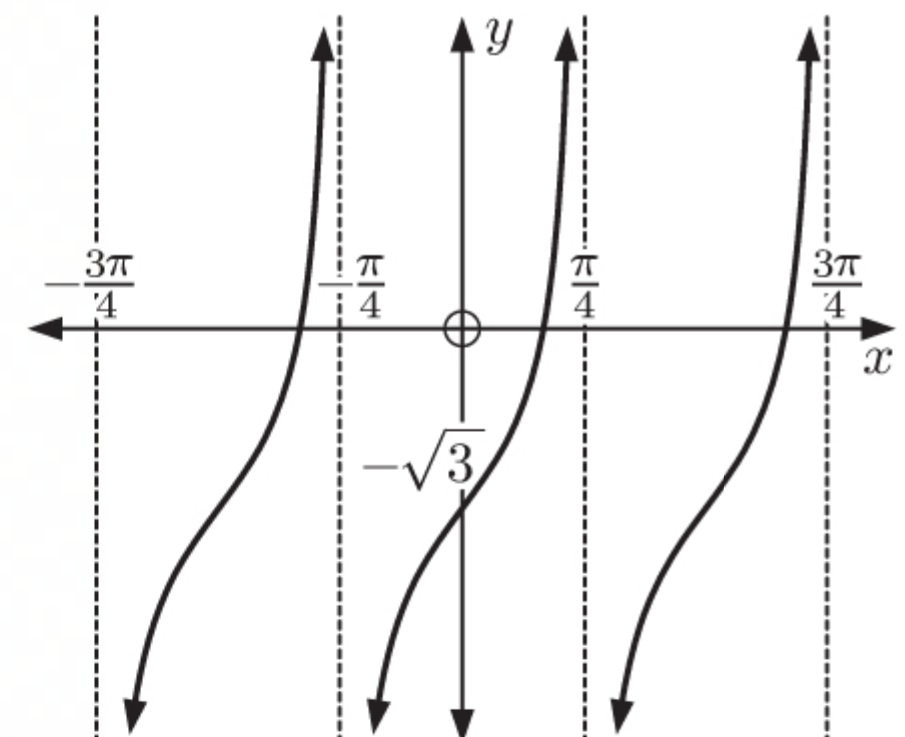
- 1** The area of the shaded region is $\frac{3}{e^6}$ units².

Find b and k .



- 2** Consider the graph of $y = \tan ax + b$ shown.

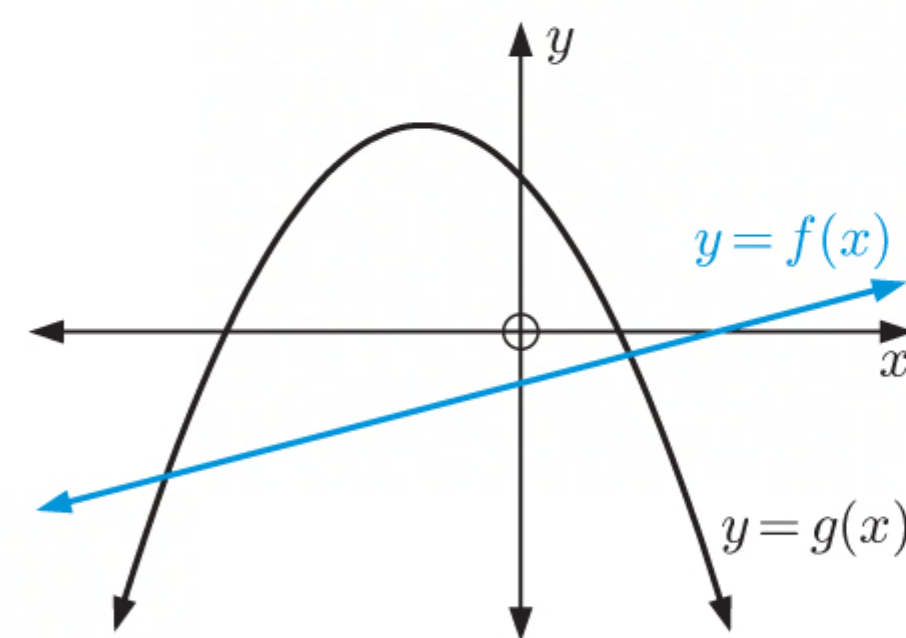
- a** Find the values of a and b .
- b** Hence find the x -intercepts of the function, for $-\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4}$.



- 3** A and B are mutually exclusive events. If $P(B) = 0.3$ and $P(A \cup B) = 0.55$, find $P(A)$.
- 4** $y = 3x^2 + 2x$ is stretched vertically with scale factor 2 and then translated by $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Find the equation of the image.
- 5** Let $f(\theta) = \frac{2 - \cos \theta}{\sin \theta}$, $0 < \theta \leq \frac{\pi}{2}$.
- a** Show that $f'(\theta) = \frac{1 - 2 \cos \theta}{\sin^2 \theta}$.
- b** Find the minimum value of $f(\theta)$.
- c** Sketch the graph of $f(\theta)$.
- 6** A bag contains 6 red balls and 4 white balls. A game is played in which the player draws 3 balls from the bag without replacement. The player wins if 3 red balls are drawn.
- a** Find the probability of the player winning a single game.
- b** Let X be the number of wins when the game is played 60 times.
- i** Find the mean μ and standard deviation σ of X .
- ii** Find $P(X = \mu)$.
- iii** Find $P(\mu - \sigma \leq X \leq \mu + \sigma)$.
- 7** A rectangle is divided by m lines parallel to one pair of opposite sides and n lines parallel to the other pair. How many rectangles are there in the figure obtained?

- 8 The graph alongside shows a linear function $f(x)$ and a quadratic function $g(x)$.

Copy the graph, and on the same set of axes sketch the graph of $y = \frac{f(x)}{g(x)}$.
Indicate clearly where the x -intercept(s) and asymptote(s) occur.



- 9 Consider the system of equations
$$\begin{cases} x + 3y - z = 2 \\ mx + y = 1 \\ 2x - 5y + (m - 2)z = -3 \end{cases}$$
 where m is a real number.

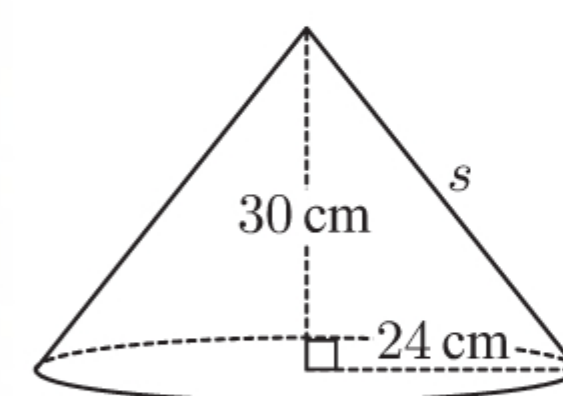
- Show that the system has no solutions for one value of m . Interpret this result geometrically.
 - Show that there are infinitely many solutions for another value of m . Interpret this result geometrically.
 - Find the value(s) of m for which the system has a unique solution. Interpret this result geometrically.
 - Find the value of m such that the unique solution has the form $x = y = z = k$ for some $k \in \mathbb{R}$.
- 10 Two roadrunners are beneath the same tree, and face 45° apart. One runs at 28 km h^{-1} and the other at 32 km h^{-1} . Find the rate at which the distance between the roadrunners is changing after 15 minutes.

MIXED QUESTIONS SET 15

- The solution of $2^{x-1} = 3^{2-x}$ is $x = \log_a b$ where $a, b \in \mathbb{Z}^+$. Find a and b .
- Two fair dice are rolled, and the difference between the scores is noted.
 - Display the possible results on a 2-dimensional grid.
 - Hence find the probability that the difference between the scores is 4.
- Consider the function $f(x) = x^3 - 3x^2 - x + 3$, where f is defined on the domain $-2 \leq x \leq 3$, $x \in \mathbb{R}$.
 - Use technology to help sketch the graph of $y = f(x)$, showing any axes intercepts and turning points.
 - Determine the range of f .

- 4 A solid right-circular cone has base radius 24 cm and vertical height 30 cm.

- Show that the slant height s is 38.4 cm, correct to 3 significant figures.
- Determine the total surface area of the cone. Give your answer in the form $a \times 10^k$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.



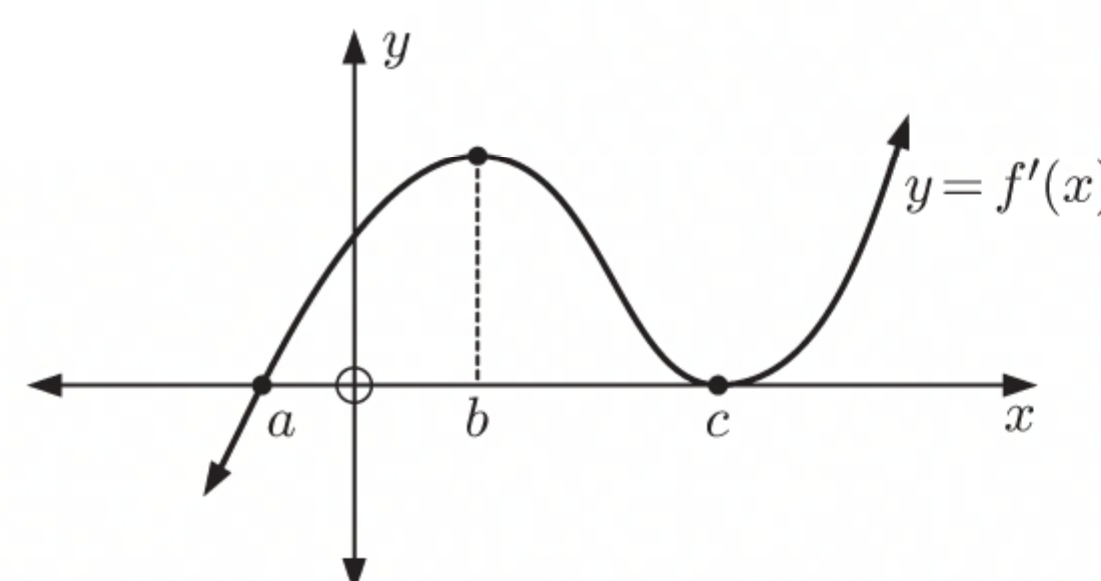
- 5 A winemaker wants to examine the effect of weed spray in his vineyard. He randomly selects 50 sample spots, each of area 1 m^2 , and counts the number of weeds in each spot. The results are shown in the table alongside.

- Determine the value of p .
- Estimate the mean number of weeds per spot.
- What percentage of sample spots had fewer than 10 weeds?

Number of weeds	Frequency
0 - 4	9
5 - 9	15
10 - 14	10
15 - 19	p
20 - 24	5
25 - 29	2

- 6 Alongside is a sketch of the gradient function $y = f'(x)$.

- Sketch a possible curve for $y = f(x)$.
- Let $y = f_1(x)$ be a curve such that $f_1'(x) = f'(x)$ for all x . Write down the form of all possible functions $f(x)$ in terms of $f_1(x)$.

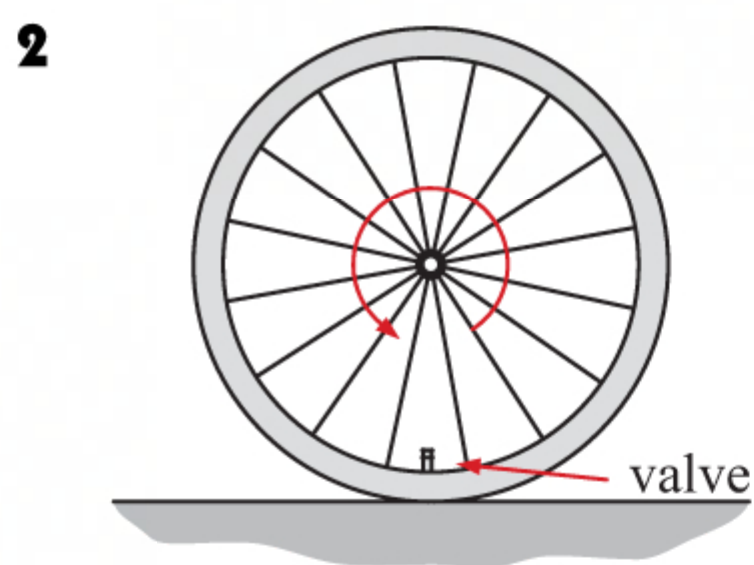


- Find the roots of $z^7 = 1$, giving your answers in polar form.
 - Hence solve $(3z - 1)^7 = 1$.
 - Explain why $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$.

- 8 a** Show that $\operatorname{cosec} 2x - \cot 2x = \tan x$. **b** Hence find the exact value of $\tan \frac{5\pi}{12}$.
- 9** Consider the continuous random variable X with probability density function $f(x) = \frac{e^{\frac{1-x}{3}}}{3(e-1)}$, $-2 \leq x \leq 1$.
- a** Verify that $f(x)$ is a valid probability density function.
- b** Find: **i** $P(0 \leq X \leq 1)$ **ii** $P(X \geq \frac{1}{4})$
- 10** Consider the series $\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \dots + \frac{1}{n(n+2)}$.
- a** Find constants A and B such that $\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$.
- b** Hence show that the sum of the series is $\frac{3}{4} - \frac{1}{2n+2} - \frac{1}{2n+4}$.
- c** Prove your answer to **b** using mathematical induction.

MIXED QUESTIONS SET 16

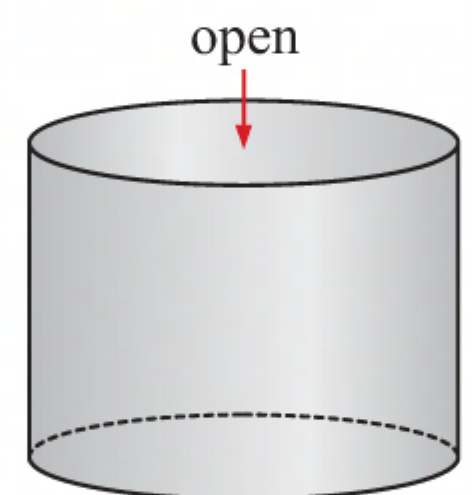
- 1** The weight of a radioactive substance after t years is given by $W(t) = 5 \times (0.965)^t$ grams, $t \geq 0$.
- a** Find the percentage decrease in weight of the substance each year.
- b** Find the weight of the substance after 300 years. Write your answer in the form $a \times 10^k$ where $1 \leq a < 10$, $k \in \mathbb{Z}$.
- c** How long will it take for the weight to fall below 1 g?



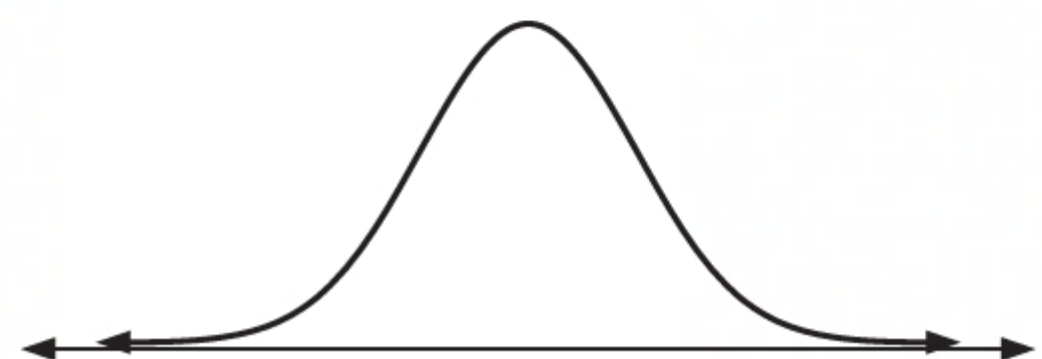
A bicycle wheel sits on the road so its valve is at the bottom. The tyre has inner radius 35 cm and outer radius 40 cm.

The wheel begins to rotate at a constant speed of 4 revolutions per second.

- a** Find the height of the valve above the road after:
- i** 0 seconds **ii** $\frac{1}{12}$ second.
- b** The height of the valve above the road after t seconds can be modelled by the function $H(t) = a \sin(b(t-c)) + d$ cm.
- Find: **i** a **ii** d **iii** b **iv** c
- c** How long does it take the valve to rise to 60 cm above the road?
- 3** An open cylindrical bin is to be made from PVC plastic and is to have capacity 500 litres. Find the dimensions of the bin which minimises the amount of PVC plastic used.



- 4** Suppose $f(x) = x^2 + 2x$, $x \leq -1$.
- a** Find $f^{-1}(x)$, and state its domain and range.
- b** Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.
- 5** The random variable X is normally distributed with mean μ and standard deviation σ . Let k be such that $P(X < k) = 0.7$.
- a** Illustrate μ and k on the normal distribution curve.
- b** Find:
- i** $P(X > k)$ **ii** $P(\mu < X < k)$
- iii** $P(\mu - \sigma < X < k)$
- c** If $P(X \geq t) = 0.2$, find $P(k \leq X \leq t)$.



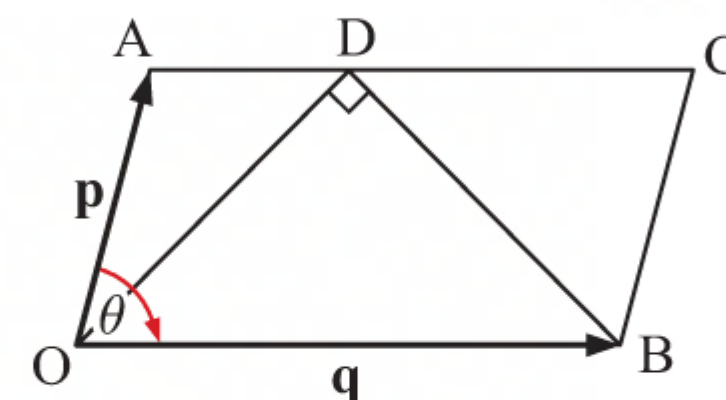
- 6 Let $f(x) = \frac{9}{2} - x^2$. The normal L at the point $P(a, f(a))$, $a > 0$, passes through the origin.
- Sketch $y = f(x)$, showing the vertex and axes intercepts.
 - Show that the equation of L is $y = \frac{1}{2a}x + 4 - a^2$.
 - Hence find the area of the region enclosed by L and $y = f(x)$.

- 7 Prove by contradiction that $\sqrt[3]{3}$ is irrational.

- 8 OACB is a parallelogram with angle AOB of measure θ . Let $\vec{OA} = \mathbf{p}$ and $\vec{OB} = \mathbf{q}$, where $|\mathbf{q}| = 2|\mathbf{p}|$.

Let $\vec{AD} = k\mathbf{q}$, $0 \leq k \leq 1$ such that angle ODB is a right angle.

- Show that $\mathbf{p} \cdot \mathbf{q} = 2|\mathbf{p}|^2 \cos \theta$.
- Find \vec{OD} and \vec{DB} in terms of \mathbf{p} , \mathbf{q} , and k .
- Hence show that $(1 - 2k)(\mathbf{p} \cdot \mathbf{q}) - (1 - 2k)^2 |\mathbf{p}|^2 = 0$.
- Hence deduce that $k = \frac{1}{2}$ or $k = \frac{1}{2} - \cos \theta$.
- Describe the position of D for which $k = \frac{1}{2}$.
- Find the smallest value of θ for which there are two possible positions of D.
 - For this value of θ , sketch the parallelogram showing the exact positions of D.



- 9 The probability distributions below show the amount of sleep, rounded to the nearest hour, that Eddy and Brett received each night in the past month.

Eddy:

Hours of sleep	5	6	7	8	9
Probability	0.4	a	0.2	0.05	0.1

Brett:

Hours of sleep	5	6	7	8	9
Probability	0.1	0.2	0.5	b	0.05

- Find the values of a and b .
 - Who generally receives the least amount of sleep per night?
 - Calculate the variance and standard deviation of each probability distribution.
 - Who has the least variation in their amount of sleep?
- 10 Solve the differential equation $\frac{dy}{dx} = \frac{xy}{x-1}$ given that $y = 2$ when $x = 2$.

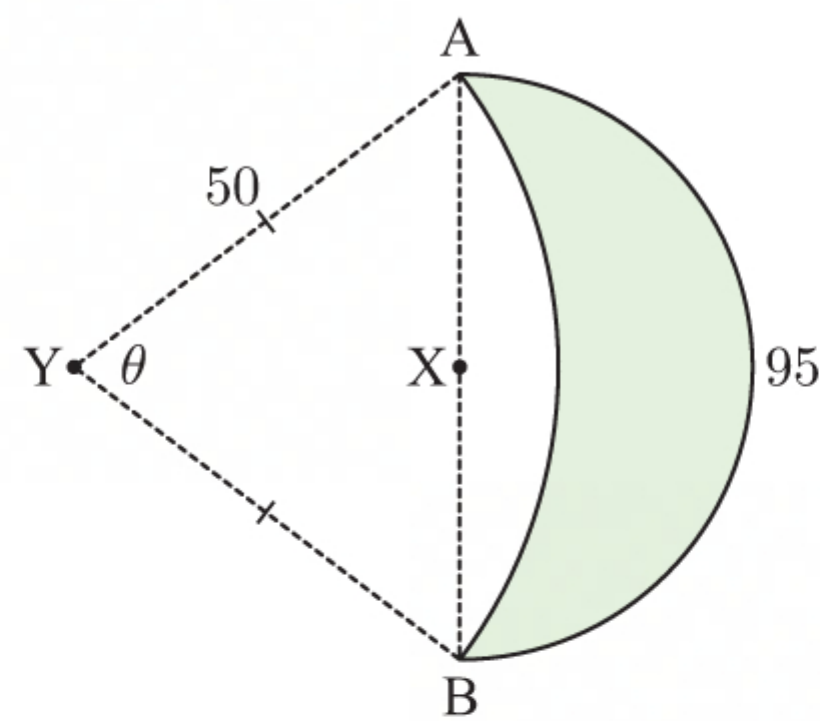
MIXED QUESTIONS SET 17

- A quadratic function has the form $f(x) = ax^2 + bx + 7$. It is known that $f(2) = 7$ and $f(4) = 23$.
 - Construct a set of simultaneous equations involving a and b .
 - Find a and b .
 - Hence calculate $f(-1)$.
- Find the equation of the tangent to $f(x) = (x^2 + 1)e^{-x}$ at the point where $x = 1$.
- Hayley and Patrick were training for a road cycling race. During the first week they both cycled 60 km. Hayley cycled an additional 20 km each subsequent week, whereas Patrick increased his distance by 20% each subsequent week.
 - How far did each of them cycle in the 5th week of training?
 - Who was the first to cycle 210 km in one week?
 - Who cycled a greater total distance in the first 12 weeks? Explain your answer.
- Suppose $f(x) = \log_3(x + 1) + 2$.
 - State the transformation which maps $y = \log_3 x$ to $y = f(x)$.
 - Find the domain and range of f .
 - Find the axes intercepts of f .
 - Sketch the graph of $y = f(x)$.
 - Find the inverse function f^{-1} .

- 5 X and Y are the centres of the two arcs AB shown.

Find:

- the length AX
- the angle θ
- the shaded area.



- 6 Suppose $P(A) = 0.35$, $P(B) = 0.7$, and $P(A \cup B) = 0.8$.

- Calculate $P(A \cap B)$.
- Represent this information on a Venn diagram.
- Find:
 - $P(A' \cap B')$
 - $P(A | B)$
- State, with a reason, whether events A and B are independent.

- 7 Suppose \mathbf{i} represents a 1 km displacement due east and \mathbf{j} represents 1 km displacement due north. A lighthouse is located at the point $(0, 10)$. A ship is moving in a straight line with parametric equations $x = 3 - 2t$, $y = 3t + 1$, $t \geq 0$, where t is the number of hours after 8:30 am.

- What was the position of the ship at 8:30 am?
- Find the ship's:
 - velocity vector
 - speed.
- Find the distance between the ship and the lighthouse at 10:30 am.
- Find the time when the ship is directly west of the lighthouse.
- Find the time when the ship is closest to the lighthouse, and the distance between the ship and the lighthouse at this time.

- 8 A random variable X has probability density function $f(x) = \begin{cases} \sin(0.5x), & 0 \leq x \leq a \\ 0, & \text{otherwise.} \end{cases}$

- Find the exact value of a .
- Find $E(X)$, $\text{Var}(X)$, and the standard deviation of X .
- Find the median and modal values of X .

- 9 Solve using an integrating factor: $\frac{dy}{dx} + e^x y = 5 - y$, $y(0) = 1$

- 10 Suppose that f and g are functions.

Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$, $\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{j=0}^n \binom{n}{j} f^{(n-j)}(x) g^{(j)}(x)$.

MIXED QUESTIONS SET 18

- 1 Find the coefficient of x^7 in the expansion of $(x - 1)(2 - x)^9$.

- 2 The following data shows Craig's weekly grocery bills, in dollars, for the last 5 months.

181, 155, 163, 200, 149, 185, 160, 159, 164, 171,
173, 212, 303, 191, 169, 161, 207, 140, 132, 165

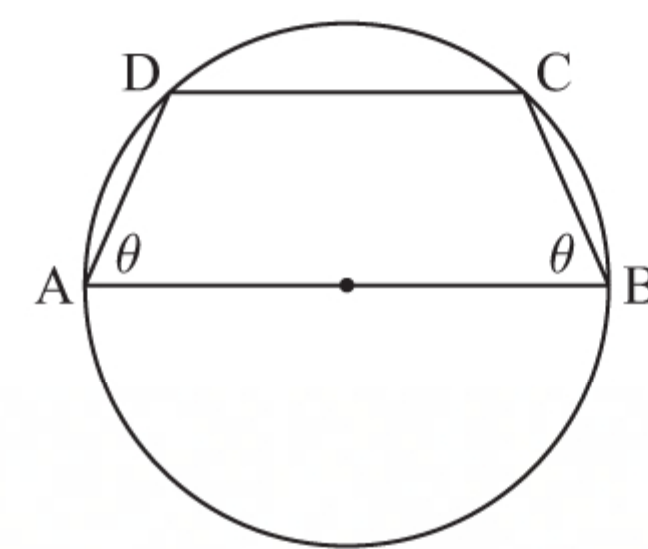
- Find the median, lower quartile, and upper quartile of the data set.
- Find the interquartile range of the data set.
- The bill of \$303 occurred when Craig bought groceries for a large Christmas lunch. Show that this value is an outlier.
- Draw a box plot of the data set.

- 3 Consider the function $f(x) = \frac{1}{x} - \frac{4}{x-2}$.

- Find the value of x for which $f(x) = 0$.
- Find and classify all stationary points.
- Find the coordinates of the point of inflection. Give your answer to two decimal places.
- Draw the graph of $y = f(x)$, showing all of the above information.
- Find the exact area enclosed by the graph, the x -axis, and the lines $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

- 4 [AB] is the diameter of a circle with radius 6 cm.

- Show that the area of trapezium ABCD is given by $A = 18(2 \sin 2\theta - \sin 4\theta)$.
- Find the angle θ which maximises the area of ABCD.



- 5 Consider $f(x) = \tan x + \cot x$, $0 \leq x \leq \frac{\pi}{2}$.

- Show that $f(x) = 2 \operatorname{cosec} 2x$.
- Find the equations of the asymptotes of $y = f(x)$ on $0 \leq x \leq \frac{\pi}{2}$.
- Without using calculus, find the least value of $f(x)$ and the corresponding value of x .
- If $\sin a = \frac{1}{3}$, find $f(2a)$ correct to 4 significant figures.

- 6 Eleven students participate in a basketball game. Their time spent training the previous week, and the number of points they scored in the game are shown in the table below.

Time spent training (t hours)	2.5	1	2.5	3.5	4	2.5	2	3	3	2	1.5
Points scored (y)	2	0	5	16	9	8	2	6	10	0	2

- Explain why it would be appropriate to use the regression line of t against y in this case.
- Find the regression line of t against y .
- Use the regression line to estimate:
 - the time spent training by a player who scored 7 points
 - the number of points scored by a player who spent 5 hours training.
- Comment on the reliability of your estimates in c.

- 7 $x = a$ is a solution of $3x^3 - 11x^2 + 8x = 12a$.

- Show that there are 3 possible values for a .
- For each value of a found in a, solve the original equation.

- 8 A mountain railway runs straight up a mountainside with the aid of a cable. The train begins at point A with position vector \mathbf{a} , and ends at point B with position vector \mathbf{b} .

After t minutes, the train is at point P with position vector $\mathbf{p} = \left(1 - \frac{t}{12}\right)\mathbf{a} + \frac{t}{12}\mathbf{b}$.

- Locate the train at time $t = 0$.
- How long does it take for the train to reach B?
- Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ where the units are kilometres.
 - Find the distance between A and B.
 - Find the average speed of the train, giving your answer exactly.

- 9 Let N be the lifetime, in years, of a panel in a solar powered calculator. The probability density function for N is $f(n) = 0.6e^{-0.6n}$, $n \geq 0$.

- Find the probability that a randomly chosen panel will last for at least one year.
- A solar calculator has 8 of these panels, each of which operates independently of each other. The calculator will continue to operate provided at least one of the panels is operating. Find the probability that a randomly chosen calculator fails within one year.

- 10 a The *inverse hyperbolic tangent function* is defined as $\operatorname{arctanh} x = \frac{1}{2}[\ln(1+x) - \ln(1-x)]$.

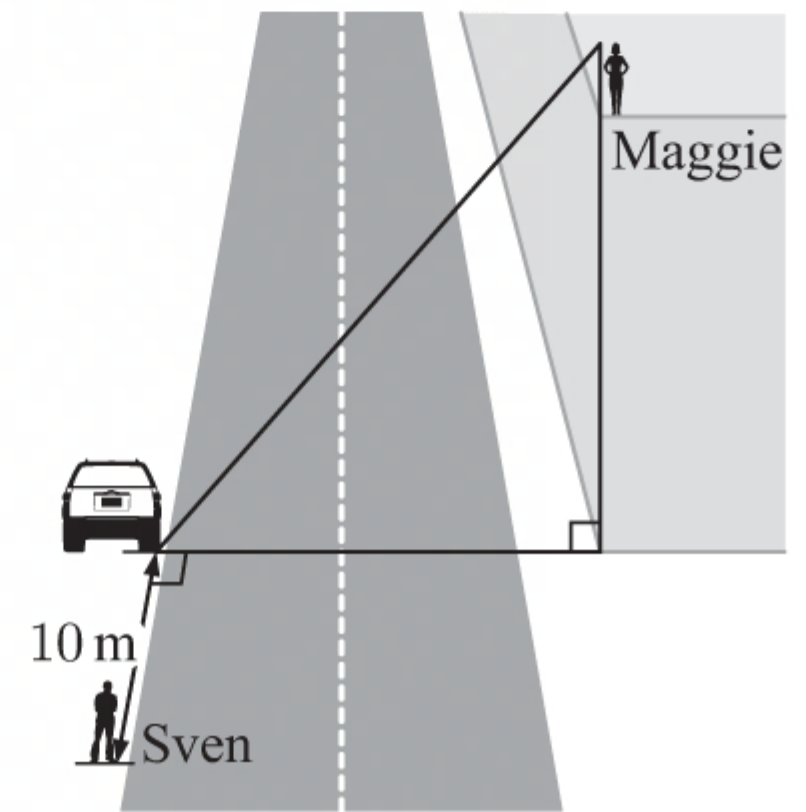
Show that the derivative of $\operatorname{arctanh} x$ is $\frac{1}{1-x^2}$.

- A function $f(x)$ has derivative $f'(x) = x(1-x^2)^{-\frac{3}{2}}$.

The volume of revolution when the region bounded by $y = f(x)$, the x -axis, $x = 0$, and $x = \frac{1}{2}$, is rotated 2π around the x -axis, is 14.589 units. Find $f(x)$.

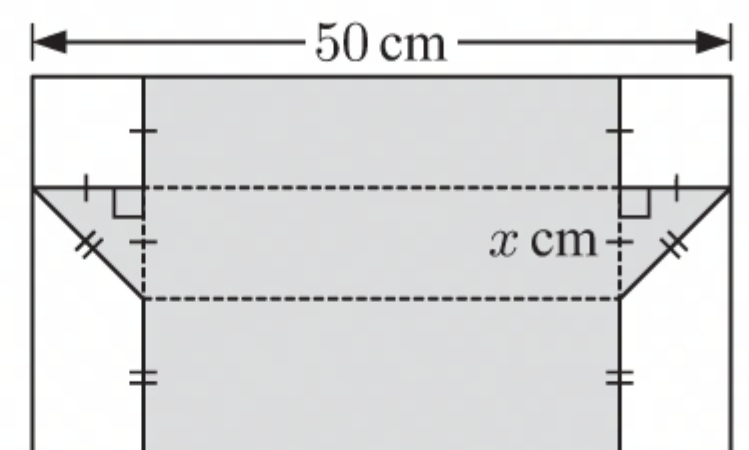
MIXED QUESTIONS SET 19

- The graph of $y = 3 - \frac{k}{x-1}$ has x -intercept $\frac{5}{3}$.
 - Find the value of k .
 - Find the y -intercept.
 - State the equations of the asymptotes.
 - Sketch the curve, showing the features you have found.
- Two year 7 students are selected each week to hoist the flag before the start of class. Year 7 has been divided into 2 classes: class A has 30 students, and class B has 27 students.
 - Find the probability that, in any given week, the two selected students selected are in the same class.
 - Over the course of 20 weeks, how many times do you expect that the two selected students are in the same class?
- Find a and b given that $2^a 8^b = \frac{1}{2}$ and $\frac{3^{-a}}{3^{b+1}} = 9$.
- Consider the lines $L_1: y = \frac{3}{4}x + 1$ and $L_2: y = -x - 1$.
 - Find the angle that each line makes with the positive x -axis.
 - Hence find the acute angle between L_1 and L_2 .
- Maggie is 155 cm tall and is standing on top of a building 50 m tall. A car is parked on the far kerb of the road, directly opposite Maggie. To see the car, Maggie looks down at an angle 67° below horizontal.
 - How far is the car from the base of the building?
 - Maggie's friend Sven is walking on the same side of the road that the car is parked. He is currently 10 m from the car.
 - Find the distance between Maggie and Sven.
 - At what angle must Sven look up to see Maggie?



- Prove that any terminating or recurring decimal is rational.
- If $\cos 2\alpha = \sin^2 \alpha$, find the exact value of $\cot \alpha$.
- Suppose $f(x) = \log_2(ax + b)$ where $f(6) = 4$ and $f'(1) = \frac{3}{\ln 2}$. Find the constants a and b .
- A triangular prism is cut from a metal sheet with width 50 cm.
 - Explain why the volume of the prism is $V(x) = \frac{1}{2}x^2(50 - 2x) \text{ cm}^3$.
 - For what values of x is it reasonable to use this function?
 - Sketch the graph of $V(x)$.
 - Determine the value(s) of x required for the volume to be 1000 cm^3 .
- The continuous random variable X has the probability density function $f(x) = \frac{1}{9}x^2$, $0 \leq x \leq 3$.
 - Verify that $f(x)$ is a valid probability density function.
 - Find the mode and median of X .
 - Find:
 - $E(X)$
 - $\text{Var}(X)$
 - $\sigma(X)$
 - Suppose $Y = aX + b$ where a and b are constants.

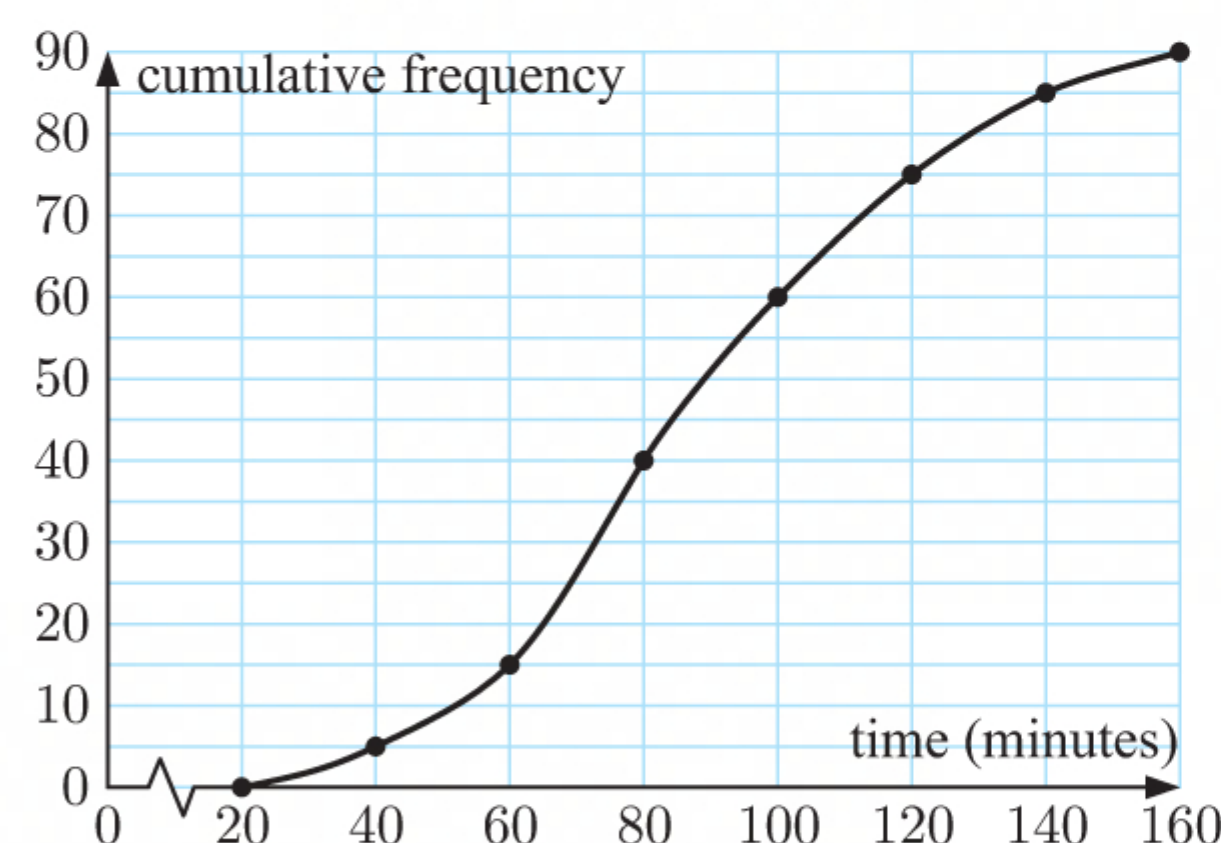
Given $E(Y) = \frac{1}{2}$ and $\text{Var}(Y) = \frac{27}{20}$, find the possible values of a and b .



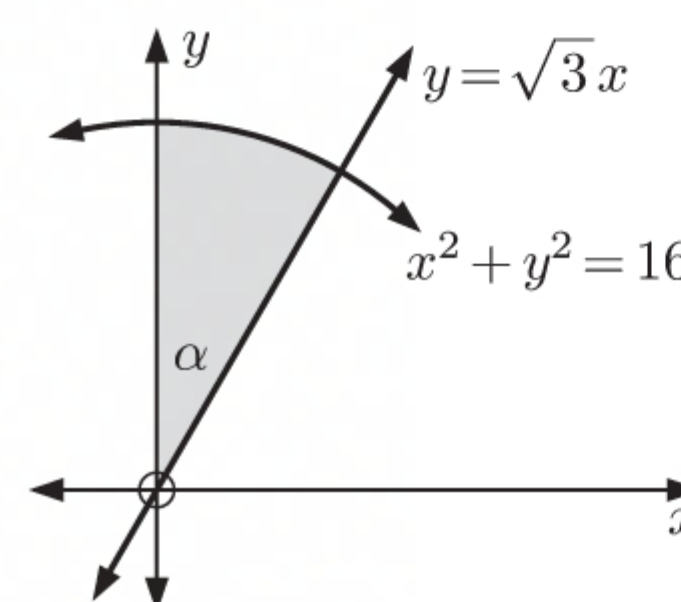
MIXED QUESTIONS SET 20

- Consider the functions $f(x) = x - 2$ and $g(x) = 3 - x - 2x^2$. Find:
 - $f^{-1}(x)$
 - $(g \circ f)(x)$
 - $(g \circ f)(-1)$

- 2** The velocity of a truck t seconds after applying its brakes is $v = \frac{20}{\sqrt{2t+1}} \text{ m s}^{-1}$, $0 \leq t \leq 10$.
- Find the speed of the truck when the brakes are applied.
 - Find the acceleration function.
 - At what time does the truck have acceleration -2.5 m s^{-2} ?
 - Find the distance travelled by the truck in the first 10 seconds after applying the brakes.
- 3** The masses of sea lions on a particular island are normally distributed with mean μ and standard deviation σ . 10% of the sea lions have mass greater than 900 kg, and 15% have mass less than 500 kg.
- Find μ and σ .
 - A randomly selected sea lion weighs more than 800 kg. Find the probability that the sea lion weighs less than 850 kg.
- 4** Suppose $f(x) = \sin^2 x - \cos^2 x$.
- Show that $f'(x) = 2 \sin 2x$.
 - Hence find $4[f'(x)]^2 + [f''(x)]^2$.
 - Given that $f(\theta)$, $f'(\theta)$, and $f''(\theta)$ form an arithmetic sequence, find the value of $\tan 2\theta$.
- 5** The lengths, in minutes, of games in a chess tournament are displayed in this cumulative frequency graph.
- How many games were played during the tournament?
 - Find the median game length.
 - Estimate the interquartile range for the data.
 - 10% of the games took less than k minutes. Estimate the value of k .
 - Draw a frequency histogram to represent the data.



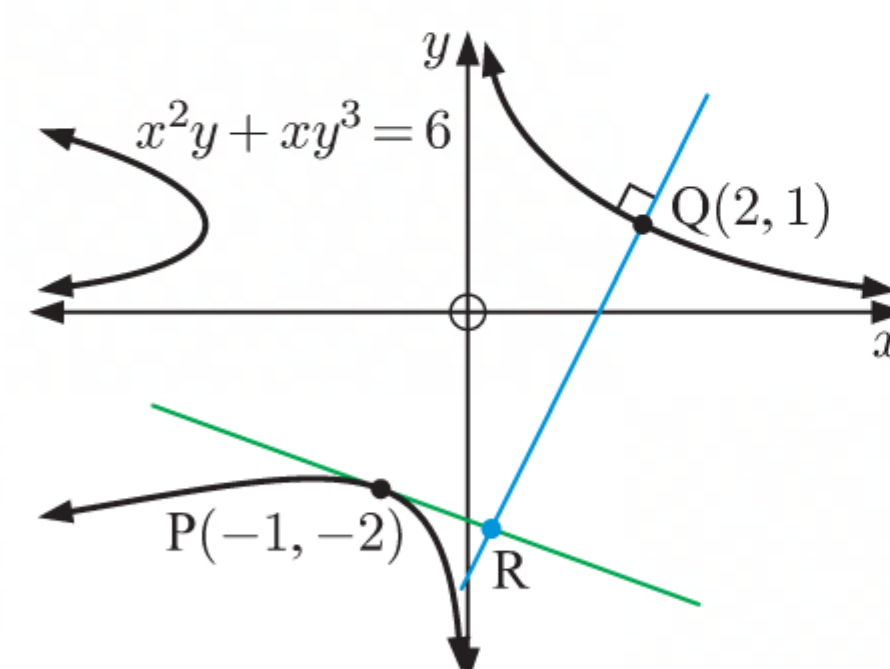
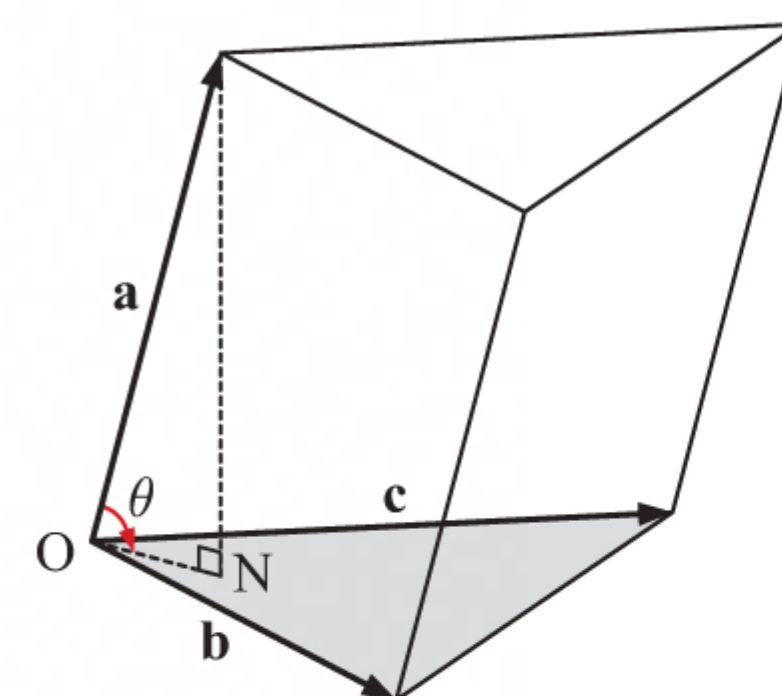
- 6** The diagram shows a line $y = \sqrt{3}x$ and an arc of the circle $x^2 + y^2 = 16$.
- Show that $\alpha = \frac{\pi}{6}$.
 - Hence find the area A of the sector shown.
 - By considering A as the area between the two curves $x^2 + y^2 = 16$ and $y = \sqrt{3}x$, show that $A = \int_0^2 \sqrt{16 - x^2} dx - 2\sqrt{3}$.
 - Show that $\int_0^2 \sqrt{16 - x^2} dx = \frac{4\pi}{3} + 2\sqrt{3}$.



- 7** One zero of $x^4 + 2x^3 + 8x^2 + 6x + 15$ has the form bi where $b \neq 0$, $b \in \mathbb{R}$. Find b and all zeros of the polynomial.
- 8** Prove that the interval $]a, b[$ has no greatest element.
- 9** Consider the 3-dimensional shape alongside with defining vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

Let θ be the angle between \mathbf{a} and the base plane.

- Find:
 - the area of the base plane
 - the perpendicular height of the shape.
 - Hence show that the volume of the shape is $\frac{1}{2} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \text{ units}^3$.
 - $A(2, 0, 1)$, $B(0, 2, 0)$, and $C(3, 1, 2)$ are vertices of this shape, adjacent to another vertex $O(0, 0, 0)$. Find the volume of this shape.
- 10** The relation $x^2y + xy^3 = 6$ is graphed alongside.
- The tangent to the graph at P , and the normal to the curve at Q , intersect at R . Find the coordinates of R .



Trial examination 1

PAPER 1

NO CALCULATOR, 120 MINUTES

SECTION A

1 [Maximum mark: 6]

Let

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Prove algebraically the recurrence relation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

2 [Maximum mark: 6]

The quadratic equation $x^2 + 5kx - (5k - 11) = 0$ has real roots α and β for which $\alpha^2 + \beta^2 = 58$. Without solving the equation, find the possible values of $k \in \mathbb{R}$.

3 [Maximum mark: 6]

a Prove the identity

$$\sin x \cos x \tan x = 1 - \cos^2 x \qquad [2]$$

b Hence, solve the equation to find the **exact** value of x , where $-\pi < x < \pi$, [4]

$$4 \sin x \cos x = \frac{1}{\tan x}$$

4 [Maximum mark: 10]

The discrete random variable X has the following probability distribution:

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	a	$2a$	$\frac{1}{6}$

a Find the value of a . [1]

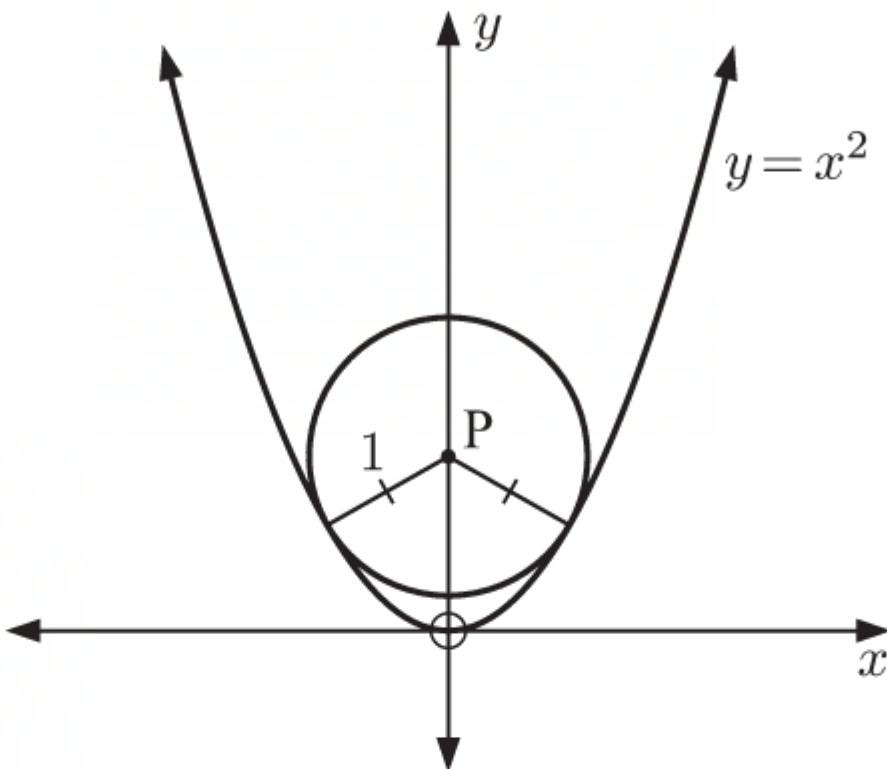
b Find $E(X)$. [2]

c Find $\text{Var}(X)$. [3]

d Let $Y = kX - k$, where k is a constant. Given that $E(Y) = \frac{8}{3}$, find $\text{Var}(Y)$. [4]

5 [Maximum mark: 7]

The figure shows a circle with equation $x^2 + (y - a)^2 = 1$ inscribed in the parabola $y = x^2$.



Find the coordinates of the centre of the circle.

6 [Maximum mark: 10]

Consider the curve with equation $x^2 + y^3 - 3xy = 0$.

a Show that the point $(4, 2)$ lies on the curve. [1]

b Use implicit differentiation to show that [4]

$$\frac{dy}{dx} = \frac{3y - 2x}{3y^2 - 3x}$$

The tangent to this curve is parallel to the x -axis at the point $x = k$, where $k > 0$.

c Find the **exact** value of k . [5]

7 [Maximum mark: 10]

a Find the roots of the equation $w^3 = 2i$ for $w \in \mathbb{C}$. Give your answers in polar form. [4]

b For two of the roots, w_1 and w_2 , $\Re(w_1) \neq 0$ and $\Re(w_2) \neq 0$. Express these roots in Cartesian form. [3]

c For the remaining root w_3 , find the exact value of w_3^{10} . [3]

SECTION B**8 [Maximum mark: 17]**

In triangle ABC, $AC = \frac{\sqrt{3}}{3}$ cm, $\widehat{ACB} = \frac{\pi}{6}$, and $\widehat{BAC} = \theta$.

a Find \widehat{ABC} in terms of θ . [2]

b Show that [3]

$$AB = \frac{\sqrt{3}}{3(\cos \theta + \sqrt{3} \sin \theta)}$$

c Find [5]

$$\frac{d(AB)}{d\theta}$$

d Hence, find the value of θ for which AB is minimised. [3]

e Find the area of triangle ABC for this value of θ . [4]

9 [Maximum mark: 19]

Consider the function $f(x) = \sin x \cos x$.

a Show that $f''(x) = -4 \sin x \cos x$. [4]

b Find $f'''(x)$. [3]

c Hence, show that $f^{(4)}(x) = 16 \sin x \cos x$. [2]

d Use mathematical induction to prove that, for $n \in \mathbb{Z}^+$, [10]

$$f^{(2n)}(x) = (-4)^n \sin x \cos x$$

10 [Maximum mark: 19]

The vertices of triangle ABC have coordinates given by $A(1, 2, 4)$, $B(4, -1, -2)$, and $C(2, -3, -3)$.

a i Find \overrightarrow{AB} and \overrightarrow{AC} . [3]

ii Hence, write down the vector equation of plane Π containing triangle ABC. [2]

b Point $P(-3, -6, a)$ lies on plane Π . Find the value of a . [5]

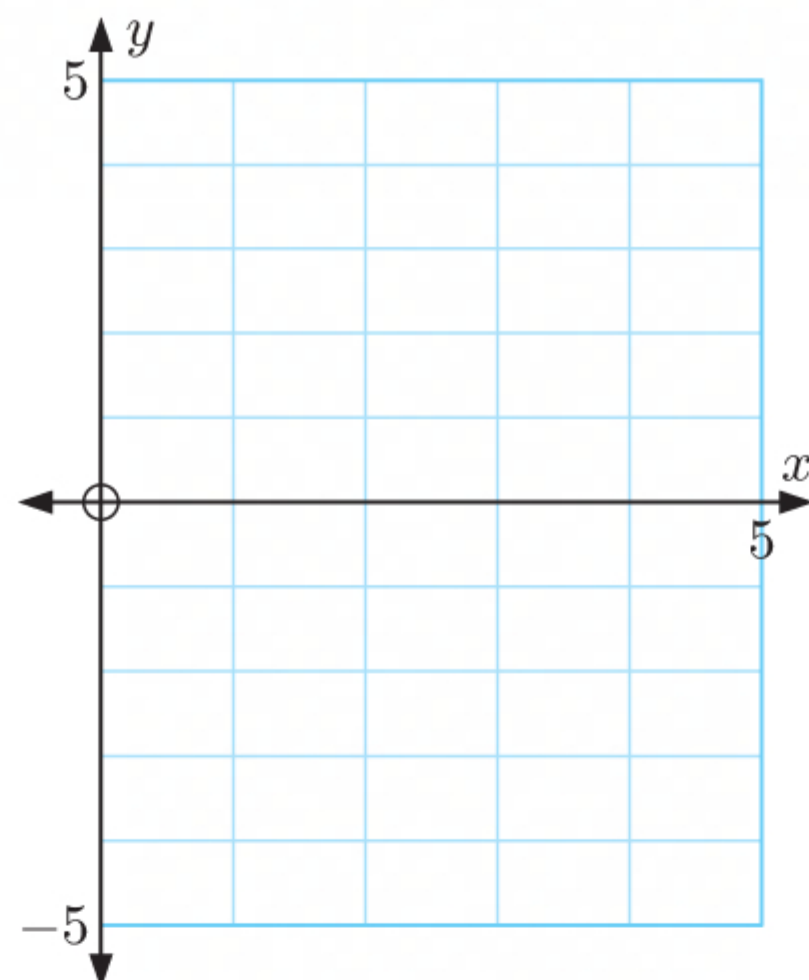
c Show that the Cartesian equation of plane Π is $3x - 5y + 4z = 9$. [3]

d Find the **exact** area of triangle ABC. [3]

e Show that $\cos \widehat{BAC} = \frac{2\sqrt{2}}{3}$. [3]

PAPER 2**CALCULATOR, 120 MINUTES****SECTION A****1 [Maximum mark: 6]**

- a** On a graph like the one below, sketch the graphs $f(x) = \sin^2 x + \log_2(x) - 3$ and $g(x) = 2(\ln x)^2 - 3$. [3]



- b** Hence, solve $f(x) \geq g(x)$ for $0 < x < 5$. [3]

2 [Maximum mark: 7]

The coefficient of x^2 in the expansion $(2x - 3)^5$ is equal to the coefficient of x^3 in the expansion of $(-3ax + 4)^4$.

- a** Find the **exact** value of a . [5]

- b** Hence, find the coefficient of x^4 in the expansion of $(2x)(-3ax + 4)^4$. [2]

3 [Maximum mark: 7]

The graph of $y = 2 + \ln[(3x + 6)^2]$, $x > -2$ is obtained from the graph of $y = \ln(x)$ by the following operations:

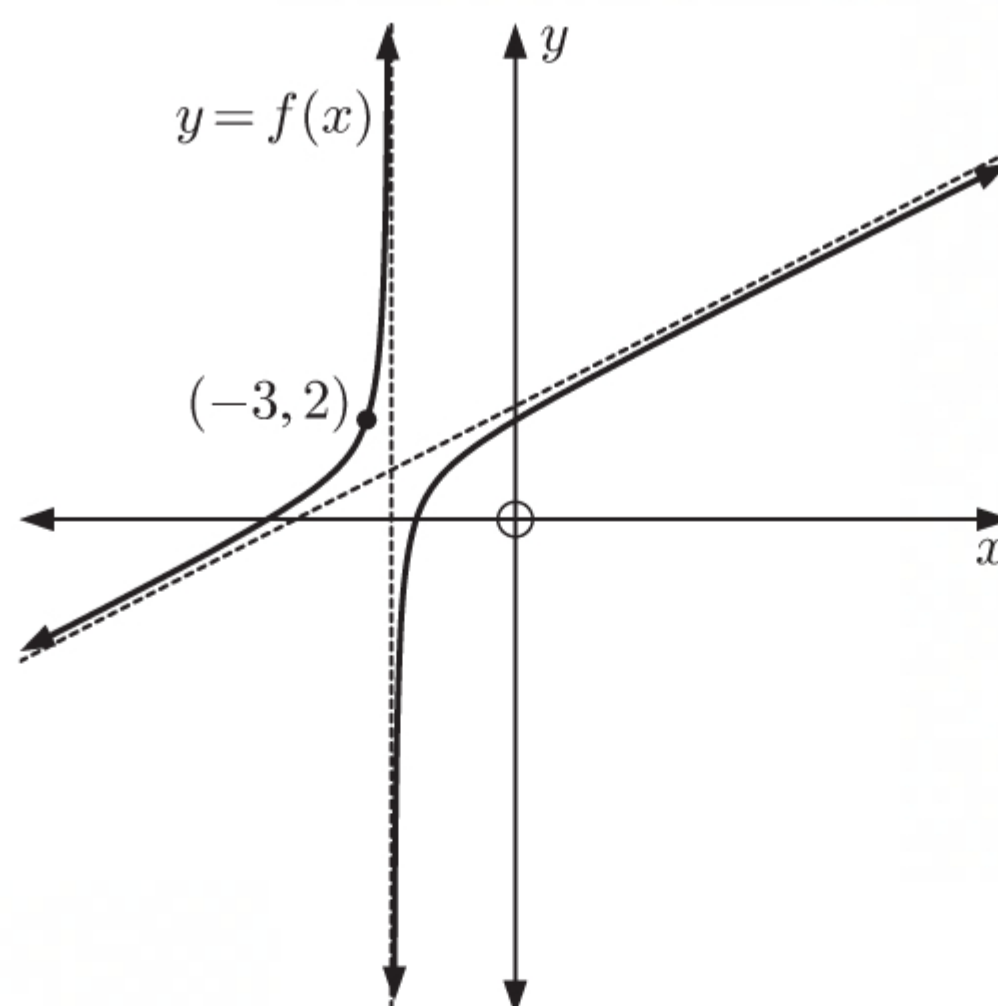
- a translation of a units in the direction of the x -axis, followed by
- a vertical stretch with scale factor b , followed by
- a translation of c units in the direction of the y -axis.

- a** Find the value of a , b , and c . [5]

- b** The region bounded by the graph of $y = 2 + \ln[(3x + 6)^2]$, the x -axis, and the lines $x = e$ and $x = e^2$ is rotated through 2π radians about the x -axis. Find the volume of the generated solid. [2]

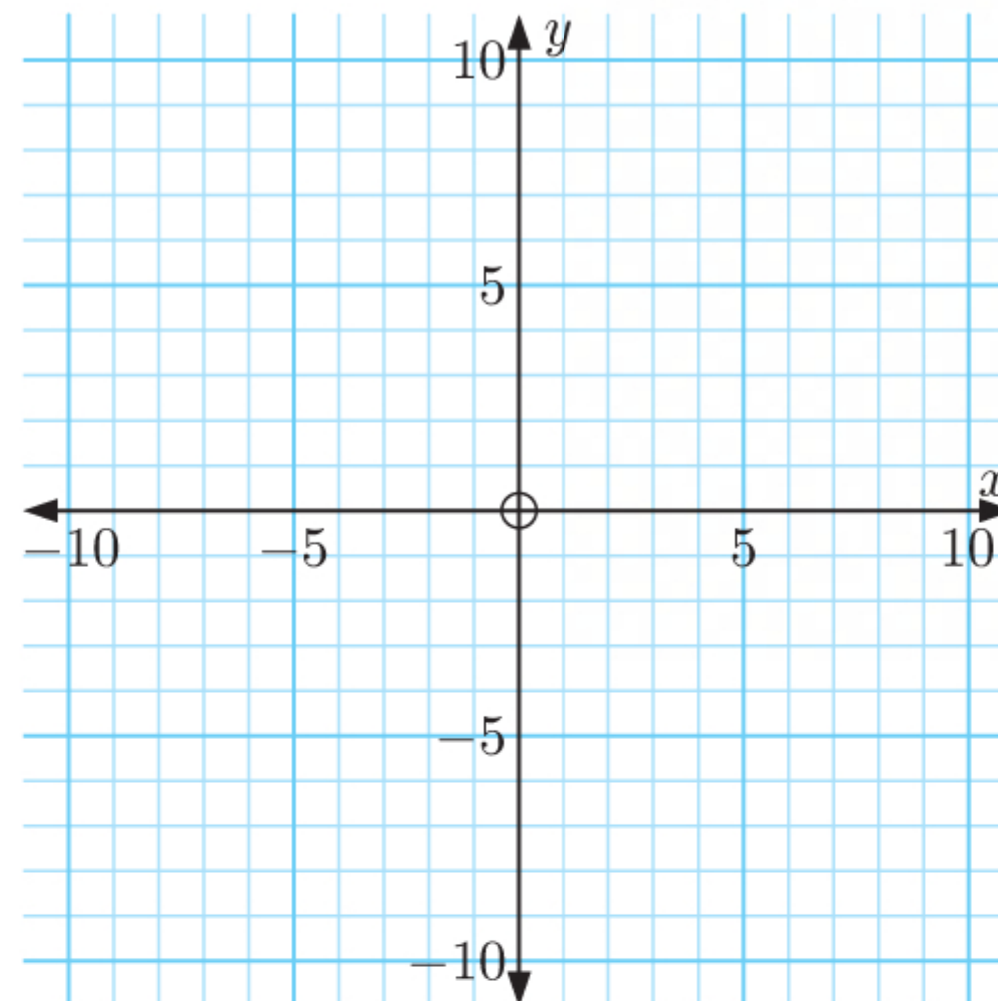
4 [Maximum mark: 8]

The graph of $f(x) = \frac{x^2 + bx + c}{2x + 5}$ is shown below. Point $(-3, 2)$ is on the graph and the equation of its oblique asymptote is $y = \frac{1}{2}x + \frac{9}{4}$.



- a** Write down the equation of the vertical asymptote. [1]

- b** Find the value of:
- b [2]
 - c . [2]
- c** Using the values of b and c found previously, sketch the graph of $g(x) = \frac{1}{f(x)}$ on an axis like the one provided [3]
below, showing clearly all intercepts and asymptotes.



5 [Maximum mark: 10]

For a 12-month period, a bike shop recorded the number of custom-made bikes sold (x) and the total profit (in dollars) made from the sale of those bikes (y). These records are shown in the following table.

Month	1	2	3	4	5	6	7	8	9	10	11	12
x	11	19	6	8	10	12	16	5	21	14	13	17
y	1425	2573	883	897	1345	1576	2123	645	3068	1705	1943	2408

- Write down the equation of the linear regression line of y against x . [2]
- Write down the value of Pearson's product-moment correlation coefficient, r . [1]
- Briefly interpret the value of r . [1]
- Use your regression line as a model to explain the meaning of the:
 - gradient [1]
 - y -intercept. [1]
- Estimate the total profit (in dollars) if the bike shop were to sell 30 custom-made bikes in a certain month. [2]
- Estimate the number of custom-made bikes that the bike shop needs to sell in order to make a profit of at least 8000 dollars from the sale of those bikes. [2]

6 [Maximum mark: 9]

The equation of the lines L_1 and L_2 are

$$L_1: \mathbf{r}_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$L_2: \mathbf{r}_2 = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}, \mu \in \mathbb{R}$$

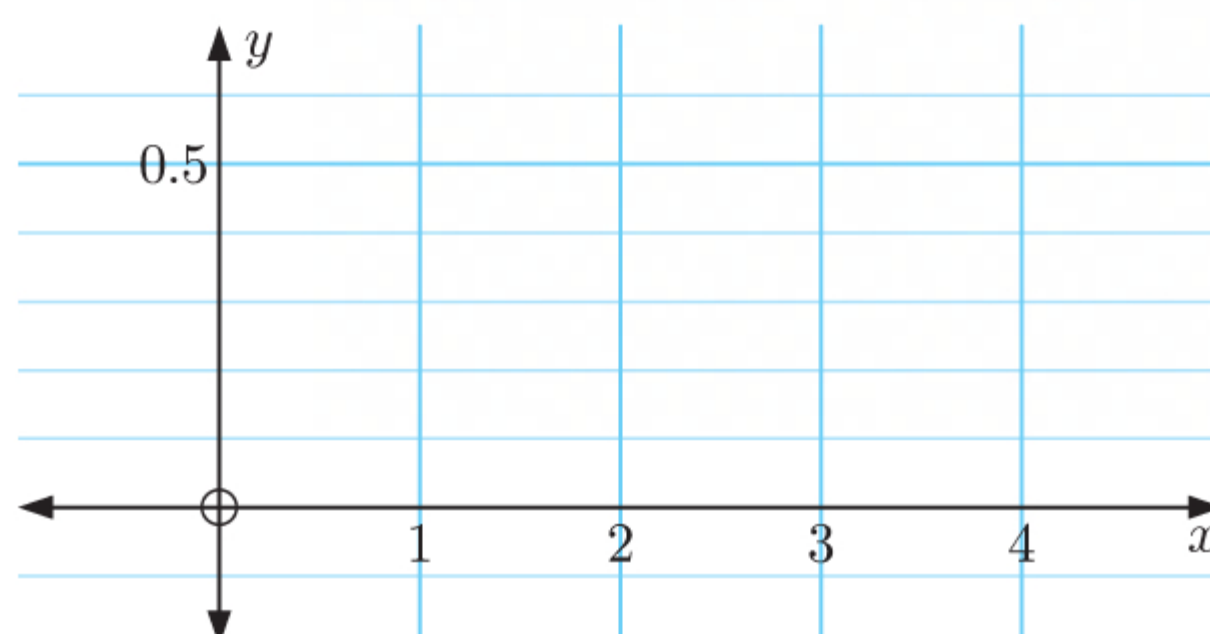
- Show that the lines L_1 and L_2 are:
 - not parallel [1]
 - not intersecting. [3]
- Find the obtuse angle between the lines L_1 and L_2 . [5]

7 [Maximum mark: 8]

A random variable X has the following probability density function:

$$f(x) = \begin{cases} \frac{1}{k}x, & 0 \leq x \leq 2 \\ \frac{1}{2k}x, & 2 < x \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

- a** Determine the value of k . [3]
b On a grid like the one below, sketch the graph of $f(x)$. [2]



- c** Find $E(X)$. [3]

SECTION B**8 [Maximum mark: 10]**

Let $f(x) = \frac{5x}{x^2 - x - 6}$ where $x \neq -2$ and $x \neq 3$.

- a** Express $f(x)$ in partial fractions. [4]
b By finding $f'(x)$, show that $f(x)$ is decreasing on its entire domain. [3]
c Use part **a** to show that [3]

$$\int_{-1}^0 f(x) dx = \ln \frac{27}{16}$$

9 [Maximum mark: 20]

Consider the curve given by $x^3 + y^3 = 6xy$, called the folium of Descartes.

- a** Find $\frac{dy}{dx}$. [4]
b Find the equation of the tangent at $(3, 3)$. [4]
c Find the **exact** coordinates of points in the first quadrant where the tangent is:
i horizontal [7]
ii vertical. [5]

10 [Maximum mark: 25]

Consider the function $f(x) = \tan x$.

- a** Write down:
i $f'(x)$ [1]
ii $f''(x)$. [2]
b Show that $f'''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x$. [2]
c Hence, obtain the first two non-zero terms in the Maclaurin expansion of $f(x)$. [4]

Consider the function $g(x) = e^{2x}$ and let $h(x) = f(x) \times g(x)$.

- d** Find the first four non-zero terms in the Maclaurin expansion of $g(x)$. [7]
e Hence, or otherwise, find the first three non-zero terms in the Maclaurin expansion of $h(x)$. [4]
f Hence, or otherwise, write down the first three non-zero terms in the Maclaurin expansion of [5]

$$2e^{2x} \tan x + e^{2x} \sec^2 x.$$

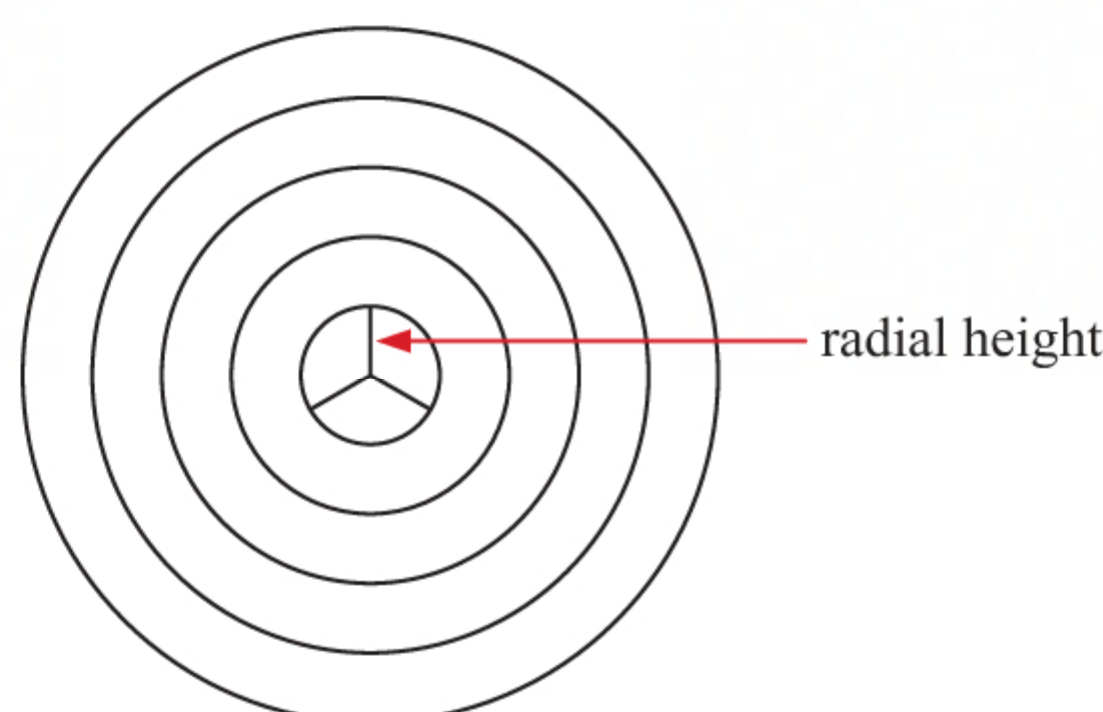
PAPER 3

CALCULATOR, 60 MINUTES

1 [Maximum mark: 32]

This question asks you to investigate a method for partitioning a unit circle in cells of equal areas and shapes, resulting in the most compact cells.

- a** A unit circle ($r = 1$) is divided into 5 rings, each with the same radial height. The *smallest* ring is then divided into three identical cells, as shown.



- i** The radial height of the *smallest* ring is indicated in the diagram above. Write down this radial height. [1]
- ii** Find, in terms of π , the **exact** area of each of the three identical cells. [2]
- b i** Find, in terms of π , the **exact** area of the *second smallest* ring. [2]
- The second smallest ring should be divided into cells such that the area of each of these cells is equal to the area of each cell found in **a** above.
- ii** Find the number of cells into which this second smallest ring should be divided. [2]
- c** Show that dividing the third smallest ring into 15 identical partitions results in cells whose area is the same as the area of cells in the smallest and second smallest rings found above. [2]

The fourth smallest (which is the second largest) ring and the largest ring are divided into a and b identical cells, respectively, such that the area of each of these cells is the same as the area of the cells in the smaller rings found above.

- d** Find the value of:
- i** a [3]
- ii** b . [2]

Now suppose the original unit circle is divided into N rings.

- e** Write down the radial height of each ring. [1]

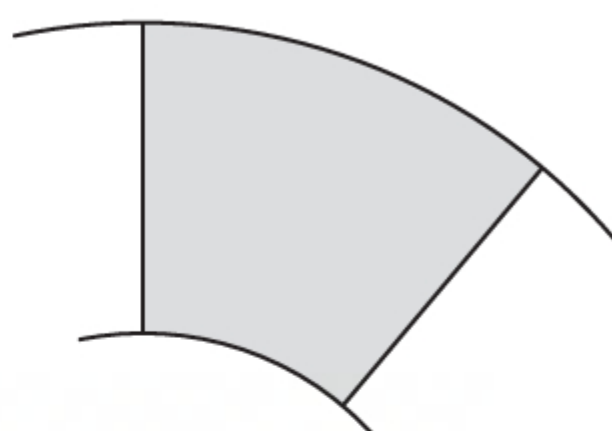
Let A_1, A_2, \dots, A_N represent the areas of the N rings, where A_1 represents the area of the smallest (innermost) ring and A_N represents the area of the largest (outermost) ring.

- f** Find, in terms of n and N , the area of the n th ring, A_n , where $1 < n \leq N$, $n \in \mathbb{Z}^+$. [2]

Suppose the innermost ring is divided into k cells, where $k \in \mathbb{Z}^+$.

- g** Find:
- i** in terms of N and k , the area of each cell in the innermost ring [2]
- ii** in terms of n and k , the number of cells in the n th ring, each of which has an area equal to the area of the innermost cells [2]
- iii** in terms of N and k , the total number of cells in the unit circle. [2]

With the exception of the cells in the innermost ring, each cell has identical quadrangle shape with two circular arcs, as shown in the diagram below.



- h** Find, in terms of N and k only, an expression for the perimeter, P , of a cell. [2]

Let β represent the shape coefficient, defined as the ratio of the square of the cell's perimeter to the cell's area.

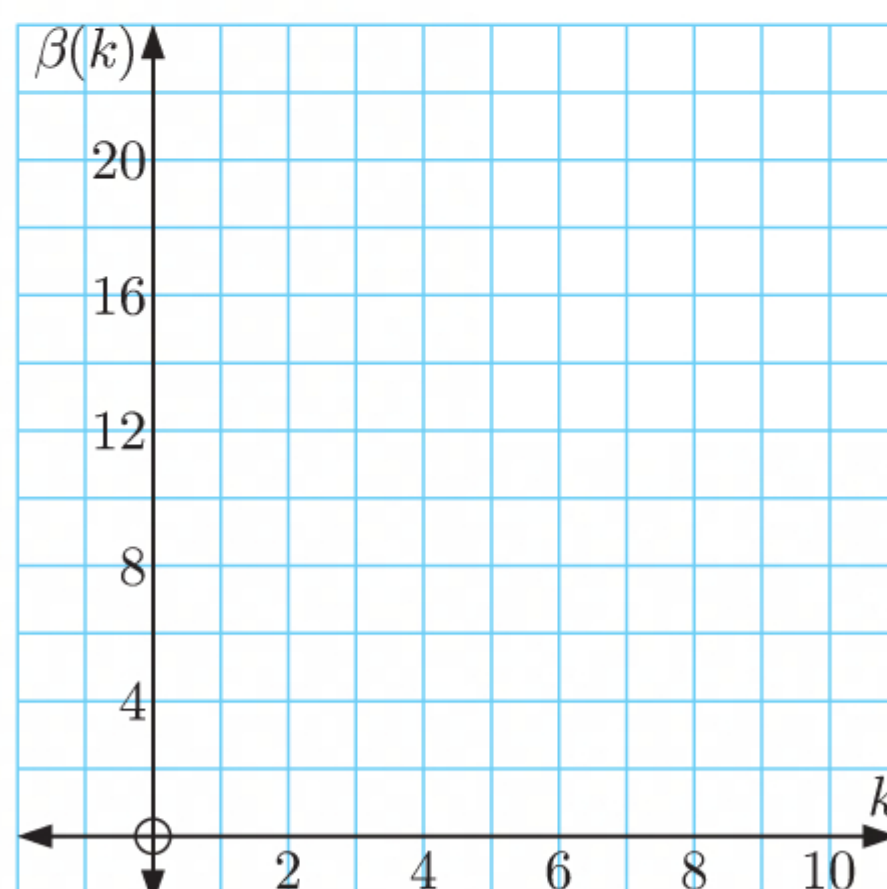
i Show that

$$\beta = \frac{4(k + \pi)^2}{k\pi}$$

[1]

j Use a coordinate grid like the one provided below to sketch the graph of $\beta(k)$.

[1]



In order to obtain the most compact cells, the shape coefficient must be as small as possible.

k i Find $\frac{d\beta}{dk}$ in the simplest form.

[2]

ii Hence, or otherwise, find the value of k for which the shape coefficient is the smallest possible.

[3]

2 [Maximum mark: 23]

This question asks you to investigate some properties of even and odd functions.

a Consider the functions $f_1(x) = \frac{3}{x^2 + 2}$ and $f_2(x) = 2x^3 - 4x$.

Determine whether the function:

i $f_1(x)$ is an even function or an odd function

[2]

ii $f_2(x)$ is an even function or an odd function.

[2]

b Suppose $f(x)$ is a function with domain $x \in \mathbb{R}$, $g(x) = \frac{f(x) + f(-x)}{2}$, and $h(x) = \frac{f(x) - f(-x)}{2}$.

i Show that $g(x) + h(x) = f(x)$.

[1]

ii Hence show that $f(x)$ can be written as the sum of an even function and an odd function.

[3]

c Using the results in **b**, find the even part ($g(x)$) and the odd part ($h(x)$) of

[3]

$$f(x) = x^3 + 2x^2 - x$$

Let $f(x)$ be differentiable at all $x \in \mathbb{R}$.

d Show that:

i if $f(x)$ is an even function, then $f'(x)$ is an odd function

[1]

ii if $f(x)$ is an odd function, then $f'(x)$ is an even function.

[1]

e Show that if $f(x)$ is an odd function defined and indefinitely differentiable at $x = 0$, then:

i $f(0) = 0$

[1]

ii $f^{(n)}(0) = 0$ when n is even

[3]

iii its Maclaurin series expansion contains only odd powers of x .

[1]

f Show that if $f(x)$ is an even function, then its Maclaurin series expansion contains only even powers of x .

[5]

Trial examination 2

PAPER 1

NO CALCULATOR, 120 MINUTES

SECTION A

1 [Maximum mark: 6]

The polynomial $f(x) = x^4 + bx^2 - 3ax + (4 - 2b)$ has linear factors of $(x - 1)$ and $(x - 2)$. Determine the value of a and b where $a, b \in \mathbb{Q}$.

2 [Maximum mark: 4]

For two independent events A and B it is known that $P(A \cup B) = 0.9$ and $P(A' \cap B) = 0.4$.

Find $P(A)$ and $P(B)$.

3 [Maximum mark: 6]

In an arithmetic series with common difference d , $S_3 = 24$ and $u_1 = d$. Given that the n th term in this series is 124, find the value of n .

4 [Maximum mark: 5]

Find all real values of x such that

$$49^{x^3-2} = \left(\frac{1}{\sqrt[3]{7}}\right)^{7-x}$$

5 [Maximum mark: 5]

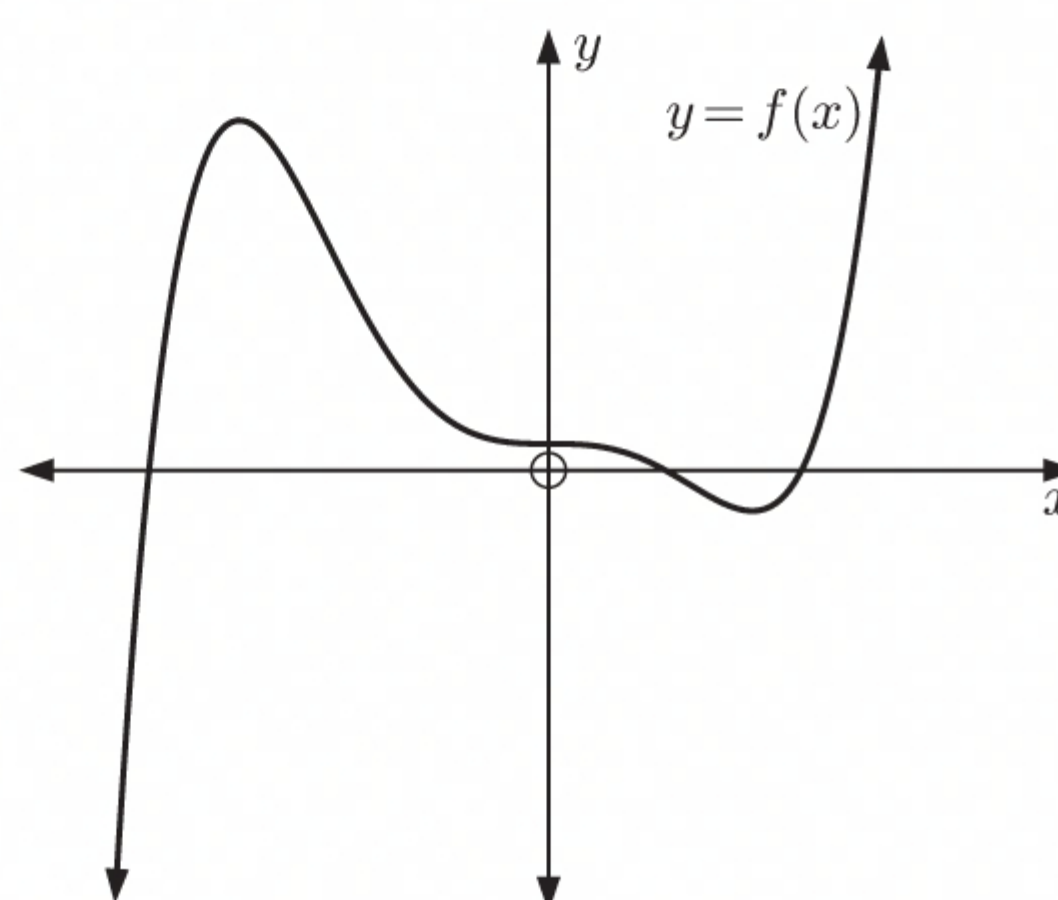
Given that $\sec \theta = \frac{5}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$, determine the exact value of $\tan 2\theta$.

6 [Maximum mark: 7]

The roots of the equation $2x^2 + 4x + k = 0$, where k is a constant, are α and β . Find, in terms of k , a quadratic equation whose roots are α^2 and β^2 .

7 [Maximum mark: 6]

The graph of $y = f(x)$ contains a local maximum for $x < 0$, a stationary inflexion point at $x = 0$ and a local minimum for $x > 0$.



On separate sets of axes, sketch the graphs of:

a $y = f(|x|)$

[2]

b $y = \frac{1}{f(|x|)}$

[4]

8 [Maximum mark: 7]

Consider the function $f(x) = x \ln(2x + 1)$.

a Find $f'(x)$.

[2]

b Hence determine $\int \ln(2x + 1) dx$.

[5]

9 [Maximum mark: 8]

Use l'Hôpital's rule to find

$$\lim_{\theta \rightarrow 0} \left(\frac{2 \tan \theta - 2 \sin \theta}{\sin 2\theta - 2 \sin \theta} \right)$$

SECTION B**10 [Maximum mark: 18]**

The equations of two lines L_1 and L_2 are:

$$L_1: \mathbf{r} = (3\mathbf{i} - \mathbf{j} - \mathbf{k}) + t(4\mathbf{i} - 3\mathbf{j}) \text{ where } t \in \mathbb{R}$$

$$L_2: \frac{x+1}{2} = \frac{3(11-2y)}{4} = z-1$$

The equations of two planes P_1 and P_2 are:

$$P_1: \mathbf{r} = (\mathbf{i} + \mathbf{j} - \mathbf{k}) + \lambda(\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{k}) \text{ where } \lambda \text{ and } \mu \text{ are scalars}$$

$$P_2: -6x + 2y - 3z = 63$$

- a** Find the vector cross product $(\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{k})$. [3]
- b i** Write down the vectors \mathbf{n}_1 and \mathbf{n}_2 which are normal to the planes P_1 and P_2 respectively. [2]
- ii** Hence, or otherwise, show that the cosine of the angle between the planes is $\frac{16}{21}$. [3]
- c** Show that L_2 is normal to P_2 . [4]

The line L_2 intersects the plane P_2 at Q.

- d** Find the coordinates of Q. [6]

11 [Maximum mark: 21]

Let $z = \cos \theta + i \sin \theta$.

- a i** Find z^4 using the binomial theorem. [3]
- ii** Use De Moivre's theorem to show that [6]

$$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$$

- b** Hence prove that $\frac{\sin 4\theta - 2 \sin 2\theta}{\cos 4\theta + 4 \cos 2\theta + 3} = -\tan^3 \theta$ [6]
- c** Use the result from part **b** to find an exact value for $\tan^3(15^\circ)$ giving your answer in the form $a + b\sqrt{3}$, where $a, b \in \mathbb{Z}$. [6]

12 [Maximum mark: 17]

- a** State the domain and range of $g(x) = \tan x$. [2]
- b** Determine the first two non-zero terms in the Maclaurin series expansion for $g(x)$. [6]
- c** Using the approximation $e^z \approx 1 + z + \frac{1}{2}z^2$, find a series for $e^{\tan x}$ up to and including the term in x^6 . [5]
- d** Hence, or otherwise find the value of [4]

$$\lim_{x \rightarrow 0} \left(\frac{e^{\tan x} - 1}{\tan x} \right)$$

PAPER 2**CALCULATOR, 120 MINUTES****SECTION A****1 [Maximum mark: 5]**

Mohammed invests 28 000 pounds in an account that pays an annual interest rate of 2.5%, compounded monthly for three years.

- a** Calculate the value of Mohammed's investment at the end of this time. Give your answer to the nearest pound. [3]
- b** The average inflation rate during this period was 1.8% per annum. Calculate the real value of Mohammed's investment to the nearest pound. [2]

2 [Maximum mark: 6]

A discrete random variable, X , has the following probability distribution.

X	1	3	6	8	x
$P(X = x)$	$\frac{1}{10}$	p	$\frac{p}{4}$	$4p^2$	$\frac{27}{80}$

Determine the value of p and of x if $E(X) = 7.95$.

3 [Maximum mark: 8]

Weights of packets of noodles, W , produced in a factory are normally distributed with mean 85 grams.

5% of the packets weigh less than 83 grams.

a i Find the standard deviation, σ of the weights of packets of noodles. [3]

ii Hence determine the probability that a randomly selected packet of noodles weighs more than 86.5 grams. [2]

b In a random sample of 60 packets, find the probability that at least 9 packets will weigh more than 86.5 grams. [3]

4 [Maximum mark: 4]

Prove that the sum of three consecutive odd integers will always be odd.

5 [Maximum mark: 6]

The equations of a plane π and a line L are as follows

$$\begin{aligned}\pi: & 3x - z = 4 \\ L: & \frac{x-1}{2} = \frac{y}{2} = 4z\end{aligned}$$

Determine the acute angle between the line and the plane to the nearest tenth of a degree.

6 [Maximum mark: 7]

Prove by mathematical induction that $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{Z}^+$.

7 [Maximum mark: 7]

The probability density function of a continuous random variable T is given by

$$f(t) = \begin{cases} 2^t - \frac{t^2}{4}, & 0 \leq t \leq k \\ 0, & \text{otherwise.} \end{cases}$$

a Find the value of k giving your answer to 5 significant figures. [5]

b Find $P\left(\frac{1}{10} \leq T \leq \frac{1}{2}\right)$. [2]

8 [Maximum mark: 7]

Given that $\arg(z + i) = \frac{\pi}{6}$ and $\arg\left(z - \frac{\sqrt{3}}{2}\right) = -\frac{\pi}{2}$, where $z \in \mathbb{C}$ and $i = \sqrt{-1}$. Find z in the form $a + bi$, $a, b \in \mathbb{R}$.

9 [Maximum mark: 6]

The school's switchboard has two operators, Anne and Billie. Anne answers calls for middle school students and Billie answers calls for senior school students. Anne fails to answer 3% of her calls and Billie fails to answer 5% of her calls. On a typical school day, Anne and Billie receive 30 and 50 calls respectively during the morning.

a Find the average number of calls Anne and Billie fail to answer in total each morning. [3]

b Find the probability that, between them Anne and Billie fail to answer 2 or more calls. [3]

SECTION B**10 [Maximum mark: 14]**

Consider a group of eight girls and two boys.

a These ten people are to be arranged in a row. Find the number of ways this can be done if:

i the two boys are together [2]

ii the two boys are not together [2]

iii there is exactly one girl separating the two boys. [3]

Teams of four are to be chosen at random from the eight girls and two boys.

b Find the number of teams which consist of:

i two girls and two boys

[2]

ii all girls.

[2]

Three of the girls in the group are sisters.

c Find the probability that a team selected contains all three sisters.

[3]

11 [Maximum mark: 15]

a Sketch the curve $y^2 = x^2 - x^4$, $-1 \leq x \leq 1$.

[3]

b Determine the exact coordinates of all stationary points of the curve for $y > 0$.

[6]

c Determine the x -coordinates of all points where the gradient to the curve is undefined.

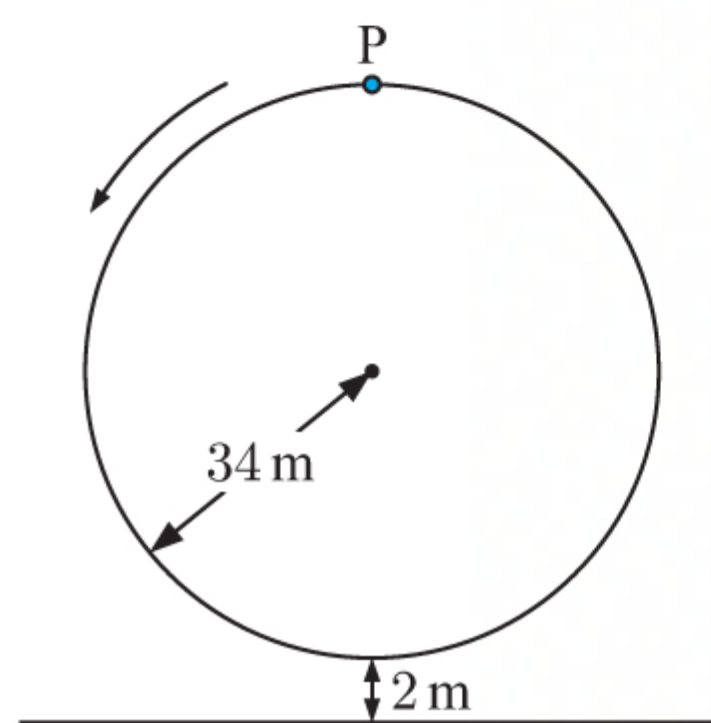
[3]

d Determine the area completely enclosed by the curve $y^2 = x^2 - x^4$, $-1 \leq x \leq 1$.

[3]

12 [Maximum mark: 11]

A large wheel sits two metres above the ground and is rotating with constant speed at a rate of one revolution per eight minutes.



Point P is initially located at the highest point of the wheel above the ground and begins rotating anti-clockwise.

a Determine how high point P is above the ground after five minutes. Give your answer to the nearest centimetre.

[5]

b Determine how fast the wheel is moving vertically upwards after five minutes.

[6]

13 [Maximum mark: 14]

a Show that the general solution of $\frac{dy}{dt} = ky(2 - ky)$, $0 < y < \frac{2}{k}$, $k \in \mathbb{R}^+$ is $y = \frac{2e^{2(kt+c)}}{1 + ke^{2(kt+c)}}$ where c is the constant of integration.

[10]

b Given that $k = \frac{1}{e}$ and $y(0) = e$, determine the exact value of $y(e)$.

[4]

PAPER 3

CALCULATOR, 60 MINUTES

1 [Maximum mark: 27]

a Consider the cubic function $p(x) = x^3 - 3x^2 - 10x + 24$.

i Show that $x = 2$ is a root of $p(x) = 0$ and hence fully factorise the cubic.

[4]

ii Find the equation of the tangent to $y = p(x)$ at $x = 3$.

[3]

iii Sketch $y = p(x)$ and the tangent at $x = 3$ on the same set of axes.

[2]

b Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, and the line $L(x) = mx + 5$.

Let x_1, x_2, x_3 be the roots of the equation $f(x) = L(x)$.

i Show that the sum of the roots is a constant independent of the choice of line L .

[2]

ii Show that the inflection point of $y = f(x)$ has the x -coordinate $x^* = \frac{x_1 + x_2 + x_3}{3}$.

[3]

- iii Suppose $y = L(x)$ is a *tangent* to $y = f(x)$ at the point $x = x_1$.
- (1) Show that $f'(x) - L'(x) = 3a(x - x_1)(x - \beta)$ for some β . [3]
- (2) Hence show that $f(x) - L(x) = \frac{a}{2}(x - x_1)^2(2x + x_1 - 3\beta)$. [5]
- (3) Assuming it exists, find the x -coordinate of the other point of intersection between the cubic and the line, in terms of x_1 and x^* . Illustrate your answer. [3]
- (4) In the case that $L(x) = 0$, interpret the result in (3) in terms of the graph of the cubic function. [2]

2 [Maximum mark: 28]

- a Use the compound angle expansion formula to expand $R \sin(x - \alpha)$, where R and α are constants. [1]

The expression $\sqrt{3} \sin x - \cos x$ can be represented as $R \sin(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

- b i Using your result from part a, show that $\sqrt{3} = R \cos \alpha$ and $1 = R \sin \alpha$. [1]
- ii Hence, find the exact value of R and α . [4]

Let $f(x) = \sqrt{3} \sin x - \cos x$, $x \in [0, \frac{\pi}{2}]$.

- c The equation $f(x) - 1 = 0$ has exactly one solution in the interval $[0, \frac{\pi}{2}]$. Find the exact value of this solution. [3]
- d i Explain why the inverse function of f exists. [1]
- ii Show that the inverse function f^{-1} is: [2]

$$f^{-1}(x) = \frac{\pi}{6} + \arcsin \frac{x}{2}$$

- e Show that $\int_0^1 f^{-1}(x) dx = \frac{\pi}{3} + \sqrt{3} - 2$. [8]

Consider the equation:

$$\int_0^1 f^{-1}(x) dx = \int_{\frac{\pi}{3}}^a \frac{4}{f(x)} dx \quad \text{where } a > \frac{\pi}{3}, a \in \mathbb{R}$$

- f Using the result $\int \operatorname{cosec} x dx = -\ln |\operatorname{cosec} x + \cot x| + C$ where C is a constant, show that: [6]

$$\left| \operatorname{cosec}\left(a - \frac{\pi}{6}\right) + \cot\left(a - \frac{\pi}{6}\right) \right| = \frac{2 + \sqrt{3}}{e^{\frac{\pi}{6} + \frac{\sqrt{3}}{2}} - 1}$$

- g Find the value of a giving your answer to 5 significant figures. [2]

Trial examination 3

PAPER 1

NO CALCULATOR, 120 MINUTES

SECTION A

1 [Maximum mark: 7]

- a** Express the binomial coefficient $\binom{2n-1}{2n-3}$ as a product of two linear polynomials in n . [4]
- b** Hence or otherwise, find the possible values of n , where $n \in \mathbb{Z}^+$, for which [3]
- $$\binom{2n-1}{2n-3} \leq 10$$

2 [Maximum mark: 6]

The following system of equations represents three planes in space.

$$\begin{aligned} 2x + 3y - 4z &= -7 \\ x - y + 2z &= 6 \\ 3x - 2y - 2z &= 11 \end{aligned}$$

Find the coordinates of the point of intersection of the three planes.

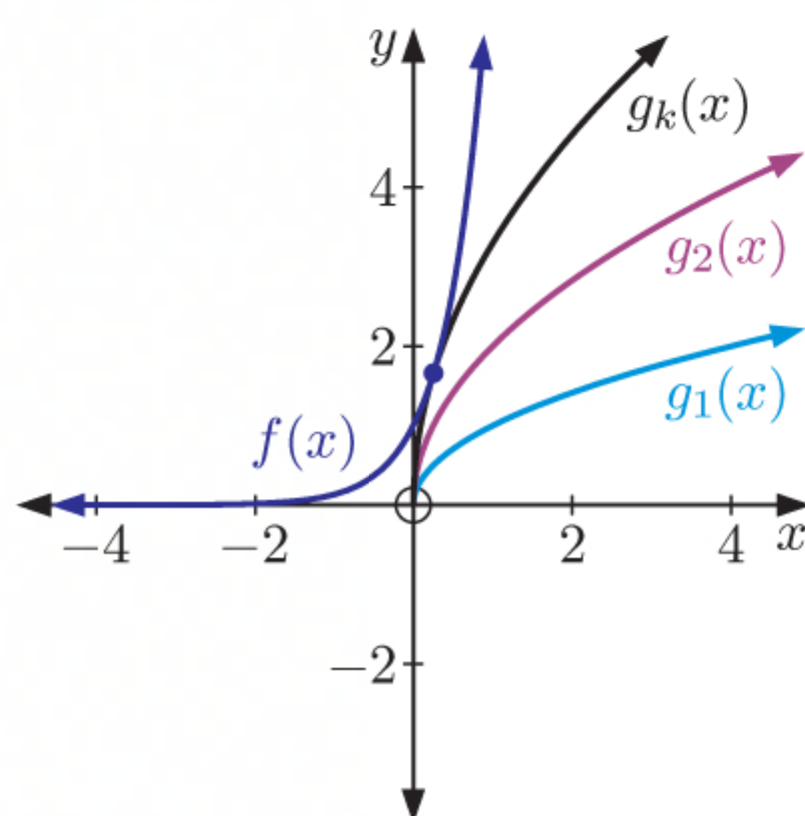
3 [Maximum mark: 10]

The sum of the first eight terms of an arithmetic sequence is 6, and the sum of the next four terms is 39.

- a** Find the value of the common difference d and the first term u_1 . [5]
- b** Hence, find an expression for u_n , the n th term of the sequence. [2]
- c** Find the smallest value of n , for which $S_n > 45$. [3]

4 [Maximum mark: 6]

Consider the graph of $f(x) = e^{2x}$ and $g_k(x) = k\sqrt{x}$ for $k = 1$, $k = 2$, and for some other value of k .



Find the **exact** value of k for which the equation $e^{2x} = k\sqrt{x}$ has **exactly one** solution.

5 [Maximum mark: 8]

The plane with equation $-3x + y + 2z = 2$ is perpendicular to the line with the parametric equations

$$\begin{aligned} x &= 2 + (t+1)\lambda \\ y &= 3 - 3\lambda \\ z &= -1 + (2-t)\lambda \end{aligned}, \quad \lambda \in \mathbb{R}$$

where t is a constant.

- a** Write down a normal \mathbf{n} to the plane. [1]
- b** Find: [3]
- i** the value of t [3]
- ii** the coordinates of the point of intersection of the plane and the line. [4]

6 [Maximum mark: 9]

Curves with equations in the form of $x^n + y^n = 1$, $n \in \{4, 6, 8, \dots\}$ are called fat circles. Consider the fat circle defined by the equation $x^4 + y^4 = 1$.

a Show that $\frac{dy}{dx} = -\frac{x^3}{y^3}$. [1]

b Show that $\frac{d^2y}{dx^2} = -\frac{3x^2}{y^7}$. [6]

c Hence or otherwise, find the points on the fat circle for which $\frac{d^2y}{dx^2} = 0$. [2]

7 [Maximum mark: 12]

A continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} k \cos x, & 0 \leq x \leq \frac{\pi}{6} \\ 0, & \text{otherwise.} \end{cases}$$

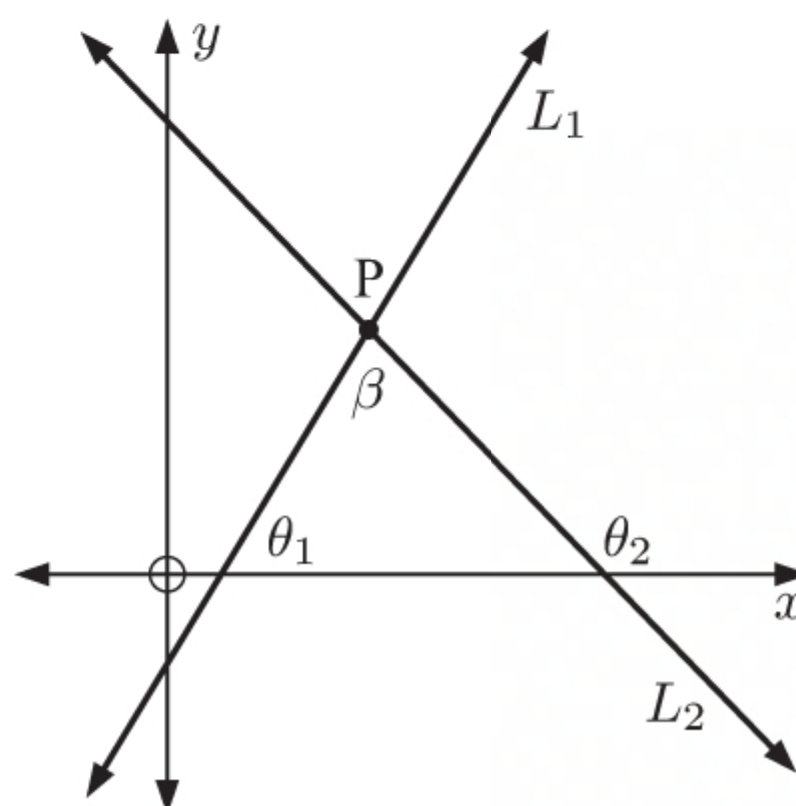
a Find the value of k . [4]

b Find the **exact** value of $E(X)$. [5]

c Find an expression to represent the upper quartile of X . [3]

SECTION B**8 [Maximum mark: 21]**

Lines L_1 and L_2 , neither of which is parallel to the y -axis, intersect at point P; θ_1 and θ_2 represent the angles of inclination that L_1 and L_2 form with the positive x -axis; and m_1 and m_2 represent the gradient of L_1 and the gradient of L_2 , respectively, as shown in the following diagram.



a Find an expression for:

i m_1 in terms of θ_1 [2]

ii m_2 in terms of θ_2 . [1]

b Show that [3]

$$\tan \beta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

c Show that the *acute* angle α formed by L_1 and L_2 satisfies [3]

$$\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|.$$

d Find the *acute* angle formed by lines with equations [6]

$$3x + y = 4 \quad \text{and} \quad y = -\frac{1}{2}x + 2$$

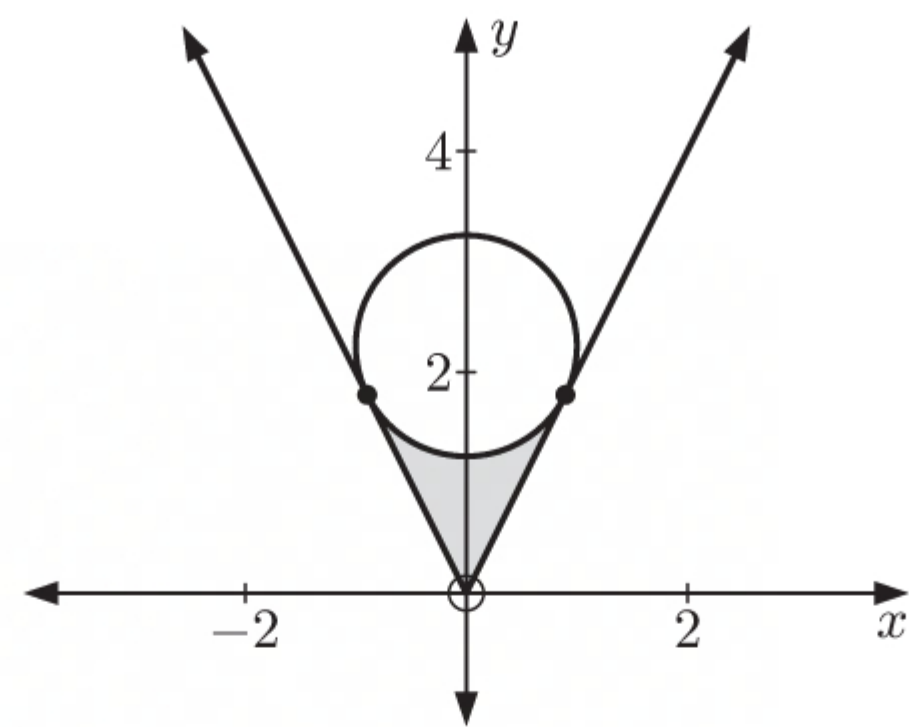
The angle of intersection of two curves is defined as the acute angle between the two tangents drawn to the curves at the point of intersection.

e Find the *tangent* of the angle of the following curves at their point of intersection. [6]

$$f(x) = x^2 \quad \text{and} \quad g(x) = (x - 3)^2 + 1$$

9 [Maximum mark: 16]

A circle with the equation $x^2 + (y - k)^2 = 1$ touches the graph of $y = |2x|$ at two distinct points. The region between the two curves is shaded, as shown.



- a Given the equation of the circle, find the equation of the lower semicircle. [3]
- b Find the **exact** value of k . [5]
- c Find the coordinates of each point of intersection for the graph of the semicircle and the graph of $y = |2x|$. [5]
- d Write an expression for the area of the shaded region. [3]

10 [Maximum mark: 15]

Two identical, unbiased discs are green on one side and red on the other.

Siobhan is playing a game where she begins a round by flipping the two discs at the same time.

If red faces are shown on both discs, one of the discs is flipped one more time, and that is the end of the round. If, however, at least one of the discs shows a green face, no disc is flipped again, and that is the end of the round.

During the round, for each red face shown Siobhan earns 2 coins, and for each green face shown she loses 1 coin.

- a Draw a tree diagram, indicating the probabilities, to represent one round of Siobhan’s game. [3]

Let X be the total number of coins earned in one round in Siobhan’s game.

- b Show that $P(X = 6) = \frac{1}{8}$. [1]
- c Find $P(X = 1)$. [2]
- d Copy and complete the probability distribution table for X . [3]

1st flip	G	G	R	R	R
2nd flip	G	R	G	R	R
3rd flip	-	-	-	G	R
x	$-1 - 1 = -2$	$-1 + 2 = 1$	$2 - 1 = 1$		
$P(X = x)$	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$		

- e Calculate the expected value of X . [2]

Suppose the game costs 1 coin per round to play.

- f Calculate the expected number of coins earned after playing 40 rounds. [2]
- g Calculate the number of rounds Siobhan needs to play in order to expect to earn 15 coins. [2]

PAPER 2

CALCULATOR, 120 MINUTES

SECTION A

1 [Maximum mark: 6]

Solve the simultaneous equations to find the **exact** value of x and y .

$\ln x^3 + \ln y^5 = 3$

$3 \ln \frac{y}{x} = \frac{1}{5}$

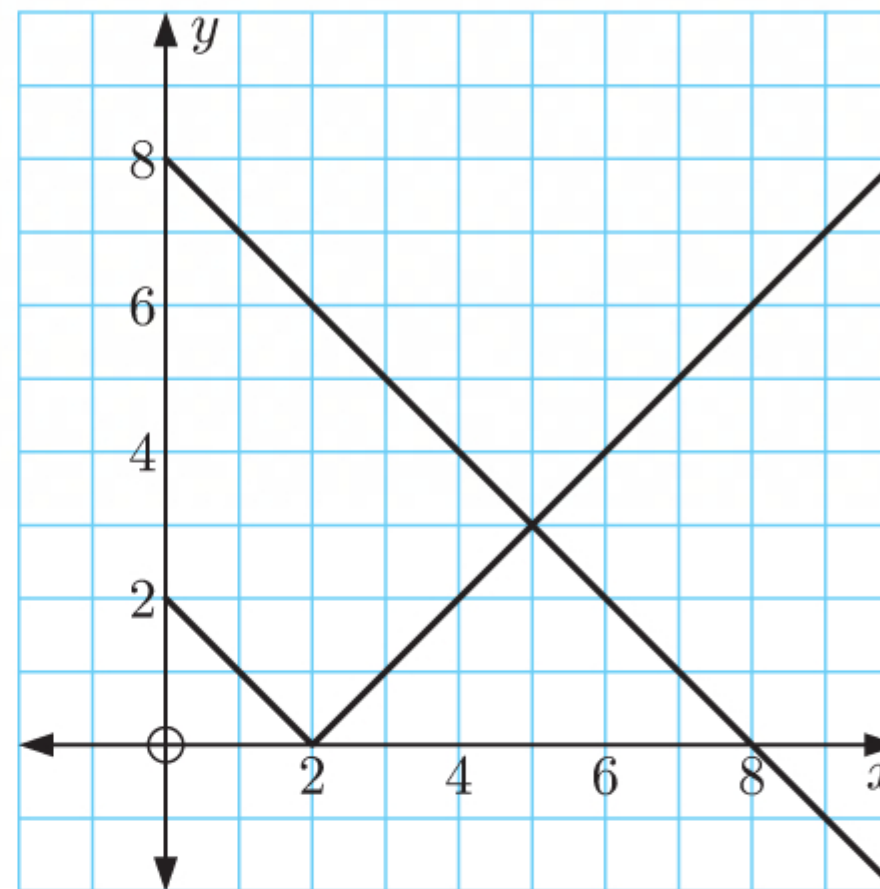
2 [Maximum mark: 6]

a Expand $(1 + 2x)^{-2}$ up to the term in x^3 . [4]

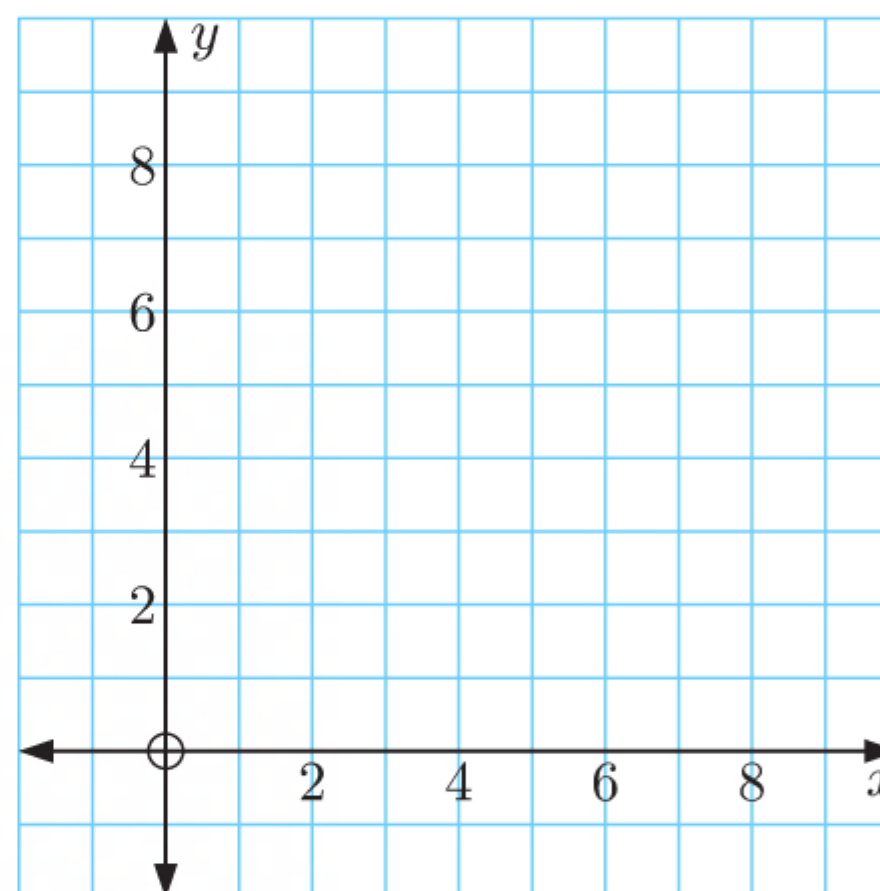
b Hence estimate the value of $\frac{1}{(1.06)^2}$. Give your estimate correct to 4 decimal places. [2]

3 [Maximum mark: 6]

The graphs of $f(x) = |2 - x|$, $x \geq 0$, and $g(x) = -x + 8$, $x \geq 0$, are shown below.



a Use a grid like the one shown below to draw the graph of $h(x) = f(x) + g(x)$. [2]



b Write down the value of:

i $h'(1)$ [1]

ii $h'(3)$. [1]

c Shade part of your graph in **a** to indicate $\int_0^3 h(x) dx$. [1]

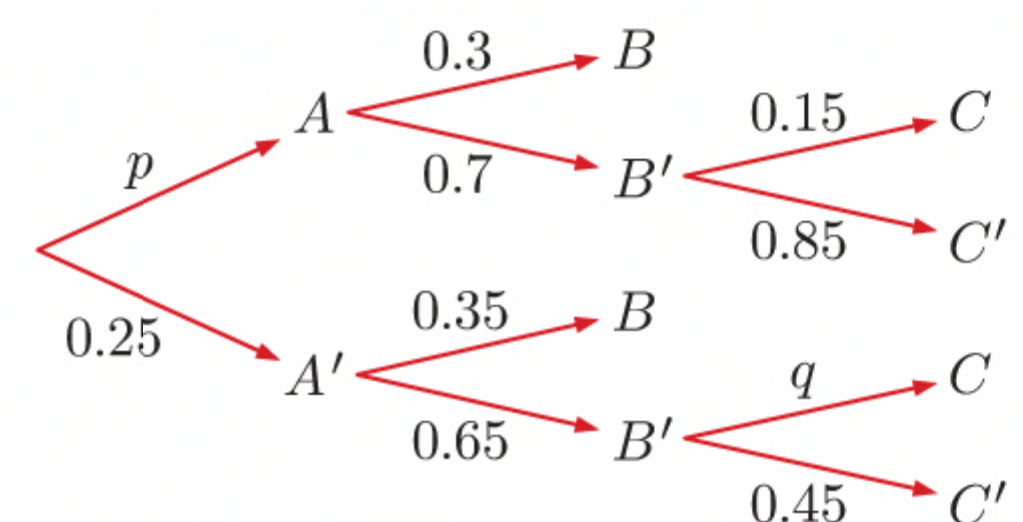
d Write down the value of $\int_0^3 h(x) dx$. [1]

4 [Maximum mark: 6]

Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$. Given that $2 + 3i$ and $-1 - \sqrt{2}i$ are roots of $p(x)$, find the values of a, b, c , and d .

5 [Maximum mark: 8]

The diagram alongside shows the tree diagram representing probabilities for events A, B , and C , where p and q are probabilities.



a Write down the value of:

i p [1]

ii q . [1]

b Find $P(B)$. [3]

c Find $P(A' \cap B' | C)$. [3]

6 [Maximum mark: 7]

A triangle has vertices $A(2, -1, 3)$, $B(5, 0, 1)$, and $C(3, -4, 7)$.

- a** Find \overrightarrow{AB} . [1]
- b** Find \overrightarrow{AC} . [1]
- c** Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [2]
- d** Hence, find the area of the triangle. [3]

7 [Maximum mark: 6]

- a** Prove the Pythagorean identity $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$. [2]
- b** Hence, solve the equation for $-\frac{\pi}{2} < x < \frac{\pi}{2}$: [4]

$$8 \operatorname{cosec}^2 \theta + 14 \cot \theta - 23 = 0$$

8 [Maximum mark: 10]

The amount of liquid in the large-size DEIT bottle of drink is normally distributed with mean 2.08 litres and standard deviation 0.05 litres.

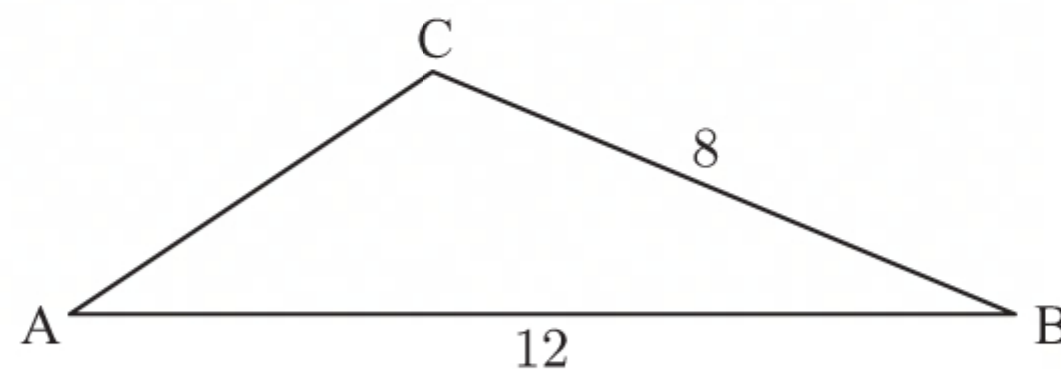
- a** Find the probability that a randomly chosen large-size DEIT bottle of drink has more than 2 litres of liquid in it. [2]

Twelve of these large-size DEIT bottles are shrink-wrapped together for easier transportation.

- b** Find the probability that at least 10 of the bottles contain more than 2 litres of liquid. [4]
- c** The company that makes large-size DEIT bottles also makes small-size DEIT cans of drinks. The amount of liquid in the small-size DEIT can of drink is normally distributed with mean 324 millilitres and standard deviation σ millilitres.
 - i** Given that 10% of small-size cans contain more than 340 millilitres, find σ . [2]
 - ii** From a batch of 50 small-size cans, how many would you expect to contain less than 320 millilitres? [2]

SECTION B**9 [Maximum mark: 19]**

The following diagram shows triangle ABC, with $AB = 12$, $BC = 8$, and $\cos \hat{ABC} = \frac{12}{13}$.



- a** Find the **exact** value of $\sin \hat{ABC}$. [3]
- b** Find AC. [2]

Points D_1 and D_2 are on line (BC) such that $CD_1 = CD_2 = 5$. Point D_1 is between points B and C, and point C is between points D_2 and B.

- c** Find:
 - i** AD_1 [2]
 - ii** AD_2 [2]
- d** Hence, find, in degrees, the measure of:
 - i** angle D_1AB [2]
 - ii** angle D_2AB [2]
- e** Show that the area of triangle ACD_1 equals the area of triangle ACD_2 . [1]
- f** Find the **exact** area of triangle ACD_2 . [2]
- g** Hence, find $\sin(\hat{ACD}_2)$. [3]

10 [Maximum mark: 19]

Consider the function

$$f(x) = \frac{3x + 11}{x^2 - x - 6}$$

- a** Find the equation of each vertical asymptote of $f(x)$. [3]
- b** Express $f(x)$ in partial fractions. [6]
- c** Hence, show that $\int_4^5 f(x) dx = \ln \frac{96}{7}$. [4]
- d** Suppose $\int_4^a f(x) dx = 2 \int_4^5 f(x) dx$ for some constant $a > 5$.
 - i** Show that $\frac{(a-3)^4}{a+2} = \frac{1536}{49}$. [5]
 - ii** Hence find a . [1]

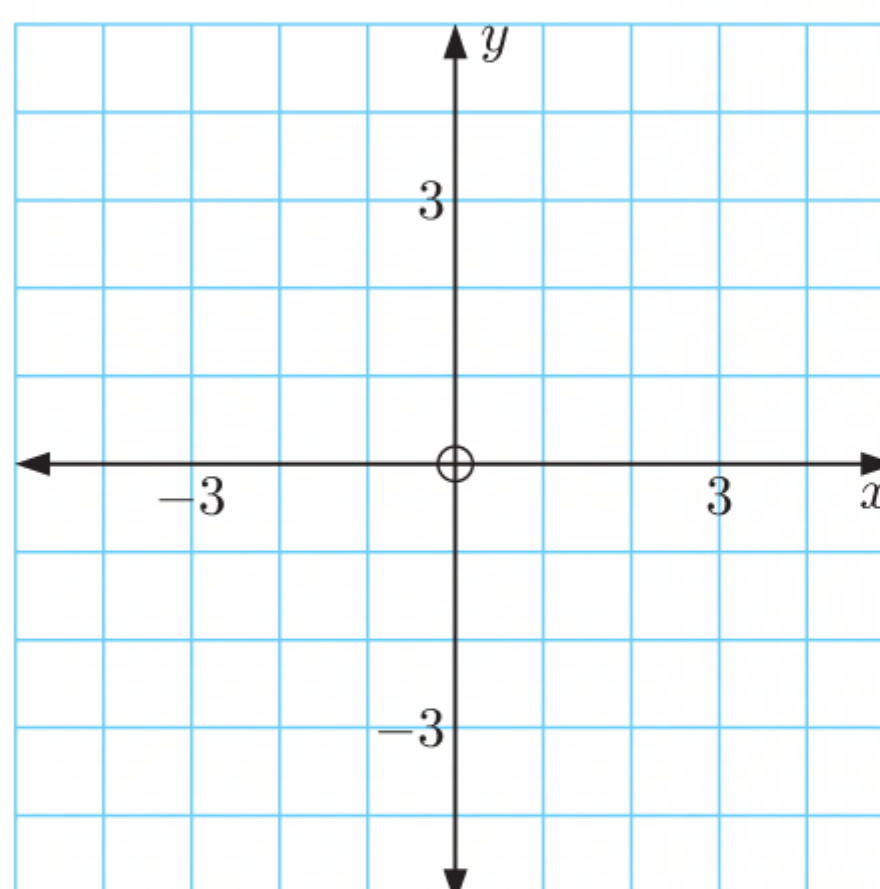
11 [Maximum mark: 17]

Consider the family of functions of the form

$$f_n(x) = \log_n \left(\frac{1}{1-x} \right)$$

for which $x \in \mathbb{R}$, $x < 1$ and $n > 1$.

- a** On a set of axes like the one shown below, sketch the graph of $y = f_2(x)$, $y = f_3(x)$, and $y = f_4(x)$. [3]



- b** Find $f_2'(x)$ and $f_2''(x)$. [3]
- c** Show that $f_2'''(x) = \frac{2}{(\ln 2)(1-x)^3}$. [1]
- d** Hence, or otherwise, find the Maclaurin series for $f_2(x)$, in the simplest form, up to and including the x^3 term. [4]
- e** Show that the Maclaurin series for $f_3(x)$, in the simplest form, up to and including the x^3 term is [4]

$$f_3(x) = \frac{1}{\ln 3}x + \frac{1}{2\ln 3}x^2 + \frac{1}{3\ln 3}x^3 + \dots$$

- f i** Write an expression for the Maclaurin series for [1]

$$f_n(x) = \log_n \left(\frac{1}{1-x} \right)$$

- ii** Hence, show that the Maclaurin series for $f_e(x) = \ln \left(\frac{1}{1-x} \right)$ is [1]

$$f_e(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

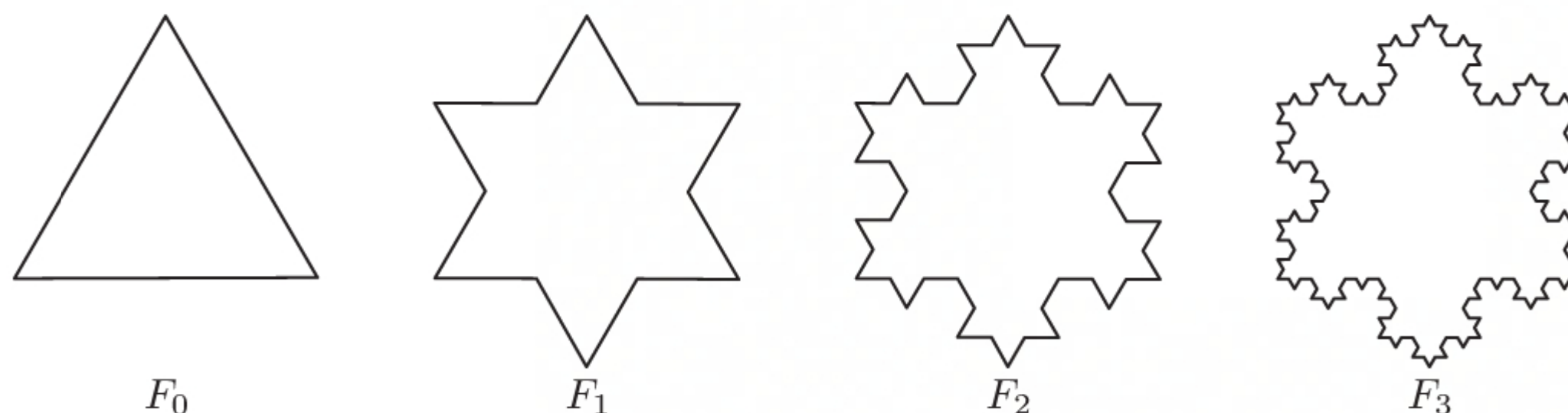
PAPER 3

CALCULATOR, 60 MINUTES

1 [Maximum mark: 28]

This question asks you to investigate the perimeter and the area of a sequence of figures $\{F_n\}$, where F_0 is an equilateral triangle with sides 1 unit in length, and each subsequent figure, for $n \geq 1$, is created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure, as n tends to infinity, is known as the Koch snowflake.

The first four figures, F_0 , F_1 , F_2 , and F_3 are shown below.



Let N_n and l_n represent the number of sides and the length of each side of figure F_n , respectively.

a Given that $N_0 = 3$ and $l_0 = 1$, find the values of N_n and l_n for:

i $n = 1$ [3]

ii $n = 2$. [2]

Let P_n represent the perimeter of figure F_n .

b Using your results in **a**, write down the value of P_n for:

i $n = 1$ [1]

ii $n = 2$. [1]

c Hence or otherwise, write an expression for:

i N_n , the number of sides of figure F_n [1]

ii l_n , the length of each side of figure F_n [1]

iii P_n , the perimeter of figure F_n . [2]

d The sequence $\{P_n\}$ is geometric. Find its common ratio. [2]

e Hence, show that the perimeter of Koch's snowflake is infinite. [1]

Let T_n and A_n represent the area of each *new* triangle created when forming F_n and the area of F_n , respectively.

f Write down the value of T_0 and A_0 . [2]

g Find an expression, for $n \geq 1$, for:

i T_n in terms of T_{n-1} [2]

ii T_n in terms of n . [2]

h Show that $A_n = A_{n-1} + \frac{3}{4} \times \frac{\sqrt{3}}{4} \times \left(\frac{4}{9}\right)^n$. [2]

i Hence show that $A_n = \frac{\sqrt{3}}{4} \left(\frac{8}{5} - \frac{3}{5} \left(\frac{4}{9}\right)^n\right)$. [4]

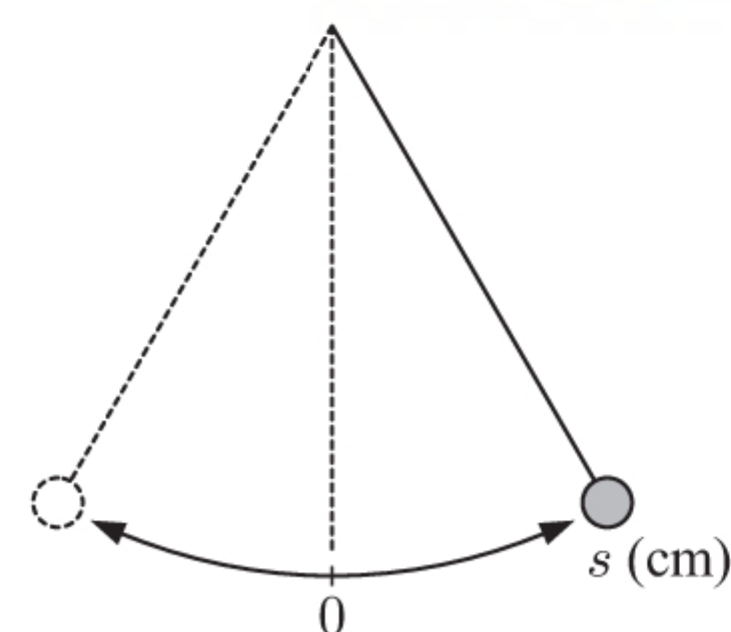
j Hence, find the area of Koch's snowflake. [2]

2 [Maximum mark: 27]

a Use integration by parts to prove that $\int e^{\frac{x}{\alpha}} \sin \beta x \, dx = \frac{e^{\frac{x}{\alpha}}}{1 + \alpha^2 \beta^2} (\alpha \sin \beta x - \alpha^2 \beta \cos \beta x) + c.$ [5]

b When this pendulum is released, its velocity along its arc of motion is given by

$$v(t) = -32e^{-\frac{t}{24}} \sin 2t \text{ cm s}^{-1}.$$



i Find the initial acceleration of the pendulum. [3]

ii Suppose that as $t \rightarrow \infty$, $s(t) \rightarrow 0$ cm. [5]

Show that the displacement of the pendulum at time t is given by $s(t) = e^{-\frac{t}{24}} (a \sin 2t + b \cos 2t)$ where $a \approx \frac{1}{3}$ and $b \approx 16$.

iii Find the first two times when the pendulum changes direction. [2]

iv Hence estimate, correct to 4 significant figures, the distance the pendulum travels in:

(1) its initial swing to the other extreme [3]

(2) the swing back towards its initial position. [2]

v (1) Estimate the percentage loss of distance with each swing of the pendulum. [2]

(2) Hence estimate the total distance the pendulum will travel before it comes to rest. [3]

vi Write down an exact expression for the total distance the pendulum will travel. [2]

Trial examination 4

PAPER 1

NO CALCULATOR, 120 MINUTES

SECTION A

1 [Maximum mark: 6]

At the point where $x = 0$, the tangent to $f(x) = e^{\sin kx} + c$ has equation $y = -x + 3$.

Find the value of:

a c

[3]

b k

[3]

2 [Maximum mark: 5]

X and Y are independent events such that $P(X \cap Y) = \frac{1}{5}$ and $P(X' \cap Y) = \frac{1}{2}$. Find $P(X \cup Y)$.

3 [Maximum mark: 4]

Consider the three digit number “ abc ”.

Prove that if $a + b + c$ is divisible by 3, then “ abc ” is also divisible by 3.

4 [Maximum mark: 5]

Consider the functions $f : x \mapsto 3 - x$ and $g : x \mapsto \frac{2x+1}{3}$.

a Show that $g^{-1}(x) = \frac{3x-1}{2}$.

[2]

b Solve for x : $(f \circ g^{-1})(x) = 4$.

[3]

5 [Maximum mark: 5]

Solve for x : $9^x + 18 = 3^{x+2}$

6 [Maximum mark: 5]

a Find the Maclaurin series for $\frac{1}{1-x}$.

[3]

b Prove that $\frac{1}{\sin^2 x} = 1 + \cos^2 x + \cos^4 x + \cos^6 x + \dots$ for all $x \neq k\pi$, $k \in \mathbb{Z}$.

[2]

7 [Maximum mark: 7]

A continuous random variable X has the probability density function $f(x) = \frac{k}{x^2+1}$, $0 \leq x \leq 1$.

a Find the value of k .

[3]

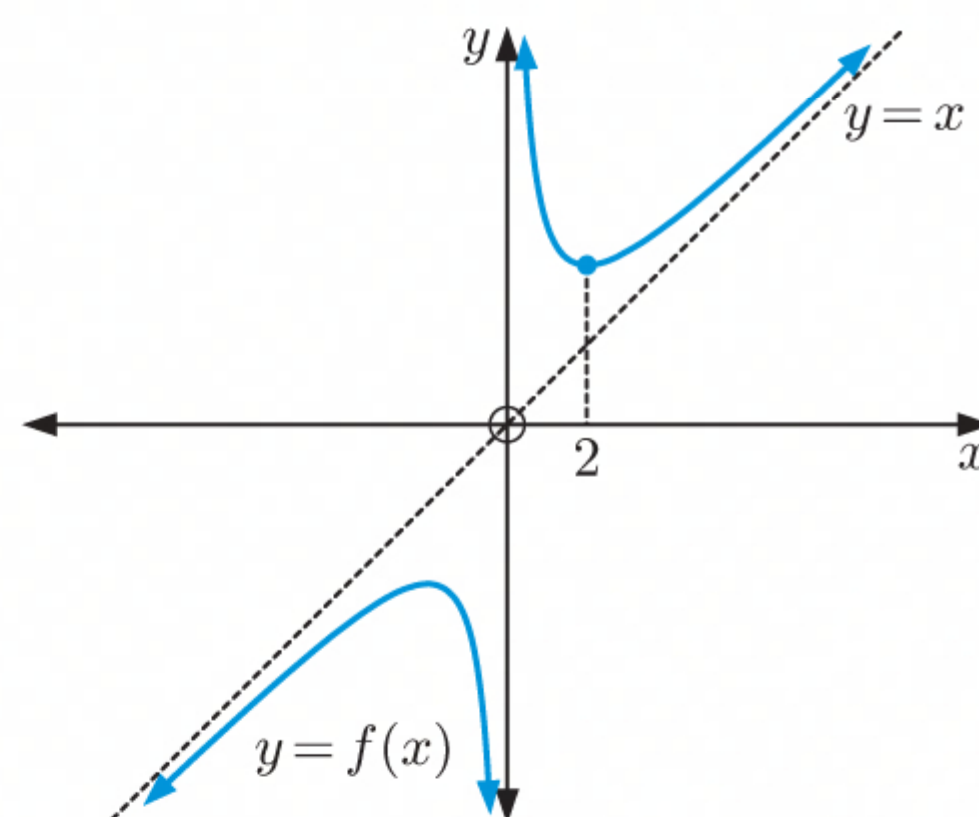
b Find $E(X)$.

[4]

8 [Maximum mark: 12]

$f(x)$ is a rational function of the form $\frac{ax^2 + bx + c}{dx + e}$.

The graph of $y = f(x)$ is shown alongside.



a Show that $f(x) = \frac{x^2 + 4}{x}$.

[3]

b Hence find the turning points of $y = f(x)$.

[2]

c Find the complex zeros of $f(x)$.

[2]

d Prove that $[f(x)]^2 > x^2 + 8$ for all x . [2]

e Graph $y = x^2 + 8$ and $y = [f(x)]^2$ on the same set of axes. Include the turning points of $y = [f(x)]^2$. [3]

9 [Maximum mark: 8]

Three planes have the equations:
$$\begin{cases} 2x + y - z = 2 \\ 2x + 2y - 3z = 3, \\ 4x + 2z = k \end{cases} \quad \text{where } k \text{ is a constant.}$$

a Write this system as an augmented matrix and apply row operations to reduce the system to row echelon form. [3]

b Find the value(s) of k for which the system has no solutions. Interpret this result geometrically. [2]

c Find the value(s) of k for which the system has infinitely many solutions. Find the solutions and interpret them geometrically. [3]

SECTION B

10 [Maximum mark: 21]

Suppose $f(x) = x^3 - 6x^2 + 9x - 2$.

a Find $f'(x)$ and draw its sign diagram. [4]

b Locate and describe the turning points of $y = f(x)$. [2]

c Find the inflection point of the function. [3]

d Sketch the graph of $y = f''(x)$. Hence discuss the *shape* of $y = f(x)$. [2]

e i Find the equation of the normal to the curve at the inflection point. [3]

ii Find the exact x -coordinates of the points where the normal meets the curve again. [7]

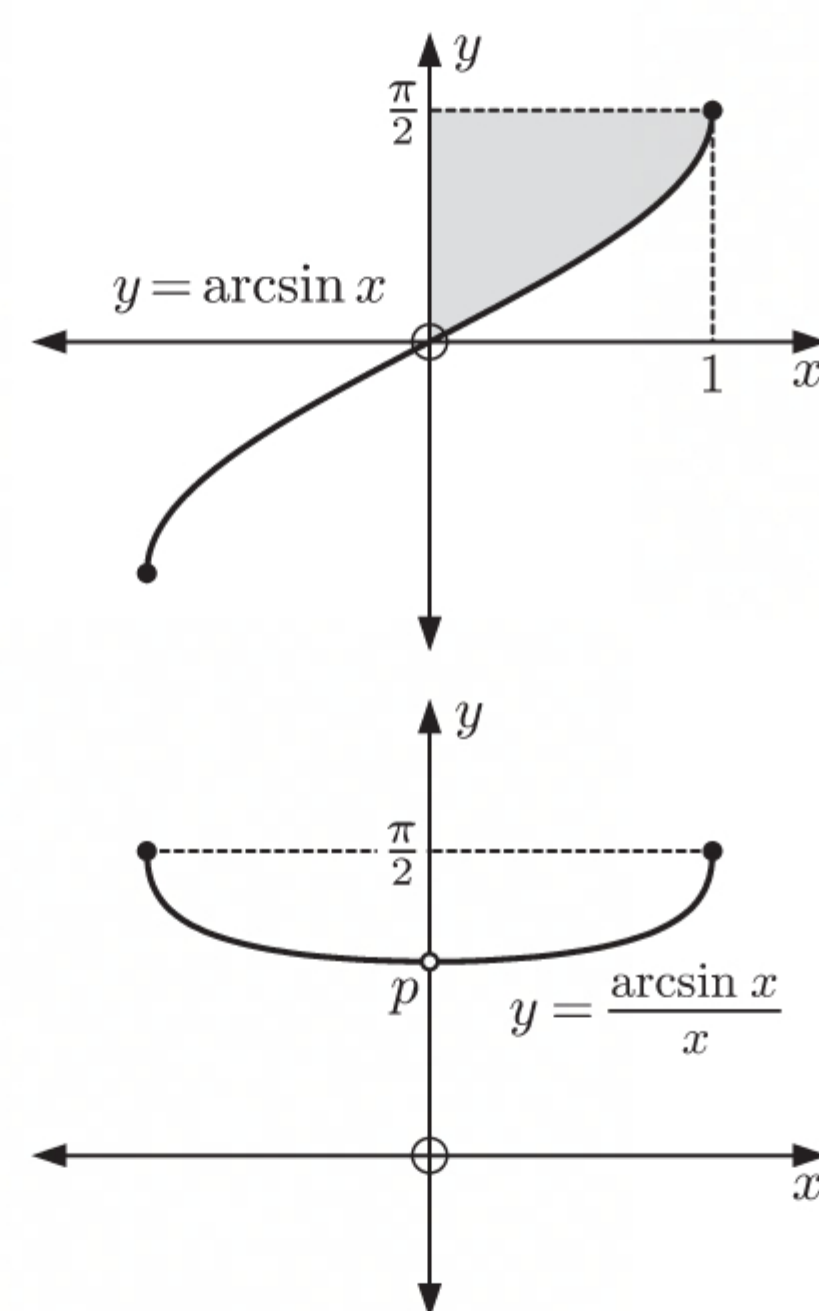
11 [Maximum mark: 15]

a i Use integration by parts to find $\int_0^1 \arcsin x \, dx$. [5]

ii Verify your answer to **i** by calculating the shaded area. [3]

b Find the volume of the solid formed by rotating the shaded area 360° about the y -axis. [4]

c The graph alongside is $y = \frac{\arcsin x}{x}$.
Find the value of p . [3]



12 [Maximum mark: 17]

a Find the cube roots of 8, giving your answers in the form $a + bi$. [5]

b Hence write $z^3 - 8$ as the product of a real linear and a real quadratic factor. [3]

c Sketch the cube roots of 8 on an Argand diagram. [3]

d Suppose the cube roots of 8 are all multiplied by $2e^{i\frac{\pi}{4}}$.

i Describe the geometric effect of this multiplication. [3]

Sketch the resulting values on an Argand diagram.

ii The resulting values are the cube roots of which number? Explain your answer. [3]

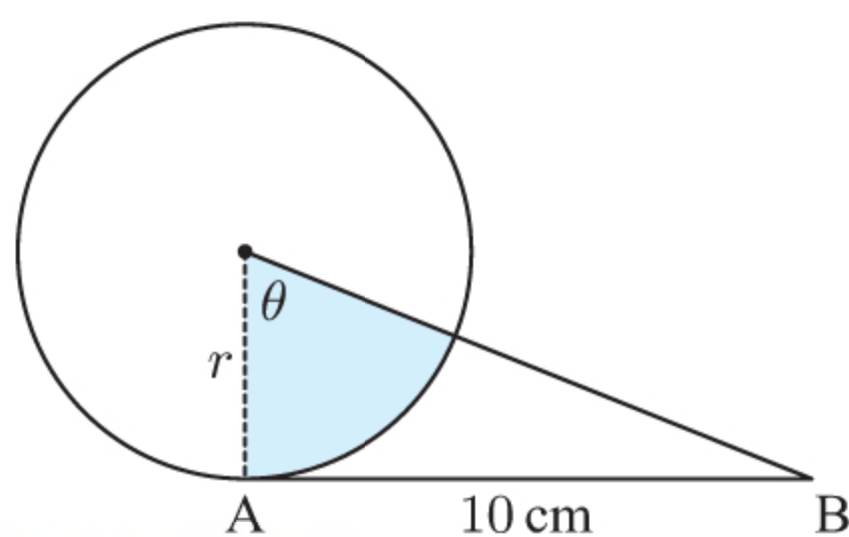
PAPER 2**CALCULATOR, 120 MINUTES****SECTION A****1 [Maximum mark: 7]**

diagram not drawn
to scale

AB is a tangent to the circle. The shaded area is 20 cm^2 . Find the radius of the circle r , and the angle at its centre θ in radians.

2 [Maximum mark: 8]

Portia has €10 000 she wants to invest. She is given two options:

A: 5% per annum simple interest paid quarterly

B: 4.4% per annum interest compounded quarterly.

- a Write a formula for the value of the simple interest investment after n quarters. [2]
- b i Write a formula for the value of the compound interest investment after n quarters. [2]
- ii Hence find the value of this investment after 7 quarters. [1]
- c Find how long Portia would need to invest her money, for the compound interest investment to be the better option. [3]

3 [Maximum mark: 5]

- a A pair of dice is rolled. Find the probability that the sum of the dice is 5. [2]
- b A pair of dice is rolled 10 times. Find the probability that their sum will be 5 at least twice. [3]

4 [Maximum mark: 5]

Jody asks 8 office workers in a big city about the distance they travel to work each day, and the time it takes them to get there. The workers record their journeys to work by GPS.

<i>Distance travelled to work (x km)</i>	2.2	6.8	15.4	3.1	5.6	9.0	17.2	1.4	4.1
<i>Travel time (y minutes)</i>	16	27	43	14	26	32	61	12	19

- a Explain why x is the independent variable. [1]
- b What Jody actually wants to know is the relationship between the straight-line distance the workers live from the office, and the time it takes them to get to work.
 - i If Jody is to use the data she has collected, explain why it is more appropriate to use the x against y regression line. [1]
 - ii Calculate the x against y regression line for the given data. [2]
 - iii Hence estimate the distance a worker lives from the office, if it takes them 50 minutes to get to work. [1]

5 [Maximum mark: 8]

Line L_1 has vector equation $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$, $s \in \mathbb{R}$.

Line L_2 has parametric equations $x = 1 + 2t$, $y = -1$, $z = 1 + t$, $t \in \mathbb{R}$.

The two lines intersect. Find:

- a the angle between the lines [4]
- b the equation of the plane containing L_1 and L_2 . [4]

6 [Maximum mark: 7]

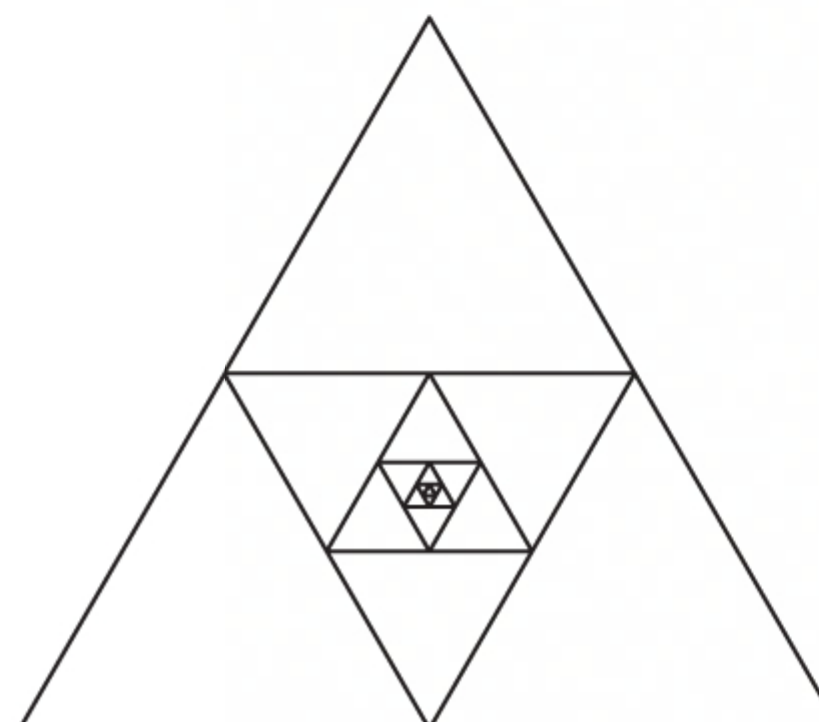
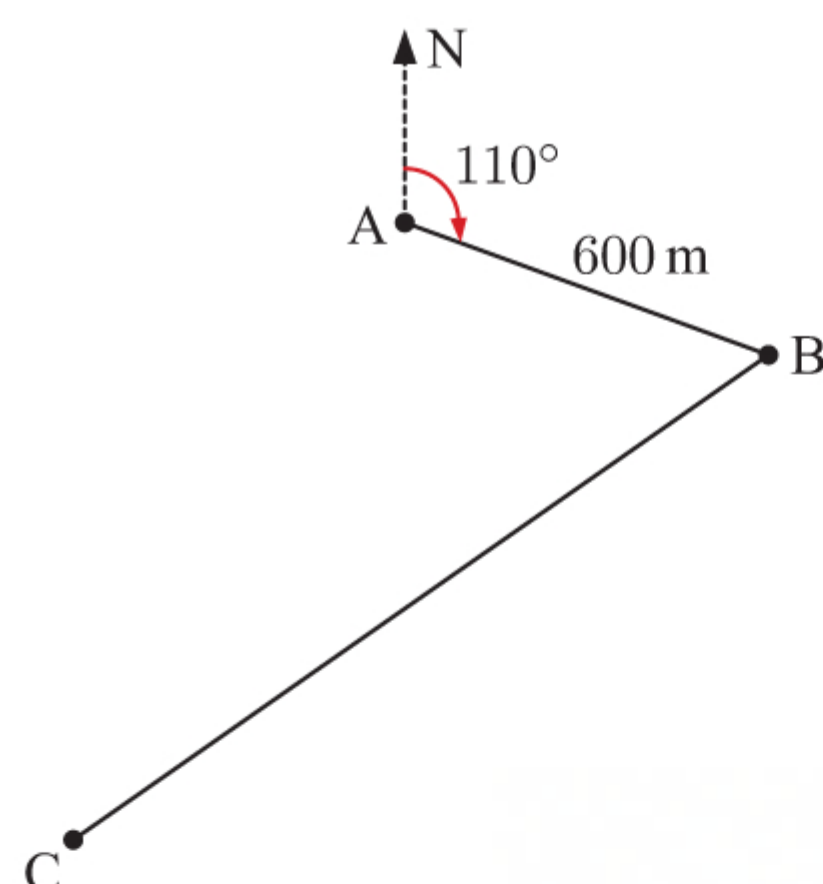
Suppose $z = a + bi$. Find the values of a and b for which $\frac{z^* - i}{z}$ is:

- a** undefined [3]
- b** real [2]
- c** purely imaginary. [2]

7 [Maximum mark: 6]

Let $S_n = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \dots + \frac{3}{4^n}$.

- a** Write a formula for S_n which is *not* the sum of terms. [2]
- b** Find S_n in the limit as $n \rightarrow \infty$. [1]
- c** Explain your result in **b** using this diagram: [3]

**8 [Maximum mark: 9]**

Alan runs at 3 m s^{-1} from A to B. At exactly the same time, Belinda starts cycling at 8 m s^{-1} on the bearing 230° from B to C.

- a** Find \widehat{ABC} . [2]
- b** Write down a formula for the distance between Alan and Belinda after t seconds, $t \geq 0$. [4]
- c** Find the minimum distance between Alan and Belinda, and the time when this occurs. [3]

9 [Maximum mark: 9]

- a** Prove that $\frac{\sin 3x}{\sin x} = 1 + 2 \cos 2x$ for all $x \neq k\pi$, $k \in \mathbb{Z}$. [3]
- b** Prove that $\frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx = \frac{\sin\left(\left(n + \frac{1}{2}\right)x\right)}{2 \sin \frac{x}{2}}$ for all $n \in \mathbb{Z}^+$. [6]

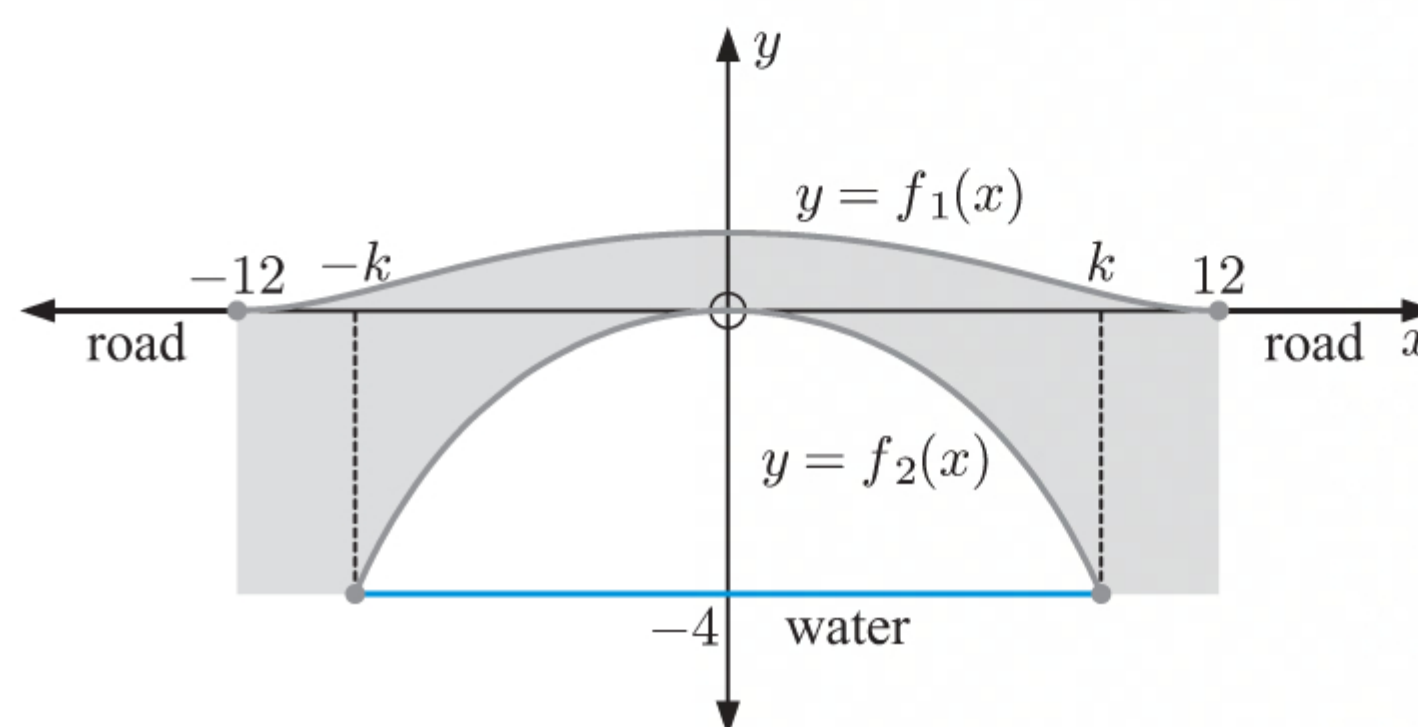
SECTION B**10 [Maximum mark: 13]**

A large sample of fallen acorns is collected from a forest. 18% of acorns are longer than 4.6 cm, and 15% of acorns are shorter than 2.8 cm.

- a** Suppose the population of acorns is normally distributed with mean μ and standard deviation σ .
 - i** Write down *two* linear equations connecting μ and σ . [4]
 - ii** Hence find μ and σ . [2]
- b** A random sample of 12 fallen acorns is chosen. Let Y represent the number of acorns longer than 4 cm.
 - i** Find $E(Y)$. [5]
 - ii** Find the probability that exactly 6 of the acorns have length greater than 4 cm. [2]

11 [Maximum mark: 19]

An arched bridge over a river is shown in the diagram.
 x and y are both in metres.



The defining functions are $f_1(x) = \ln(\cos \frac{\pi x}{12} + 2)$, $-12 \leq x \leq 12$

and $f_2(x) = 2 \ln(\cos \frac{\pi x}{12} + 1) + a$, $-k \leq x \leq k$.

- a Find the value of a . [2]
- b i Show that $\cos \frac{k\pi}{12} = \frac{2}{e^2} - 1$. [4]
- ii Hence find k . [1]
- c Find exactly the maximum gradient of the road, and when this occurs. [8]
- d Find the shaded cross-sectional area of the bridge. [4]

12 [Maximum mark: 14]

A species of lizard is released on an island. Conservationists are aware that the winters will be hard for the lizards, and it will take time for the species to adapt. However, they believe their program will benefit the species in the long term.

500 lizards are initially released, and the population P grows according to the differential equation

$$\frac{dP}{dt} = \frac{1}{10}P \left(1 - \frac{P}{10\,000}\right) (1 - 2 \cos 2\pi t),$$

where t is the time in years.

- a Describe what happens to the population immediately after release. [2]
- b Show that the differential equation is separable, and find its particular solution. [6]
- c Sketch the particular solution, with the aid of technology. [2]
- d Describe what happens to the population in the long term. [2]
- e Find the year in which the population will first reach 5000. [2]

PAPER 3**CALCULATOR, 60 MINUTES****1 [Maximum mark: 29]**

- a Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.
 - i Find ω^3 . [2]
 - ii Show that $\omega + \omega^2 = -1$. [3]
- b Consider a cubic equation of the form $x^3 - 3px + 2q = 0$ for some $p, q \in \mathbb{R}$.
 - i Suppose $x^3 - 3px + 2q = (x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b)$ where a, b are not necessarily real. [4]
 Show that $a^3 + b^3 = 2q$ and $ab = p$.
 - ii Solve the system $\begin{cases} a^3 + b^3 = 2q \\ ab = p \end{cases}$ simultaneously to write a and b in terms of p and q . [6]
 - iii Hence prove Cardano's formula, which states that $x = -\sqrt[3]{q + \sqrt{q^2 - p^3}} - \sqrt[3]{q - \sqrt{q^2 - p^3}}$ is a solution to the equation $x^3 - 3px + 2q = 0$. [2]
 - iv Use Cardano's formula to find a solution to $x^3 - 6x + 6 = 0$. [3]
- c Let $p(z) = az^3 + bz^2 + cz + d$.
 - i Find α such that $p(x - \alpha)$ has no x^2 coefficient. [4]
 - ii Use Cardano's formula to find a solution to $z^3 + 3z^2 - 4 = 0$. [5]

2 [Maximum mark: 26]

For a discrete random variable X and any function $f(x)$ whose domain includes the set $\{x_i\}$ of possible values of X , the **expected value** of $f(X)$ is defined as $E(f(X)) = \sum_i P(X = x_i) f(x_i)$.

The **probability generating function** of X is the function $G(z)$ which, for any value of z , is defined as the expected value of z^X . So, $G(z) = E(z^X) = \sum_i P(X = x_i) z^{x_i}$.

- a** Find the value of $G(1)$. [2]
- b** Show that:
 - i** $G'(1) = E(X)$ [2]
 - ii** $G''(1) = E(X^2 - X)$ [2]
 - iii** $G''(1) + G'(1) - [G'(1)]^2 = \text{Var}(X)$ [2]
- c** Suppose $X \sim B(n, p)$.
 - i** Show that the probability generating function of X is $G(z) = (pz + 1 - p)^n$. [3]
 - ii** Hence find $E(X)$ and $\text{Var}(X)$. [7]
- d** Suppose X is a uniform discrete random variable with possible values $1, 2, 3, \dots, n$.
 - i** Show that the probability generating function of X is $G(z) = \frac{1}{n}(z + z^2 + \dots + z^n)$. [2]
 - ii** Given that $\sum_{i=2}^n i(i-1) = \frac{(n-1)n(n+1)}{3}$ for all $n \in \mathbb{Z}^+$, $n \geq 2$, find $E(X)$ and $\text{Var}(X)$. [6]

Trial examination 5

PAPER 1

NO CALCULATOR, 120 MINUTES

SECTION A

1 [Maximum mark: 5]

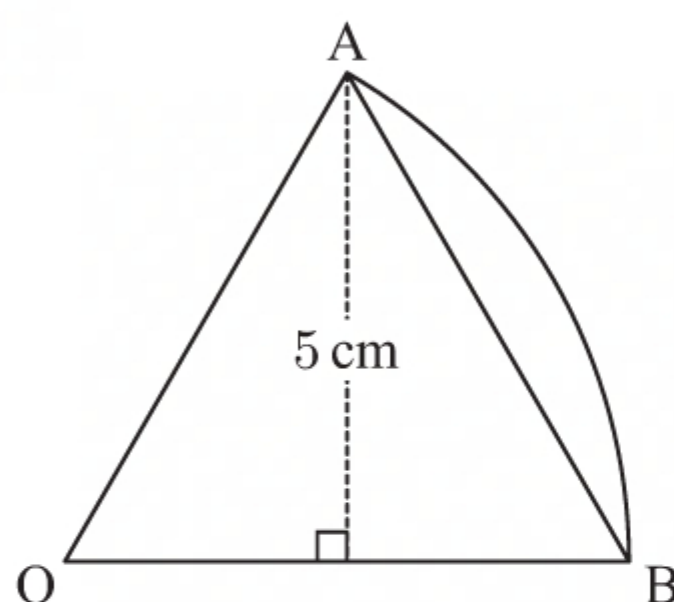
The probability distribution of a discrete random variable, X , is given by the following table, where A and p are constants.

x	A	5	7	9
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{4}{9}$	p

- a** Find the value of p . [2]
b Given that $E(X) = 6$, find the value of A . [3]

2 [Maximum mark: 5]

The diagram shows a sector AOB of a circle with centre O and radius r . The triangle AOB is equilateral and has perpendicular height 5 cm.



- a** Show that the radius of the sector is $\frac{10\sqrt{3}}{3}$ cm. [2]
b Find in terms of π , the perimeter P of the sector. [3]

3 [Maximum mark: 5]

Events A and B are such that $P(A) = \frac{3}{10}$, $P(B) = \frac{2}{5}$, and $P(A | B) = \frac{1}{4}$.

- a** Find $P(A \cap B)$. [2]
b Find $P(A \cup B)$. [2]
c State with a reason whether or not events A and B are independent. [1]

4 [Maximum mark: 7]

The function $f(x) = \frac{4x-1}{ax+b}$ has asymptotes $x = \frac{1}{2}$ and $y = 2$.

- a** Find the values a and b . [2]
b Find $f^{-1}(x)$. [3]
c Find the domain and range of $f^{-1}(x)$. [2]

5 [Maximum mark: 4]

Given that $\cos(\theta - 30^\circ) = 2 \sin \theta$, find the exact value of $\tan \theta$.

6 [Maximum mark: 8]

Consider the vectors $\vec{OA} = \begin{pmatrix} k \\ 3 \\ -7 \end{pmatrix}$ and $\vec{OB} = \begin{pmatrix} k-1 \\ k \\ 5 \end{pmatrix}$.

- a** Given that \vec{OA} is perpendicular to \vec{OB} , find the possible values of k . [3]
b Given that $k > 0$, find a unit vector in the opposite direction to \vec{AB} . [5]

7 [Maximum mark: 8]

The graphs of $y = (x + 3)^2$ and $y = -x^2 + bx + c$ touch at a single point P.

- a** Show that $b^2 - 12b + 8c = 36$. [4]
- b** Given that P has coordinates $(-4, 1)$, show that $4b - c + 17 = 0$. [1]
- c** Hence find the values of b and c . [3]

8 [Maximum mark: 5]

Consider the curve C given by the equation $y^3 - 5xy = 7 + e^{\sin x}$. By differentiating implicitly, find an expression for $\frac{dy}{dx}$.

9 [Maximum mark: 9]

- a** Use integration by parts twice to show that $\int e^{3x} \cos x \, dx = Ae^{3x} \sin x + Be^{3x} \cos x + c$ where A and B are constants to be found. [6]
- b** Hence find the equation of the curve which passes through the point $(0, 3)$ and for which $\frac{dy}{dx} = e^{3x} \cos x$. [3]

SECTION B**10 [Maximum mark: 16]**

- a** A sequence is given by the recurrence relation $u_1 = 8$, $u_{n+1} = 2u_n - 3$ for $n \geq 1$. Prove by mathematical induction that the general formula for the sequence is $u_n = 5 \times 2^{n-1} + 3$. [6]
- b** By considering the recurrence relation as a function of two separate sequences, one of which is geometric, find an expression for the sum of the first eight terms of the recurrence relation. [4]
- c** Consider now the arithmetic sequence with first three terms: $\frac{1}{\log_3 2x}$, $\frac{1}{\log_{27} 2x}$, $\frac{1}{\log_{243} 2x}$. Find the value of x if the sum of the first 30 terms of this sequence is 450. [6]

11 [Maximum mark: 19]

Let $y = kx(x + 1)^2$ where $k > 0$ is a constant.

- a** By differentiating from first principles, prove that $\frac{dy}{dx} = 3kx^2 + 4kx + k$. [5]
- b** Hence show that y has two turning points at $x = -1$ and $x = -\frac{1}{3}$, and state the coordinates at these points. [4]
- c** By finding the second derivative of y with respect to x , classify the nature of each turning point. [3]
- d** Find the values of x for which y is a strictly increasing function. [3]
- e** Given that $v = 2y$ and $\frac{dv}{dx} = 20$ at $x = -2$, find the value of the constant k . [4]

12 [Maximum mark: 19]

Let $f(x) = \frac{-6x^2 + 12x + 4}{(2 - 3x)(1 + 2x)}$, $x \neq \frac{2}{3}$, $x \neq -\frac{1}{2}$.

- a** Show that $f(x)$ can be expressed in the form $A + \frac{B}{2 - 3x} + \frac{C}{1 + 2x}$ where A , B , and C are constants to be found. [6]
- b** Hence, or otherwise, find $\int f(x) \, dx$. [3]
- c** Using the result in part **a**, find the first four terms of the binomial expansion for $f(x)$. Give your answer in ascending powers of x , up to and including the term in x^3 . [8]
- d** State the values of x for which the expansion is valid. [2]

PAPER 2**CALCULATOR, 120 MINUTES****SECTION A****1 [Maximum mark: 7]**

Peter has a 35% chance of winning a game of table tennis. He enters a tournament, in which he plays 15 games. Let X be the number of games that Peter wins in the tournament.

- a** Find the probability that Peter wins at least half of his games. [3]
- b** If μ and σ denote the mean and standard deviation of X , respectively, find $P(\mu - \sigma < X < \mu + \sigma)$. [4]

2 [Maximum mark: 6]

A particle P moves in a straight line with velocity function $v(t) = (2t + 3)(5t^{\frac{3}{2}} + 8)$ m/s at time t (in seconds), where $t \geq 0$. The particle is initially located at the origin.

- a** Find the displacement function $s(t)$. [4]
- b** Find the initial acceleration of P. [2]

3 [Maximum mark: 5]

The times taken for a group of runners to complete a half-marathon are found to follow a normal distribution with mean μ and standard deviation σ . Given that 24% of runners complete the race in less than 80 minutes and one in three runners take more than 120 minutes, find the values of μ and σ .

4 [Maximum mark: 6]

Let $z = 1 + \sqrt{2}i$ and $w = 3 - 2\sqrt{2}i$.

- a** Find the value of $\operatorname{Re}(2z^*w)$. [3]
- b** Find the value of $\operatorname{Im}\left(\frac{z}{w^*} + i\right)$. [3]

5 [Maximum mark: 7]

A manufacturer is producing paper cups that are able to hold 240 cm^3 of liquid when filled to the top. The cups are to be modelled as an open-topped cylinder with base radius r cm.

- a** Show that the total surface area $A \text{ cm}^2$ of paper required per cup is $A = \frac{480}{r} + \pi r^2$. [3]
- b** Show that the surface area of paper required per cup is minimised when $r = h$. [4]

6 [Maximum mark: 8]

The random variable X has the probability density function

$$f(x) = \begin{cases} kxe^{-2x} & \text{for } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

- a** Find the exact value of k . [5]
- b** Use technology to find $\operatorname{Var}(X)$. [3]

7 [Maximum mark: 5]

Find the exact solutions to the equation $3(4^x) - 10(2^x) = -3$.

8 [Maximum mark: 5]

Evaluate the limit $\lim_{x \rightarrow 0} \left(\frac{e^{\sin x} - x - 1}{x^2} \right)$.

9 [Maximum mark: 7]

Solve the differential equation

$$\frac{dy}{dx} = \frac{(x+y)^2 - xy}{x^2}$$

subject to the boundary condition $y(1) = \sqrt{3}$.

SECTION B

10 [Maximum mark: 15]

The functions $f(x) = x^3 - x$ and $g(x) = e^{2x}$ are defined for all $x \in \mathbb{R}$.

The function $g(x)$ is translated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then vertically stretched with scale factor 3, then further translated by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. The resultant function is denoted $h(x)$.

- a** Find the algebraic form of the function $h(x)$. [3]
- b** Find the exact coordinates of where the graph $y = h(x)$ crosses the x -axis. [4]
- c** Find the equation of the normal to the function $y = f(x)$ at $x = 2$. [4]
- d** Write down the composite function $gf(x)$, and find the total area of the regions enclosed by $y = f(x)$ and $y = gf(x) - 1$. [4]

11 [Maximum mark: 14]

In this question, all quantities can be assumed to be in SI units. Hence, you are not required to give units in your answers.

A particle P is initially located at $(-1, 5, 1)$ and moves with velocity $(\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$.

- a** Write down an equation for the position vector of P at time t . [1]

The magnetic force \mathbf{F} on a particle of unit charge moving with velocity \mathbf{v} in a magnetic field \mathbf{B} is given by $\mathbf{F} = \mathbf{v} \times \mathbf{B}$.

A particle Q of unit charge is initially located at $(1, -3, 4)$. It experiences a magnetic force $\mathbf{F} = (-3\mathbf{i} + 10\mathbf{j} + 41\mathbf{k})$ in a magnetic field $\mathbf{B} = (3\mathbf{i} + 5\mathbf{j} - \mathbf{k})$, and moves with constant velocity.

- b** Given that the velocity vector of particle Q takes the form $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + \mathbf{k}$, find the values of a and b , and hence write down an equation for the position vector of Q at time t . [5]
- c** Find the angle between the paths of particles P and Q. [4]

Both particles move towards the plane $2x + 5y + z = 50$.

- d** Find the coordinates of the point where particle P intersects the plane. [4]

12 [Maximum mark: 25]

- a** Prove that $\frac{1}{1 + \sin x} = \sec^2 x - \sec x \tan x$. [3]

Consider the differential equation

$$\frac{dy}{dx} = \frac{3y}{1 + \sin x}$$

where $x \geq 0$ and $y(0) = 1$.

- b i** Without solving the differential equation, show that the first three non-zero terms in the Maclaurin expansion of the solution are $y = 1 + 3x + 3x^2$. [6]
- ii** Use this result to estimate the value of $y(1.5)$. [1]
- c** Use Euler's method with a step size of 0.5 to estimate the value of $y(1.5)$ to three decimal places. [4]
- d** Find the exact solution of the differential equation. [5]
- e** Hence, find the percentage error in your estimate of $y(1.5)$ using the approximate methods in parts **b** and **c**. State how the error could be decreased in each case. [6]

PAPER 3**CALCULATOR, 60 MINUTES****1 [Maximum mark: 26]**

This question considers the Bernoulli differential equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad \dots (1)$$

for a function $y = y(x)$ and $x \geq \frac{\pi}{2}$. In this question, $p(x)$ and $q(x)$ are continuous functions throughout the domain, and $n \in \mathbb{N}$.

We will initially let $p(x) = \frac{1}{x}$ and $q(x) = \cos x$.

a Show that, when $n = 0$, (1) can be solved using an integrating factor of x . [2]

b Hence show that, in this case, the general solution to (1) is [5]

$$y(x) = \sin x + \frac{A + \cos x}{x}$$

where A is a constant.

c Show that, when $n = 1$, (1) is separable, and hence find $y = y(x)$ given that $y(\frac{\pi}{2}) = 1$. [6]

Now consider the case of general continuous functions $p(x)$ and $q(x)$.

d By using implicit differentiation, show that if $u = y^{1-n}$, then [3]

$$\frac{dy}{dx} = \frac{u^{\frac{n}{1-n}}}{1-n} \frac{du}{dx}.$$

e Hence show that if we make the substitution $u = y^{1-n}$, (1) reduces to the equation [4]

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x).$$

f Hence find the general solution to (1) in the case when $p(x) = q(x) = x$ and $n = 2$. [6]

2 [Maximum mark: 29]

This question asks you to investigate some properties of the Legendre polynomials $P_n(x)$, where $n \in \mathbb{N}$ and $-1 \leq x \leq 1$. The first two Legendre polynomials are $P_0(x) = 1$ and $P_1(x) = x$.

In this question, you may use the result $\int \cos^2 t \, dt = \frac{1}{4} \sin 2t + \frac{1}{2}t + c$ without proof.

a Use the formula $P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$ to find $P_2(x)$. [3]

b Legendre polynomials can also be obtained from the formula [4]

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2 - 1} \cos t)^n \, dt.$$

Show that this formula generates the same function $P_2(x)$ as in part **a**.

c Use the recursion relation [3]

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

or otherwise, to find $P_3(x)$.

d Sketch the graphs of $P_n(x)$ for $n = 0, 1, 2, 3$ on the same set of axes, clearly labelling the exact x and y -intercepts of each polynomial. [4]

e Find the exact x -coordinates of the points where $P_2(x)$ and $P_3(x)$ intersect. [5]

f Solve $4P_2(\cos \theta) - P_0(\cos \theta) = 0$ for $0 \leq \theta \leq 2\pi$. [3]

g Prove that the identity [3]

$$\frac{\sin((n+1)\theta)}{\sin \theta} = \sum_{m=0}^n P_m(\cos \theta) P_{n-m}(\cos \theta)$$

holds when $n = 1$.

h Use the identity in part **g** to show that $\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$. [4]

Paper 3 practice

The questions in this section are intended to give students practice at answering the style of questions that will appear in Paper 3 examinations. The questions do not have a markscheme attached to them, and may not necessarily be of a similar length to questions that will appear in the Paper 3 examination.

- 1 a** Show that $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.
- b** Use mathematical induction to prove that $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$ for $n \in \mathbb{Z}^+$.
- c** Consider the conjecture
- $$1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} \text{ for } n \in \mathbb{Z}^+.$$
- i** Given that $P_3(k) = \frac{k(k+1)(k+2)(k+3)}{4}$, write $P_3(k+1) - P_3(k)$ in fully factorised form.
- ii** Hence prove by induction that the conjecture is true.
- d** Let $P_l(k) = \frac{k(k+1)(k+2)\dots(k+l)}{l+1}$.
- i** Find $P_l(k+1) - P_l(k)$ in fully factorised form.
- ii** Hence prove by induction that for a positive integer l ,
- $$(1 \times 2 \times 3 \times \dots \times l) + (2 \times 3 \times 4 \times \dots \times (l+1)) + \dots + (n(n+1)(n+2)\dots(n+l-1))$$
- $$= \frac{n(n+1)(n+2)\dots(n+l)}{l+1} \text{ for } n \in \mathbb{Z}^+.$$
- e** Consider the sum
- $$S_n = 1 \times 2 \times \dots \times 10 + 2 \times 3 \times \dots \times 11 + 3 \times 4 \times 5 \times \dots \times 12 + \dots + n(n+1)(n+2)\dots(n+9)$$
- which has n terms.
- i** Explain why S_n is divisible by 9 for all $n \in \mathbb{Z}$, $n \geq 1$.
- ii** Find the smallest value of n such that S_n is divisible by 11.
- 2** Let $z_n = \frac{1}{(1+i)^0} + \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^n}$ where $i = \sqrt{-1}$.
- a** Write $\frac{1}{1+i}$ in the form $a + bi$ where $a, b \in \mathbb{R}$.
- b** Hence calculate z_0, z_1, z_2 , and z_3 .
- c** Plot z_0, z_1, z_2 , and z_3 on an Argand diagram.
- d** Prove that $z_n = 1 - i + \frac{\text{cis}(\frac{\pi}{2} - \frac{\pi}{4}n)}{\sqrt{2}^n}$ for all $n \in \mathbb{Z}^+$.
- e** Hence find z_{10} .
- f** Find $z = \sum_{k=0}^{\infty} \frac{1}{(1+i)^k}$.

3 In this question we consider series of the form $\sum_{n=1}^N nr^n$.

a Evaluate $\sum_{n=1}^4 n \times 3^n$.

b Find the derivative with respect to r of $\sum_{n=1}^5 r^n$.

c i Show that $\sum_{n=1}^N nr^n - r \sum_{n=1}^N nr^n = \sum_{n=1}^N r^n - Nr^{N+1}$.

ii Hence or otherwise, show that $\sum_{n=1}^N nr^n = \frac{r - (N+1)r^{N+1} + Nr^{N+2}}{(1-r)^2}$.

iii Use this formula to verify your answer to **a**.

d i Show that $\frac{d}{dr} \left(\sum_{n=1}^N r^n \right) = \sum_{n=1}^N nr^{n-1}$.

ii Explain why $\sum_{n=1}^N nr^n = r \frac{d}{dr} \left(\frac{r - r^{N+1}}{1-r} \right)$.

iii Hence establish the formula for $\sum_{n=1}^N nr^n$ found in **c ii**.

e Use mathematical induction to show that $\sum_{n=1}^N nr^n = \frac{r - (N+1)r^{N+1} + Nr^{N+2}}{(1-r)^2}$ for all $N \in \mathbb{Z}^+$.

f Use l'Hôpital's rule to find $\lim_{r \rightarrow 1} \left(\frac{r - (N+1)r^{N+1} + Nr^{N+2}}{(1-r)^2} \right)$. Verify that this result is consistent with the value of $\sum_{n=1}^N n$.

g Now consider the *infinite* series $\sum_{n=1}^{\infty} nr^n$.

i Show that $\sum_{n=1}^{\infty} nr^n$ can be written as the sum of the infinite geometric series

$$(r + r^2 + r^3 + r^4 + r^5 + \dots) + (r^2 + r^3 + r^4 + r^5 + r^6 + \dots) + (r^3 + r^4 + r^5 + r^6 + r^7 + \dots) + \dots$$

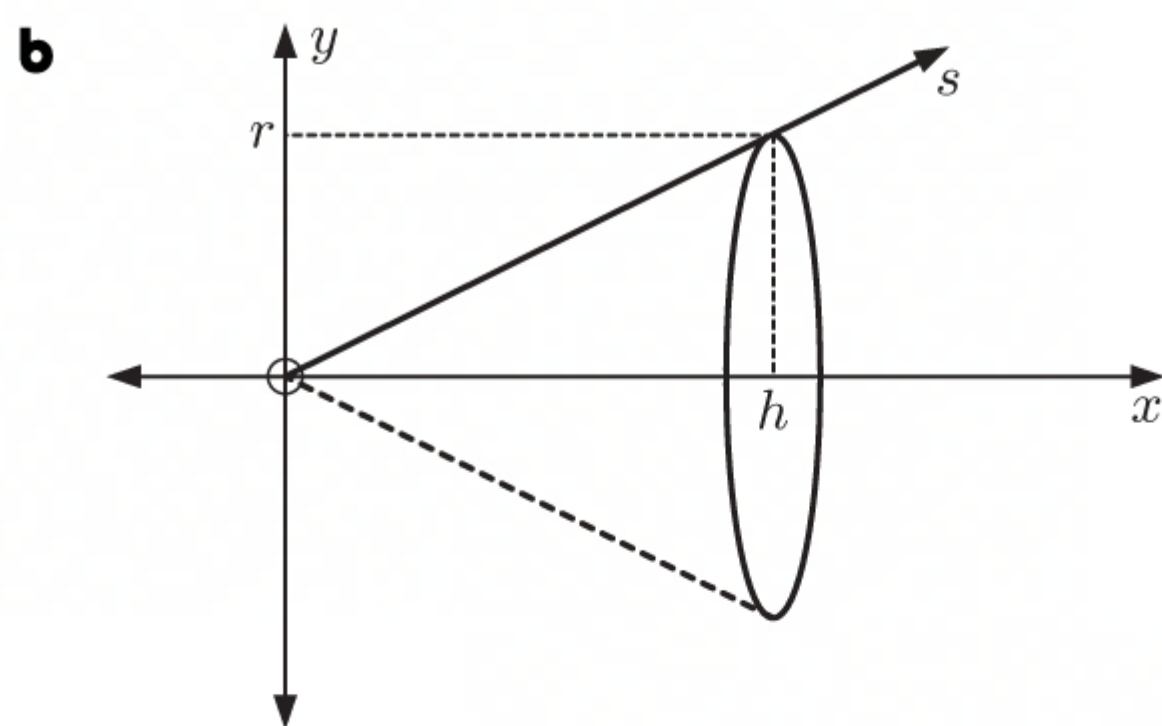
ii Hence show that, for $|r| < 1$, $\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}$.

iii Evaluate $\frac{1}{\sqrt{2}} + \frac{2}{2} + \frac{3}{2\sqrt{2}} + \frac{4}{4} + \frac{5}{4\sqrt{2}} + \dots$. Write your answer in the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

iv Suppose the third term of $\sum_{n=1}^{\infty} nr^n$ is the largest in the series.

Find the range of possible values that $\sum_{n=1}^{\infty} nr^n$ can take.

- 4 a** Write a formula for the surface area of the *curved* surface of a cone with radius r and height h .



In the diagram alongside, the straight line $f(x) = \frac{r}{h}x$ has revolved about the x -axis to produce a cone.

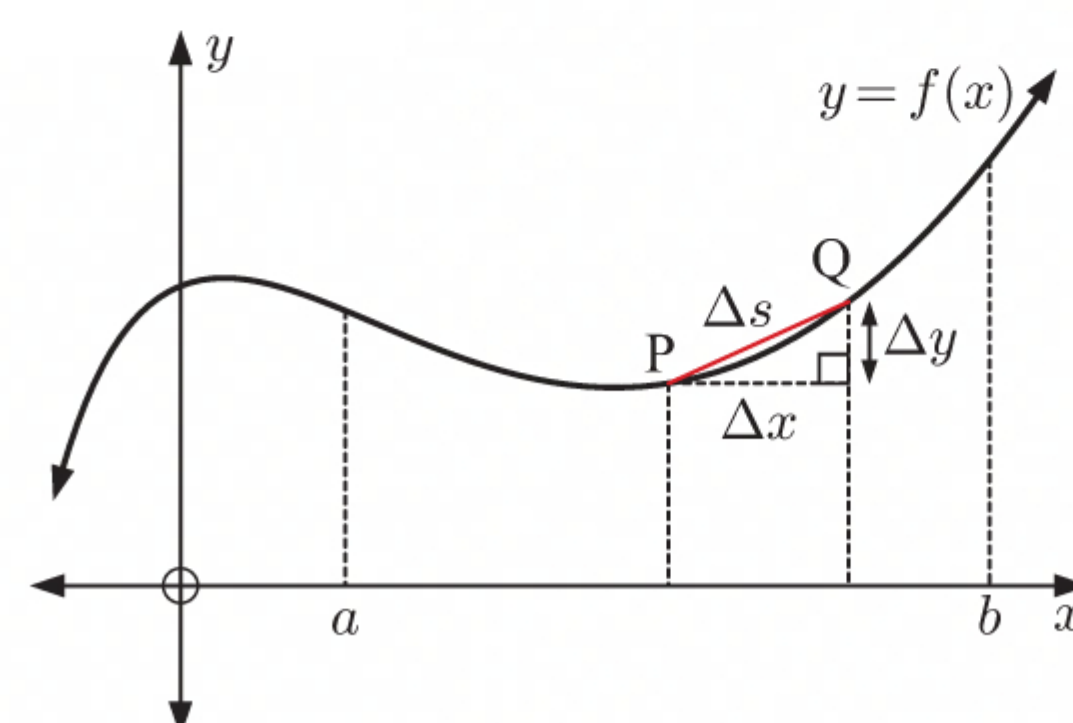
- i** Explain why the curved surface of the cone is *not* given by $\text{Area} = \int_0^h 2\pi f(x) dx$.
- ii** An extra axis has been added along the slant edge of the cone in the Cartesian plane. Explain why the surface area of the curved surface of the cone is given by

$$\text{Area} = \int_0^{\sqrt{r^2+h^2}} 2\pi r \frac{s}{\sqrt{r^2+h^2}} ds$$

and show that this integral produces the same formula as **a**.

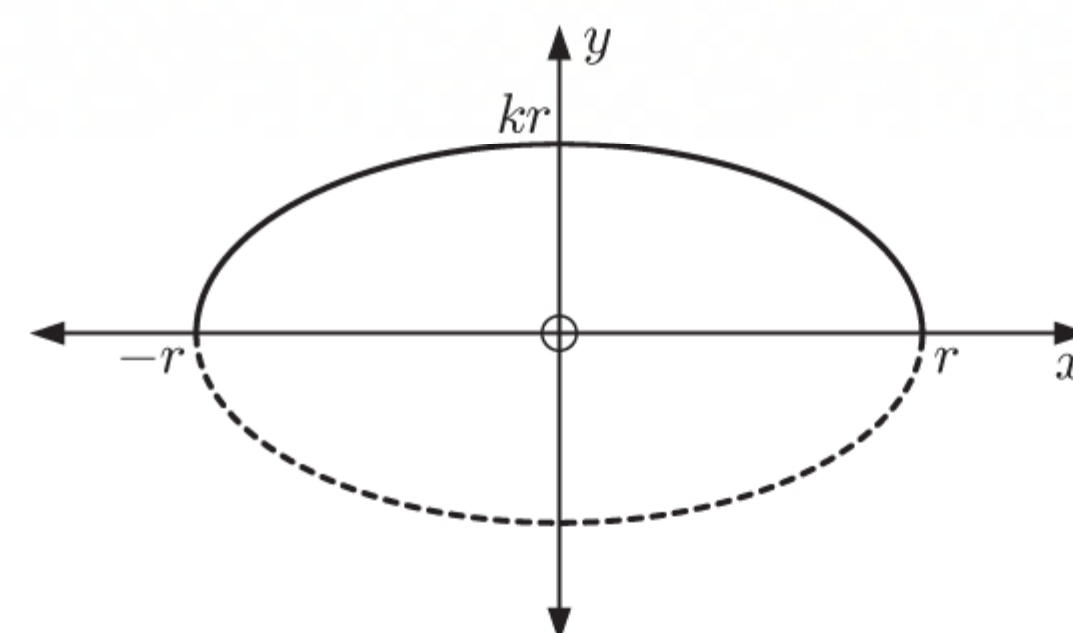
- c** Let P and Q be points on the curve $y = f(x)$ as shown.

- i** Show that $\Delta s = \sqrt{1 + \left[\frac{\Delta y}{\Delta x}\right]^2} \Delta x$.
- ii** Hence explain why the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$.



- d** Consider the top half of the ellipse $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{kr}\right)^2 = 1$.

- i** Show that $\frac{dy}{dx} = -\frac{kx}{\sqrt{r^2 - x^2}}$.
- ii** For $k = 1$, use the result in **c ii** to verify that the length of the semi-circle is πr .
- iii** For $k = 2$, $r = 1$, use your graphics calculator to calculate the length of the semi-ellipse correct to 3 decimal places.



- e i** The function $y = f(x)$, $x \in [a, b]$ is revolved around the x -axis to produce a solid of revolution. Explain why the surface area of the curved surface of the solid is given by

$$\text{Area} = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx.$$

- ii** Show that this integral produces the same formula as **a** for a cone with radius r and height h .
- iii** Use the integral in **a** to derive the formula for the surface area of a sphere with radius r .

- 5** The Bessel functions defined by **Daniel Bernoulli** and **Friedrich Bessel** are the solutions of Bessel's differential equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0$ where α is an arbitrary constant.

For a given real value of α , the Bessel's functions are labelled $J_\alpha(x)$.

- a** Show that if $J_\alpha(x)$ is a solution to the differential equation for a given value of α , then any constant multiple of $J_\alpha(x)$ is also a solution.
- b** Show that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ is a solution to the differential equation for $\alpha = \frac{1}{2}$.
- c** Given that $J_0(x)$ is a solution to the differential equation for $\alpha = 0$, show that $J_1(x) = -J'_0(x)$ is a solution to the differential equation for $\alpha = 1$.

When a drum is struck with a mallet or drumstick, the sound is produced by vibrations in the circular skin. These can be described using *modes* which are analogous to the harmonic frequencies of other instruments. However, a major difference is that the vibrations of a drum occur in two dimensions across the circular skin.

For any $n \in \mathbb{N}$, the *radial* component of the n th mode of vibration is given by the Bessel function $J_n(x)$. This is the shape of the vibration if we treat any radius of the drum skin as the x -axis.

Suppose the Maclaurin series for $J_n(x)$, $n \in \mathbb{N}$, is $\sum_{k=0}^{\infty} a_k x^k$.

- d** Show that $-n^2 a_0 + (1 - n^2)a_1 x + \sum_{k=2}^{\infty} [(k^2 - n^2)a_k + a_{k-2}]x^k = 0$.
- e** Explain why the first (non-zero) term in the Maclaurin series expansion for $J_n(x)$ is the x^n term.
- f** Explain why $J_n(x)$ is an odd function if n is odd, and an even function if n is even.
- g** We define $J_0(x)$ as the solution to the differential equation for which $J_0(0) = 1$.
 - i** Use the result from **d** to show that the Maclaurin series for $J_0(x)$ is $1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m (m!)^2} x^{2m}$.
 - ii** Hence write down the Maclaurin series for $J_1(x)$.
- h** Use technology to generate the 8th order Maclaurin series polynomial for $J_0(x)$. Sketch the polynomial for $0 \leq x \leq 6$. Hence estimate the first two zeros of the polynomial, and compare your answers with the first two zeros of $J_0(x)$, which are 2.405 and 5.520 to 4 significant figures.

Worked solutions

TOPIC 1 SKILL BUILDER QUESTIONS

- 1 a** Amount of fluoride = concentration \times volume

$$= (3 \times 10^{-4}) \times (5.6 \times 10^8)$$

$$= 1.68 \times 10^5 \text{ g}$$
- b** Volume = $\frac{\text{amount of fluoride}}{\text{concentration}}$

$$= \frac{4.13 \times 10^7}{3 \times 10^{-4}}$$

$$\approx 1.38 \times 10^{11} \text{ litres}$$
- 2 a** If $-2, k+4, k^2+11$ are consecutive terms of an arithmetic sequence, then

$$k+4 - (-2) = k^2+11 - (k+4) \quad \{\text{equating differences}\}$$

$$\therefore k+6 = k^2+11 - k-4$$

$$\therefore k^2 - 2k + 1 = 0$$

$$\therefore (k-1)^2 = 0$$

$$\therefore k = 1$$
- b** If $k-5, 2k, 2k^2$ are consecutive terms of an arithmetic sequence, then

$$2k - (k-5) = 2k^2 - 2k \quad \{\text{equating differences}\}$$

$$\therefore k+5 = 2k^2 - 2k$$

$$\therefore 2k^2 - 3k - 5 = 0$$

$$\therefore 2k^2 + 2k - 5k - 5 = 0$$

$$\therefore 2k(k+1) - 5(k+1) = 0$$

$$\therefore (k+1)(2k-5) = 0$$

$$\therefore k = -1 \text{ or } k = \frac{5}{2}$$
- 3 a** Francesca adds \$0.50 in the first week, \$1 the next, \$1.50 the next, adding an additional \$0.50 each subsequent week.
 \therefore in the n th week, Francesca adds $0.50n$ dollars to her money box.
 Now the last week before her 11th birthday is the 51st week.
 \therefore in the last week before her 11th birthday, Francesca added $\$0.50 \times 51 = \25.50 to her money box.
- b** Let $P(n)$ dollars be the amount Pierre had added to his money box after n weeks, and $F(n)$ dollars be the amount Francesca had added to her money box after n weeks.
 Pierre adds \$10 each week, so after n weeks he has added $10n$ dollars.
 So, $P(n) = 10n$
 $\therefore P(8) = 10 \times 8 = 80$
 After 8 weeks Pierre had added \$80 to his money box.
 From **a**, Francesca adds $0.50n$ dollars in the n th week, so after n weeks she has added $0.50 + 1 + 1.50 + \dots + 0.50n$ dollars.
 Now $0.50 + 1 + 1.50 + \dots + 0.50n$ is an arithmetic series with $u_1 = 0.5$ and $d = 0.5$.

$$\therefore 0.50 + 1 + 1.50 + \dots + 0.50n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$= \frac{n}{2}(2 \times 0.5 + (n-1) \times 0.5)$$

$$= \frac{n}{2}(1 + 0.5n - 0.5)$$

$$= \frac{n}{2}(0.5 + 0.5n)$$

$$= 0.25n + 0.25n^2$$

 So, $F(n) = 0.25n + 0.25n^2$
 $\therefore F(8) = 0.25 \times 8 + 0.25 \times 8^2 = 18$
 After 8 weeks, Francesca had added \$18 to her money box.

- c** There are 52 weeks in 1 year.

Now $P(52) = 10 \times 52 = 520$

and $F(52) = 0.25 \times 52 + 0.25 \times 52^2 = 689$

\therefore after 1 year, Pierre had $\$520 + \$100 = \$620$ in his money box, and Francesca had $\$687 + \$100 = \$789$ in her money box.

So, Francesca had more money in her money box after 1 year.

4 a $u_5 = u_1 r^4 = 324 \quad \dots (1)$

and $u_{10} = u_1 r^9 = 78\,732 \quad \dots (2)$

Now $\frac{u_1 r^9}{u_1 r^4} = \frac{78\,732}{324} \quad \{(2) \div (1)\}$

$\therefore r^5 = 243$

$\therefore r = \sqrt[5]{243}$

$\therefore r = 3$

Using (1), $u_1(3)^4 = 324$

$\therefore 81u_1 = 324$

$\therefore u_1 = 4$

Thus $u_n = 4 \times 3^{n-1}$

b $u_8 = u_1 r^7 = -10 \quad \dots (1)$

and $u_{12} = u_1 r^{11} = -160 \quad \dots (2)$

Now $\frac{u_1 r^{11}}{u_1 r^7} = \frac{-160}{-10} \quad \{(2) \div (1)\}$

$\therefore r^4 = 16$

$\therefore r = \pm \sqrt[4]{16}$

$\therefore r = \pm 2$

If $r = 2$, then using (1), $u_1(2)^7 = -10$

$\therefore 128u_1 = -10$

$\therefore u_1 = \frac{-10}{128} = -\frac{5}{64}$

Thus $u_n = -\frac{5}{64} \times 2^{n-1}$

If $r = -2$, then using (1), $u_1(-2)^7 = -10$

$\therefore -128u_1 = -10$

$\therefore u_1 = \frac{10}{128} = \frac{5}{64}$

Thus $u_n = \frac{5}{64} \times (-2)^{n-1}$

5 $2, 2\sqrt{3}, 6, 6\sqrt{3}$

a $\frac{2\sqrt{3}}{2} = \sqrt{3}, \quad \frac{6}{2\sqrt{3}} = \frac{\cancel{2} \times 3}{\cancel{2}\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}, \quad \frac{6\sqrt{3}}{6} = \sqrt{3}$

Consecutive terms have a common ratio of $\sqrt{3}$.

\therefore the sequence is geometric with $u_1 = 2$ and $r = \sqrt{3}$.

b $u_n = u_1 r^{n-1}$
 $= 2(\sqrt{3})^{n-1}$
 $= 2 \times 3^{\frac{n-1}{2}}$

c $u_{10} = 2 \times 3^{\frac{10-1}{2}}$
 $= 2 \times 3^{\frac{9}{2}}$
 $= 2 \times 3^4 \times 3^{\frac{1}{2}}$
 $= 2 \times 81 \times \sqrt{3}$
 $= 162\sqrt{3}$

d We need to find n such that $u_n = 2 \times 3^{\frac{n-1}{2}} > 1000$.

Using a graphics calculator with $Y_1 = 2 \times 3 \wedge ((X-1) \div 2)$, we view a table of values:

X	Y1
11	486
12	841.77
13	1458
14	2525.3

The first term to exceed 1000 is $u_{13} = 1458$.

- 6** There is a fixed percentage increase each year, so the population forms a geometric sequence with $u_0 = 217$ and $r = 1.42$.

\therefore the population after n years is $u_n = 217 \times (1.42)^n$.

a i $u_5 = 217 \times (1.42)^5$
 ≈ 1252.86

The expected population size after 5 years is approximately 1250 birds.

ii $u_{10} = 217 \times (1.42)^{10}$
 ≈ 7233.41

The expected population size after 10 years is approximately 7230 birds.

$$\begin{aligned}
 \mathbf{b} \quad & 217 \times (1.42)^n = 30\,000 \\
 & \therefore (1.42)^n = \frac{30\,000}{217} \\
 & \therefore n \log 1.42 = \log \left(\frac{30\,000}{217} \right) \\
 & \therefore n = \frac{\log \left(\frac{30\,000}{217} \right)}{\log 1.42} \\
 & \therefore n \approx 14.1
 \end{aligned}$$

It will take approximately 14.1 years for the population to reach 30 000.

$$\mathbf{7} \quad \mathbf{a} \quad \text{If the interest rate per annum is 7.2\%, then the interest rate per month } i = \frac{7.2\%}{12} = 0.6\% = 0.006.$$

$$\begin{aligned}
 r &= 1 + i \\
 &= 1 + 0.006 \\
 &= 1.006
 \end{aligned}$$

\mathbf{b} The interest is calculated monthly, so $n = 3 \times 12 = 36$ time periods.

$$\begin{aligned}
 u_{36} &= u_0 \times r^{36} \\
 &= 500 \times (1.006)^{36} \\
 &\approx 620.15
 \end{aligned}$$

The value of the account after 3 years is €620.15.

$$\begin{aligned}
 \mathbf{c} \quad & \text{real value} \times (1.02)^3 = \text{€}620.15 \\
 & \therefore \text{real value} = \frac{\text{€}620.15}{(1.02)^3} \\
 & = \text{€}584.38
 \end{aligned}$$

$\mathbf{8} \quad \mathbf{a}$ The series is arithmetic with $u_1 = 11$, $d = 4$, and $n = 20$.

$$\begin{aligned}
 \text{Now } S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\
 \therefore S_{20} &= \frac{20}{2}(2 \times 11 + 19 \times 4) \\
 &= 10(22 + 76) \\
 &= 980
 \end{aligned}$$

$$\mathbf{b} \quad 7 + 12.5 + 18 + 23.5 + \dots + 106$$

The series is arithmetic with $u_1 = 7$, $d = 5.5$, and $u_n = 106$.

First we need to find n .

$$\begin{aligned}
 \text{Now } u_n &= 106 \\
 \therefore u_1 + (n-1)d &= 106 \\
 \therefore 7 + 5.5(n-1) &= 106 & \text{Using } S_n = \frac{n}{2}(u_1 + u_n), \\
 \therefore 5.5(n-1) &= 99 & S_{19} = \frac{19}{2}(7 + 106) \\
 \therefore n-1 &= 18 & = \frac{19}{2} \times 113 \\
 \therefore n &= 19 & = 1073.5
 \end{aligned}$$

\mathbf{c} $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$ to 100 terms can be expressed as two separate arithmetic series:

$$\begin{aligned}
 & 1 + 3 + 5 + 7 + \dots \quad \text{where } u_1 = 1, d = 2, n = 50 \\
 \text{and } & -2 - 4 - 6 - 8 - \dots \quad \text{where } u_1 = -2, d = -2, n = 50
 \end{aligned}$$

$$\begin{aligned}
 \text{Using } S_n = \frac{n}{2}(2u_1 + (n-1)d), \quad \text{the sum of the first series} &= \frac{50}{2}(2(1) + 49(2)) \\
 &= 25(2 + 98) \\
 &= 2500
 \end{aligned}$$

$$\begin{aligned}
 \text{and the sum of the second series} &= \frac{50}{2}(2(-2) + 49(-2)) \\
 &= 25(-4 - 98) \\
 &= -2550
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{the sum of both series} &= 2500 + (-2550) \\
 &= -50
 \end{aligned}$$

So, $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$ to 100 terms is -50 .

- d** The integers from 1 to 200 which are not divisible by 3 are 1, 2, 4, 5, 7, 8, ..., 200.

The sum of these integers can be expressed as two separate arithmetic series A and B :

$$S_A = 1 + 4 + 7 + \dots + 196 + 199 \quad \text{where } u_1 = 1, d = 3, u_n = 199$$

$$\text{and } S_B = 2 + 5 + 8 + \dots + 197 + 200 \quad \text{where } u_1 = 2, d = 3, u_n = 200$$

$$\text{Now for } S_A, u_n = u_1 + (n-1)d \quad \text{and for } S_B, u_n = u_1 + (n-1)d$$

$$\therefore 199 = 1 + 3(n-1) \qquad \qquad \qquad \therefore 200 = 2 + 3(n-1)$$

$$\therefore 198 = 3(n-1) \qquad \qquad \qquad \therefore 198 = 3(n-1)$$

$$\therefore 66 = n-1 \qquad \qquad \qquad \therefore 66 = n-1$$

$$\therefore n = 67 \qquad \qquad \qquad \therefore n = 67$$

$$\text{Using } S_n = \frac{n}{2}(u_1 + u_n), \quad S_A = \frac{67}{2}(1 + 199) = 6700 \quad \text{and} \quad S_B = \frac{67}{2}(2 + 200) = 6767$$

$$\begin{aligned} \text{The total sum} &= S_A + S_B \\ &= 6700 + 6767 \\ &= 13\,467 \end{aligned}$$

9 a $u_7 = 1 \qquad \therefore u_1 + 6d = 1 \qquad \dots (1) \quad \{\text{using } u_n = u_1 + (n-1)d\}$
 $u_{15} = -23 \quad \therefore u_1 + 14d = -23 \quad \dots (2)$

We now solve (1) and (2) simultaneously:

$$\begin{array}{rcl} -u_1 - 6d & = & -1 \quad \{\text{multiplying both sides of (1) by } -1\} \\ u_1 + 14d & = & -23 \\ \hline 8d & = & -24 \quad \{\text{adding the equations}\} \\ \therefore d & = & -3 \end{array}$$

$$\begin{aligned} \text{So, in (1):} \quad u_1 + 6(-3) &= 1 \\ \therefore u_1 - 18 &= 1 \\ \therefore u_1 &= 19 \end{aligned}$$

b $u_n = u_1 + (n-1)d$
 $\therefore u_{27} = 19 + 26(-3) \quad \{\text{using a}\}$
 $= -59$

c $S_n = \frac{n}{2}(u_1 + u_n)$
 $\therefore S_{27} = \frac{27}{2}(19 + (-59)) \quad \{\text{from b}\}$
 $= \frac{27}{2} \times (-40)$
 $= -540$

10 a $S_n = \frac{n}{2}(2u_1 + (n-1)d)$
 $\therefore -210 = \frac{n}{2}(2 \times 18 - 3(n-1))$
 $\therefore \frac{n}{2}(36 - 3n + 3) = -210$
 $\therefore \frac{n}{2}(39 - 3n) = -210$

b From **a**, $\frac{n}{2}(39 - 3n) = -210$
 $\therefore n(39 - 3n) = -420$
 $\therefore 39n - 3n^2 = -420$
 $\therefore 3n^2 - 39n - 420 = 0$
 $\therefore n^2 - 13n - 140 = 0$
 $\therefore (n-20)(n+7) = 0$
 $\therefore n = 20 \quad \{n > 0\}$

11 a $r = \frac{0.25}{0.125} = 2$

b Using $u_n = u_1 \times r^{n-1}$,
 $u_{20} = 0.125 \times 2^{19}$
 $= 65\,536$

c Using $S_n = \frac{u_1(r^n - 1)}{r - 1}$,
 $S_{10} = \frac{0.125(2^{10} - 1)}{2 - 1}$
 $= 127.875$

- 12 a** The interest is calculated annually, so $n = 7$ time periods, and $i = 8.25\%$.

$$\begin{aligned} u_7 &= u_0 \times (1 + i)^7 \\ &= 2000 \times (1.0825)^7 \quad \{8.25\% = 0.0825\} \\ &\approx 3483.58 \end{aligned}$$

The total value of Kapil's investment on January 1st 2019 is 3484 rupees.

- b** There are $n = 7 \times 12 = 84$ time periods.

Each time period the investment increases by $i = \frac{8\%}{12} \approx 0.667\%$

$$\begin{aligned} \therefore \text{the value after 7 years is } u_{84} &= u_0 \times (1 + i)^{84} \\ &\approx 2000 \times (1.00667)^{84} \quad \{0.667\% = 0.00667\} \\ &\approx 3494.84 \end{aligned}$$

The total value of Kapil's investment on January 1st 2019 for this account is 3495 rupees, which is 11 rupees more than the account in **a**.

\therefore investing in the account paying 8% per annum interest compounded monthly is the better option.

- 13 a i** $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k = 2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^3 + \dots$ is an infinite geometric series with $u_1 = 2\left(\frac{2}{3}\right) = \frac{4}{3}$ and $r = \frac{2}{3}$.

ii $S = \frac{u_1}{1-r}$

$$\therefore S = \frac{\frac{4}{3}}{1 - \frac{2}{3}}$$

$$= \frac{\frac{4}{3}}{\frac{1}{3}}$$

$$= 4$$

- b i** $\sum_{k=1}^n (k-4) = -3 - 2 - 1 + 0 + 1 + \dots + (n-4)$ is an arithmetic series with $u_1 = -3$ and $d = 1$.

ii $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$$\therefore S_n = \frac{n}{2}(2(-3) + (n-1))$$

$$= \frac{n}{2}(-6 + n - 1)$$

$$= \frac{n}{2}(n - 7)$$

c $S_n = S$

$$\therefore \frac{n}{2}(n - 7) = 4$$

$$\therefore n(n - 7) = 8$$

$$\therefore n^2 - 7n - 8 = 0$$

$$\therefore (n - 8)(n + 1) = 0$$

$$\therefore n = 8 \quad \{n > 0\}$$

- 14 a** $S = 1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots$ is an infinite geometric series with $u_1 = 1$ and $r = 0.6$.

$$\therefore S = \frac{u_1}{1-r}$$

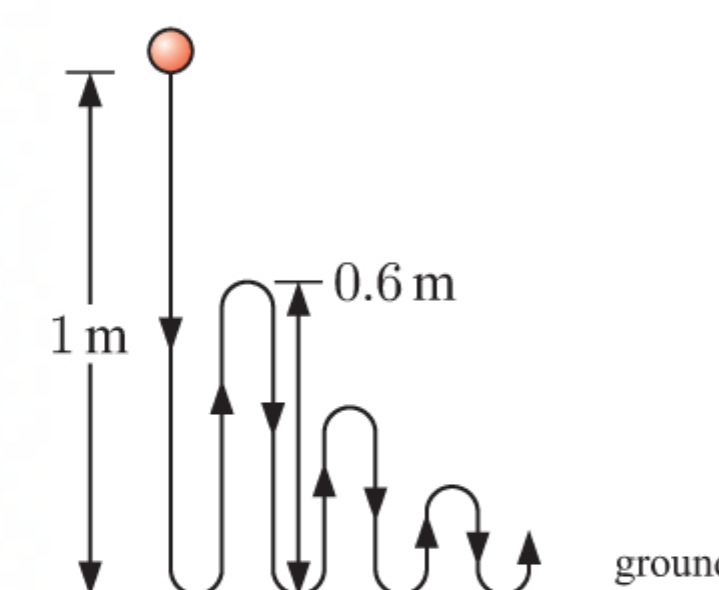
$$= \frac{1}{1-0.6}$$

$$= \frac{1}{0.4}$$

$$= 2.5$$

- b** Each time the ball bounces upward, it must travel the same distance on its way downward.

$$\begin{aligned} \therefore \text{total distance} &= 1 + 2(0.6) + 2(0.6)^2 + 2(0.6)^3 + \dots \\ &= 1 + 0.6 + (0.6)^2 + (0.6)^3 + \dots \\ &\quad + 0.6 + (0.6)^2 + (0.6)^3 + \dots \\ &= S + (S - 1) \\ &= 2S - 1 \\ &= 2(2.5) - 1 \quad \{\text{using a}\} \\ &= 5 - 1 \\ &= 4 \text{ m} \end{aligned}$$



15 a The series will converge if $|\text{common ratio}| < 1$

$$\begin{aligned}\therefore |x - 2| &< 1 \\ \therefore -1 < x - 2 < 1 \\ \therefore 1 < x < 3\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \sum_{k=1}^{\infty} 12(x-2)^{k-1} &= \frac{12}{1-(x-2)} \quad \{u_1 = 12, r = x-2\} \\ &= \frac{12}{3-x}\end{aligned}$$

and $x = \sqrt{5}$ satisfies $1 < x < 3$

$$\therefore \text{when } x = \sqrt{5}, \text{ the sum of the series} = \frac{12}{3-\sqrt{5}} \approx 15.7.$$

$$\mathbf{16} \quad \sum_{k=1}^{\infty} \left(\frac{4x}{3}\right)^{k-1} = \frac{5}{2}$$

$$\therefore \frac{1}{1-\frac{4x}{3}} = \frac{5}{2} \quad \left\{u_1 = 1, r = \frac{4x}{3}\right\}$$

$$\therefore \frac{3}{3-4x} = \frac{5}{2}$$

$$\therefore 6 = 5(3-4x)$$

$$\therefore 6 = 15 - 20x$$

$$\therefore 20x = 9$$

$$\therefore x = \frac{9}{20}$$

$$\begin{aligned}\mathbf{17 a} \quad 4^{\frac{5}{2}} &= (2^2)^{\frac{5}{2}} \\ &= 2^5 \\ &= 32\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 49^{-\frac{3}{2}} &= (7^2)^{-\frac{3}{2}} \\ &= 7^{-3} \\ &= \frac{1}{7^3} \\ &= \frac{1}{343}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad 27^{\frac{5}{3}} &= (3^3)^{\frac{5}{3}} \\ &= 3^5 \\ &= 243\end{aligned}$$

$$\begin{aligned}\mathbf{18 a} \quad x^{\frac{1}{2}}(x^{-\frac{1}{2}} + 2x - x^{\frac{1}{2}}) \\ &= x^{\frac{1}{2}} \times x^{-\frac{1}{2}} + x^{\frac{1}{2}} \times 2x - x^{\frac{1}{2}} \times x^{\frac{1}{2}} \\ &= x^0 + 2x^{\frac{3}{2}} - x^1 \\ &= 1 + 2x^{\frac{3}{2}} - x\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 5^x(5^{-x} + 5^{3x}) \\ &= 5^x \times 5^{-x} + 5^x \times 5^{3x} \\ &= 5^0 + 5^{4x} \\ &= 1 + 5^{4x}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad 2^{-2x}(2^{2x+3} - 2^{-4x} + 3) \\ &= 2^{-2x} \times 2^{2x+3} - 2^{-2x} \times 2^{-4x} + 3 \times 2^{-2x} \\ &= 2^3 - 2^{-6x} + 3 \times 2^{-2x} \\ &= 8 - 2^{-6x} + 3 \times 2^{-2x}\end{aligned}$$

$$\begin{aligned}\mathbf{19 a} \quad 5 \times 2^x &= 160 \\ \therefore 2^x &= 32 \\ \therefore 2^x &= 2^5 \\ \therefore x &= 5 \quad \{\text{equating indices}\}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 8^{2x-3} &= 16^{2-x} \\ \therefore (2^3)^{2x-3} &= (2^4)^{2-x} \\ \therefore 2^{6x-9} &= 2^{8-4x} \\ \therefore 6x-9 &= 8-4x \quad \{\text{equating indices}\} \\ \therefore 10x &= 17 \\ \therefore x &= \frac{17}{10}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \left(\frac{1}{3}\right)^{2x-5} &= 27 \\ \therefore (3^{-1})^{2x-5} &= 3^3 \\ \therefore 3^{5-2x} &= 3^3 \\ \therefore 5-2x &= 3 \quad \{\text{equating indices}\} \\ \therefore -2x &= -2 \\ \therefore x &= 1\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad 25^x + 2(5^x) &= 35 \\ \therefore (5^x)^2 + 2(5^x) - 35 &= 0 \\ \therefore (5^x - 5)(5^x + 7) &= 0 \\ \therefore 5^x &= 5 \quad \text{or} \quad 5^x = -7 \\ \therefore 5^x &= 5^1 \quad \{5^x > 0 \text{ for all } x\} \\ \therefore x &= 1 \quad \{\text{equating indices}\}\end{aligned}$$

$$\begin{aligned}
 \mathbf{20} \quad \mathbf{a} \quad & \log_4 8 \\
 &= \log_4 (2 \times 4) \\
 &= \log_4 (\sqrt{4} \times 4) \\
 &= \log_4 (4^{\frac{1}{2}} \times 4^1) \\
 &= \log_4 (4^{\frac{3}{2}}) \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_9 \left(\frac{1}{27} \right) \\
 &= \log_9 \left(\frac{1}{9 \times 3} \right) \\
 &= \log_9 \left(\frac{1}{9 \times \sqrt{9}} \right) \\
 &= \log_9 \left(\frac{1}{9^{\frac{3}{2}}} \right) \\
 &= \log_9 (9^{-\frac{3}{2}}) \\
 &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \log_9 \left(\frac{1}{3\sqrt{3}} \right) \\
 &= \log_9 \left(\frac{1}{3^{\frac{3}{2}}} \right) \\
 &= \log_9 \left(\frac{1}{(\sqrt{9})^{\frac{3}{2}}} \right) \\
 &= \log_9 \left(\frac{1}{9^{\frac{3}{4}}} \right) \\
 &= \log_9 (9^{-\frac{3}{4}}) \\
 &= -\frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{21} \quad \mathbf{a} \quad & \log_3 x = 2 \\
 & \therefore x = 3^2 = 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_x 27 = 3 \\
 & \therefore 27 = x^3 \\
 & \therefore x = 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \log_5 (2x - 1) = 1 \\
 & \therefore 2x - 1 = 5^1 \\
 & \therefore 2x - 1 = 5 \\
 & \therefore 2x = 6 \\
 & \therefore x = 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{22} \quad & \log_a (x + 2) = \log_a x + 2 \\
 \therefore & \log_a (x + 2) - \log_a x = 2 \\
 \therefore & \log_a \left(\frac{x + 2}{x} \right) = 2 \\
 \therefore & \frac{x + 2}{x} = a^2 \\
 \therefore & x + 2 = a^2 x \\
 \therefore & (1 - a^2)x = -2 \\
 \therefore & x = \frac{-2}{1 - a^2} = \frac{2}{a^2 - 1} \quad \{a > 1\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{23} \quad \mathbf{a} \quad & \frac{1}{4} \ln 81 + \ln 12 - \ln 4 \\
 &= \frac{1}{4} \ln(3^4) + \ln(3 \times 4) - \ln 4 \\
 &= \frac{1}{4} (4 \ln 3) + \ln 3 + \cancel{\ln 4} - \cancel{\ln 4} \\
 &= \ln 3 + \ln 3 \\
 &= 2 \ln 3 \\
 &= \ln(3^2) \\
 &= \ln 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 3 \log_9 2 - \log_9 24 \\
 &= 3 \log_9 2 - \log_9 (8 \times 3) \\
 &= 3 \log_9 2 - \log_9 (2^3 \times 9^{\frac{1}{2}}) \\
 &= 3 \log_9 2 - (\log_9 (2^3) + \log_9 (9^{\frac{1}{2}})) \\
 &= 3 \log_9 2 - (3 \log_9 2 + \frac{1}{2}) \\
 &= -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 5 + \log_2 3 - \frac{1}{2} \log_2 49 = 5 + \log_2 3 - \log_2 (49^{\frac{1}{2}}) \\
 &= \log_2 (2^5) + \log_2 3 - \log_2 7 \\
 &= \log_2 (32 \times 3) - \log_2 7 \\
 &= \log_2 \left(\frac{96}{7} \right)
 \end{aligned}$$

$$\mathbf{24} \quad x = \log_a 5$$

$$\begin{aligned}
 \mathbf{a} \quad & \log_a (5a) = \log_a 5 + \log_a a \\
 &= x + 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_a \left(\frac{125}{a^2} \right) \\
 &= \log_a 125 - \log_a (a^2) \\
 &= \log_a (5^3) - 2 \\
 &= 3 \log_a 5 - 2 \\
 &= 3x - 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \log_{25a} 5 \\
 &= \frac{\log_a 5}{\log_a (25a)} \\
 &\quad \{\text{change of base rule}\} \\
 &= \frac{x}{\log_a 25 + \log_a a} \\
 &= \frac{x}{\log_a (5^2) + 1} \\
 &= \frac{x}{2 \log_a 5 + 1} \\
 &= \frac{x}{2x + 1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{25 \quad a} \quad & \frac{\log_2 9}{\log_2 3} \\
 &= \log_3 9 \quad \{\text{change of base rule}\} \\
 &= \log_3 (3^2) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{\log_5 8}{\log_5 4} \\
 &= \log_4 8 \quad \{\text{change of base rule}\} \\
 &= \log_4 (4 \times 2) \\
 &= \log_4 (4 \times 4^{\frac{1}{2}}) \\
 &= \log_4 (4^{\frac{3}{2}}) \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{\log_3 (0.25)}{\log_3 64} \\
 &= \log_{64} (0.25) \\
 &\quad \{\text{change of base rule}\} \\
 &= \log_{64} \left(\frac{1}{4}\right) \\
 &= \log_{64} (4^{-1}) \\
 &= \log_{64} (64^{-\frac{1}{3}}) \\
 &= -\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{26 \quad a} \quad & 3 \log_5 x = \log_5 24 + \log_5 \left(\frac{1}{3}\right) \\
 \therefore \log_5 (x^3) &= \log_5 \left(\frac{24}{3}\right) \\
 \therefore \log_5 (x^3) &= \log_5 8 \\
 \therefore x^3 &= 8 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_2 x = \log_2 12 - \log_2 (7 - x) \\
 \therefore \log_2 x &= \log_2 \left(\frac{12}{7 - x}\right) \\
 \therefore x &= \frac{12}{7 - x} \\
 \therefore x(7 - x) &= 12 \\
 \therefore 7x - x^2 &= 12 \\
 \therefore x^2 - 7x + 12 &= 0 \\
 \therefore (x - 3)(x - 4) &= 0 \\
 \therefore x &= 3 \text{ or } 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \ln(x^2 - 3) - \ln(2x) = 0 \\
 \therefore \ln \left(\frac{x^2 - 3}{2x}\right) &= 0 \\
 \therefore \frac{x^2 - 3}{2x} &= e^0 = 1 \\
 \therefore x^2 - 3 &= 2x \\
 \therefore x^2 - 2x - 3 &= 0 \\
 \therefore (x - 3)(x + 1) &= 0 \\
 \therefore x &= 3 \text{ or } -1
 \end{aligned}$$

But $x = -1$ does not satisfy the original equation, as $\ln(-2)$ is undefined.

\therefore the only solution is $x = 3$.

$$\begin{aligned}
 \mathbf{d} \quad & \log_3 x + \log_3 (x - 2) = 1 \\
 \therefore \log_3 (x(x - 2)) &= 1 \\
 \therefore x(x - 2) &= 3 \\
 \therefore x^2 - 2x - 3 &= 0 \\
 \therefore (x - 3)(x + 1) &= 0 \\
 \therefore x &= 3 \text{ or } -1
 \end{aligned}$$

But $x = -1$ does not satisfy the original equation, as $\log_3(-1)$ is undefined.

\therefore the only solution is $x = 3$.

$$\begin{aligned}
 \mathbf{27 \quad a} \quad & \log_{\frac{1}{9}} x = \log_9 5 \\
 \therefore \frac{\log_9 x}{\log_9 \left(\frac{1}{9}\right)} &= \log_9 5 \quad \{\text{change of base rule}\} \\
 \therefore \frac{\log_9 x}{\log_9 (9^{-1})} &= \log_9 5 \\
 \therefore \frac{\log_9 x}{-1} &= \log_9 5 \\
 \therefore \log_9 x &= -\log_9 5 \\
 \therefore \log_9 x &= \log_9 (5^{-1}) \\
 \therefore x &= 5^{-1} = \frac{1}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_2 x - \log_8 x = 3 \\
 \therefore \log_2 x - \frac{\log_2 x}{\log_2 8} &= 3 \quad \{\text{change of base rule}\} \\
 \therefore \log_2 x - \frac{\log_2 x}{3} &= 3 \\
 \therefore \frac{2}{3} \log_2 x &= 3 \\
 \therefore \log_2 x &= \frac{9}{2} \\
 \therefore x &= 2^{\frac{9}{2}} \\
 \therefore x &= 2^4 \times 2^{\frac{1}{2}} = 16\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \log_{27} (x^4) = \log_9 x - \log_3 (\sqrt[5]{9}) \\
 \therefore \frac{\log_3 (x^4)}{\log_3 27} &= \frac{\log_3 x}{\log_3 9} - \log_3 (\sqrt[5]{9}) \quad \{\text{change of base rule}\} \\
 \therefore \frac{\log_3 (x^4)}{\log_3 (3^3)} &= \frac{\log_3 x}{\log_3 (3^2)} - \log_3 (3^{\frac{2}{5}}) \\
 \therefore \frac{4 \log_3 x}{3} &= \frac{\log_3 x}{2} - \frac{2}{5} \\
 \therefore \log_3 x \left(\frac{4}{3} - \frac{1}{2}\right) &= -\frac{2}{5} \\
 \therefore \frac{5}{6} \log_3 x &= -\frac{2}{5} \\
 \therefore \log_3 x &= -\frac{2}{5} \times \frac{6}{5} = -\frac{12}{25} \\
 \therefore x &= 3^{-\frac{12}{25}} = \frac{1}{3^{\frac{12}{25}}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{28} \quad \frac{8}{\log_5 9} &= \frac{8}{\left(\frac{\log_3 9}{\log_3 5}\right)} \quad \{\text{change of base rule}\} \\
 &= \frac{8 \log_3 5}{\log_3(3^2)} \\
 &= \frac{8 \log_3 5}{2} \\
 &= 4 \log_3 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{29} \quad \mathbf{a} \quad 3^x &= 15 \\
 \therefore \log(3^x) &= \log 15 \\
 \therefore x \log 3 &= \log 15 \\
 \therefore x &= \frac{\log 15}{\log 3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad 3 \times 4^x - 2^x &= 0 \\
 \therefore 3 \times (2^x)^2 - 2^x &= 0 \\
 \therefore 2^x(3 \times 2^x - 1) &= 0 \\
 \therefore 3 \times 2^x - 1 &= 0 \quad \{2^x > 0\} \\
 \therefore 3 \times 2^x &= 1 \\
 \therefore 2^x &= \frac{1}{3} \\
 \therefore \log(2^x) &= \log\left(\frac{1}{3}\right) = -\log 3 \\
 \therefore x \log 2 &= -\log 3 \\
 \therefore x &= -\frac{\log 3}{\log 2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{30} \quad \mathbf{a} \quad 9^x - 6(3^x) + 8 &= 0 \\
 \therefore (3^x)^2 - 6(3^x) + 8 &= 0 \\
 \therefore (3^x - 4)(3^x - 2) &= 0 \\
 \therefore 3^x &= 4 \text{ or } 2 \\
 \therefore \log(3^x) &= \log 4 \text{ or } \log 2 \\
 \therefore x \log 3 &= \log 4 \text{ or } \log 2 \\
 \therefore x &= \frac{\log 4}{\log 3} \text{ or } \frac{\log 2}{\log 3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad 2 \times 3^{2x} + 3^{x+1} &= 5 \\
 \therefore 2 \times (3^x)^2 + 3(3^x) - 5 &= 0 \\
 \therefore 2 \times (3^x)^2 - 2(3^x) + 5(3^x) - 5 &= 0 \\
 \therefore 2(3^x)(3^x - 1) + 5(3^x - 1) &= 0 \\
 \therefore (3^x - 1)(2(3^x) + 5) &= 0 \\
 \therefore 3^x &= 1 \quad \{3^x > 0\} \\
 \therefore \log(3^x) &= \log 1 \\
 \therefore x \log 3 &= 0 \\
 \therefore x &= 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{31} \quad \mathbf{a} \quad K(0) &= 3200 \times (0.85)^0 \\
 &= 3200 \times 1 \\
 &= 3200
 \end{aligned}$$

\therefore the initial population was 3200 kangaroos.

$$\mathbf{c} \quad \text{We need to find when } 3200 \times (0.85)^t = 1000.$$

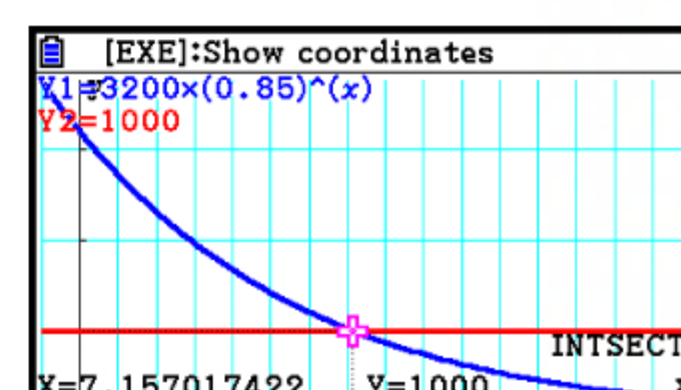
Using technology, $t \approx 7.16$.

\therefore it will take approximately 7.16 years for the kangaroo population to fall to 1000.

$$\begin{aligned}
 \mathbf{b} \quad e^{2x} - 20 &= e^x \\
 \therefore e^{2x} - e^x - 20 &= 0 \\
 \therefore (e^x)^2 - e^x - 20 &= 0 \\
 \therefore (e^x - 5)(e^x + 4) &= 0 \\
 \therefore e^x &= 5 \quad \{e^x > 0\} \\
 \therefore x &= \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad 25^x - 5^{x+1} + 6 &= 0 \\
 \therefore (5^x)^2 - 5(5^x) + 6 &= 0 \\
 \therefore (5^x - 2)(5^x - 3) &= 0 \\
 \therefore 5^x &= 2 \text{ or } 3 \\
 \therefore \log(5^x) &= \log 2 \text{ or } \log 3 \\
 \therefore x \log 5 &= \log 2 \text{ or } \log 3 \\
 \therefore x &= \frac{\log 2}{\log 5} \text{ or } \frac{\log 3}{\log 5}
 \end{aligned}$$

\therefore after 5 years, there were about 1420 kangaroos.



d $3200 \times (0.85)^t = 1000$

$$\therefore (0.85)^t = \frac{1000}{3200} = 0.3125$$

$$\therefore t \log(0.85) = \log(0.3125)$$

$$\therefore t = \frac{\log(0.3125)}{\log(0.85)}$$

$$\approx 7.16 \quad \checkmark$$

32 a The statement “If $x > 1$ then $\frac{1}{x} < 1$ ” is true, since the function $f(x) = \frac{1}{x}$ is decreasing for $x > 0$.

b The statement “If $\frac{1}{x} < 1$ then $x > 1$ ” is false, since $-\frac{1}{2} < 1$, but $-2 \not> 1$.

c The statement “ $x > 1$ if and only if $\frac{1}{x} < 1$ ” is false, since $\frac{1}{x} < 1$ does not imply $x > 1$ (from **b**).

33 Let the middle number be x .

\therefore the sum of the three consecutive odd integers is $(x-2) + x + (x+2) = 3x$ which is divisible by 3.

34 $2x^3 \geq x$

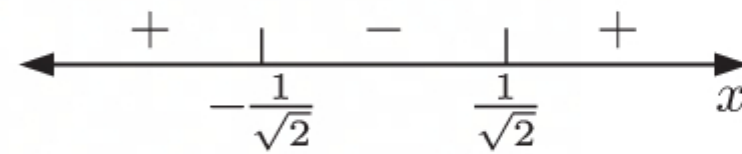
$\therefore 2x^2 \geq 1$ ← incorrect step, if $x = 0$, then we cannot divide both sides by x and if $x < 0$, then the inequality is reversed.

$$\therefore x^2 \geq \frac{1}{2}$$

$$\therefore x^2 - \frac{1}{2} \geq 0$$

$$\therefore \left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right) \geq 0$$

$$\therefore x \geq \frac{1}{\sqrt{2}} \quad \text{or} \quad x \leq -\frac{1}{\sqrt{2}}$$



Correct solution: $2x^3 \geq x$

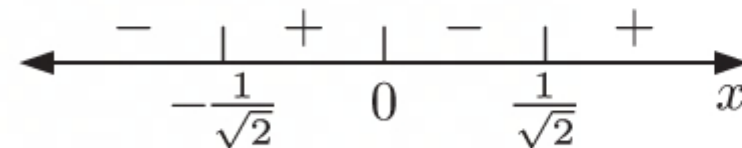
$$\therefore 2x^3 - x \geq 0$$

$$\therefore x(2x^2 - 1) \geq 0$$

$$\therefore x\left(x^2 - \frac{1}{2}\right) \geq 0$$

$$\therefore x\left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right) \geq 0$$

$$\therefore -\frac{1}{\sqrt{2}} \leq x \leq 0 \quad \text{or} \quad x \geq \frac{1}{\sqrt{2}}$$



35 a $(a+b)^3 - (a-b)^3 = (a^3 + 3a^2b + 3ab^2 + b^3) - (a^3 - 3a^2b + 3ab^2 - b^3)$ {binomial theorem}

$$= \cancel{a^3} + 3a^2b + \cancel{3ab^2} + b^3 - \cancel{a^3} + 3a^2b - \cancel{3ab^2} + b^3$$

$$= 6a^2b + 2b^3$$

$$= 2b(3a^2 + b^2) \quad \checkmark$$

b If $a = 2$ and $b = 1$,	$\text{LHS} = (a+b)^3 - (a-b)^3$	$\text{RHS} = 2b(3a^2 + b^2)$
	$= (2+1)^3 - (2-1)^3$	$= 2(1)(3(2)^2 + (1)^2)$
	$= 3^3 - 1^3$	$= 2(12 + 1)$
	$= 27 - 1$	$= 2(13)$
	$= 26$	$= 26$
		$= \text{LHS} \quad \checkmark$

36 (\Rightarrow) Suppose $xyz = 1$, then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{x^2z + y^2x + z^2y}{xyz}$

$$= x^2z + y^2x + z^2y$$

(\Leftarrow) Suppose $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = x^2z + y^2x + z^2y$

$$\therefore \frac{x^2z + y^2x + z^2y}{xyz} = x^2z + y^2x + z^2y$$

$$\therefore xyz = \frac{x^2z + y^2x + z^2y}{x^2z + y^2x + z^2y}$$

$$= 1 \quad \text{since } x, y, z > 0 \Rightarrow x^2z + y^2x + z^2y > 0.$$

37 P_n is: $5n^3 - 3n^2 - 2n$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $5(1)^3 - 3(1)^2 - 2(1) = 0 = 0 \times 6 \quad \therefore P_1$ is true.

(2) If P_k is true, then $5k^3 - 3k^2 - 2k = 6A$ where A is an integer.

$$\begin{aligned}
 \text{Now } & 5(k+1)^3 - 3(k+1)^2 - 2(k+1) \\
 &= 5(k^3 + 3k^2 + 3k + 1) - 3(k^2 + 2k + 1) - 2(k+1) \\
 &= 5k^3 + 15k^2 + 15k + 5 - 3k^2 - 6k - 3 - 2k - 2 \\
 &= (5k^3 - 3k^2 - 2k) + 15k^2 + 9k \\
 &= 6A + 3k(5k + 3) \quad \{\text{using } P_k\} \\
 &= 6A + 3(2B), \quad B \in \mathbb{Z} \quad \{k(5k + 3) \text{ is even since either } k \text{ is even, or } k \text{ is odd} \Rightarrow 5k + 3 \text{ is even}\} \\
 &= 6(A + B), \quad \text{where } A, B \in \mathbb{Z}
 \end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

38 Proof by exhaustion:

If we divide n by 5 then the remainder will be 0, 1, 2, 3, or 4. Hence every integer can be written in one of the forms $5k$, $5k + 1$, $5k + 2$, $5k + 3$, or $5k + 4$, for some $k \in \mathbb{Z}$.

If $n = 5k$, then $(5k)^2 + 2 = 25k^2 + 2 = 5(5k^2) + 2$ which is not divisible by 5.

$$\begin{aligned}
 \text{If } n = 5k + 1, \text{ then } & (5k + 1)^2 + 2 = 25k^2 + 10k + 1 + 2 \\
 &= 25k^2 + 10k + 3 \\
 &= 5(5k^2 + 2k) + 3 \quad \text{which is not divisible by 5.}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } n = 5k + 2, \text{ then } & (5k + 2)^2 + 2 = 25k^2 + 20k + 4 + 2 \\
 &= 25k^2 + 20k + 6 \\
 &= 5(5k^2 + 4k + 1) + 1 \quad \text{which is not divisible by 5.}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } n = 5k + 3, \text{ then } & (5k + 3)^2 + 2 = 25k^2 + 30k + 9 + 2 \\
 &= 25k^2 + 30k + 11 \\
 &= 5(5k^2 + 6k + 2) + 1 \quad \text{which is not divisible by 5.}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } n = 5k + 4, \text{ then } & (5k + 4)^2 + 2 = 25k^2 + 40k + 16 + 2 \\
 &= 25k^2 + 40k + 18 \\
 &= 5(5k^2 + 8k + 3) + 3 \quad \text{which is not divisible by 5.}
 \end{aligned}$$

In all cases, $n^2 + 2$ is not divisible by 5.

$\therefore 5$ never divides $n^2 + 2$ for all $n \in \mathbb{Z}$.

39 5^3 and 6^3 are consecutive cubes with difference $6^3 - 5^3 = 216 - 125 = 91 = 7 \times 13$ which is not prime.

\therefore the conjecture is false.

40 For $n = 5$, we have $5^2 + 2^5 = 25 + 32$

$$= 57$$

$$= 3 \times 19 \quad \text{which is not prime.}$$

\therefore the conjecture is false.

41 Proof by contrapositive:

The contrapositive is: If p does not divide m or n , then p does not divide mn .

If p does not divide m or n , then p is not a prime factor of m or n .

$\therefore p$ is not a prime factor of mn .

$\therefore p$ does not divide mn .

This proves the contrapositive and hence the original statement.

\therefore if p divides mn , then p divides at least one of m or n .

42 Proof by contrapositive:

The contrapositive is: If a is rational, then a^3 is rational.

Let $a = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

$$\therefore a^3 = \left(\frac{p}{q}\right)^3 = \frac{p^3}{q^3} \in \mathbb{Q} \quad \{p^3, q^3 \in \mathbb{Z}, \quad q^3 \neq 0\}$$

This proves the contrapositive and hence the original statement.

\therefore if a^3 is irrational then a is irrational.

43 Proof by contradiction:

Suppose $\log_5 9$ is rational.

$$\therefore \log_5 9 = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}, \quad q \neq 0$$

$$\therefore 9 = 5^{\frac{p}{q}}$$

$$\therefore 9^q = 5^p$$

The left hand side is always divisible by 9 and the right hand side is never divisible by 9.

This is a contradiction.

\therefore our original supposition is false, and $\log_5 9$ is irrational.

44 $p, q \in \mathbb{Q}$, $p < q$

a Let $r = \frac{p+q}{2}$.

Since $p < q$ we have $p+q < 2q$ and $2p < p+q$

$$\therefore \frac{p+q}{2} < q \quad \text{and} \quad p < \frac{p+q}{2}$$

$$\therefore r < q \quad \text{and} \quad p < r$$

So, $p < r < q$.

Since $p, q \in \mathbb{Q}$, $p = \frac{m_1}{m_2}$ and $q = \frac{n_1}{n_2}$ where $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $m_2, n_2 \neq 0$.

$$\text{Now } r = \frac{\frac{m_1}{m_2} + \frac{n_1}{n_2}}{2} = \frac{m_1 n_2 + m_2 n_1}{2m_2 n_2} \in \mathbb{Q} \quad \{m_1 n_2 + m_2 n_1, 2m_2 n_2 \in \mathbb{Z}, 2m_2 n_2 \neq 0\}$$

$$\therefore r = \frac{p+q}{2} \text{ is a rational number such that } p < r < q.$$

b Proof by contradiction:

Suppose there are a finite number of rational numbers between p and q .

Let N be the greatest rational number between p and q .

Since $N \in \mathbb{Q}$ and $p < N < q$, there is a rational number r such that $p < N < r < q$. {using **a**}

This is a contradiction as N is the greatest rational number between p and q .

\therefore our original supposition is false, and there are infinitely many rational numbers between p and q .

45 P_n is: $1 \times 2 + 2 \times 5 + 3 \times 8 + \dots + n(3n-1) = n^2(n+1)$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $1 \times 2 = 2$ and RHS = $1^2(1+1) = 2 \therefore P_1$ is true.

(2) If P_k is true, then $1 \times 2 + 2 \times 5 + 3 \times 8 + \dots + k(3k-1) = k^2(k+1)$.

$$\begin{aligned} \text{Now } & 1 \times 2 + 2 \times 5 + 3 \times 8 + \dots + k(3k-1) + (k+1)(3(k+1)-1) \\ &= k^2(k+1) + (k+1)(3(k+1)-1) \quad \{\text{using } P_k\} \\ &= (k+1)(k^2 + 3(k+1) - 1) \\ &= (k+1)(k^2 + 3k + 2) \\ &= (k+1)(k+1)(k+2) \\ &= (k+1)^2([k+1] + 1) \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

46 P_n is: $\sum_{i=1}^n i \times 2^{i-1} = (n-1)2^n + 1$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = 1 and RHS = $(1-1)2^1 + 1 = 1$ $\therefore P_1$ is true.

(2) If P_k is true, then $\sum_{i=1}^k i \times 2^{i-1} = (k-1)2^k + 1$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} i \times 2^{i-1} &= \sum_{i=1}^k i \times 2^{i-1} + (k+1) \times 2^k \\ &= (k-1)2^k + 1 + (k+1) \times 2^k \quad \{\text{using } P_k\} \\ &= k \times 2^k - \cancel{2^k} + 1 + k \times 2^k + \cancel{2^k} \\ &= 2 \times k \times 2^k + 1 \\ &= ([k+1] - 1)2^{k+1} + 1 \end{aligned}$$

$\therefore P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

47 P_n is: $3^n > n^2 + n$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $3^1 = 3$ and RHS = $1^2 + 1 = 2$ $\therefore P_1$ is true.

(2) If P_k is true, then $3^k > k^2 + k$.

$$\begin{aligned} \text{Now } 3^{k+1} &= 3 \times 3^k \\ \therefore 3^{k+1} &> 3(k^2 + k) \quad \{\text{using } P_k\} \\ \therefore 3^{k+1} &> 3k^2 + 3k \\ \therefore 3^{k+1} &> 2k^2 + k^2 + 3k \\ \therefore 3^{k+1} &> 2 + k^2 + 3k \quad \{k \geq 1\} \\ \therefore 3^{k+1} &> k^2 + 2k + 1 + k + 1 \\ \therefore 3^{k+1} &> (k+1)^2 + (k+1) \end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

48 P_n is: $\sin x + \sin 3x + \sin 5x + \dots + \sin[(2n-1)x] = \frac{\sin^2(nx)}{\sin x}$ for all $n \in \mathbb{Z}^+$, $0 < x < \pi$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $\sin x$ and RHS = $\frac{\sin^2 x}{\sin x} = \sin x$ $\{\sin x \neq 0 \text{ for } 0 < x < \pi\}$ $\therefore P_1$ is true.

(2) If P_k is true, then $\sin x + \sin 3x + \sin 5x + \dots + \sin[(2k-1)x] = \frac{\sin^2(kx)}{\sin x}$.

$$\begin{aligned} \text{Now } \sin x + \sin 3x + \dots + \sin[(2k-1)x] + \sin[(2k+1)x] &= \frac{\sin^2(kx)}{\sin x} + \sin[(2k+1)x] \quad \{\text{using } P_k\} \\ &= \frac{\sin^2(kx) + \sin[(2k+1)x] \sin x}{\sin x} \\ &= \frac{\sin^2(kx) + (-\frac{1}{2})[\cos[(2k+2)x] - \cos(2kx)]}{\sin x} \quad \{\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]\} \\ &= \frac{\sin^2(kx) - \frac{1}{2} \cos[2(k+1)x] + \frac{1}{2} \cos(2kx)}{\sin x} \\ &= \frac{\frac{1}{2} - \frac{1}{2} \cos(2kx) - \frac{1}{2} \cos[2(k+1)x] + \frac{1}{2} \cos(2kx)}{\sin x} \quad \{\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta\} \\ &= \frac{\frac{1}{2} - \frac{1}{2} \cos[2(k+1)x]}{\sin x} \\ &= \frac{\sin^2[(k+1)x]}{\sin x} \end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

49 a i P_n is: $\sum_{r=1}^n r^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $1^3 = 1$ and RHS = $\frac{1^2 \times 2^2}{4} = 1 \quad \therefore P_1$ is true.

(2) If P_k is true, then $\sum_{r=1}^k r^3 = \frac{k^2(k+1)^2}{4}$

$$\begin{aligned} \text{Now } \sum_{r=1}^{k+1} r^3 &= \sum_{r=1}^k r^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \{\text{using } P_k\} \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2([k+1] + 1)^2}{4} \end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

ii $1^3 + 2^3 + \dots + 100^3 = \sum_{r=1}^{100} r^3$

$$\begin{aligned} &= \frac{100^2(101)^2}{4} \quad \{\text{using a i}\} \\ &= 25\,502\,500 \end{aligned}$$

b $\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n$ which is arithmetic with $u_1 = 1$, $d = 1$.

$$\begin{aligned} \therefore \sum_{r=1}^n r &= \frac{n}{2}[2u_1 + (n-1)d] \\ &= \frac{n}{2}[2 + n - 1] \\ &= \frac{n(n+1)}{2} \end{aligned}$$

So, $\left(\sum_{r=1}^n r\right)^2 = \frac{n^2(n+1)^2}{4} = \sum_{r=1}^n r^3 \quad \{\text{from a i}\}$

50 P_n is: $\sum_{r=1}^n (r^2 - r) = \frac{n(n^2 - 1)}{3}$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $1^2 - 1 = 0$ and RHS = $\frac{(1)(1^2 - 1)}{3} = 0 \quad \therefore P_1$ is true.

(2) If P_k is true, then $\sum_{r=1}^k (r^2 - r) = \frac{k(k^2 - 1)}{3}$.

$$\begin{aligned} \text{Now } \sum_{r=1}^{k+1} (r^2 - r) &= \sum_{r=1}^k (r^2 - r) + [(k+1)^2 - (k+1)] \\ &= \frac{k(k^2 - 1)}{3} + (k+1)^2 - (k+1) \quad \{\text{using } P_k\} \\ &= \frac{k(k+1)(k-1) + 3(k+1)^2 - 3(k+1)}{3} \\ &= \frac{(k+1)[k(k-1) + 3(k+1) - 3]}{3} \\ &= \frac{(k+1)(k^2 - k + 3k + 3 - 3)}{3} \\ &= \frac{(k+1)(k^2 + 2k)}{3} \\ &= \frac{(k+1)(k^2 + 2k + 1 - 1)}{3} \quad \{\text{completing the square}\} \\ &= \frac{(k+1)[(k+1)^2 - 1]}{3} \end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
 P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

51 P_n is: $\frac{d^n}{dx^n} (xe^x) = (x+n)e^x$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $\frac{d}{dx} (xe^x) = 1e^x + xe^x$
 $= e^x(x+1)$
 $= (x+1)e^x \quad \therefore P_1$ is true.

(2) If P_k is true, then $\frac{d^k}{dx^k} (xe^x) = (x+k)e^x$.

$$\begin{aligned} \text{Now } \frac{d^{k+1}}{dx^{k+1}} (xe^x) &= \frac{d}{dx} \left[\frac{d^k}{dx^k} (xe^x) \right] \\ &= \frac{d}{dx} [(x+k)e^x] \quad \{\text{using } P_k\} \\ &= 1e^x + (x+k)e^x \\ &= e^x(x+k+1) \\ &= e^x(x+[k+1]) \end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
 P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

52 P_n is: $\frac{d^n}{dx^n} (\ln x) = (-1)^{n-1} \frac{(n-1)!}{x^n}$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $\frac{d}{dx} (\ln x) = \frac{1}{x}$
 $= (-1)^{1-1} \frac{(1-1)!}{x^1} \quad \therefore P_1$ is true.

(2) If P_k is true, then $\frac{d^k}{dx^k}(\ln x) = (-1)^{k-1} \frac{(k-1)!}{x^k}$.

$$\begin{aligned}\text{Now } \frac{d^{k+1}}{dx^{k+1}}(\ln x) &= \frac{d}{dx} \left[\frac{d^k}{dx^k}(\ln x) \right] \\ &= \frac{d}{dx} \left[(-1)^{k-1} \frac{(k-1)!}{x^k} \right] \quad \{\text{using } P_k\} \\ &= (-1)^{k-1} (-k) \frac{(k-1)!}{x^{k+1}} \\ &= (-1)^k \frac{k(k-1)!}{x^{k+1}} \\ &= (-1)^{(k+1)-1} \frac{([k+1]-1)!}{x^{k+1}}\end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

53 a $\text{cis } \theta_1 \text{ cis } \theta_2 = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$
 $= [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] + i[\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2]$
 $= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$
 $= \text{cis}(\theta_1 + \theta_2)$

b P_n is: $\text{cis}(\theta_1 + \theta_2 + \dots + \theta_n) = \text{cis } \theta_1 \text{ cis } \theta_2 \text{ cis } \theta_3 \dots \text{cis } \theta_n$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $\text{cis } \theta_1 = \text{cis } \theta_1 \therefore P_1$ is true.

(2) If P_k is true, then $\text{cis}(\theta_1 + \theta_2 + \dots + \theta_k) = \text{cis } \theta_1 \text{ cis } \theta_2 \text{ cis } \theta_3 \dots \text{cis } \theta_k$.

$$\begin{aligned}\text{Now } \text{cis}(\theta_1 + \theta_2 + \dots + \theta_{k+1}) &= \text{cis}((\theta_1 + \theta_2 + \dots + \theta_k) + \theta_{k+1}) \\ &= \text{cis}(\theta_1 + \theta_2 + \dots + \theta_k) \text{ cis } \theta_{k+1} \quad \{\text{using a}\} \\ &= \text{cis } \theta_1 \text{ cis } \theta_2 \text{ cis } \theta_3 \dots \text{cis } \theta_k \text{ cis } \theta_{k+1} \quad \{\text{using } P_k\}\end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

54 P_n is: $2n^3 - 3n^2 + n + 31 \geq 0$ for all $n \in \mathbb{Z}$, $n \geq -2$.

Proof: (By the principle of mathematical induction)

$$\begin{aligned}(1) \text{ If } n = -2, \quad 2(-2)^3 - 3(-2)^2 + (-2) + 31 &= 2(-8) - 3(4) - 2 + 31 \\ &= -16 - 12 + 29 \\ &= 1 \\ &\geq 0\end{aligned}$$

$\therefore P_{-2}$ is true.

(2) If P_k is true, then $2k^3 - 3k^2 + k + 31 \geq 0$.

$$\begin{aligned}\text{Now } 2(k+1)^3 - 3(k+1)^2 + (k+1) + 31 &= 2(k^3 + 3k^2 + 3k + 1) - 3(k^2 + 2k + 1) + (k+1) + 31 \\ &= (2k^3 - 3k^2 + k + 31) + 6k^2 + \cancel{6k} + 2 - \cancel{6k} - 3 + 1 \\ &\geq 6k^2 \quad \{\text{using } P_k\} \\ &\geq 0\end{aligned}$$

So, P_{k+1} is true.

Since P_{-2} is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}$, $n \geq -2$. {principle of mathematical induction}

55 On the menu given, there are 2 entrées, 4 main courses, and 3 desserts available.

Using the product principle, there are $2 \times 4 \times 3 = 24$ different combinations for a person to order one of each.

- 56** There are 26 English letters and 10 digits from 0 to 9, so there are $26 + 10 = 36$ characters to choose from.

First consider passwords with 4 characters.

If there are no restrictions, there are 36^4 possible passwords.

If no digit from 0 to 9 is used, there are 26 characters to choose from.

\therefore there are 26^4 passwords without any of the digits 0 to 9.

So, there are $36^4 - 26^4 = 1\,222\,640$ valid 4-character passwords.

Similarly, there are $36^5 - 26^5 = 48\,584\,800$ 5 character passwords,

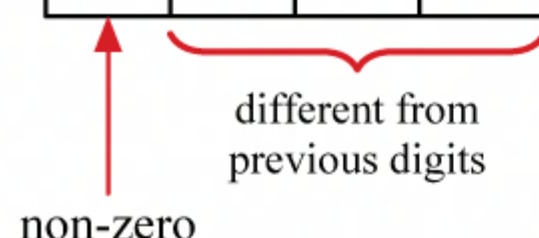
and there are $36^6 - 26^6 = 1\,867\,866\,560$ 6 character passwords.

So, there are $1\,222\,640 + 48\,584\,800 + 1\,867\,866\,560 = 1\,917\,674\,000$ valid passwords.

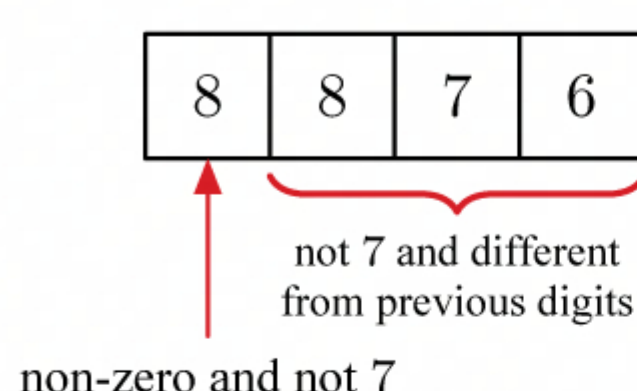
- 57 a**

9	9	8	7
---	---	---	---

 \therefore there are $9 \times 9 \times 8 \times 7 = 4536$ numbers.



- b** Consider the numbers that do *not* have a 7 as one of the four digits:



\therefore there are $8 \times 8 \times 7 \times 6 = 2688$ numbers that do not contain a 7.

\therefore there are $4536 - 2688 = 1848$ numbers that *do* contain a 7.

- 58 a**

floor manager	cleaner	cashier
20	19	18

 \therefore there are $20 \times 19 \times 18 = 6840$ ways for the positions to be filled.

- b**

floor manager	cleaner	cashier
5	19	18

 \therefore there are $5 \times 19 \times 18 = 1710$ ways for the positions to be filled.

- c**

floor manager	cleaner	cashier
11	9	10


 \therefore there are $11 \times 9 \times 10 = 990$ ways for the positions to be filled.

- 59 a** There are $\binom{11}{2} = 55$ ways of choosing 2 people from 11.

\therefore the total number of handshakes is 55.

- b**

6	5
---	---

 \therefore there are $6 \times 5 = 30$ handshakes between a man and a woman.
- 

60

$$\binom{n}{3} = 3\binom{n-1}{2} - \binom{n-1}{1}$$

$$\therefore \frac{n!}{3!(n-3)!} = 3 \times \frac{(n-1)!}{2!(n-3)!} - \frac{(n-1)!}{1!(n-2)!}$$

$$\therefore \frac{n(n-1)(n-2)}{6} = \frac{3(n-1)(n-2)}{2} - (n-1)$$

$$\therefore n(n-1)(n-2) = 9(n-1)(n-2) - 6(n-1)$$

$$\therefore n(n-1)(n-2) - 9(n-1)(n-2) + 6(n-1) = 0$$

$$\therefore (n-1)(n(n-2) - 9(n-2) + 6) = 0$$

$$\therefore (n-1)(n^2 - 2n - 9n + 18 + 6) = 0$$

$$\therefore (n-1)(n^2 - 11n + 24) = 0$$

$$\therefore (n-1)(n-3)(n-8) = 0$$

$$\therefore n = 1, 3, \text{ or } 8$$

- 61 a** The captain must come from school A, and we need any 10 of the remaining $22 - 1 = 21$ students from schools A and B.

This can be done in $11 \times \binom{21}{10} = 3\,879\,876$ different ways.

- b** Consider the number of ways the team can be made up of k students from school A, where $k = 1, 2, 3, \dots, 11$.

The captain must be chosen from these k students, and the remainder of the team is made up of $11 - k$ students from school B.

This can be done in $\binom{k}{1} \times \binom{11}{k} \times \binom{11}{11-k} = k \binom{11}{k}^2$ different ways.

$$\begin{aligned} \text{So, the team can be formed in } \sum_{k=1}^{11} k \binom{11}{k}^2 &= 1 \binom{11}{1}^2 + 2 \binom{11}{2}^2 + 3 \binom{11}{3}^2 + \dots + 11 \binom{11}{11}^2 \\ &= 3\,879\,876 \text{ different ways.} \end{aligned}$$

This is the same as the answer calculated in **a**.

- c** Suppose the squad is made up of n students each from schools A and B, and a combined team of n students is to be selected. The captain must be chosen from school A.

So, the other $n - 1$ positions must be chosen from the remaining $2n - 1$ students from schools A and B.

This can be done in $\binom{n}{1} \times \binom{2n-1}{n-1} = n \binom{2n-1}{n-1}$ different ways.

Now consider the number of ways the team can be made up of k students from school A, where $k = 1, 2, 3, \dots, n$.

The captain must be chosen from these k students, and the remainder of the team is made up of $n - k$ students from school B.

This can be done in $\binom{k}{1} \times \binom{n}{k} \times \binom{n}{n-k} = k \binom{n}{k}^2$ different ways.

$$\text{So, the team can be formed in } \sum_{k=1}^n k \binom{n}{k}^2 = 1 \binom{n}{1}^2 + 2 \binom{n}{2}^2 + 3 \binom{n}{3}^2 + \dots + n \binom{n}{n}^2 \text{ different ways}$$

$$\therefore 1 \binom{n}{1}^2 + 2 \binom{n}{2}^2 + 3 \binom{n}{3}^2 + \dots + n \binom{n}{n}^2 = n \binom{2n-1}{n-1} \text{ as required.}$$

62 a

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ & 1 & 3 & & 3 & 1 & \\ 1 & 4 & & 6 & & 4 & 1 \\ 1 & 5 & 10 & & 10 & 5 & 1 \end{array}$$

$$\begin{aligned} \text{b i } \left(x + \frac{1}{x}\right)^5 &= x^5 + 5(x)^4\left(\frac{1}{x}\right) + 10(x)^3\left(\frac{1}{x}\right)^2 + 10(x)^2\left(\frac{1}{x}\right)^3 + 5(x)\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5 \\ &= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5} \end{aligned}$$

$$\begin{aligned} \text{ii } (1 - \sqrt{2})^5 &= 1^5 + 5(1)^4(-\sqrt{2}) + 10(1)^3(-\sqrt{2})^2 + 10(1)^2(-\sqrt{2})^3 + 5(1)(-\sqrt{2})^4 + (-\sqrt{2})^5 \\ &= 1 - 5\sqrt{2} + 20 - 20\sqrt{2} + 20 - 4\sqrt{2} \\ &= 41 - 29\sqrt{2} \end{aligned}$$

63

$$(a + bx)^n = 1 - 12x + 54x^2 - \dots, \quad a > 0, \quad n \in \mathbb{Z}^+$$

$$\therefore \sum_{r=0}^n \binom{n}{r} a^{n-r} (bx)^r = 1 - 12x + 54x^2 - \dots \quad \{\text{binomial theorem}\}$$

$$\therefore \binom{n}{0} a^n + \binom{n}{1} a^{n-1} bx + \binom{n}{2} a^{n-2} b^2 x^2 + \dots = 1 - 12x + 54x^2 - \dots$$

$$\therefore a^n + na^{n-1}bx + \frac{n(n-1)}{2} a^{n-2} b^2 x^2 + \dots = 1 - 12x + 54x^2 - \dots$$

Equating coefficients:

$$a^n = 1 \quad \dots (1)$$

$$na^{n-1}b = -12 \quad \dots (2)$$

$$\frac{n(n-1)}{2} a^{n-2} b^2 = 54 \quad \dots (3)$$

From (1), $a = 1$ since $a > 0$.

Substituting into (2) we have $nb = -12$

$$\therefore b = \frac{-12}{n} \quad \dots (4)$$

$$\begin{aligned}
 \text{Substituting into (3) gives } \frac{n(n-1)}{2} \left(\frac{144}{n^2} \right) &= 54 \\
 \therefore \frac{144(n-1)}{n} &= 108 \\
 \therefore 144(n-1) &= 108n \\
 \therefore 144n - 144 &= 108n \\
 \therefore 36n &= 144 \\
 \therefore n &= \frac{144}{36} = 4 \\
 \therefore b &= \frac{-12}{4} = -3 \quad \{\text{using (4)}\}
 \end{aligned}$$

64 a The general term of the expansion is $T_{r+1} = \binom{6}{r} a^{6-r} b^r$, $r = 0, 1, 2, \dots, 6$.

b For $a^4 b^2$, $r = 2$

$$\begin{aligned}
 \therefore T_3 &= \binom{6}{2} a^4 b^2 \\
 &= \binom{6}{4} a^4 b^2 \quad \left\{ \binom{6}{2} = \binom{6}{4} \right\} \\
 &= 15a^4 b^2
 \end{aligned}$$

\therefore the coefficient of $a^4 b^2$ is 15.

65 $(x+2)(1-x)^{10}$

$$\begin{aligned}
 &= (x+2) \left[1^{10} + \binom{10}{1}(1)^9(-x) + \binom{10}{2}(1)^8(-x)^2 + \binom{10}{3}(1)^7(-x)^3 + \binom{10}{4}(1)^6(-x)^4 + \binom{10}{5}(1)^5(-x)^5 + \dots \right] \\
 &= (x+2) \left[1 - \binom{10}{1}x + \binom{10}{2}x^2 - \binom{10}{3}x^3 + \binom{10}{4}x^4 - \binom{10}{5}x^5 + \dots \right]
 \end{aligned}$$

So, the terms containing x^5 are $\binom{10}{4}x^5$ from (1)
and $-2\binom{10}{5}x^5$ from (2)

\therefore the coefficient of x^5 is $\binom{10}{4} - 2\binom{10}{5} = -294$

66 a

r	0	1	2	3	4	5	6	7
7C_r	1	7	21	35	35	21	7	1

b ${}^7C_r = 35$
 $\therefore r = 3$ or 4 {from **a**}

c The coefficient of x^3 is ${}^7C_4 2^3 k^4 = 35 \times 8k^4 = 280k^4$ (1)

The coefficient of x is ${}^7C_6 2k^6 = 7 \times 2k^6 = 16k^6$ (2)

(1) is 10 times larger than (2), so $280k^4 = 10 \times 16k^6$

$$\therefore 28k^4 = 16k^6$$

$$\therefore k^2 = \frac{7}{4} \quad \{k \neq 0\}$$

$$\therefore k = \frac{\sqrt{7}}{2} \quad \{k > 0\}$$

67 $(3x^2 - 7)(x - 2)^3$

$$= (3x^2 - 7)(x^3 - 6x^2 + 12x - 8)$$

So, the terms containing x^3 are $36x^3$ from (1)
and $-7x^3$ from (2)

\therefore the coefficient of x^3 is $36 - 7 = 29$.

68 a Using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $\binom{6}{2} = \frac{6!}{2!4!}$

$$\begin{aligned}
 &= \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{2 \times 1 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\
 &= \frac{6 \times 5}{2 \times 1} \\
 &= \frac{30}{2} \\
 &= 15
 \end{aligned}$$

$$\mathbf{b} \quad \binom{6}{4} = \frac{6!}{4!2!} = \binom{6}{2} = 15$$

$$\begin{aligned} \mathbf{c} \quad (x-2)^6 &= x^6 + \binom{6}{1}(x)^5(-2)^1 + \binom{6}{2}(x)^4(-2)^2 + \binom{6}{3}(x)^3(-2)^3 + \binom{6}{4}(x)^2(-2)^4 + \binom{6}{5}(x)^1(-2)^5 + (-2)^6 \\ &= x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64 \end{aligned}$$

$$\begin{aligned} \mathbf{69} \quad \mathbf{a} \quad (3x+5)^{\frac{2}{5}} &= 5^{\frac{2}{5}} \left(1 + \frac{3x}{5}\right)^{\frac{2}{5}} \\ &= \sqrt[5]{25} \sum_{r=0}^{\infty} \binom{\frac{2}{5}}{r} \left(\frac{3x}{5}\right)^r \\ &= \sqrt[5]{25} \left(\binom{\frac{2}{5}}{0} \left(\frac{3x}{5}\right)^0 + \binom{\frac{2}{5}}{1} \left(\frac{3x}{5}\right)^1 + \binom{\frac{2}{5}}{2} \left(\frac{3x}{5}\right)^2 + \binom{\frac{2}{5}}{3} \left(\frac{3x}{5}\right)^3 + \dots \right) \\ &= \sqrt[5]{25} \left(1 + \left(\frac{2}{5}\right) \left(\frac{3x}{5}\right) + \frac{\left(\frac{2}{5}\right)\left(-\frac{3}{5}\right)}{2!} \left(\frac{3x}{5}\right)^2 + \frac{\left(\frac{2}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)}{3!} \left(\frac{3x}{5}\right)^3 + \dots \right) \\ &= \sqrt[5]{25} \left(1 + \frac{6}{25}x - \frac{27}{625}x^2 + \frac{216}{15625}x^3 + \dots \right) \\ &= \sqrt[5]{25} + \frac{6\sqrt[5]{25}}{25}x - \frac{27\sqrt[5]{25}}{625}x^2 + \frac{216\sqrt[5]{25}}{15625}x^3 + \dots \end{aligned}$$

\mathbf{b} The series converges provided $\left| \frac{3x}{5} \right| < 1$, which is the interval $-\frac{5}{3} < x < \frac{5}{3}$.

$$\begin{aligned} \mathbf{c} \quad \text{Letting } x = \frac{1}{3}, \quad (3(\frac{1}{3}) + 5)^{\frac{2}{5}} &\approx \sqrt[5]{25} + \frac{6\sqrt[5]{25}}{25}(\frac{1}{3}) + \frac{27\sqrt[5]{25}}{625}(\frac{1}{3})^2 + \frac{216\sqrt[5]{25}}{15625}(\frac{1}{3})^3 \\ \therefore 6^{\frac{2}{5}} &\approx 2.047783 \\ \therefore 36^{\frac{1}{5}} &\approx 2.047783 \end{aligned}$$

Using technology, $36^{\frac{1}{5}} \approx 2.047672$

$$\begin{aligned} \mathbf{70} \quad \mathbf{a} \quad \frac{1}{(2+3x)^3} &= (2+3x)^{-3} \\ &= 2^{-3} \left(1 + \frac{3x}{2}\right)^{-3} \\ &= \frac{1}{8} \sum_{r=0}^{\infty} \binom{-3}{r} \left(\frac{3x}{2}\right)^r \\ &= \frac{1}{8} \left(\binom{-3}{0} \left(\frac{3x}{2}\right)^0 + \binom{-3}{1} \left(\frac{3x}{2}\right)^1 + \binom{-3}{2} \left(\frac{3x}{2}\right)^2 + \binom{-3}{3} \left(\frac{3x}{2}\right)^3 + \dots \right) \\ &= \frac{1}{8} \left(1 + (-3) \left(\frac{3x}{2}\right) + \frac{(-3)(-4)}{2!} \left(\frac{3x}{2}\right)^2 + \frac{(-3)(-4)(-5)}{3!} \left(\frac{3x}{2}\right)^3 + \dots \right) \\ &= \frac{1}{8} \left(1 - \frac{9}{2}x + \frac{27}{2}x^2 - \frac{135}{4}x^3 + \dots \right) \\ &= \frac{1}{8} - \frac{9}{16}x + \frac{27}{16}x^2 - \frac{135}{32}x^3 + \dots \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (ax+1)^4 &= (ax)^4 + 4(ax)^3 + 6(ax)^2 + 4(ax) + 1 \\ &= a^4x^4 + 4a^3x^3 + 6a^2x^2 + 4ax + 1 \end{aligned}$$

$$\text{Now } \frac{(ax+1)^4}{(2+3x)^3} = (a^4x^4 + 4a^3x^3 + 6a^2x^2 + 4ax + 1) \left(\frac{1}{8} - \frac{9}{16}x + \frac{27}{16}x^2 - \frac{135}{32}x^3 + \dots \right) \quad \{\text{using } \mathbf{a}\}$$

So, the terms containing x^2 are $\frac{3}{4}a^2x^2$ from (1)
 $-\frac{9}{4}ax^2$ from (2)
 and $\frac{27}{16}x^2$ from (3)

\therefore the coefficient of x^2 is $\frac{3}{4}a^2 - \frac{9}{4}a + \frac{27}{16}$.

$$\text{But the coefficient of } x^2 \text{ is } \frac{243}{16} \quad \therefore \frac{3}{4}a^2 - \frac{9}{4}a + \frac{27}{16} = \frac{243}{16}$$

$$\therefore 12a^2 - 36a + 27 = 243$$

$$\therefore 12a^2 - 36a - 216 = 0$$

$$\therefore a^2 - 3a - 18 = 0$$

$$\therefore (a-6)(a+3) = 0$$

$$\therefore a = 6 \text{ or } -3$$

$$71 \quad \mathbf{a} \quad x^2 + x - 2 = (x - 1)(x + 2)$$

$$\text{Let } \frac{3}{x^2 + x - 2} = \frac{A}{x - 1} + \frac{B}{x + 2}$$

$$\therefore 3 = A(x + 2) + B(x - 1)$$

$$\text{Substituting } x = -2, \quad 3 = -3B$$

$$\therefore B = -1$$

$$\text{Substituting } x = 1, \quad 3 = 3A$$

$$\therefore A = 1$$

$$\therefore \frac{3}{x^2 + x - 2} = \frac{1}{x - 1} - \frac{1}{x + 2}$$

$$\mathbf{c} \quad 6x^2 - 13x + 6 = (3x - 2)(2x - 3)$$

$$\text{Let } \frac{1 - 9x}{6x^2 - 13x + 6} = \frac{A}{3x - 2} + \frac{B}{2x - 3}$$

$$\therefore 1 - 9x = A(2x - 3) + B(3x - 2)$$

$$\text{Substituting } x = \frac{3}{2}, \quad 1 - 9\left(\frac{3}{2}\right) = B\left(3\left(\frac{3}{2}\right) - 2\right)$$

$$\therefore 1 - \frac{27}{2} = B\left(\frac{9}{2} - 2\right)$$

$$\therefore -\frac{25}{2} = \frac{5}{2}B$$

$$\therefore B = -5$$

$$\text{Substituting } x = \frac{2}{3}, \quad 1 - 9\left(\frac{2}{3}\right) = A\left(2\left(\frac{2}{3}\right) - 3\right)$$

$$\therefore 1 - 6 = A\left(\frac{4}{3} - 3\right)$$

$$\therefore -5 = -\frac{5}{3}A$$

$$\therefore A = 3$$

$$\therefore \frac{1 - 9x}{6x^2 - 13x + 6} = \frac{3}{3x - 2} - \frac{5}{2x - 3}$$

$$72 \quad \text{Let } \frac{3x^2 + 4x + 12}{(x - 3)(x^2 + 2x + 2)} = \frac{A}{x - 3} + \frac{Bx + C}{x^2 + 2x + 2}$$

$$\therefore 3x^2 + 4x + 12 = A(x^2 + 2x + 2) + (Bx + C)(x - 3)$$

$$\text{Substituting } x = 3, \quad 3(3)^2 + 4(3) + 12 = A(3^2 + 2(3) + 2)$$

$$\therefore 51 = 17A$$

$$\therefore A = 3$$

$$\text{Substituting } x = 0, \quad 3(0)^2 + 4(0) + 12 = 3(0^2 + 2(0) + 2) + (B(0) + C)(-3)$$

$$\therefore 12 = 6 - 3C$$

$$\therefore 3C = -6$$

$$\therefore C = -2$$

$$\text{Substituting } x = 1, \quad 3(1)^2 + 4(1) + 12 = 3(1^2 + 2(1) + 2) + (B(1) - 2)(-2)$$

$$\therefore 19 = 15 - 2B + 4$$

$$\therefore 2B = 0$$

$$\therefore B = 0$$

$$\therefore \frac{3x^2 + 4x + 12}{(x - 3)(x^2 + 2x + 2)} = \frac{3}{x - 3} - \frac{2}{x^2 + 2x + 2}$$

$$73 \quad \mathbf{a} \quad 3x^2 = -27$$

$$\therefore x^2 = -9$$

$$\therefore x = \pm 3i$$

$$\mathbf{b} \quad x^2 + x + 1 = 0$$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \quad \{\text{quadratic formula}\}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\mathbf{b} \quad 2x^2 + 3x - 2 = (2x - 1)(x + 2)$$

$$\text{Let } \frac{7x + 5}{2x^2 + 3x - 2} = \frac{A}{2x - 1} + \frac{B}{x + 2}$$

$$\therefore 7x + 5 = A(x + 2) + B(2x - 1)$$

$$\text{Substituting } x = -2, \quad 7(-2) + 5 = B(2(-2) - 1)$$

$$\therefore -14 + 5 = B(-4 - 1)$$

$$\therefore -9 = -5B$$

$$\therefore B = \frac{9}{5}$$

$$\text{Substituting } x = \frac{1}{2}, \quad 7\left(\frac{1}{2}\right) + 5 = A\left(\frac{1}{2} + 2\right)$$

$$\therefore \frac{7}{2} + 5 = \frac{5}{2}A$$

$$\therefore 7 + 10 = 5A$$

$$\therefore A = \frac{17}{5}$$

$$\therefore \frac{7x + 5}{2x^2 + 3x - 2} = \frac{17}{5(2x - 1)} + \frac{9}{5(x + 2)}$$

$$\mathbf{c} \quad 2x^2 + x + 5 = 0$$

$$\begin{aligned}\therefore x &= \frac{-1 \pm \sqrt{1^2 - 4(2)(5)}}{2(2)} \quad \{\text{quadratic formula}\} \\ &= \frac{-1 \pm \sqrt{-39}}{4} \\ &= -\frac{1}{4} \pm \frac{\sqrt{39}}{4}i\end{aligned}$$

$$\mathbf{74} \quad z = 3 - 5i, \quad w = 7 + 2i$$

$$\begin{aligned}\mathbf{a} \quad 3z - 2w &= 3(3 - 5i) - 2(7 + 2i) \\ &= 9 - 15i - 14 - 4i \\ &= -5 - 19i\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad zw^2 &= (3 - 5i)(7 + 2i)^2 \\ &= (3 - 5i)(49 + 28i + 4i^2) \\ &= (3 - 5i)(45 + 28i) \\ &= 135 + 84i - 225i - 140i^2 \\ &= 275 - 141i\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \frac{3i}{zw} &= \frac{3i}{(3 - 5i)(7 + 2i)} \\ &= \frac{3i}{21 + 6i - 35i - 10i^2} \\ &= \left(\frac{3i}{31 - 29i} \right) \times \left(\frac{31 + 29i}{31 + 29i} \right) \\ &= \frac{93i + 87i^2}{961 - 841i^2} \\ &= -\frac{87}{1802} + \frac{93}{1802}i\end{aligned}$$

$$\mathbf{75} \quad z = 2 + i, \quad w = 3 + 5i$$

$$\begin{aligned}\mathbf{a} \quad z - 3w &= (2 + i) - 3(3 + 5i) \\ &= 2 + i - 9 - 15i \\ &= -7 - 14i \\ \therefore \operatorname{Re}(z - 3w) &= -7\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad iw^2 &= i(3 + 5i)^2 \\ &= i(9 + 30i + 25i^2) \\ &= i(-16 + 30i) \\ &= -16i + 30i^2 \\ &= -30 - 16i\end{aligned}$$

$$\therefore \operatorname{Im}(iw^2) = -16$$

$$\begin{aligned}\mathbf{c} \quad \frac{z}{w} &= \left(\frac{2 + i}{3 + 5i} \right) \times \left(\frac{3 - 5i}{3 - 5i} \right) \\ &= \frac{6 - 10i + 3i - 5i^2}{9 - 25i^2} \\ &= \frac{11 - 7i}{34} \\ &= \frac{11}{34} - \frac{7}{34}i \\ \therefore \operatorname{Re}\left(\frac{z}{w}\right) &= \frac{11}{34}\end{aligned}$$

$$\begin{aligned}\mathbf{76} \quad \mathbf{a} \quad \left(\frac{3 + 4i}{1 - 3i} \right) \times \left(\frac{1 + 3i}{1 + 3i} \right) &= \frac{3 + 9i + 4i + 12i^2}{1 - 9i^2} \\ &= \frac{-9 + 13i}{10} \\ &= -\frac{9}{10} + \frac{13}{10}i\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \frac{3}{i} \left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}} \right)^2 &= \frac{3}{i} \left(\frac{1 - 2i}{\sqrt{5}} \right)^2 \\ &= \frac{3}{i} \left(\frac{(1 - 2i)^2}{5} \right) \\ &= \frac{3}{i} \left(\frac{1 - 4i + 4i^2}{5} \right) \\ &= \frac{3}{i} \left(\frac{-3 - 4i}{5} \right) \\ &= \left(\frac{-9 - 12i}{5i} \right) \times \frac{i}{i} \\ &= \frac{-9i - 12i^2}{5i^2} \\ &= \frac{12 - 9i}{-5} \\ &= -\frac{12}{5} + \frac{9}{5}i\end{aligned}$$

$$\begin{aligned}
 77 \quad & \frac{z+2}{z-2} = i \\
 \therefore & z+2 = i(z-2) \\
 \therefore & z+2 = iz-2i \\
 \therefore & z-iz = -2-2i \\
 \therefore & z(1-i) = -2-2i \\
 \therefore & z = \left(\frac{-2-2i}{1-i} \right) \times \left(\frac{1+i}{1+i} \right) \\
 & = \frac{-2-2i-2i-2i^2}{1^2-i^2} \\
 & = \frac{-4i}{2} \\
 & = -2i
 \end{aligned}$$

$$\begin{aligned}
 78 \quad & z^2 - z + 1 + i = 0 \\
 \therefore & z = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1+i)}}{2(1)} \quad \{\text{quadratic formula}\} \\
 & = \frac{1 \pm \sqrt{1-4-4i}}{2} \\
 & = \frac{1 \pm \sqrt{-3-4i}}{2}
 \end{aligned}$$

$$\text{Let } a+bi = \sqrt{-3-4i}, \quad a, b \in \mathbb{R}$$

$$\therefore (a+bi)^2 = -3-4i$$

$$\therefore a^2 + 2abi - b^2 = -3-4i$$

Equating real and imaginary parts,

$$a^2 - b^2 = -3 \quad \dots (1) \qquad 2ab = -4 \quad \dots (2)$$

$$\text{From (2), } ab = -2, \text{ and so } b = -\frac{2}{a}$$

$$\text{Substituting into (1), } a^2 - \left(\frac{-2}{a} \right)^2 = -3$$

$$\therefore a^2 + 3 - \frac{4}{a^2} = 0$$

$$\therefore a^4 + 3a^2 - 4 = 0$$

$$\therefore (a^2 + 4)(a^2 - 1) = 0$$

$$\therefore a = \pm 1 \quad \{a^2 + 4 > 0\}$$

$$\therefore b = \mp 2$$

$$\begin{aligned} \therefore \sqrt{-3-4i} &= \pm 1 \mp 2i \\ &= \pm(1-2i) \end{aligned}$$

$$\begin{aligned} \text{So, } z &= \frac{1 \pm (1-2i)}{2} \\ &= 1-i \text{ or } i \end{aligned}$$

$$79 \quad \mathbf{a} \quad (3-2i)(x-yi) = -i$$

$$\therefore 3x - 3yi - 2xi + 2yi^2 = -i$$

$$\therefore (3x-2y) + (-3y-2x)i = -i$$

Equating real and imaginary parts,

$$3x - 2y = 0 \quad \text{and} \quad -3y - 2x = -1$$

$$\therefore 3x = 2y \quad \text{and} \quad 3y + 2x = 1$$

$$\therefore x = \frac{2y}{3} \quad \text{and} \quad 3y + 2x = 1$$

$$\therefore 3y + 2\left(\frac{2y}{3}\right) = 1$$

$$\therefore 9y + 4y = 3$$

$$\therefore 13y = 3$$

$$\therefore y = \frac{3}{13}$$

$$\text{and } x = \frac{2(\frac{3}{13})}{3} = \frac{2}{13}$$

$$\mathbf{b} \quad (x+yi)^2 - (x-yi)^2 = x-y+16i$$

$$\therefore x^2 + 2xyi + y^2i^2 - x^2 + 2xyi - y^2i^2 = x-y+16i$$

$$\therefore 4xyi = x-y+16i$$

Equating real and imaginary parts,

$$x - y = 0 \quad \text{and} \quad 4xy = 16$$

$$\therefore x = y \quad \text{and} \quad 4xy = 16$$

$$\therefore 4x^2 = 16$$

$$\therefore x^2 = 4$$

$$\therefore x = \pm 2 \text{ and } y = \pm 2$$

$$\begin{aligned}
 \mathbf{80} \quad & z = iz^* \\
 \therefore & x + iy = i(x - iy) \\
 \therefore & x + iy = ix + y \\
 \therefore & x = y \quad \{\text{equating real and imaginary parts}\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{81} \quad & \text{Let } z = a + bi \text{ and } w = c + di \\
 & zw = (a + bi)(c + di) \\
 & = ac + adi + bci + bdi^2 \\
 & = (ac - bd) + (ad + bc)i \\
 \therefore & (zw)^* = (ac - bd) - (ad + bc)i \\
 \text{Now } & z^*w^* = (a - bi)(c - di) \\
 & = ac - adi - bci + bdi^2 \\
 & = (ac - bd) - (ad + bc)i \\
 & = (zw)^*
 \end{aligned}$$

82 Suppose $z \neq 0$.

$$\begin{aligned}
 \frac{z}{z^*} + \frac{z^*}{z} &= \frac{z^2 + (z^*)^2}{z^*z} \\
 &= \frac{z^2 + (z^2)^*}{|z|^2} \quad \text{which is real} \quad \left\{ w + w^* \text{ and } |w|^2 \text{ are real for all } w \right\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{83} \quad & (w + 3z^*) + (z - w^*)^* = w + 3z^* + z^* - (w^*)^* \quad \{(z + w)^* = z^* + w^*\} \\
 & = w + 4z^* - w \quad \{(w^*)^* = w\} \\
 & = 4z^*
 \end{aligned}$$

$$\mathbf{84} \quad \frac{w}{z} = 1 + i \quad \dots (1)$$

$$\therefore w - 2z^* = -1 - 5i \quad \dots (2)$$

$$\text{From (1), } w = (1 + i)z \quad \dots (3)$$

$$\text{Substituting (3) into (2), } (1 + i)z - 2z^* = -1 - 5i$$

$$\text{Letting } z = a + bi, \quad (1 + i)(a + bi) - 2(a - bi) = -1 - 5i$$

$$\therefore a + bi + ai + bi^2 - 2a + 2bi = -1 - 5i$$

$$\therefore (-a - b) + (a + 3b)i = -1 - 5i$$

$$\begin{aligned}
 \text{Equating real and imaginary parts,} \quad & -a - b = -1 \quad \text{and} \quad a + 3b = -5 \\
 & \therefore b = 1 - a \quad \text{and} \quad a + 3b = -5 \\
 & \therefore a + 3(1 - a) = -5 \\
 & \therefore a + 3 - 3a = -5 \\
 & \therefore -2a = -8 \\
 & \therefore a = 4 \\
 & \text{and } b = 1 - 4 = -3
 \end{aligned}$$

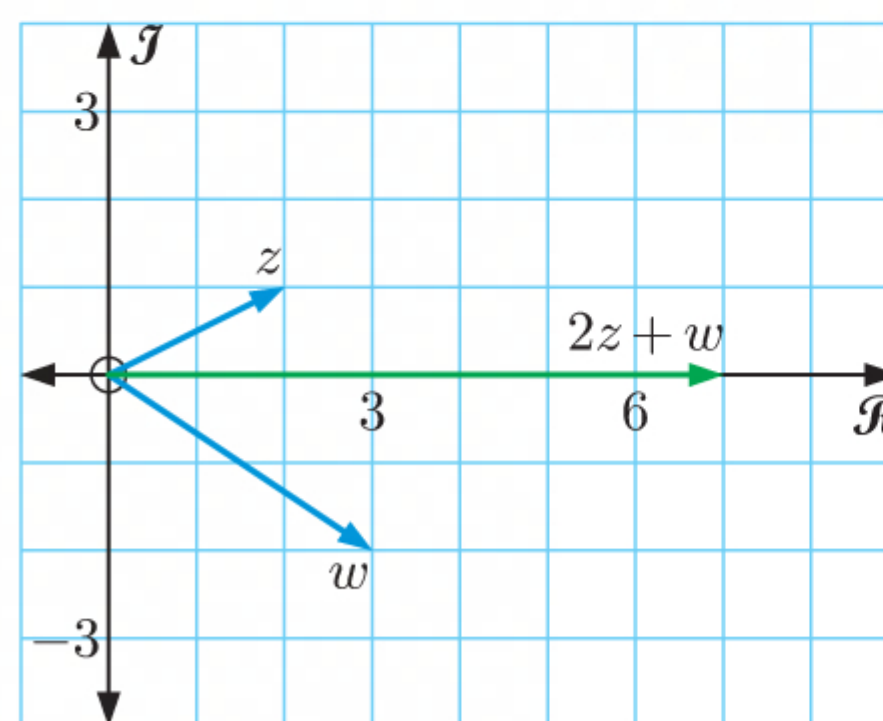
$$\begin{aligned}
 \text{So, } z &= 4 - 3i \quad \text{and} \quad w = (1 + i)(4 - 3i) \quad \{\text{using (3)}\} \\
 &= 4 - 3i + 4i - 3i^2 \\
 &= 7 + i
 \end{aligned}$$

$$\mathbf{85} \quad \sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^k \text{ is a geometric series with } u_1 = 1 \text{ and } r = \frac{i}{2}.$$

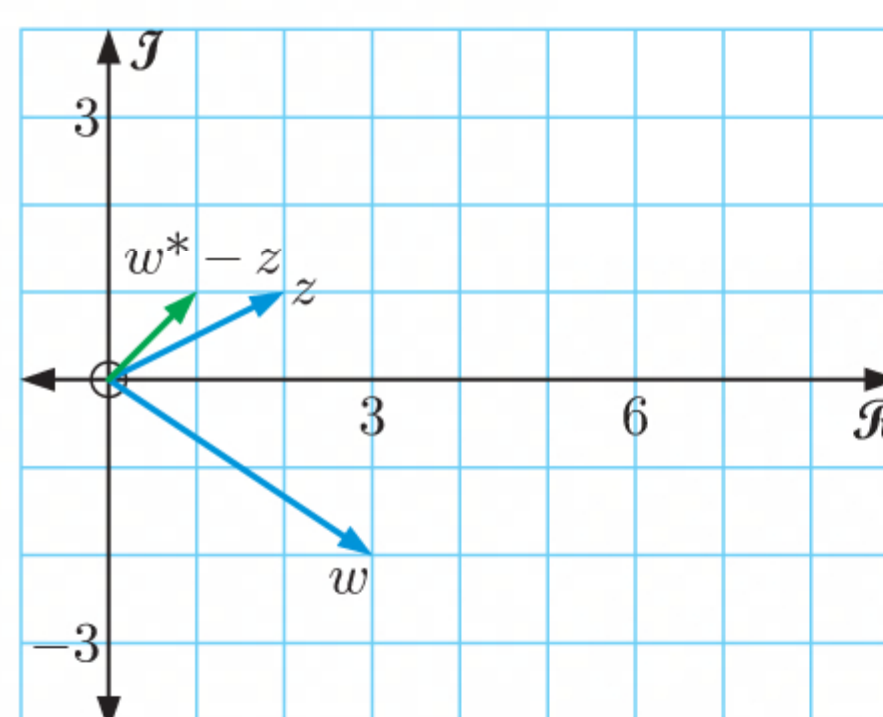
$$\begin{aligned}
 \text{Since } |r| = \left|\frac{i}{2}\right| = \frac{1}{2} < 1, \text{ the series converges to } & \frac{1}{1 - \frac{i}{2}} = \left(\frac{2}{2 - i}\right) \times \left(\frac{2 + i}{2 + i}\right) \\
 &= \frac{4 + 2i}{4 - i^2} \\
 &= \frac{4 + 2i}{5} \\
 &= \frac{4}{5} + \frac{2}{5}i
 \end{aligned}$$

86 $z = 2 + i, \quad w = 3 - 2i$

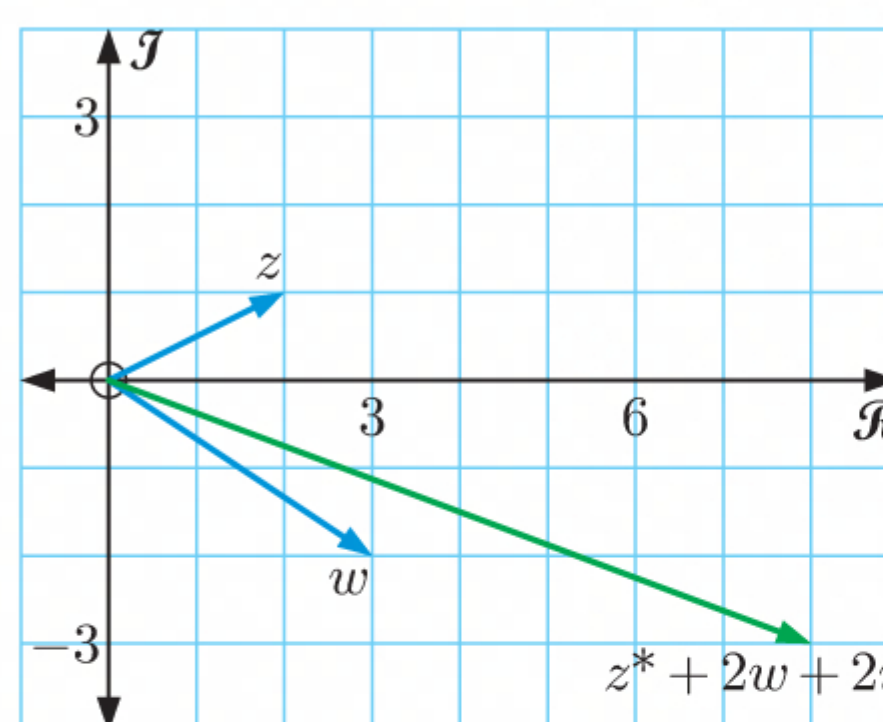
a $2z + w = 2(2 + i) + (3 - 2i)$
 $= 4 + 2i + 3 - 2i$
 $= 7$



b $w^* - z = 3 + 2i - (2 + i)$
 $= 3 + 2i - 2 - i$
 $= 1 + i$



c $z^* + 2w + 2i = 2 - i + 2(3 - 2i) + 2i$
 $= 2 - i + 6 - 4i + 2i$
 $= 8 - 3i$



87 $|z - 3| = |z - 1|$
 $\therefore |(x - 3) + yi| = |(x - 1) + yi|$
 $\therefore \sqrt{(x - 3)^2 + y^2} = \sqrt{(x - 1)^2 + y^2}$
 $\therefore (x - 3)^2 + \cancel{y^2} = (x - 1)^2 + \cancel{y^2}$
 $\therefore (x - 3)^2 = (x - 1)^2$
 $\therefore x^2 - 6x + 9 = x^2 - 2x + 1$
 $\therefore -4x = -8$
 $\therefore x = 2$ as required

88 $|z| = 3$

a $|3z| = 3|z|$
 $= 3(3)$
 $= 9$

b $|(2 + i)z| = |2 + i||z|$
 $= \sqrt{2^2 + 1^2}(3)$
 $= 3\sqrt{5}$

c $\left| \frac{2i}{z^2} \right| = \frac{|2i|}{|z^2|}$
 $= \frac{2|i|}{|z|^2}$
 $= \frac{2(1)}{3^2}$
 $= \frac{2}{9}$

89 $\frac{10}{z} + \frac{15}{z^*} = 5 + 2i$
 $\therefore 10z^* + 15z = (5 + 2i)zz^*$
 $\therefore 10z^* + 15z = (5 + 2i)|z|^2$
 $\therefore 10z^* + 15z = (5 + 2i) \times 5 \quad \{ |z| = \sqrt{5} \}$
 $\therefore 2z^* + 3z = 5 + 2i$

Letting $z = a + bi$, $2(a - bi) + 3(a + bi) = 5 + 2i$
 $\therefore 2a - 2bi + 3a + 3bi = 5 + 2i$
 $\therefore 5a + bi = 5 + 2i$

Equating real and imaginary coefficients, $5a = 5$ and $b = 2$
 $\therefore a = 1$ and $b = 2$

So, $z = 1 + 2i$.

90 $\arg\left(\frac{z}{z+2}\right) = \arg z - \arg(z+2) = \frac{\pi}{4}$, so we have the Argand diagram:

Let $x = |z+2|$.

Using the cosine rule, $2^2 = |z|^2 + x^2 - 2|z|x \cos \frac{\pi}{4}$

$$\therefore 4 = 1^2 + x^2 - \frac{2}{\sqrt{2}}x$$

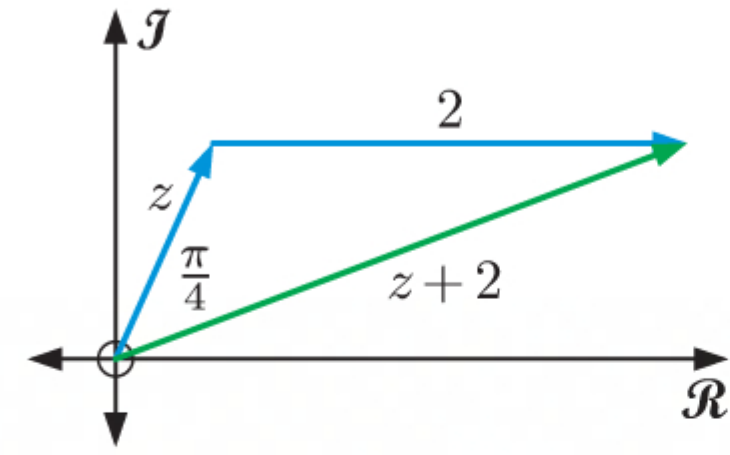
$$\therefore x^2 - \sqrt{2}x - 3 = 0$$

$$\therefore x = \frac{\sqrt{2} \pm \sqrt{2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{\sqrt{2} \pm \sqrt{14}}{2}$$

$$= \frac{\sqrt{2} + \sqrt{14}}{2} \quad \{x > 0\}$$

$$\text{So, } |z+2| = \frac{\sqrt{2} + \sqrt{14}}{2}.$$

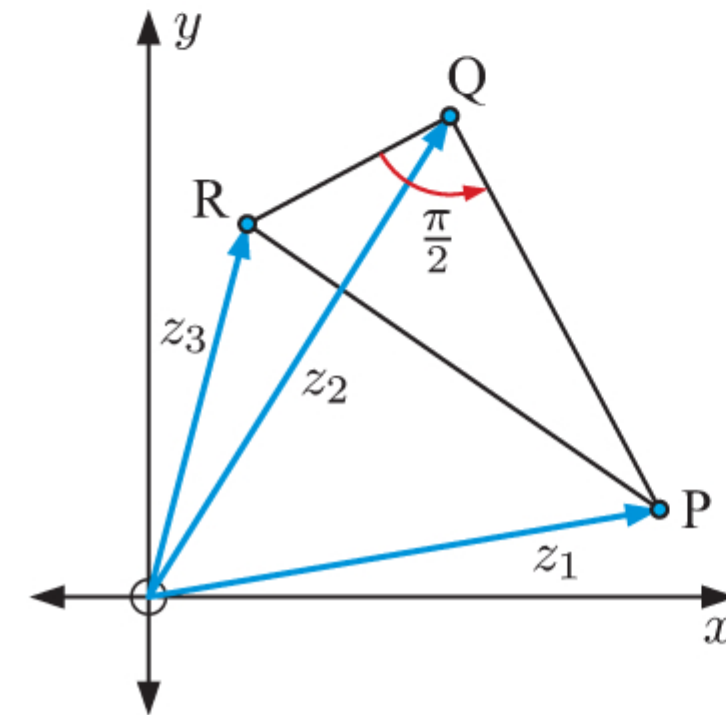


91 \overrightarrow{QP} represents $z_1 - z_2$ and \overrightarrow{QR} represents $z_3 - z_2$.

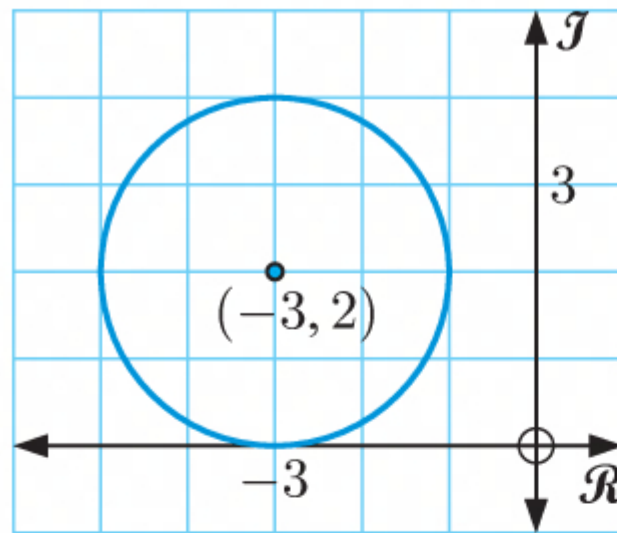
Geometrically, multiplication by i represents an anticlockwise rotation through $\frac{\pi}{2}$.

So, if $i(z_3 - z_2) = z_1 - z_2$, then $\widehat{PQR} = \frac{\pi}{2}$.

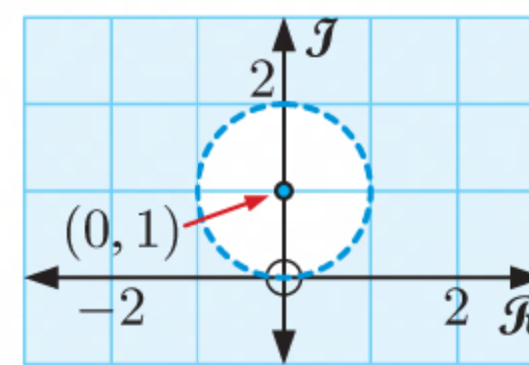
\therefore triangle PQR is right angled at Q.



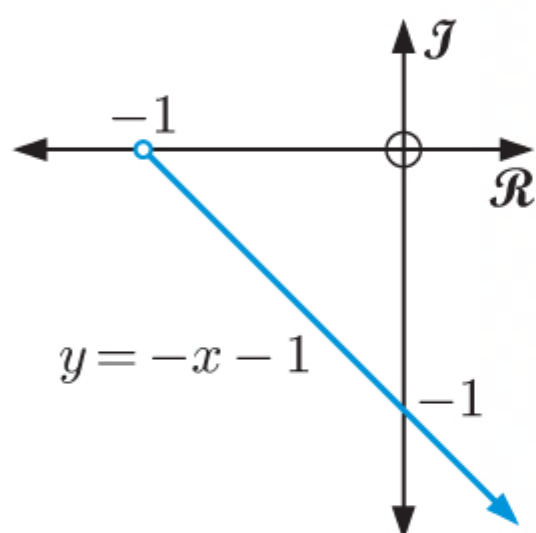
92 a $|z + 3 - 2i| = 2$
 $\therefore |z - (-3 + 2i)| = 2$



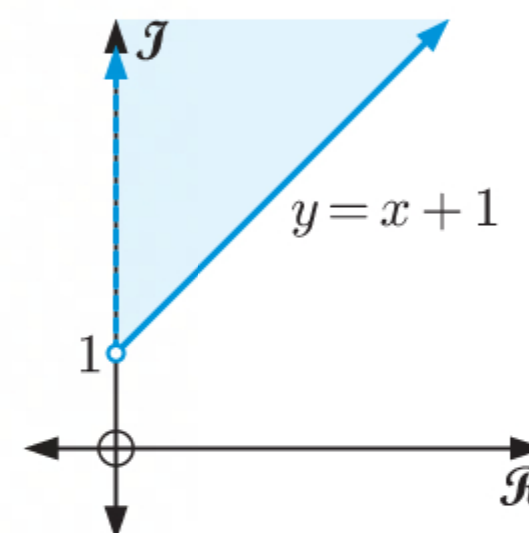
b $|z - i| > 1$



c $\arg(z + 1) = -\frac{\pi}{4}$



d $\frac{\pi}{4} \leq \arg(z - i) < \frac{\pi}{2}$



93 $z = r \operatorname{cis} \theta$

So, $z^4 = r^4 \operatorname{cis} 4\theta$ {De Moivre}

$$\frac{1}{z} = z^{-1}$$

$$= r^{-1} \operatorname{cis}(-\theta)$$
 {De Moivre}

and $iz^* = \operatorname{cis} \frac{\pi}{2} \times r \operatorname{cis}(-\theta)$

$$= r \operatorname{cis}\left(\frac{\pi}{2} - \theta\right) \quad \{\operatorname{cis} \theta \operatorname{cis} \phi = \operatorname{cis}(\theta + \phi)\}$$

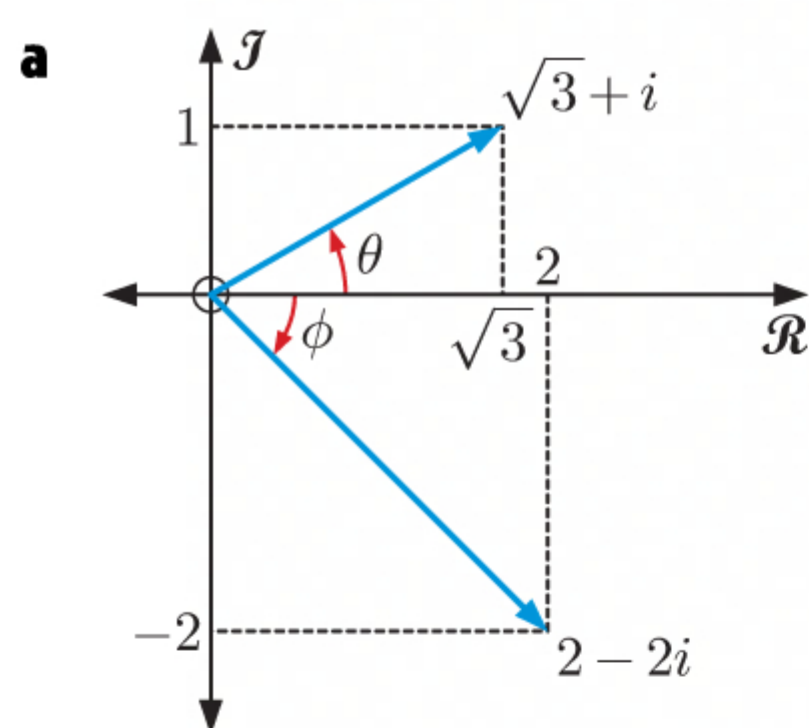
$$\begin{aligned}
 \text{94 a} \quad & 2 \operatorname{cis} \frac{\pi}{7} \operatorname{cis} \frac{6\pi}{7} \\
 &= 2 \operatorname{cis} \left(\frac{\pi}{7} + \frac{6\pi}{7} \right) \quad \{ \operatorname{cis} \theta \operatorname{cis} \phi = \operatorname{cis}(\theta + \phi) \} \\
 &= 2 \operatorname{cis} \pi \\
 &= 2(\cos \pi + i \sin \pi) \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{\sqrt{8} \operatorname{cis} \frac{3\pi}{16}}{\sqrt{2} \operatorname{cis}(-\frac{5\pi}{16})} \\
 &= \frac{2\sqrt{2}}{\sqrt{2}} \operatorname{cis} \left(\frac{3\pi}{16} - \left(-\frac{5\pi}{16} \right) \right) \quad \left\{ \frac{\operatorname{cis} \theta}{\operatorname{cis} \phi} = \operatorname{cis}(\theta - \phi) \right\} \\
 &= 2 \operatorname{cis} \frac{8\pi}{16} \\
 &= 2 \operatorname{cis} \frac{\pi}{2} \\
 &= 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \\
 &= 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (\operatorname{cis} \frac{5\pi}{12})^2 \\
 &= \operatorname{cis} \left(2 \times \frac{5\pi}{12} \right) \quad \{ (\operatorname{cis} \theta)^n = \operatorname{cis} n\theta \} \\
 &= \operatorname{cis} \frac{5\pi}{6} \\
 &= \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \\
 &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \operatorname{cis}(\theta + 15\pi) \\
 &= \operatorname{cis} \theta \operatorname{cis} 15\pi \quad \{ \operatorname{cis}(\theta + \phi) = \operatorname{cis} \theta \operatorname{cis} \phi \} \\
 &= \operatorname{cis} \theta (\operatorname{cis} \pi)^{15} \quad \{ \operatorname{cis} n\theta = (\operatorname{cis} \theta)^n \} \\
 &= (\cos \theta + i \sin \theta)(\cos \pi + i \sin \pi)^{15} \\
 &= (\cos \theta + i \sin \theta)(-1)^{15} \\
 &= -\cos \theta - i \sin \theta
 \end{aligned}$$

$$\text{95} \quad z = \sqrt{3} + i, \quad w = 2 - 2i$$



$$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\text{Now} \quad \tan \theta = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = \frac{\pi}{6}$$

$$\therefore \arg z = \frac{\pi}{6}$$

$$\therefore z = 2 \operatorname{cis} \frac{\pi}{6}$$

$$|w| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\text{Now} \quad \tan \phi = \frac{2}{2} = 1$$

$$\therefore \phi = \frac{\pi}{4}$$

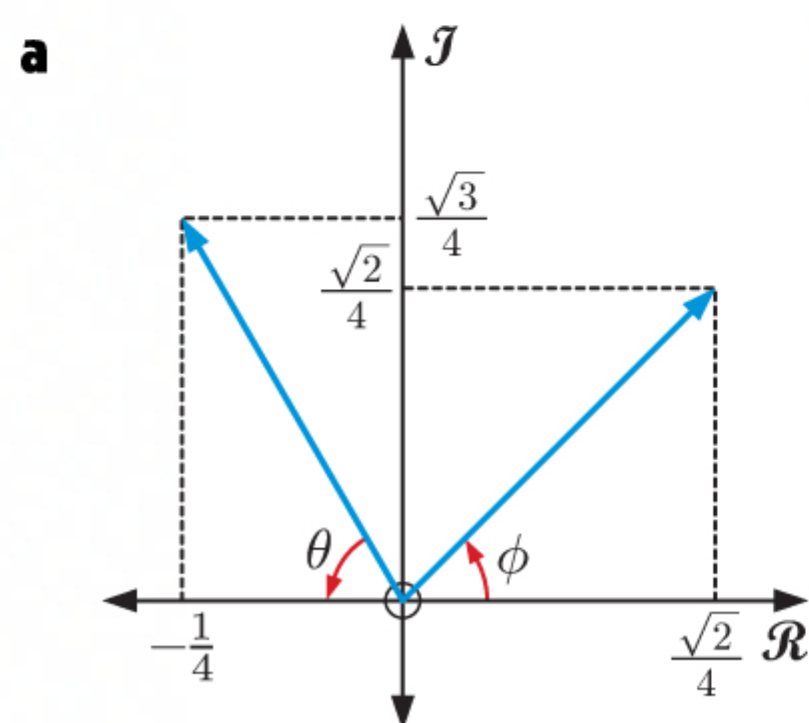
$$\therefore \arg w = -\frac{\pi}{4}$$

$$\therefore w = 2\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right)$$

$$\begin{aligned}
 \text{b} \quad & zw = 2 \operatorname{cis} \frac{\pi}{6} \times 2\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \\
 &= 4\sqrt{2} \operatorname{cis} \left(\frac{\pi}{6} - \frac{\pi}{4} \right) \\
 &= 4\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{12} \right)
 \end{aligned}$$

c When z is multiplied by w , it is dilated with scale factor $2\sqrt{2}$, then rotated clockwise through $\frac{\pi}{4}$ about the origin.

$$\text{96} \quad z = \frac{-1 + i\sqrt{3}}{4}, \quad w = \frac{\sqrt{2} + i\sqrt{2}}{4}$$



$$|z| = \sqrt{\left(-\frac{1}{4} \right)^2 + \left(\frac{\sqrt{3}}{4} \right)^2} = \sqrt{\frac{4}{16}} = \frac{2}{4} = \frac{1}{2}$$

$$\text{Now} \quad \tan \theta = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{4}}$$

$$= \sqrt{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore \arg z = \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

$$\therefore z = \frac{1}{2} \operatorname{cis} \frac{2\pi}{3}$$

$$|w| = \sqrt{\left(\frac{\sqrt{2}}{4} \right)^2 + \left(\frac{\sqrt{2}}{4} \right)^2} = \sqrt{\frac{4}{16}} = \frac{2}{4} = \frac{1}{2}$$

$$\text{Now} \quad \tan \phi = \frac{\frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{4}} = 1$$

$$\therefore \phi = \frac{\pi}{4}$$

$$\arg w = \frac{\pi}{4}$$

$$\therefore w = \frac{1}{2} \operatorname{cis} \frac{\pi}{4}$$

$$\begin{aligned}
 \text{b } zw &= \frac{1}{2} \operatorname{cis} \frac{2\pi}{3} \times \frac{1}{2} \operatorname{cis} \frac{\pi}{4} \\
 &= \frac{1}{4} \operatorname{cis} \left(\frac{2\pi}{3} + \frac{\pi}{4} \right) \\
 &= \frac{1}{4} \operatorname{cis} \frac{11\pi}{12} \\
 &= \frac{1}{4} \left(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right) \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } zw &= \left(\frac{-1 + i\sqrt{3}}{4} \right) \left(\frac{\sqrt{2} + i\sqrt{2}}{4} \right) \\
 &= \frac{-\sqrt{2} - i\sqrt{2} + i\sqrt{6} + i^2\sqrt{6}}{16} \\
 &= \frac{-\sqrt{2} - \sqrt{6}}{16} + \frac{\sqrt{6} - \sqrt{2}}{16}i
 \end{aligned}$$

$$\text{But } zw = \frac{1}{4} \left(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right) \quad \{\text{from b}\}$$

$$\therefore \frac{-\sqrt{2} - \sqrt{6}}{16} + \frac{\sqrt{6} - \sqrt{2}}{16}i = \frac{1}{4} \cos \frac{11\pi}{12} + \frac{1}{4}i \sin \frac{11\pi}{12}$$

Equating real and imaginary parts,

$$\begin{aligned}
 \frac{1}{4} \cos \frac{11\pi}{12} &= \frac{-\sqrt{2} - \sqrt{6}}{16} \quad \text{and} \quad \frac{1}{4} \sin \frac{11\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{16} \\
 \therefore \cos \frac{11\pi}{12} &= -\frac{\sqrt{2} + \sqrt{6}}{4} \quad \therefore \sin \frac{11\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{97 } z &= \cos \theta + i \sin \theta \\
 \therefore z^2 &= (\cos \theta + i \sin \theta)^2 \\
 &= \cos 2\theta + i \sin 2\theta
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } |z| &= \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \quad \text{and} \quad 0 < \theta < \frac{\pi}{4} \\
 \therefore |z^2| &= |z|^2 = 1 \quad \therefore 0 < 2\theta < \frac{\pi}{2}
 \end{aligned}$$

So, z and z^2 lie on the unit circle, and are in the first quadrant.

Let \overrightarrow{OP} represent z^2 ,
 \overrightarrow{OQ} represent 1, and
 \overrightarrow{OR} represent $1 - z^2$.

\therefore quadrilateral $OPQR$ is a parallelogram.

$$\begin{aligned}
 \text{Now } \widehat{OQR} &= \widehat{POQ} \quad \{\text{alternate angles}\} \\
 &= \theta + \theta \\
 &= 2\theta
 \end{aligned}$$

Using the cosine rule in $\triangle OQR$,

$$\begin{aligned}
 |1 - z^2|^2 &= 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos 2\theta \\
 &= 2 - 2 \cos 2\theta \\
 &= 2(1 - \cos 2\theta) \\
 &= 2(2 \sin^2 \theta) \\
 &= 4 \sin^2 \theta
 \end{aligned}$$

$$\therefore |1 - z^2| = 2 \sin \theta \quad \{|1 - z^2| > 0\}$$

$$\text{Using the sine rule in } \triangle OQR, \quad \frac{\sin \phi}{1} = \frac{\sin 2\theta}{2 \sin \theta}$$

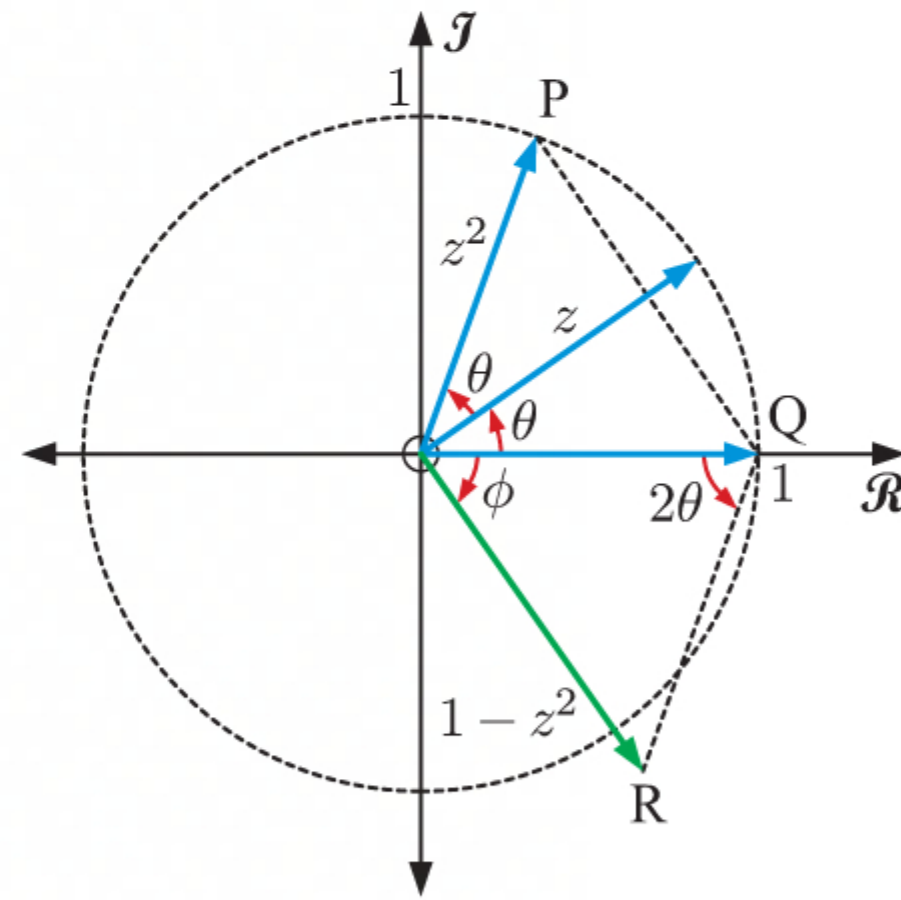
$$\therefore \sin \phi = \frac{2 \sin \theta \cos \theta}{2 \sin \theta}$$

$$\therefore \sin \phi = \cos \theta$$

$$\therefore \sin \phi = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\therefore \phi = \frac{\pi}{2} - \theta \quad \left\{ 0 < \phi < \frac{\pi}{2} \quad \text{and} \quad 0 < \frac{\pi}{2} - \theta < \frac{\pi}{2} \right\}$$

$$\therefore \arg(1 - z^2) = \theta - \frac{\pi}{2}$$



98 a $z_1 - z_2$ represents \overrightarrow{BA}

$z_3 - z_2$ represents \overrightarrow{BC}

b OABC is a cyclic quadrilateral.

$$\therefore \widehat{AOC} + \widehat{ABC} = \pi$$

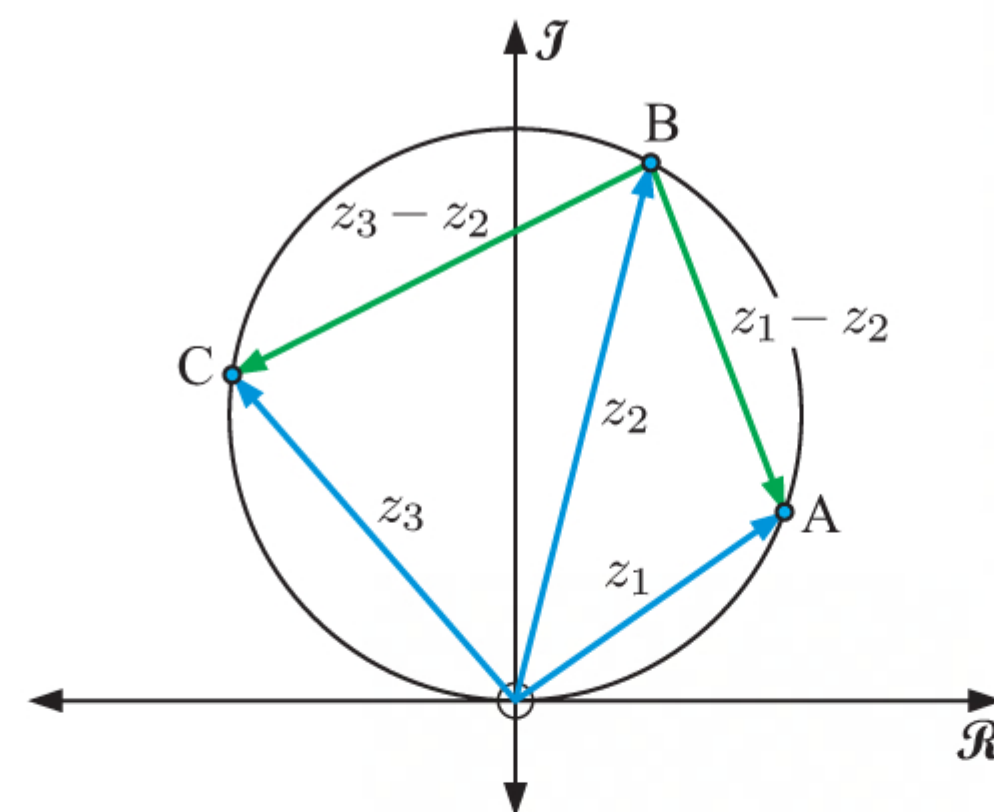
$$\text{Now } \widehat{AOC} = \arg z_3 - \arg z_1$$

$$= \arg\left(\frac{z_3}{z_1}\right)$$

$$\text{and } \widehat{ABC} = \arg(z_1 - z_2) - \arg(z_3 - z_2) \quad \{\text{using a}\}$$

$$= \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$$

$$\therefore \arg\left(\frac{z_3}{z_1}\right) + \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \pi$$



99 Sum of roots $= 3 \operatorname{cis} \frac{5\pi}{6} + 3 \operatorname{cis} \frac{7\pi}{6}$

$$= 3\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) + 3\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right)$$

$$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + 3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= -\frac{3\sqrt{3}}{2} + \frac{3}{2}i - \frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

$$= -3\sqrt{3}$$

Product of roots $= 3 \operatorname{cis} \frac{5\pi}{6} \times 3 \operatorname{cis} \frac{7\pi}{6}$

$$= 9 \operatorname{cis}\left(\frac{5\pi}{6} + \frac{7\pi}{6}\right)$$

$$= 9 \operatorname{cis} 2\pi$$

$$= 9(1 + 0i)$$

$$= 9$$

$$\therefore \text{the equations are } a(x^2 - (-3\sqrt{3})x + 9) = 0$$

$$\therefore a(x^2 + 3\sqrt{3}x + 9) = 0, \quad a \neq 0, \quad a \in \mathbb{R}.$$

100 a i $\widehat{AOD} = \frac{3\pi}{5}$ {OABCD is a regular pentagon}

$$\therefore z_1 = \operatorname{cis} \frac{3\pi}{5}$$

ii $z_2 - z_1$ represents \overrightarrow{AB}

$$\text{Now } \widehat{OAC} = \pi - \widehat{AOD} \quad \{\text{supplementary angles}\}$$

$$= \pi - \frac{3\pi}{5}$$

$$= \frac{2\pi}{5}$$

$$\text{and } \widehat{BAC} = \widehat{OAB} - \widehat{OAC}$$

$$= \frac{3\pi}{5} - \frac{2\pi}{5}$$

$$= \frac{\pi}{5}$$

$$\therefore z_2 - z_1 = \operatorname{cis} \frac{\pi}{5}$$

iii $\widehat{COD} = \widehat{OCD}$ {base angles of isosceles $\triangle OCD$ }

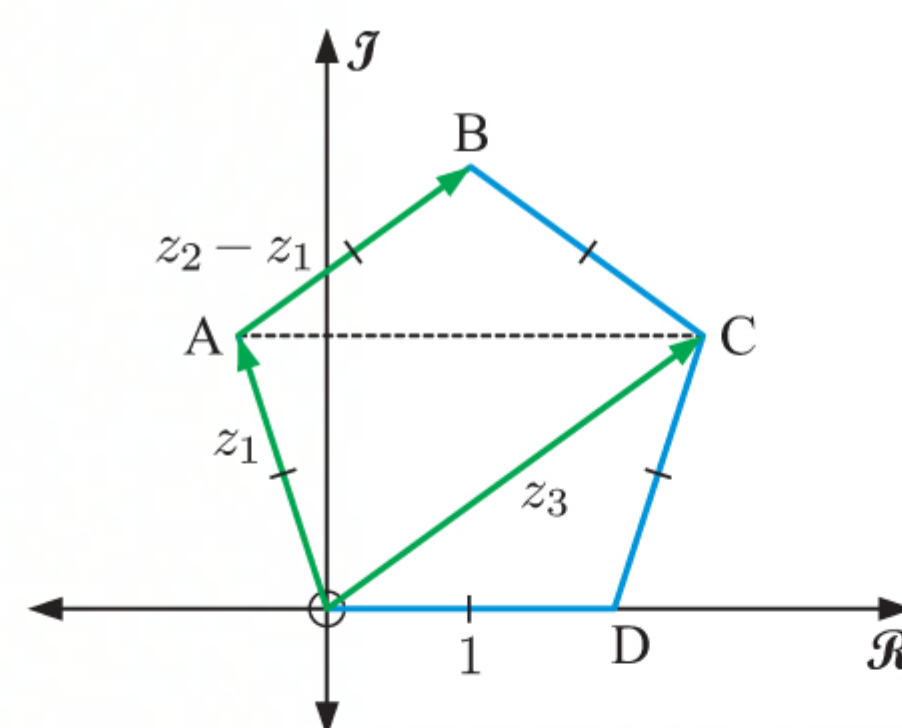
$$\text{Now } \widehat{COD} = \pi - \widehat{ODC} - \widehat{OCD}$$

$$\therefore \widehat{COD} = \pi - \frac{3\pi}{5} - \widehat{COD}$$

$$\therefore 2\widehat{COD} = \frac{2\pi}{5}$$

$$\therefore \widehat{COD} = \frac{\pi}{5}$$

$$\text{So, } \arg(z_3) = \frac{\pi}{5}$$



Using the cosine rule in $\triangle OCD$, $OC^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos \frac{3\pi}{5}$

$$\begin{aligned} &= 2 - 2 \cos \frac{3\pi}{5} \\ &= 2(1 - \cos(2 \times \frac{3\pi}{10})) \\ &= 2(2 \sin^2(\frac{3\pi}{10})) \\ &= 4 \sin^2(\frac{3\pi}{10}) \end{aligned}$$

$$\therefore \text{OC} = 2 \sin \frac{3\pi}{10} \quad \{\text{OC} > 0\}$$

So, $|z_3| = 2 \sin \frac{3\pi}{10}$

$$\therefore z_3 = 2 \sin \frac{3\pi}{10} \operatorname{cis} \frac{\pi}{5}$$

b $\widehat{AOD} = \frac{3\pi}{5} = \widehat{OAB}$

$$\hat{A}\hat{O}B = \hat{A}\hat{B}O \quad \{\text{base angles of isosceles } \triangle OAB\}$$

Now $\hat{A}\hat{O}\hat{B} = \pi - \hat{O}\hat{A}\hat{B} - \hat{A}\hat{B}\hat{O}$

$$\therefore \hat{A\hat{O}B} = \pi - \frac{3\pi}{5} - \hat{A\hat{O}B}$$

$$\therefore 2 \angle \hat{AOB} = \frac{2\pi}{5}$$

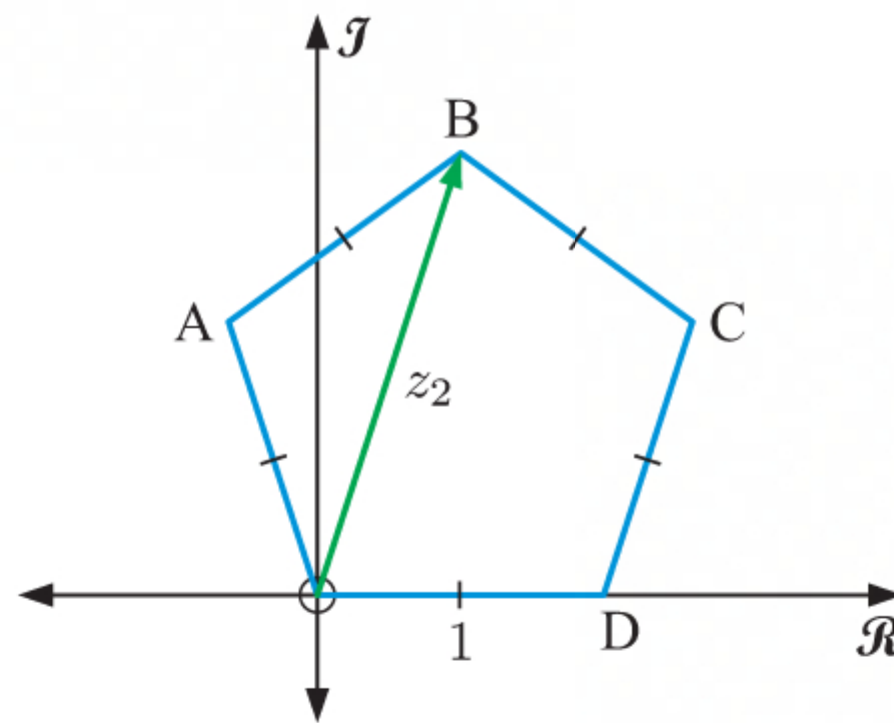
$$\therefore \hat{A\hat{O}B} = \frac{\pi}{5}$$

So, $\widehat{BOD} = \widehat{AOD} - \widehat{AOB}$

$$= \frac{3\pi}{5} - \frac{\pi}{5}$$

$$= \frac{2\pi}{5}$$

$$\therefore \arg(z_2) = \frac{2\pi}{5}$$



Using the cosine rule in $\triangle OAB$, $OB^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos \frac{3\pi}{5}$

$$= 2 - 2 \cos \frac{3\pi}{5}$$

$$= 4 \sin^2 \left(\frac{3\pi}{10} \right) \quad \{\text{from **a iii**}\}$$

$$\therefore \text{OB} = 2 \sin \frac{3\pi}{10} \quad \{\text{OB} > 0\}$$

So, $|z_2| = 2 \sin \frac{3\pi}{10}$

$$\therefore z_2 = 2 \sin \frac{3\pi}{10} \operatorname{cis} \frac{2\pi}{5}$$

$$\begin{aligned}\text{Now } z_2^n &= \left(2 \sin \frac{3\pi}{10} \operatorname{cis} \frac{2\pi}{5}\right)^n \\ &= \left(2 \sin \frac{3\pi}{10}\right)^n \operatorname{cis} \frac{2\pi n}{5} \\ &= \left(2 \sin \frac{3\pi}{10}\right)^n \left(\cos \frac{2\pi n}{5} + i \sin \frac{2\pi n}{5}\right)\end{aligned}$$

So, z_2^n is real when $\sin \frac{2\pi n}{5} = 0$

$$\therefore \frac{2\pi n}{5} = k\pi, \quad k \in \mathbb{R}$$

$$\therefore n = \frac{5}{2}k, \quad k \in \mathbb{R}$$

\therefore the smallest positive integer n such that z_2^n is a real number is $n = 5$ when $k = 2$.

c We have $z_1^5 = \left(\text{cis } \frac{3\pi}{5}\right)^5$ {from **a i**}

$$= \text{cis } 3\pi$$

$$= \cos 3\pi + i \sin 3\pi$$

$$= -1$$

$$z_2^5 = \left(2 \sin \frac{3\pi}{10} \operatorname{cis} \frac{2\pi}{5}\right)^5 \quad \{\text{from } \mathbf{b}\}$$

$$= 32 \sin^5\left(\frac{3\pi}{10}\right) \operatorname{cis} 2\pi$$

$$= 32 \sin^5\left(\frac{3\pi}{10}\right)(\cos 2\pi + i \sin 2\pi)$$

$$= 32 \sin^5\left(\frac{3\pi}{10}\right)$$

$$z_3^5 = \left(2 \sin \frac{3\pi}{10} \operatorname{cis} \frac{\pi}{5}\right)^5$$

$$= 32 \sin^5\left(\frac{3\pi}{10}\right) \operatorname{cis} \pi$$

$$= 32 \sin^5\left(\frac{3\pi}{10}\right)(\cos \pi + i \sin \pi)$$

$$= -32 \sin^5\left(\frac{3\pi}{10}\right)$$

$$z_4^5 = 1^5$$

$$= 1$$

$$\therefore z_1^5 + z_2^5 + z_3^5 + z_4^5 = -1 + 32 \sin^5\left(\frac{3\pi}{10}\right) + (-32 \sin^5\left(\frac{3\pi}{10}\right)) + 1 = 0 \quad \text{as required}$$

$$\begin{aligned}
 \mathbf{101} \quad \mathbf{a} \quad & \left| \sqrt{3} + i \right| = 2, \quad \arg(\sqrt{3} + i) = \frac{\pi}{6} \\
 & \therefore \sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6} \quad (\text{polar form}) \\
 & \quad = 2e^{i\frac{\pi}{6}} \quad (\text{Euler form})
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 5e^{-i\frac{\pi}{4}} = 5 \cos\left(-\frac{\pi}{4}\right) + 5i \sin\left(-\frac{\pi}{4}\right) \\
 & = \frac{5}{\sqrt{2}} - \frac{5}{\sqrt{2}}i \quad (\text{Cartesian form}) \\
 & = 5 \operatorname{cis}\left(-\frac{\pi}{4}\right) \quad (\text{polar form})
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{102} \quad \mathbf{a} \quad & |1 + i| = \sqrt{2}, \quad \arg(1 + i) = \frac{\pi}{4} \\
 & \therefore 1 + i = \sqrt{2}e^{i\frac{\pi}{4}} \\
 & \left| \sqrt{3} - i \right| = 2, \quad \arg(\sqrt{3} - i) = -\frac{\pi}{6} \\
 & \therefore \sqrt{3} - i = 2e^{-i\frac{\pi}{6}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 2 \operatorname{cis} \frac{5\pi}{6} = 2 \cos \frac{5\pi}{6} + 2i \sin \frac{5\pi}{6} \\
 & = -\sqrt{3} + i \quad (\text{Cartesian form}) \\
 & = 2e^{i\frac{5\pi}{6}} \quad (\text{Euler form})
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & z = \frac{-1 - i}{\sqrt{3} - i} \\
 & = \frac{-(1 + i)}{\sqrt{3} - i} \\
 & = -\frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{6}}} \quad \{\text{using a}\} \\
 & = -\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4} + i\frac{\pi}{6}} \\
 & = -\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & z = -\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}} \\
 & \therefore z^n = \left(-\frac{1}{\sqrt{2}}\right)^n e^{i\frac{5\pi n}{12}} \\
 & \quad = \left(-\frac{1}{\sqrt{2}}\right)^n \left(\cos \frac{5\pi n}{12} + i \sin \frac{5\pi n}{12}\right)
 \end{aligned}$$

So, z^n is real when $\sin \frac{5\pi n}{12} = 0$

$$\therefore \frac{5\pi n}{12} = k\pi, \quad k \in \mathbb{Z}$$

$$\therefore n = \frac{12}{5}k, \quad k \in \mathbb{Z}$$

\therefore the smallest positive integer n such that z^n is a real number is $n = 12$ when $k = 5$.

$$\mathbf{103} \quad \mathbf{a} \quad \left| 1 + i\sqrt{3} \right| = 2, \quad \arg(1 + i\sqrt{3}) = \frac{\pi}{3}$$

$$\begin{aligned}
 \therefore 1 + i\sqrt{3} &= 2 \operatorname{cis} \frac{\pi}{3} \\
 |1 + i| &= \sqrt{2}, \quad \arg(1 + i) = \frac{\pi}{4} \\
 \therefore 1 + i &= \sqrt{2} \operatorname{cis} \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } z &= \frac{1 + i\sqrt{3}}{1 + i} \\
 &= \frac{2 \operatorname{cis} \frac{\pi}{3}}{\sqrt{2} \operatorname{cis} \frac{\pi}{4}} \\
 &= \sqrt{2} \operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
 &= \sqrt{2} \operatorname{cis} \frac{\pi}{12}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & z = \sqrt{2} \operatorname{cis} \frac{\pi}{12} \\
 \therefore z^n &= \left(\sqrt{2} \operatorname{cis} \frac{\pi}{12}\right)^n \\
 &= (\sqrt{2})^n \operatorname{cis} \frac{n\pi}{12} \\
 &= (\sqrt{2})^n \left(\cos \frac{n\pi}{12} + i \sin \frac{n\pi}{12}\right)
 \end{aligned}$$

\mathbf{i} z^n is real when $\sin \frac{n\pi}{12} = 0$

$$\therefore \frac{n\pi}{12} = k\pi, \quad k \in \mathbb{Z}$$

$$\therefore n = 12k, \quad k \in \mathbb{Z}$$

\therefore the smallest positive value of n such that z^n is real is $n = 12$ when $k = 1$.

\mathbf{ii} z^n is purely imaginary when $\cos \frac{n\pi}{12} = 0$

$$\therefore \frac{n\pi}{12} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore n = 6 + 12k, \quad k \in \mathbb{Z}$$

\therefore the smallest positive value of n such that z^n is purely imaginary is $n = 6$ when $k = 0$.

$$\begin{aligned}
 104 \quad z &= \left(\frac{-1+5i}{2+3i} \right) \times \left(\frac{2-3i}{2-3i} \right) \\
 &= \frac{-2+3i+10i-15i^2}{4-9i^2} \\
 &= \frac{13+13i}{13} \\
 &= 1+i
 \end{aligned}$$

Now $|1+i| = \sqrt{2}$ and $\arg(1+i) = \frac{\pi}{4}$

$$\begin{aligned}
 \therefore z &= \sqrt{2} \operatorname{cis} \frac{\pi}{4} \\
 &= 2^{\frac{1}{2}} \operatorname{cis} \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 \therefore z^{12} &= \left(2^{\frac{1}{2}} \operatorname{cis} \frac{\pi}{4} \right)^{12} \\
 &= 2^6 \operatorname{cis} \left(12 \times \frac{\pi}{4} \right) \quad \{\text{De Moivre}\} \\
 &= 64 \operatorname{cis} 3\pi \\
 &= 64(\cos 3\pi + i \sin 3\pi) \\
 &= -64 \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 105 \quad a \quad \left(\sqrt{5} \operatorname{cis} \frac{\pi}{8} \right)^6 &= \left(\sqrt{5} \right)^6 \operatorname{cis} \frac{6\pi}{8} \quad \{\text{De Moivre}\} \\
 &= 5^3 \operatorname{cis} \frac{3\pi}{4} \\
 &= 125 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \\
 &= 125 \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \\
 &= -\frac{125}{\sqrt{2}} + \frac{125}{\sqrt{2}}i
 \end{aligned}$$

$$\begin{aligned}
 b \quad |\sqrt{3}-i| &= 2, \quad \arg(\sqrt{3}-i) = -\frac{\pi}{6} \\
 \therefore \sqrt{3}-i &= 2 \operatorname{cis} \left(-\frac{\pi}{6} \right) \\
 \therefore (\sqrt{3}-i)^5 &= \left(2 \operatorname{cis} \left(-\frac{\pi}{6} \right) \right)^5 \\
 &= 2^5 \operatorname{cis} \left(-\frac{5\pi}{6} \right) \quad \{\text{De Moivre}\} \\
 &= 32 \left(\cos \left(-\frac{5\pi}{6} \right) + i \sin \left(-\frac{5\pi}{6} \right) \right) \\
 &= 32 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) \\
 &= -16\sqrt{3} - 16i
 \end{aligned}$$

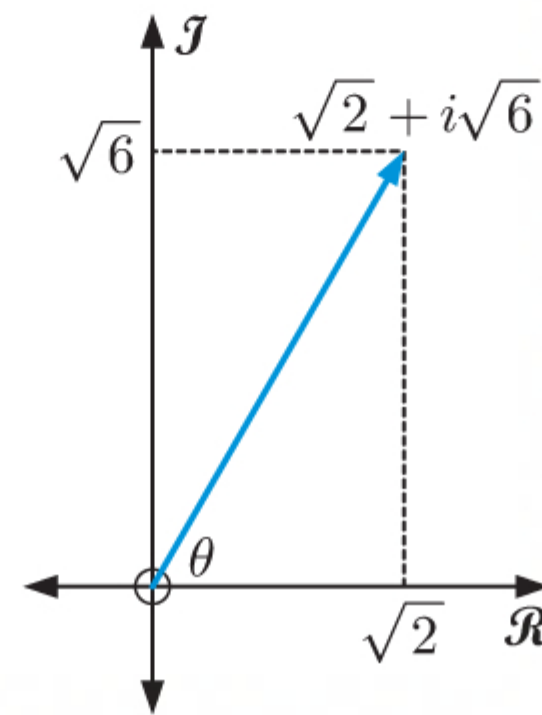
$$c \quad |\sqrt{2} + i\sqrt{6}| = \sqrt{(\sqrt{2})^2 + (\sqrt{6})^2} = \sqrt{8} = 2\sqrt{2}$$

$$\begin{aligned}
 \text{Now } \tan \theta &= \frac{\sqrt{6}}{\sqrt{2}} \\
 &= \sqrt{3} \\
 \therefore \theta &= \frac{\pi}{3}
 \end{aligned}$$

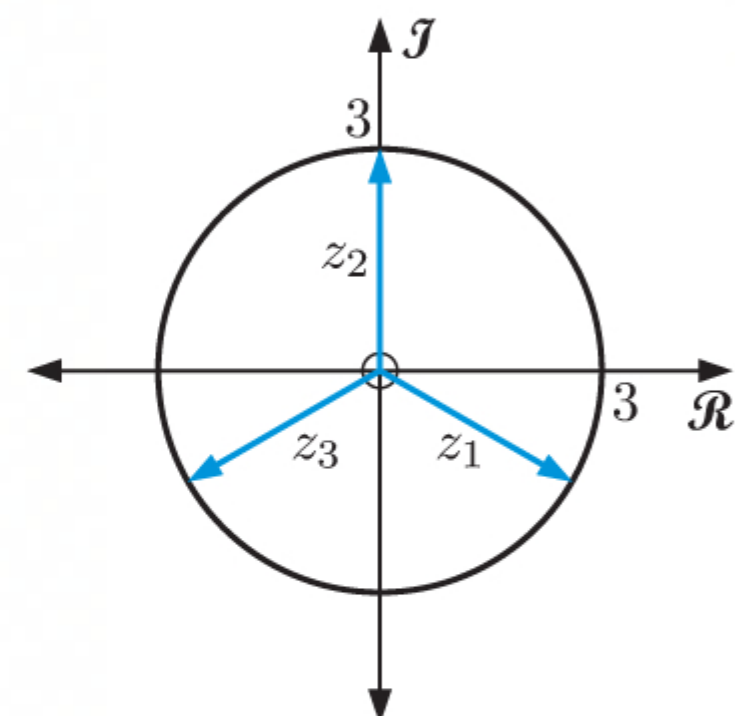
$$\therefore \arg(\sqrt{2} + i\sqrt{6}) = \frac{\pi}{3}$$

$$\text{So, } \sqrt{2} + i\sqrt{6} = 2\sqrt{2} \operatorname{cis} \frac{\pi}{3}$$

$$\begin{aligned}
 \therefore \left(\sqrt{2} + i\sqrt{6} \right)^{\frac{1}{2}} &= \left(2\sqrt{2} \operatorname{cis} \frac{\pi}{3} \right)^{\frac{1}{2}} \\
 &= \left(2^{\frac{3}{2}} \right)^{\frac{1}{2}} \operatorname{cis} \frac{\pi}{6} \quad \{\text{De Moivre}\} \\
 &= 2^{\frac{3}{4}} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \\
 &= 2^{\frac{3}{4}} \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \\
 &= 2^{-\frac{1}{4}}\sqrt{3} + 2^{-\frac{1}{4}}i
 \end{aligned}$$



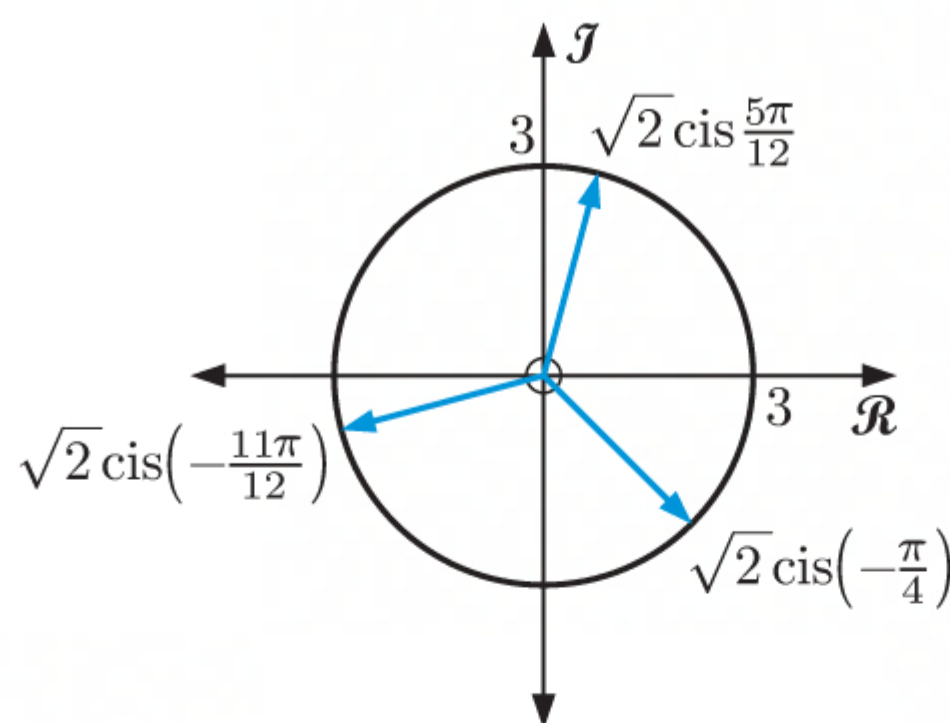
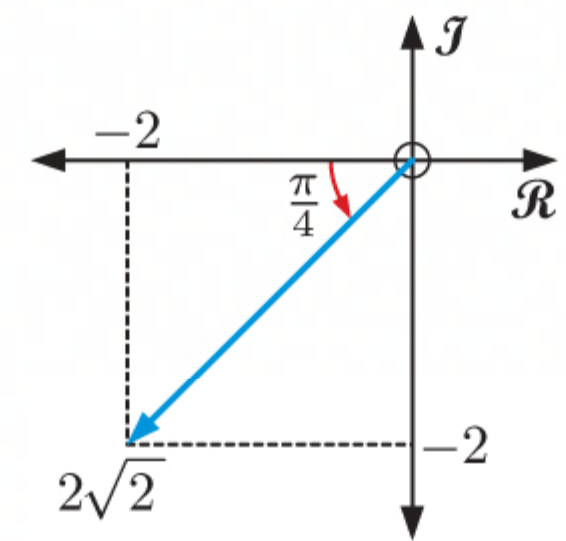
$$\begin{aligned}
 106 \quad a \quad z^3 &= -27i \\
 \therefore z^3 &= 27 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\} \\
 \therefore z &= \left(27 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right) \right)^{\frac{1}{3}} \\
 \therefore z &= 27^{\frac{1}{3}} \operatorname{cis} \left(-\frac{\pi}{6} + \frac{k2\pi}{3} \right) \quad \{\text{De Moivre}\} \\
 \therefore z &= 3 \operatorname{cis} \left(-\frac{\pi}{6} \right), 3 \operatorname{cis} \frac{\pi}{2}, 3 \operatorname{cis} \frac{7\pi}{6} \quad \{\text{letting } k = 0, 1, 2\} \\
 \therefore z &= 3 \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right), 3(0+i), 3 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) \\
 \therefore z &= \frac{3\sqrt{3}}{2} - \frac{3}{2}i, 3i, -\frac{3\sqrt{3}}{2} - \frac{3}{2}i \\
 \text{So, } z_1 &= \frac{3\sqrt{3}}{2} - \frac{3}{2}i, z_2 = 3i, \text{ and } z_3 = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{b} \quad z_2 z_3 &= 3i \left(-\frac{3\sqrt{3}}{2} - \frac{3}{2}i \right) \\
 &= -\frac{9\sqrt{3}}{2}i - \frac{9}{2}i^2 \\
 &= \frac{9}{2} - \frac{9\sqrt{3}}{2}i \\
 z_1^2 &= \left(\frac{3\sqrt{3}}{2} - \frac{3}{2}i \right)^2 \\
 &= \frac{27}{4} - \frac{9\sqrt{3}}{2}i + \frac{9}{4}i^2 \\
 &= \frac{18}{4} - \frac{9\sqrt{3}}{2}i \\
 &= \frac{9}{2} - \frac{9\sqrt{3}}{2}i \\
 &= z_2 z_3 \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad z_1 z_2 z_3 &= z_1 (z_2 z_3) \\
 &= z_1 (z_1^2) \quad \{\text{from } \mathbf{b}\} \\
 &= z_1^3 \\
 &= -27i \quad \{\text{as } z_1 \text{ is a solution to } z^3 = -27i\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{107} \quad \mathbf{a} \quad z^3 &= -2 - 2i \\
 \therefore z^3 &= 2\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4} + k2\pi\right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\} \\
 \therefore z &= \left[2\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4} + k2\pi\right) \right]^{\frac{1}{3}} \\
 \therefore z &= \left(2\sqrt{2} \right)^{\frac{1}{3}} \operatorname{cis}\left(-\frac{\pi}{4} + \frac{k2\pi}{3}\right) \quad \{\text{De Moivre}\} \\
 \therefore z &= \sqrt{2} \operatorname{cis}\left(-\frac{11\pi}{12}\right), \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right), \sqrt{2} \operatorname{cis}\frac{5\pi}{12} \quad \{\text{letting } k = -1, 0, 1\}
 \end{aligned}$$



b The sum of the cube roots of any complex number is 0.

$$\begin{aligned}
 \therefore \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right) + \sqrt{2} \operatorname{cis}\frac{5\pi}{12} + \sqrt{2} \operatorname{cis}\left(-\frac{11\pi}{12}\right) &= 0 \\
 \therefore \operatorname{cis}\left(-\frac{\pi}{4}\right) + \operatorname{cis}\frac{5\pi}{12} + \operatorname{cis}\left(-\frac{11\pi}{12}\right) &= 0 \\
 \therefore \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) + \cos\frac{5\pi}{12} + i \sin\frac{5\pi}{12} + \cos\left(-\frac{11\pi}{12}\right) + i \sin\left(-\frac{11\pi}{12}\right) &= 0 \\
 \therefore \left(\cos\left(-\frac{\pi}{4}\right) + \cos\frac{5\pi}{12} + \cos\left(-\frac{11\pi}{12}\right) \right) + i \left(\sin\left(-\frac{\pi}{4}\right) + \sin\frac{5\pi}{12} + \sin\left(-\frac{11\pi}{12}\right) \right) &= 0 \\
 \text{Equating real parts, } \cos\left(-\frac{\pi}{4}\right) + \cos\frac{5\pi}{12} + \cos\left(-\frac{11\pi}{12}\right) &= 0 \\
 \therefore \cos\frac{\pi}{4} + \cos\frac{5\pi}{12} + \cos\frac{13\pi}{12} &= 0 \quad \{\cos(-\theta) = \cos\theta, \cos\theta = \cos(\theta + 2\pi)\}
 \end{aligned}$$

$$\mathbf{108} \quad \mathbf{a} \quad \begin{cases} x - y = 3 \\ 3x - 3y = 5 \end{cases} \quad \text{which is} \quad \begin{cases} x - y = 3 \\ x - y = \frac{5}{3} \end{cases}$$

This system is inconsistent as $x - y$ cannot be equal to both 3 and $\frac{5}{3}$ simultaneously.

$$\mathbf{b} \quad \begin{cases} 2x + y + z = 4 \\ x - y - z = 2 \end{cases}$$

This system is consistent as $x = 2, y = 0, z = 0$ is a solution.

$$\text{Check: } 2(2) + 0 + 0 = 4 \quad \text{and} \quad 2 - 0 - 0 = 2 \quad \checkmark$$

109 a
$$\begin{cases} x + 3y = 6 \\ 2x + 7y = 13 \end{cases}$$

In augmented matrix form, the system is

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 2 & 7 & | & 13 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 6 \\ 0 & 1 & | & 1 \end{pmatrix} \quad R_2 - 2R_1 \rightarrow R_2 \leftarrow \left\{ \begin{array}{ccc} 2 & 7 & 13 \\ -2 & -6 & -12 \\ \hline 0 & 1 & 1 \end{array} \right\}$$

Using row 2, $y = 1$

Substituting into row 1, $x + 3(1) = 6$

$$\therefore x = 3$$

\therefore the solution is $x = 3, y = 1$.

b
$$\begin{cases} x - 4y = 2 \\ 3x + 5y = -11 \end{cases}$$

In augmented matrix form, the system is

$$\begin{pmatrix} 1 & -4 & | & 2 \\ 3 & 5 & | & -11 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & | & 2 \\ 0 & 17 & | & -17 \end{pmatrix} \quad R_2 - 3R_1 \rightarrow R_2 \leftarrow \left\{ \begin{array}{ccc} 3 & 5 & -11 \\ -3 & 12 & -6 \\ \hline 0 & 17 & -17 \end{array} \right\}$$

$$\sim \begin{pmatrix} 1 & -4 & | & 2 \\ 0 & 1 & | & -1 \end{pmatrix} \quad \frac{1}{17}R_2 \rightarrow R_2$$

Using row 2, $y = -1$

Substituting into row 1, $x - 4(-1) = 2$

$$\therefore x = -2$$

\therefore the solution is $x = -2, y = -1$.

c
$$\begin{cases} 5x + y = 2 \\ 2x - 3y = 7 \end{cases}$$

In augmented matrix form, the system is

$$\begin{pmatrix} 5 & 1 & | & 2 \\ 2 & -3 & | & 7 \end{pmatrix} \sim \begin{pmatrix} 5 & 1 & | & 2 \\ 0 & -17 & | & 31 \end{pmatrix} \quad 5R_2 - 2R_1 \rightarrow R_2 \leftarrow \left\{ \begin{array}{ccc} 10 & -15 & 35 \\ -10 & -2 & -4 \\ \hline 0 & -17 & 31 \end{array} \right\}$$

$$\sim \begin{pmatrix} 5 & 1 & | & 2 \\ 0 & 1 & | & -\frac{31}{17} \end{pmatrix} \quad -\frac{1}{17}R_2 \rightarrow R_2$$

Using row 2, $y = -\frac{31}{17}$

Substituting into row 1, $5x + (-\frac{31}{17}) = 2$

$$\therefore 5x = \frac{65}{17}$$

$$\therefore x = \frac{13}{17}$$

\therefore the solution is $x = \frac{13}{17}, y = -\frac{31}{17}$.

$$110 \quad \begin{cases} 2x - y = 4 \\ 6x + ky = 12 \end{cases}$$

a The system has augmented matrix $\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 6 & k & 12 \end{array} \right)$.

$$\begin{aligned} \mathbf{b} \quad & \left(\begin{array}{cc|c} 2 & -1 & 4 \\ 6 & k & 12 \end{array} \right) \\ & \sim \left(\begin{array}{cc|c} 2 & -1 & 4 \\ 0 & k+3 & 0 \end{array} \right) \quad R_2 - 3R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 6 & k & 12 \\ -6 & 3 & -12 \\ 0 & k+3 & 0 \end{array} \right\} \end{aligned}$$

The system has infinitely many solutions if the last row is all zeros. This occurs when $k = -3$.

In this case, we let $y = t$.

Using row 1, $2x - t = 4$

$$\therefore 2x = t + 4$$

$$\therefore x = \frac{1}{2}t + 2$$

\therefore the solutions have the form $x = \frac{1}{2}t + 2$, $y = t$, where $t \in \mathbb{R}$.

c Suppose $k \neq 3$.

Using row 2, $(k+3)y = 0$

$$\therefore y = 0 \quad \{k+3 \neq 0\}$$

Substituting into row 1, $2x - 0 = 4$

$$\therefore x = 2$$

\therefore the unique solution is $x = 2$, $y = 0$, for $k \neq 3$, $k \in \mathbb{R}$.

$$111 \quad \mathbf{a} \quad \begin{cases} x + 3y - 4z = -5 \\ 2x + y + z = 7 \\ x - 4y + 2z = -1 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left(\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 2 & 1 & 1 & 7 \\ 1 & -4 & 2 & -1 \end{array} \right) \\ & \sim \left(\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & -5 & 9 & 17 \\ 0 & -7 & 6 & 4 \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 2 & 1 & 1 & 7 \\ -2 & -6 & 8 & 10 \\ 0 & -5 & 9 & 17 \end{array} \right\} \\ & \sim \left(\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & 1 & -\frac{9}{5} & -\frac{17}{5} \\ 0 & 0 & -33 & -99 \end{array} \right) \quad \begin{array}{l} -\frac{1}{5}R_2 \rightarrow R_2 \\ 5R_3 - 7R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 1 & -4 & 2 & -1 \\ -1 & -3 & 4 & 5 \\ 0 & -7 & 6 & 4 \end{array} \right\} \\ & \sim \left(\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & 1 & -\frac{9}{5} & -\frac{17}{5} \\ 0 & 0 & 1 & 3 \end{array} \right) \quad \begin{array}{l} -\frac{1}{33}R_3 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 0 & -35 & 30 & 20 \\ 0 & 35 & -63 & -119 \\ 0 & 0 & -33 & -99 \end{array} \right\} \end{aligned}$$

Using row 3, $z = 3$

Substituting into row 2, $y - \frac{9}{5}(3) = -\frac{17}{5}$

$$\therefore y - \frac{27}{5} = -\frac{17}{5}$$

$$\therefore y = \frac{10}{5} = 2$$

Substituting into row 1, $x + 3(2) - 4(3) = -5$

$$\therefore x = 1$$

\therefore the unique solution is $x = 1$, $y = 2$, $z = 3$.

$$\mathbf{b} \quad \begin{cases} 2x - 3y - z = -8 \\ 3x + y - 2z = 1 \\ 5x - 2y - 3z = -7 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left(\begin{array}{ccc|c} 2 & -3 & -1 & -8 \\ 3 & 1 & -2 & 1 \\ 5 & -2 & -3 & -7 \end{array} \right) \\ & \sim \left(\begin{array}{ccc|c} 2 & -3 & -1 & -8 \\ 0 & 11 & -1 & 26 \\ 0 & 11 & -1 & 26 \end{array} \right) \quad \begin{array}{l} 2R_2 - 3R_1 \rightarrow R_2 \\ 2R_3 - 5R_1 \rightarrow R_3 \end{array} \\ & \sim \left(\begin{array}{ccc|c} 2 & -3 & -1 & -8 \\ 0 & 11 & -1 & 26 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad R_3 - R_2 \rightarrow R_3 \end{aligned}$$

$$\left\{ \begin{array}{ccc|c} 6 & 2 & -4 & 2 \\ -6 & 9 & 3 & 24 \\ \hline 0 & 11 & -1 & 26 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 10 & -4 & -6 & -14 \\ -10 & 15 & 5 & 40 \\ \hline 0 & 11 & -1 & 26 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 0 & 11 & -1 & 26 \\ 0 & -11 & 1 & -26 \\ \hline 0 & 0 & 0 & 0 \end{array} \right\}$$

Row 3 indicates there are infinitely many solutions.

If we let $z = t$, then using row 2, $11y - t = 26$

$$\therefore 11y = t + 26$$

$$\therefore y = \frac{1}{11}t + \frac{26}{11}$$

Substituting into row 1, $2x - 3\left(\frac{1}{11}t + \frac{26}{11}\right) - t = -8$

$$\therefore 2x - \frac{3}{11}t - \frac{78}{11} - t = -8$$

$$\therefore 2x = \frac{14}{11}t - \frac{10}{11}$$

$$\therefore x = \frac{7}{11}t - \frac{5}{11}$$

\therefore the solutions have the form $x = \frac{7}{11}t - \frac{5}{11}$, $y = \frac{1}{11}t + \frac{26}{11}$, $z = t$, where $t \in \mathbb{R}$.

$$\mathbf{112} \quad \begin{cases} x - 2y + 3z = 4 \\ 2x - 3y + 2z = 1 \\ 3x - 4y + kz = -2 \end{cases} \quad \text{where } k \text{ is a constant.}$$

a The system has augmented matrix

$$\begin{aligned} & \left(\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -3 & 2 & 1 \\ 3 & -4 & k & -2 \end{array} \right) \\ & \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -4 & -7 \\ 0 & 2 & k-9 & -14 \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array} \\ & \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & k-1 & 0 \end{array} \right) \quad R_3 - 2R_2 \rightarrow R_3 \end{aligned}$$

$$\left\{ \begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ -2 & 4 & -6 & -8 \\ \hline 0 & 1 & -4 & -7 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 3 & -4 & k & -2 \\ -3 & 6 & -9 & -12 \\ \hline 0 & 2 & k-9 & -14 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 0 & 2 & k-9 & -14 \\ 0 & -2 & 8 & 14 \\ \hline 0 & 0 & k-1 & 0 \end{array} \right\}$$

There is a unique solution when $k \neq 1$, so $k_1 = 1$.

In this case, using row 3, $(k-1)z = 0$

$$\therefore z = 0 \quad \{k-1 \neq 0\}$$

Substituting into row 2, $y - 4(0) = -7$

$$\therefore y = -7$$

Substituting into row 1, $x - 2(-7) + 3(0) = 4$

$$\therefore x = -10$$

\therefore the unique solution is $x = -10$, $y = -7$, $z = 0$, where $k \neq 1$.

b Suppose $k = 1$.

Row 3 indicates there are infinitely many solutions.

If we let $z = t$, then using row 2, $y - 4t = -7$

$$\therefore y = 4t - 7$$

Substituting into row 1, $x - 2(4t - 7) + 3t = 4$

$$\therefore x - 8t + 14 + 3t = 4$$

$$\therefore x = 5t - 10$$

\therefore if $k = 1$, the solutions have the form $x = 5t - 10$, $y = 4t - 7$, $z = t$, where $t \in \mathbb{R}$.

$$113 \quad \begin{cases} 3x - ay + 2z = 4 \\ x + 2y - 3z = 1 \\ -x - y + z = 12 \end{cases} \quad \text{where } a \in \mathbb{R}$$

a The system has augmented matrix

$$\begin{aligned} & \left(\begin{array}{ccc|c} 3 & -a & 2 & 4 \\ 1 & 2 & -3 & 1 \\ -1 & -1 & 1 & 12 \end{array} \right) \\ & \sim \left(\begin{array}{ccc|c} 3 & -a & 2 & 4 \\ 0 & a+6 & -11 & -1 \\ 0 & -(a+3) & 5 & 40 \end{array} \right) \quad \begin{array}{l} 3R_2 - R_1 \rightarrow R_2 \\ 3R_3 + R_1 \rightarrow R_3 \end{array} \\ & \sim \left(\begin{array}{ccc|c} 3 & -a & 2 & 4 \\ 0 & a+6 & -11 & -1 \\ 0 & 0 & -6a-3 & 39a+237 \end{array} \right) \quad (a+6)R_3 + (a+3)R_2 \rightarrow R_3 \\ & \sim \left(\begin{array}{ccc|c} 3 & -a & 2 & 4 \\ 0 & a+6 & -11 & -1 \\ 0 & 0 & 2a+1 & -13a-79 \end{array} \right) \quad \begin{array}{l} -\frac{1}{3}R_3 \rightarrow R_3 \end{array} \end{aligned}$$

$$\left\{ \begin{array}{ccc|c} 3 & 6 & -9 & 3 \\ -3 & a & -2 & -4 \\ 0 & a+6 & -11 & -1 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} -3 & -3 & 3 & 36 \\ 3 & -a & 2 & 4 \\ 0 & -(a+3) & 5 & 40 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 0 & -(a+3)(a+6) & 5(a+6) & 40(a+6) \\ 0 & (a+6)(a+3) & -11(a+3) & -(a+3) \\ 0 & 0 & -6a-3 & 39a+237 \end{array} \right\}$$

If $a = -\frac{1}{2}$, the last row gives $0x + 0y + 0z = -\frac{145}{2}$ which is not possible.

\therefore there are no solutions if $a = -\frac{1}{2}$.

b Suppose $a \neq -\frac{1}{2}$.

Using row 3, $(2a+1)z = -13a-79$

$$\therefore z = \frac{-13a-79}{2a+1} \quad \{a \neq -\frac{1}{2}\}$$

Substituting into row 2, $(a+6)y - 11\left(\frac{-13a-79}{2a+1}\right) = -1$

$$\therefore (2a+1)(a+6)y + 143a + 869 = -(2a+1)$$

$$\therefore (2a+1)(a+6)y = -145a - 870$$

$$\begin{aligned} \text{For } a \neq -6, \quad y &= \frac{-145(a+6)}{(2a+1)(a+6)} \quad \{a \neq -\frac{1}{2}\} \\ &= \frac{-145}{2a+1} \end{aligned}$$

Substituting into row 1, $3x - a\left(\frac{-145}{2a+1}\right) + 2\left(\frac{-13a-79}{2a+1}\right) = 4$

$$\therefore 3(2a+1)x + 145a - 26a - 158 = 4(2a+1)$$

$$\therefore 3(2a+1)x = -111a + 162$$

$$\begin{aligned} \therefore x &= \frac{3(54-37a)}{3(2a+1)} \quad \{a \neq -\frac{1}{2}\} \\ &= \frac{54-37a}{2a+1} \end{aligned}$$

\therefore the unique solution is $x = \frac{54-37a}{2a+1}$, $y = -\frac{145}{2a+1}$, $z = \frac{-13a-79}{2a+1}$, where $a \in \mathbb{R}$, $a \neq -\frac{1}{2}$ or -6 .

If $a = -6$, then

$$\begin{aligned}
 & \left(\begin{array}{ccc|c} 3 & 6 & 2 & 4 \\ 1 & 2 & -3 & 1 \\ -1 & -1 & 1 & 12 \end{array} \right) \\
 & \sim \left(\begin{array}{ccc|c} 3 & 6 & 2 & 4 \\ 0 & 0 & -11 & -1 \\ 0 & 3 & 5 & 40 \end{array} \right) \quad \{\text{from } \mathbf{a}\} \\
 & \sim \left(\begin{array}{ccc|c} 3 & 6 & 2 & 4 \\ 0 & 3 & 5 & 40 \\ 0 & 0 & -11 & -1 \end{array} \right) \quad R_3 \leftrightarrow R_2 \\
 & \sim \left(\begin{array}{ccc|c} 3 & 6 & 2 & 4 \\ 0 & 1 & \frac{5}{3} & \frac{40}{3} \\ 0 & 0 & 1 & \frac{1}{11} \end{array} \right) \quad \begin{array}{l} \frac{1}{3}R_2 \rightarrow R_2 \\ -\frac{1}{11}R_3 \rightarrow R_3 \end{array}
 \end{aligned}$$

Using row 3, $z = \frac{1}{11}$

$$\begin{aligned}
 \text{Substituting into row 2, } y + \frac{5}{3}\left(\frac{1}{11}\right) &= \frac{40}{3} \\
 \therefore y &= \frac{145}{11}
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting into row 1, } 3x + 6\left(\frac{145}{11}\right) + 2\left(\frac{1}{11}\right) &= 4 \\
 \therefore 3x &= -\frac{828}{11} \\
 \therefore x &= -\frac{276}{11}
 \end{aligned}$$

\therefore the unique solution is $x = -\frac{276}{11}$, $y = \frac{145}{11}$, $z = \frac{1}{11}$.

TOPIC 2 SKILL BUILDER QUESTIONS

1 a i $4x - 3y + 2 = 0$

$$\therefore 3y = 4x + 2$$

$$\therefore y = \frac{4}{3}x + \frac{2}{3} \text{ has gradient } \frac{4}{3}$$

ii The perpendicular bisector has gradient $-\frac{3}{4}$.

b The equation of the perpendicular bisector is $3x + 4y = 3(4) + 4(6)$

$$\text{which is } 3x + 4y = 36$$

$$\text{or } 3x + 4y - 36 = 0$$

2 a The line is parallel to $2x - y = -3$ or $y = 2x + 3$ which has gradient 2.

$$\therefore \text{ the line has gradient 2 and passes through } (5, 3).$$

$$\therefore \text{ the equation of the line is } y - 3 = 2(x - 5)$$

$$\therefore y - 3 = 2x - 10$$

$$\therefore y = 2x - 7$$

b The line is perpendicular to $y = -4x + 3$, which has gradient -4 .

$$\therefore \text{ the line has gradient } \frac{1}{4} \text{ and passes through } (-1, 5).$$

$$\therefore \text{ the equation of the line is } y - 5 = \frac{1}{4}(x - (-1))$$

$$\therefore y - 5 = \frac{1}{4}(x + 1)$$

$$\therefore y - 5 = \frac{1}{4}x + \frac{1}{4}$$

$$\therefore y = \frac{1}{4}x + \frac{21}{4}$$

3 $L: y = 3 - 2x$

a Substituting $x = 3$ and $y = k$ into the equation gives $k = 3 - 2(3)$

$$\therefore k = 3 - 6$$

$$\therefore k = -3$$

b Line L has gradient -2 .

c From **b**, the line is perpendicular to L , which has gradient -2 .

$$\therefore \text{ the line has gradient } \frac{1}{2} \text{ and passes through } P(3, -3). \text{ \{from a\}}$$

$$\therefore \text{ the equation of the line is } y - (-3) = \frac{1}{2}(x - 3)$$

$$\therefore y + 3 = \frac{1}{2}x - \frac{3}{2}$$

$$\therefore y = \frac{1}{2}x - \frac{9}{2}$$

4 a x adult tickets at \$30 each and y child tickets at \$15 costs \$120 in total.

$$\therefore 30x + 15y = 120$$

b When $y = 4$, $30x + 15(4) = 120$

$$\therefore 30x + 60 = 120$$

$$\therefore 30x = 60$$

$$\therefore x = 2$$

$$\therefore \text{ Tammy bought 2 adult tickets.}$$

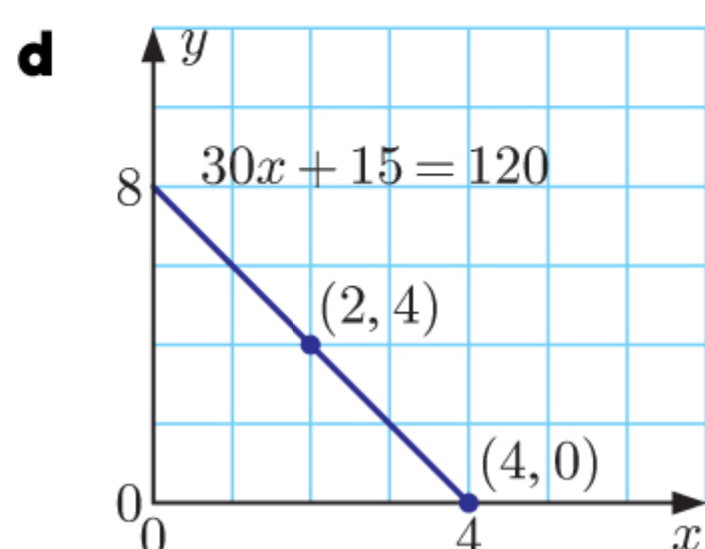
c When $y = 0$, $30x + 15(0) = 120$

$$\therefore 30x = 120$$

$$\therefore x = 4$$

$$\therefore \text{ the } x\text{-intercept of the line } 30x + 15y = 120 \text{ is 4.}$$

If Tammy did not buy any child tickets, then she bought 4 adult tickets.



$$5 \quad \mathbf{a} \quad 2x^2 - 9x = 0$$

$$\therefore x(2x - 9) = 0$$

$$\therefore x = 0 \text{ or } 2x - 9 = 0$$

$$\therefore x = 0 \text{ or } \frac{9}{2}$$

$$\mathbf{c} \quad 4x^2 + 11x = 3$$

$$\therefore 4x^2 + 11x - 3 = 0$$

$$\therefore (4x - 1)(x + 3) = 0$$

$$\therefore x = \frac{1}{4} \text{ or } -3$$

$$\mathbf{b} \quad x^2 + 8x - 20 = 0$$

$$\therefore (x + 10)(x - 2) = 0$$

$$\therefore x = -10 \text{ or } 2$$

$$\mathbf{d} \quad (x + 3)(1 - 2x) = -9$$

$$\therefore x - 2x^2 + 3 - 6x = -9$$

$$\therefore -2x^2 - 5x + 12 = 0$$

$$\therefore 2x^2 + 5x - 12 = 0$$

$$\therefore (2x - 3)(x + 4) = 0$$

$$\therefore x = \frac{3}{2} \text{ or } -4$$

$$6 \quad \mathbf{a} \quad x = -2 \text{ is a solution to } x^2 + bx + (b - 2) = 0$$

$$\therefore (-2)^2 + b(-2) + b - 2 = 0$$

$$\therefore 4 - 2b + b - 2 = 0$$

$$\therefore -b = -2$$

$$\therefore b = 2$$

$$\mathbf{b} \quad x^2 + 2x = 0 \quad \{\text{using } \mathbf{a}\}$$

$$\therefore x(x + 2) = 0$$

$$\therefore x = 0 \text{ or } -2$$

$$\therefore \text{the other solution to the equation is } x = 0.$$

$$7 \quad mx^2 + (m - 2)x + m = 0 \quad \text{has } a = m, \quad b = m - 2, \quad \text{and } c = m \quad \therefore \Delta = b^2 - 4ac$$

$$= (m - 2)^2 - 4(m)(m)$$

$$= m^2 - 4m + 4 - 4m^2$$

$$= -3m^2 - 4m + 4$$

For a repeated root, $\Delta = 0$

$$\therefore 0 = -3m^2 - 4m + 4$$

$$\therefore 3m^2 + 4m - 4 = 0$$

$$\therefore (3m - 2)(m + 2) = 0$$

$$\therefore m = \frac{2}{3} \text{ or } -2$$

$$8 \quad \mathbf{a} \quad \mathbf{i} \quad x^2 - 8x + 15 = 0 \quad \text{has } a = 1, \quad b = -8, \quad \text{and } c = 15$$

$$\therefore \text{sum} = -\frac{b}{a} = -\frac{(-8)}{1} = 8$$

$$\text{product} = \frac{c}{a} = \frac{15}{1} = 15$$

$$\mathbf{ii} \quad x^2 - 8x + 15 = 0$$

$$\therefore (x - 3)(x - 5) = 0$$

$$\therefore x = 3 \text{ or } 5$$

So the quadratic has roots 3 and 5 which have $\text{sum} = 3 + 5 = 8$ ✓

and $\text{product} = (3)(5) = 15$ ✓

$$\mathbf{b} \quad \mathbf{i} \quad 3x^2 - 4x - 2 = 0 \quad \text{has } a = 3, \quad b = -4, \quad \text{and } c = -2$$

$$\therefore \text{sum} = -\frac{b}{a} = -\frac{(-4)}{3} = \frac{4}{3}$$

$$\text{product} = \frac{c}{a} = -\frac{2}{3}$$

$$\mathbf{ii} \quad 3x^2 - 4x - 2 = 0 \quad \text{has roots } \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)} = \frac{4 \pm \sqrt{40}}{6}$$

$$= \frac{4 \pm 2\sqrt{10}}{6}$$

$$= \frac{2 \pm \sqrt{10}}{3}$$

$$\text{These have sum} = \frac{2 + \sqrt{10}}{3} + \frac{2 - \sqrt{10}}{3} = \frac{4}{3} \quad \checkmark$$

$$\text{and product} = \left(\frac{2 + \sqrt{10}}{3}\right)\left(\frac{2 - \sqrt{10}}{3}\right) = \frac{4 - 10}{9} = \frac{-6}{9} = -\frac{2}{3} \quad \checkmark$$

9 a $\frac{2}{x} = 5x - 3$
 $\therefore 2 = 5x^2 - 3x$

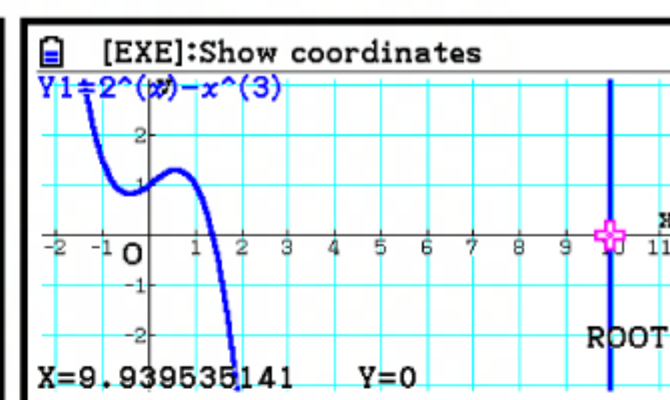
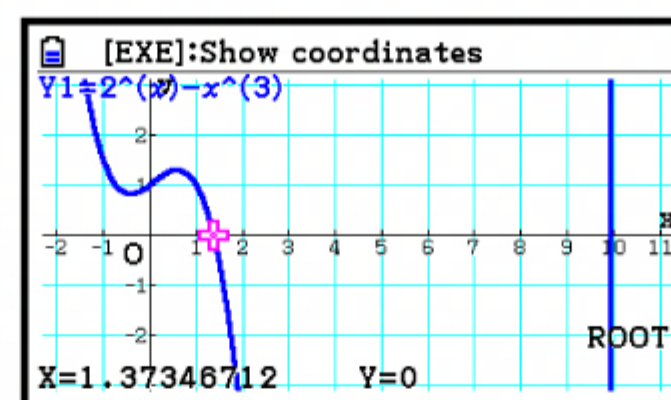
$\therefore 5x^2 - 3x - 2 = 0$

Using technology, $x = -0.4$ or 1

b We graph $y = 2^x - x^3$

The x -intercepts are ≈ 1.37 and 9.94 .

\therefore the solutions are $x \approx 1.37$ or 9.94 .



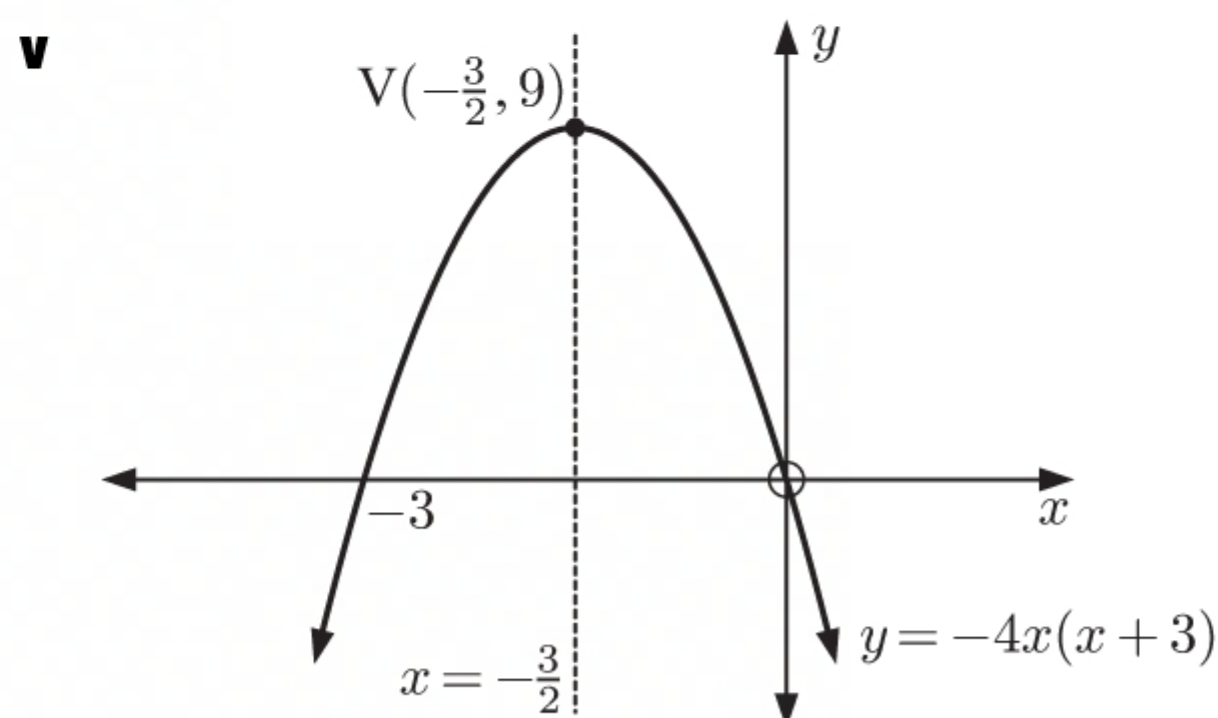
10 a $y = -4x(x + 3)$

i When $y = 0$, $-4x(x + 3) = 0$
 $\therefore x = 0$ or -3

\therefore the x -intercepts are 0 and -3 .

iii When $x = -\frac{3}{2}$, $y = -4(-\frac{3}{2})(-\frac{3}{2} + 3)$
 $= -4(-\frac{3}{2})(\frac{3}{2})$
 $= 9$

\therefore the vertex is $(-\frac{3}{2}, 9)$.



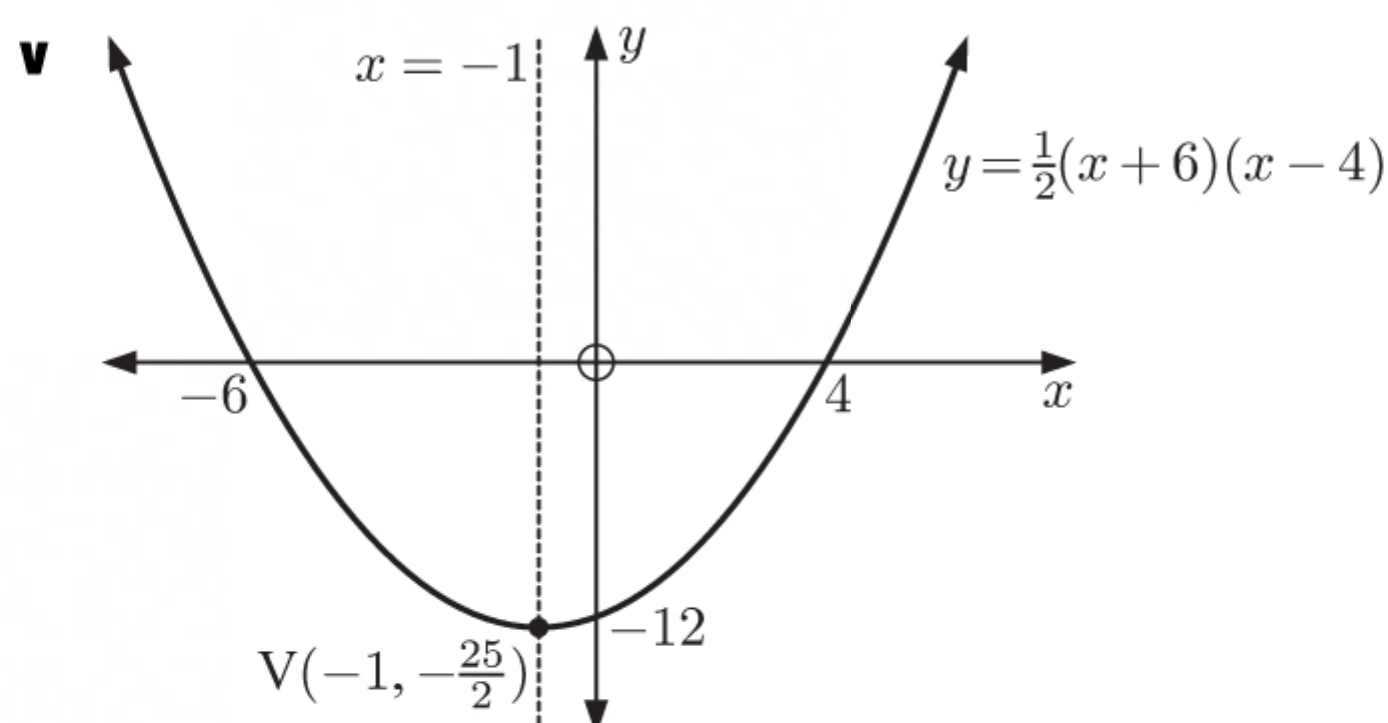
b $y = \frac{1}{2}(x + 6)(x - 4)$

i When $y = 0$, $\frac{1}{2}(x + 6)(x - 4) = 0$
 $\therefore x = -6$ or 4

\therefore the x -intercepts are -6 and 4 .

iii When $x = -1$, $y = \frac{1}{2}(-1 + 6)(-1 - 4)$
 $= \frac{1}{2}(5)(-5)$
 $= -\frac{25}{2}$

\therefore the vertex is $(-1, -\frac{25}{2})$.



ii The axis of symmetry is midway between the x -intercepts.

\therefore the axis of symmetry is $x = \frac{0 + (-3)}{2} = -\frac{3}{2}$.

iv When $x = 0$, $y = 0$

\therefore the y -intercept is 0 .

ii The axis of symmetry is midway between the x -intercepts.

\therefore the axis of symmetry is $x = \frac{-6 + 4}{2} = -1$.

iv When $x = 0$, $y = \frac{1}{2}(6)(-4)$
 $= -12$

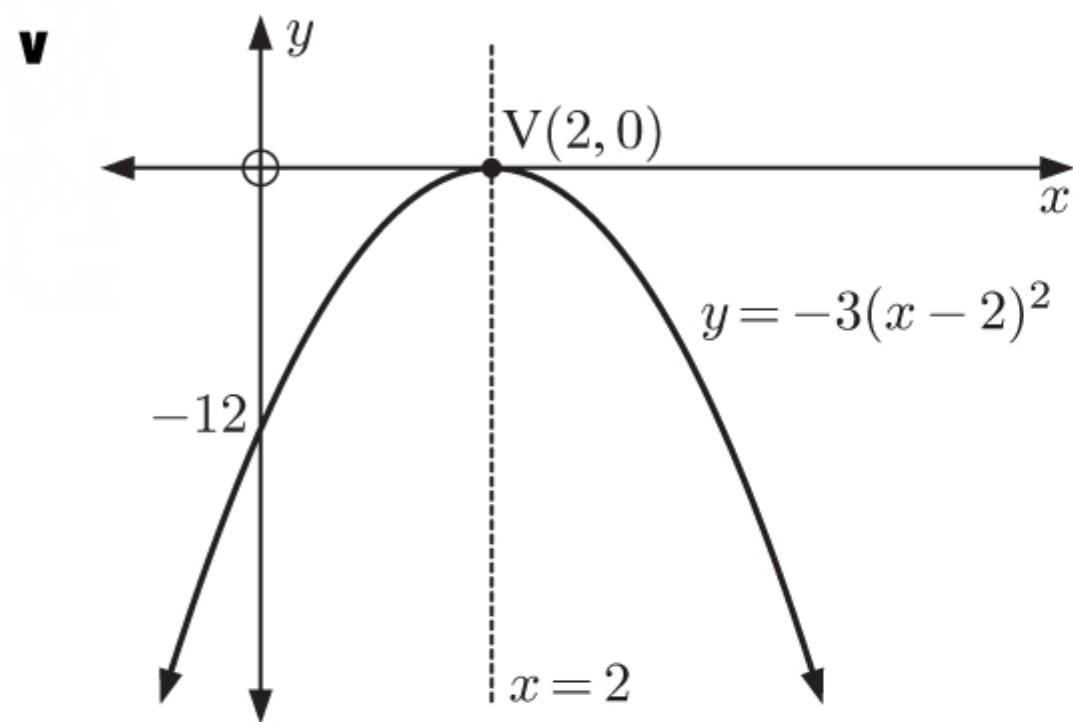
\therefore the y -intercept is -12 .

c $y = -3(x - 2)^2$

i When $y = 0$, $-3(x - 2)^2 = 0$
 $\therefore x = 2$

\therefore the x -intercept is 2.

iii The vertex is $(2, 0)$.



ii The axis of symmetry is $x = 2$.

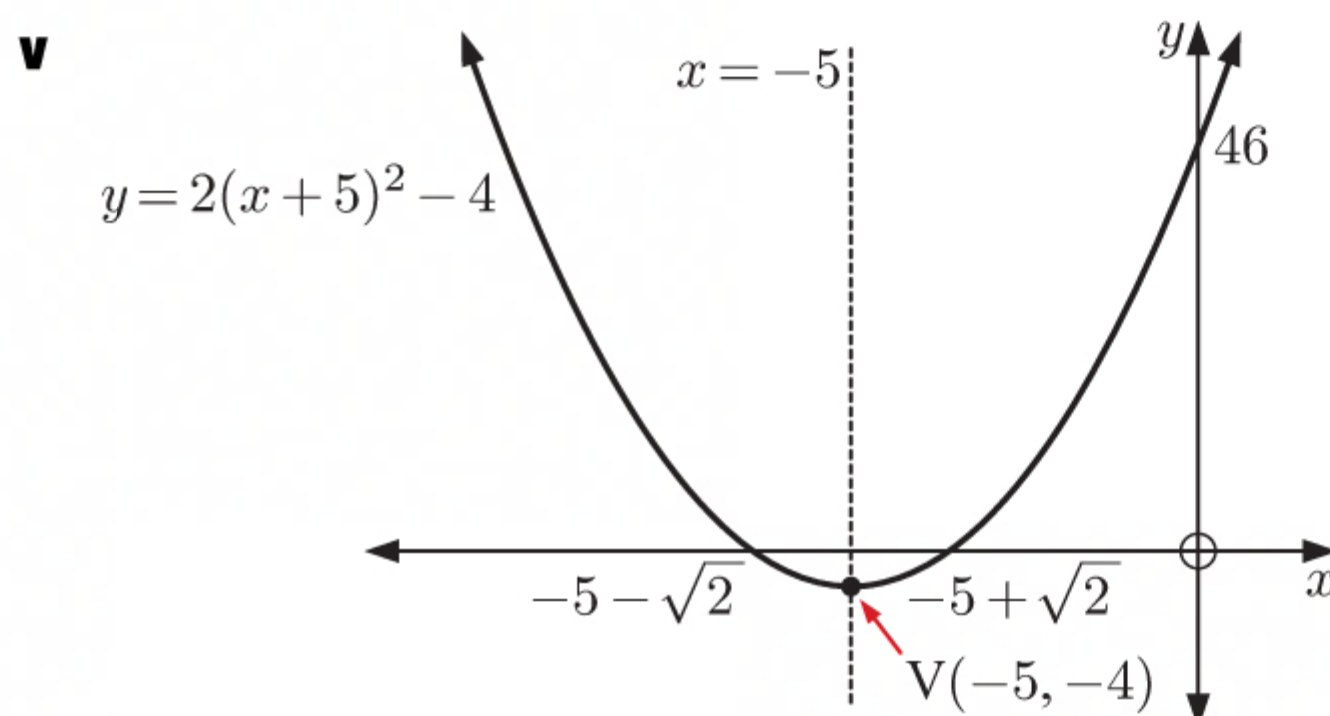
iv When $x = 0$, $y = -3(-2)^2$
 $= -12$

\therefore the y -intercept is -12 .

d $y = 2(x + 5)^2 - 4$

i When $y = 0$, $2(x + 5)^2 - 4 = 0$
 $\therefore 2(x + 5)^2 = 4$
 $\therefore (x + 5)^2 = 2$
 $\therefore x + 5 = \pm\sqrt{2}$
 $\therefore x = -5 \pm \sqrt{2}$

\therefore the x -intercepts are $-5 - \sqrt{2}$ and $-5 + \sqrt{2}$.



ii The axis of symmetry is $x = -5$.

iii The vertex is $(-5, -4)$.

iv When $x = 0$, $y = 2(5)^2 - 4$
 $= 50 - 4$
 $= 46$

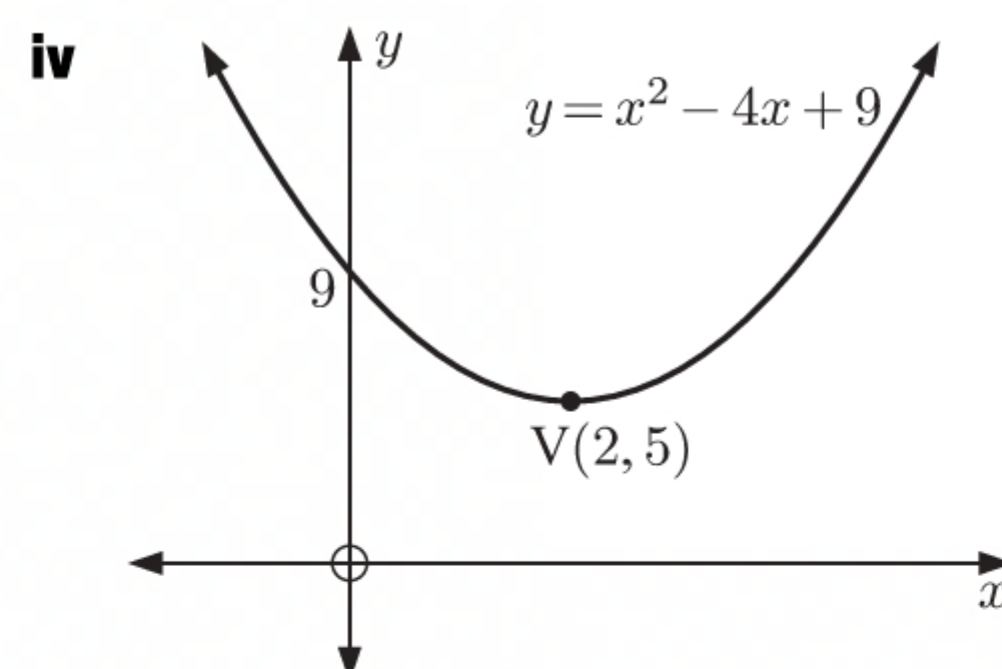
\therefore the y -intercept is 46.

11 a $y = x^2 - 4x + 9$

i The y -intercept is 9.

ii $y = x^2 - 4x + 9$
 $= x^2 - 4x + (-2)^2 + 9 - (-2)^2$
 $\therefore y = (x - 2)^2 + 5$

iii The vertex is $(2, 5)$.

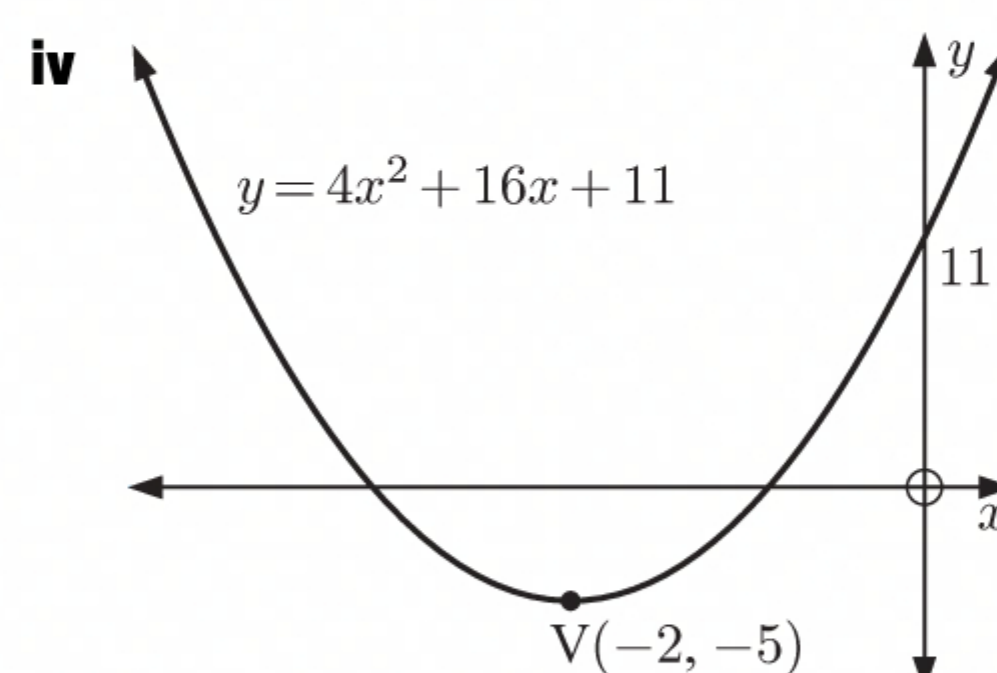


b $y = 4x^2 + 16x + 11$

i The y -intercept is 11.

ii $y = 4x^2 + 16x + 11$
 $= 4\left[x^2 + 4x + \frac{11}{4}\right]$
 $= 4\left[x^2 + 4x + 2^2 + \frac{11}{4} - 2^2\right]$
 $= 4\left[(x + 2)^2 - \frac{5}{4}\right]$
 $\therefore y = 4(x + 2)^2 - 5$

iii The vertex is $(-2, -5)$.

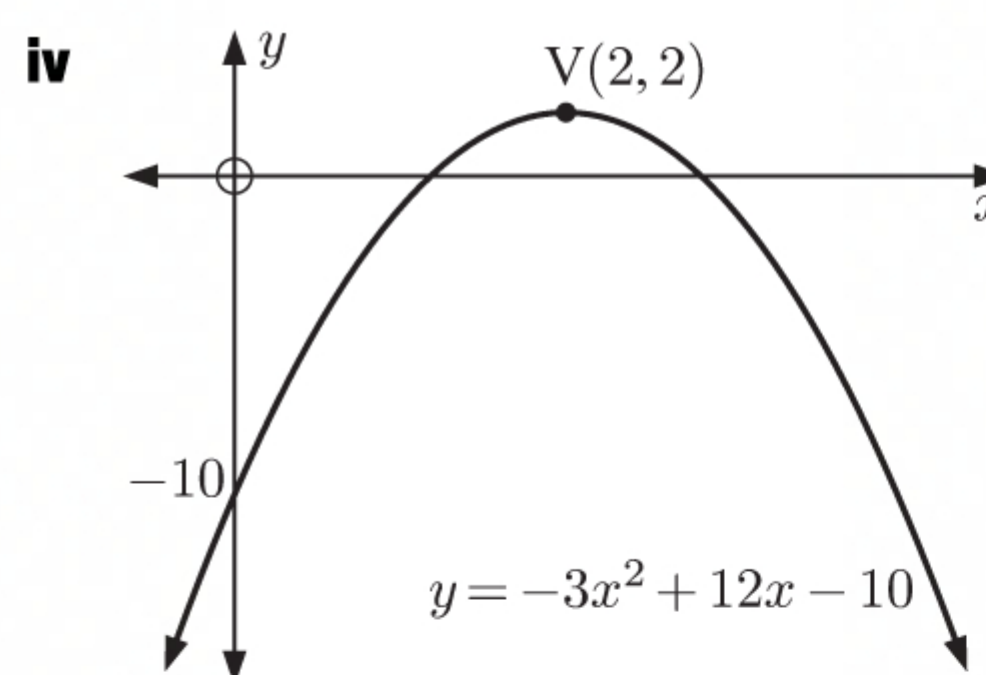


c $y = -3x^2 + 12x - 10$

i The y -intercept is -10 .

ii
$$\begin{aligned} y &= -3x^2 + 12x - 10 \\ &= -3\left[x^2 - 4x + \frac{10}{3}\right] \\ &= -3\left[x^2 - 4x + (-2)^2 + \frac{10}{3} - (-2)^2\right] \\ &= -3\left[(x-2)^2 - \frac{2}{3}\right] \\ \therefore y &= -3(x-2)^2 + 2 \end{aligned}$$

iii The vertex is $(2, 2)$.



12 a i $y = x^2 - 3x - 4$ has $a = 1$, $b = -3$, and $c = -4$


Now $\frac{-b}{2a} = \frac{-(-3)}{2(1)} = \frac{3}{2}$

\therefore the axis of symmetry is $x = \frac{3}{2}$.

When $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 4$

$$\begin{aligned} &= \frac{9}{4} - \frac{9}{2} - 4 \\ &= -\frac{25}{4} \end{aligned}$$

\therefore the vertex is $\left(\frac{3}{2}, -\frac{25}{4}\right)$.

ii $a > 0$, so the shape is  \therefore the vertex $\left(\frac{3}{2}, -\frac{25}{4}\right)$ is a minimum.

iv The y -intercept is -4 .

When $y = 0$,

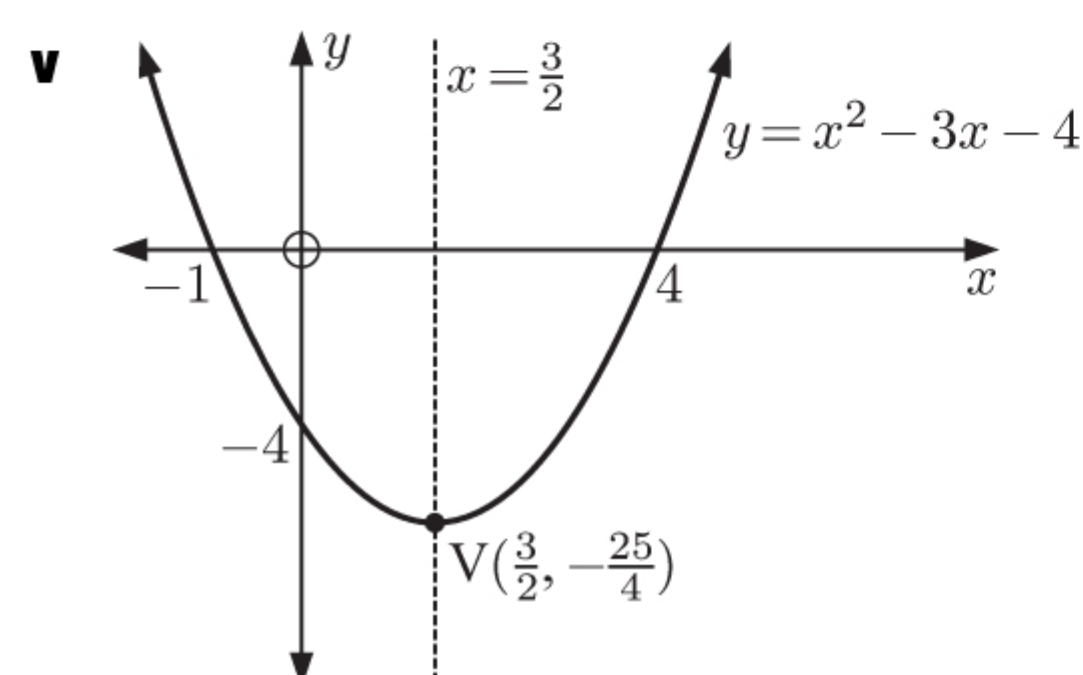
$$x^2 - 3x - 4 = 0$$

$$\therefore (x-4)(x+1) = 0$$

$$\therefore x = 4 \text{ or } -1$$

$$\therefore \text{the } x\text{-intercepts are } 4 \text{ and } -1.$$

iii The range is $\{y \mid y \geq -\frac{25}{4}\}$.



b i $y = -2x^2 - 5x + 7$ has $a = -2$, $b = -5$, and $c = 7$


Now $\frac{-b}{2a} = \frac{-(-5)}{2(-2)} = -\frac{5}{4}$

\therefore the axis of symmetry is $x = -\frac{5}{4}$.

When $x = -\frac{5}{4}$, $y = -2\left(-\frac{5}{4}\right)^2 - 5\left(-\frac{5}{4}\right) + 7$

$$\begin{aligned} &= -\frac{25}{8} + \frac{25}{4} + 7 \\ &= \frac{81}{8} \end{aligned}$$

\therefore the vertex is $\left(-\frac{5}{4}, \frac{81}{8}\right)$.

ii $a < 0$, so the shape is  \therefore the vertex $\left(-\frac{5}{4}, \frac{81}{8}\right)$ is a maximum.

iv The y -intercept is 7 .

When $y = 0$,

$$-2x^2 - 5x + 7 = 0$$

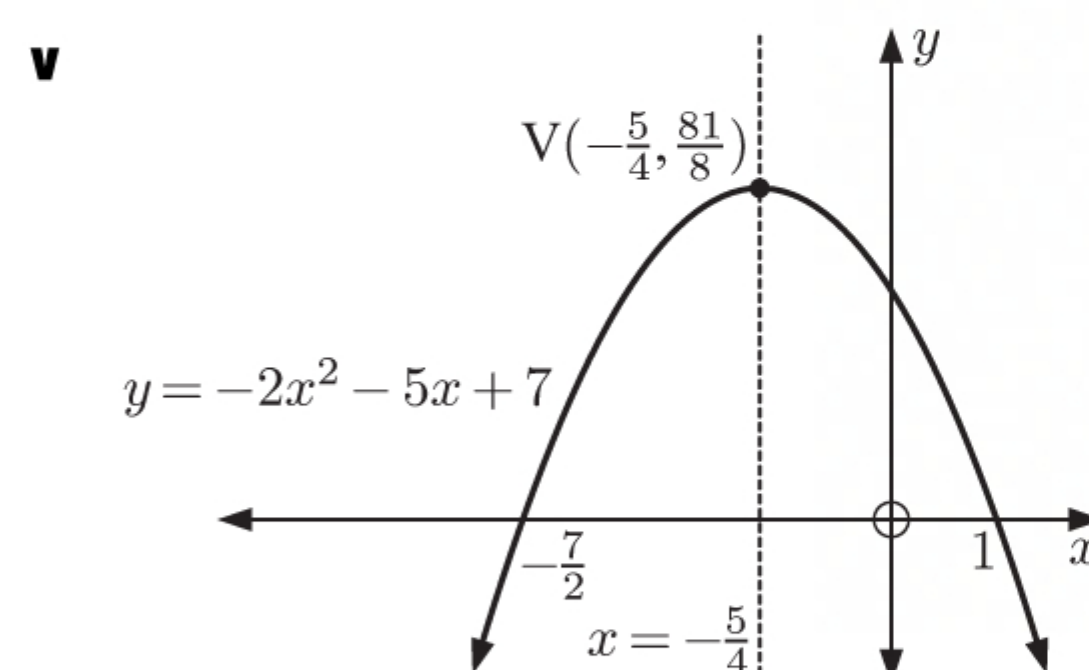
$$\therefore -(2x^2 + 5x - 7) = 0$$

$$\therefore -(2x+7)(x-1) = 0$$

$$\therefore x = -\frac{7}{2} \text{ or } 1$$

$$\therefore \text{the } x\text{-intercepts are } -\frac{7}{2} \text{ and } 1.$$

iii The range is $\{y \mid y \leq \frac{81}{8}\}$.



13 $y = (k+3)x^2 - 2kx + (k-2)$ has $a = k+3$, $b = -2k$, $c = k-2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-2k)^2 - 4(k+3)(k-2) \\ &= 4k^2 - 4(k^2 + k - 6) \\ &= 4k^2 - 4k^2 - 4k + 24 \\ &= 24 - 4k\end{aligned}$$

Also, if $k = -3$, then the function is the line $y = 6x - 5$, in which case it cuts the x -axis only *once* at $x = \frac{5}{6}$.

a The graph cuts the x -axis twice if $\Delta > 0$.

$$\begin{aligned}\therefore 24 - 4k &> 0 \\ \therefore 4k &< 24 \\ \therefore k &< 6, \quad k \neq -3\end{aligned}$$

b The graph touches the x -axis if $\Delta = 0$.

$$\begin{aligned}\therefore 24 - 4k &= 0 \\ \therefore 4k &= 24 \\ \therefore k &= 6\end{aligned}$$

c The graph misses the x -axis if $\Delta < 0$.

$$\begin{aligned}\therefore 24 - 4k &< 0 \\ \therefore 4k &> 24 \\ \therefore k &> 6\end{aligned}$$

14 a $y = f(x) = a(x-p)(x-q)$ has x -intercepts -1 and 5 .

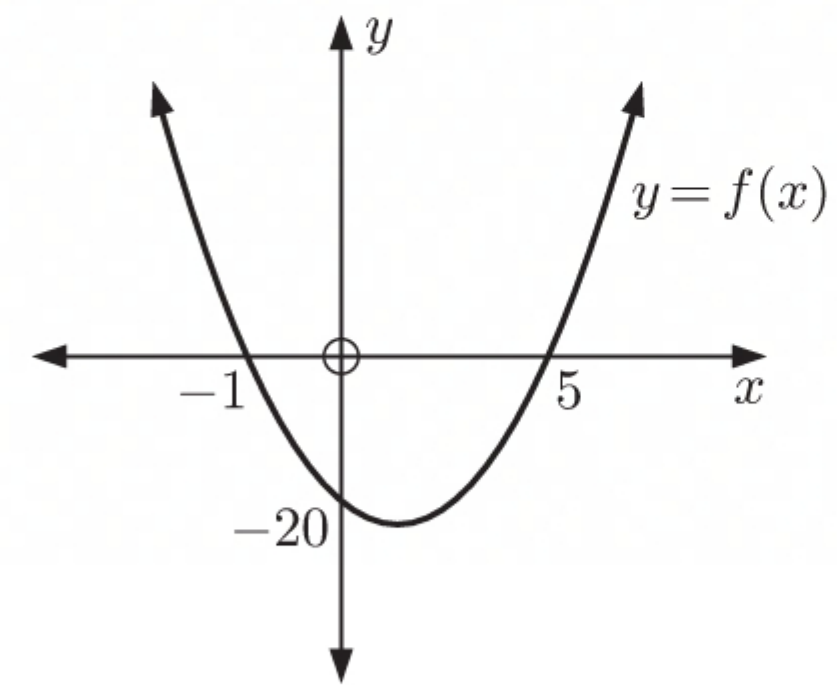
$$\therefore p = 5 \text{ and } q = -1 \quad \{p > q\}$$

b From **a**, $f(x) = a(x-5)(x+1)$.

$$\begin{aligned}\text{From the graph, } f(0) &= -20 \\ \therefore -20 &= a(-5)(1) \\ \therefore -20 &= -5a \\ \therefore a &= 4\end{aligned}$$

c The axis of symmetry is midway between the x -intercepts.

$$\therefore \text{the axis of symmetry is } x = \frac{-1+5}{2} = 2.$$



15 $y = -3x^2 - x + 1$ has $a = -3$, $b = -1$, and $c = 1$

a Since $a < 0$, the graph is concave down.

$$\begin{aligned}\mathbf{b} \quad \Delta &= b^2 - 4ac \\ &= (-1)^2 - 4(-3)(1) \\ &= 13\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

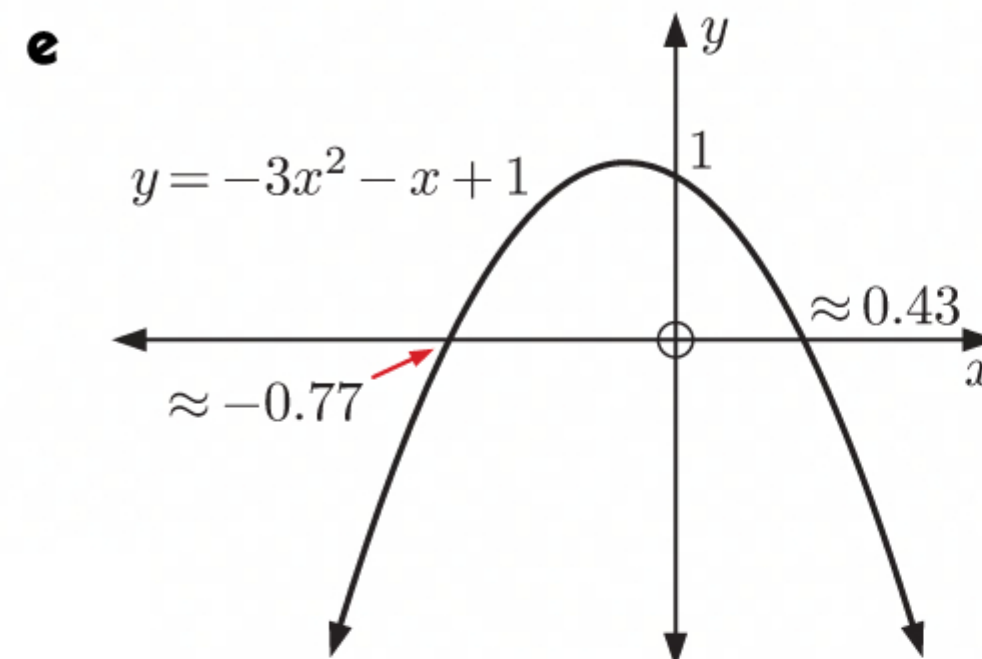


c When $y = 0$, $-3x^2 - x + 1 = 0$

$$\begin{aligned}\therefore x &= \frac{-b \pm \sqrt{\Delta}}{2a} \\ \therefore x &= \frac{-(-1) \pm \sqrt{13}}{2(-3)} \\ \therefore x &= -\frac{1}{6} \pm \frac{\sqrt{13}}{6} \\ \therefore x &\approx 0.43 \text{ or } -0.77\end{aligned}$$

\therefore the x -intercepts are ≈ 0.43 or ≈ -0.77 .

d The y -intercept is 1.



16 $y = mx^2 + 4x + 6$ has $a = m$, $b = 4$, and $c = 6$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 4^2 - 4(m)(6) \\ &= 16 - 24m\end{aligned}$$

The graph lies entirely above the x -axis if it is positive definite.

$$\begin{aligned}\therefore a &> 0 \quad \text{and} \quad \Delta < 0 \\ \therefore m &> 0 \quad \text{and} \quad 16 - 24m < 0 \\ \therefore m &> 0 \quad \text{and} \quad 24m > 16 \\ \therefore m &> 0 \quad \text{and} \quad m > \frac{2}{3} \\ \therefore m &> \frac{2}{3}\end{aligned}$$

- 17** $y = kx - 2$ is a tangent to $y = 3x^2 + x + 1$ if they meet at exactly one point (touch).

$$y = 3x^2 + x + 1 \text{ meets } y = kx - 2 \text{ where } 3x^2 + x + 1 = kx - 2$$

$$\therefore 3x^2 + (1 - k)x + 3 = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$

$$\begin{aligned}\therefore (1 - k)^2 - 4(3)(3) &= 0 \\ \therefore 1 - 2k + k^2 - 36 &= 0 \\ \therefore k^2 - 2k - 35 &= 0 \\ \therefore (k - 7)(k + 5) &= 0 \\ \therefore k &= 7 \text{ or } -5\end{aligned}$$

- 18 a** $y = x^2 - 4x - 5$ meets $y = 3x - 11$ where $x^2 - 4x - 5 = 3x - 11$

$$\begin{aligned}\therefore x^2 - 7x + 6 &= 0 \\ \therefore (x - 6)(x - 1) &= 0 \\ \therefore x &= 6 \text{ or } 1\end{aligned}$$

Substituting into $y = 3x - 11$, when $x = 6$, $y = 7$,
and when $x = 1$, $y = -8$.

\therefore the graphs meet at $(6, 7)$ and $(1, -8)$.

- b** $y = -2x^2 + 5x$ meets $y = 5 - 2x$ where $-2x^2 + 5x = 5 - 2x$

$$\begin{aligned}\therefore -2x^2 + 7x - 5 &= 0 \\ \therefore -(2x^2 - 7x + 5) &= 0 \\ \therefore -(2x - 5)(x - 1) &= 0 \\ \therefore x &= \frac{5}{2} \text{ or } 1\end{aligned}$$

Substituting into $y = 5 - 2x$, when $x = \frac{5}{2}$, $y = 0$,
and when $x = 1$, $y = 3$.

\therefore the graphs meet at $(\frac{5}{2}, 0)$ and $(1, 3)$.

- 19** $y = 2x + c$ meets $y = 3x^2 + 5x + 7$ where $3x^2 + 5x + 7 = 2x + c$

$$\therefore 3x^2 + 3x + (7 - c) = 0$$

The graphs will never meet if this equation has no real roots $\therefore \Delta < 0$

$$\begin{aligned}\therefore 3^2 - 4(3)(7 - c) &< 0 \\ \therefore 9 - 84 + 12c &< 0 \\ \therefore 12c &< 75 \\ \therefore c &< \frac{25}{4}\end{aligned}$$

- 20 a** The graph touches the x -axis, and has axis of symmetry $x = -3$.

\therefore the graph touches the x -axis at -3 .

\therefore the quadratic has the form $f(x) = a(x + 3)^2$, $a \neq 0$.

The y -intercept is -3 .

So, $f(0) = -3$

$$\begin{aligned}\therefore -3 &= a(3)^2 \\ \therefore 9a &= -3 \\ \therefore a &= -\frac{1}{3}\end{aligned}$$

The quadratic is $f(x) = -\frac{1}{3}(x + 3)^2$
 $= -\frac{1}{3}(x^2 + 6x + 9)$

$$\therefore f(x) = -\frac{1}{3}x^2 - 2x - 3$$

b $y = kx - \frac{9}{4}$ is a tangent to $y = -\frac{1}{3}x^2 - 2x - 3$ if they meet at exactly one point (touch).

$$y = -\frac{1}{3}x^2 - 2x - 3 \text{ meets } y = kx - \frac{9}{4} \text{ where } -\frac{1}{3}x^2 - 2x - 3 = kx - \frac{9}{4}$$

$$\therefore -\frac{1}{3}x^2 - (k+2)x - \frac{3}{4} = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (-(k+2))^2 - 4(-\frac{1}{3})(-\frac{3}{4}) = 0$$

$$\therefore k^2 + 4k + 4 - 1 = 0$$

$$\therefore k^2 + 4k + 3 = 0$$

$$\therefore (k+3)(k+1) = 0$$

$$\therefore k = -3 \text{ or } -1$$

When $k = -3$, the tangent is $y = -3x - \frac{9}{4}$.

The tangent meets the curve where $-\frac{1}{3}x^2 - (-1)x - \frac{3}{4} = 0$

$$\therefore -\frac{1}{3}x^2 + x - \frac{3}{4} = 0$$

$$\therefore 4x^2 - 12x + 9 = 0$$

$$\therefore (2x - 3)^2 = 0$$

$$\therefore x = \frac{3}{2}$$

Substituting into $y = -3x - \frac{9}{4}$, when $x = \frac{3}{2}$, $y = -\frac{27}{4}$.

\therefore the tangent $y = -3x - \frac{9}{4}$ meets the curve at $(\frac{3}{2}, -\frac{27}{4})$.

When $k = -1$, the tangent is $y = -x - \frac{9}{4}$.

The tangent meets the curve where $-\frac{1}{3}x^2 - x - \frac{3}{4} = 0$

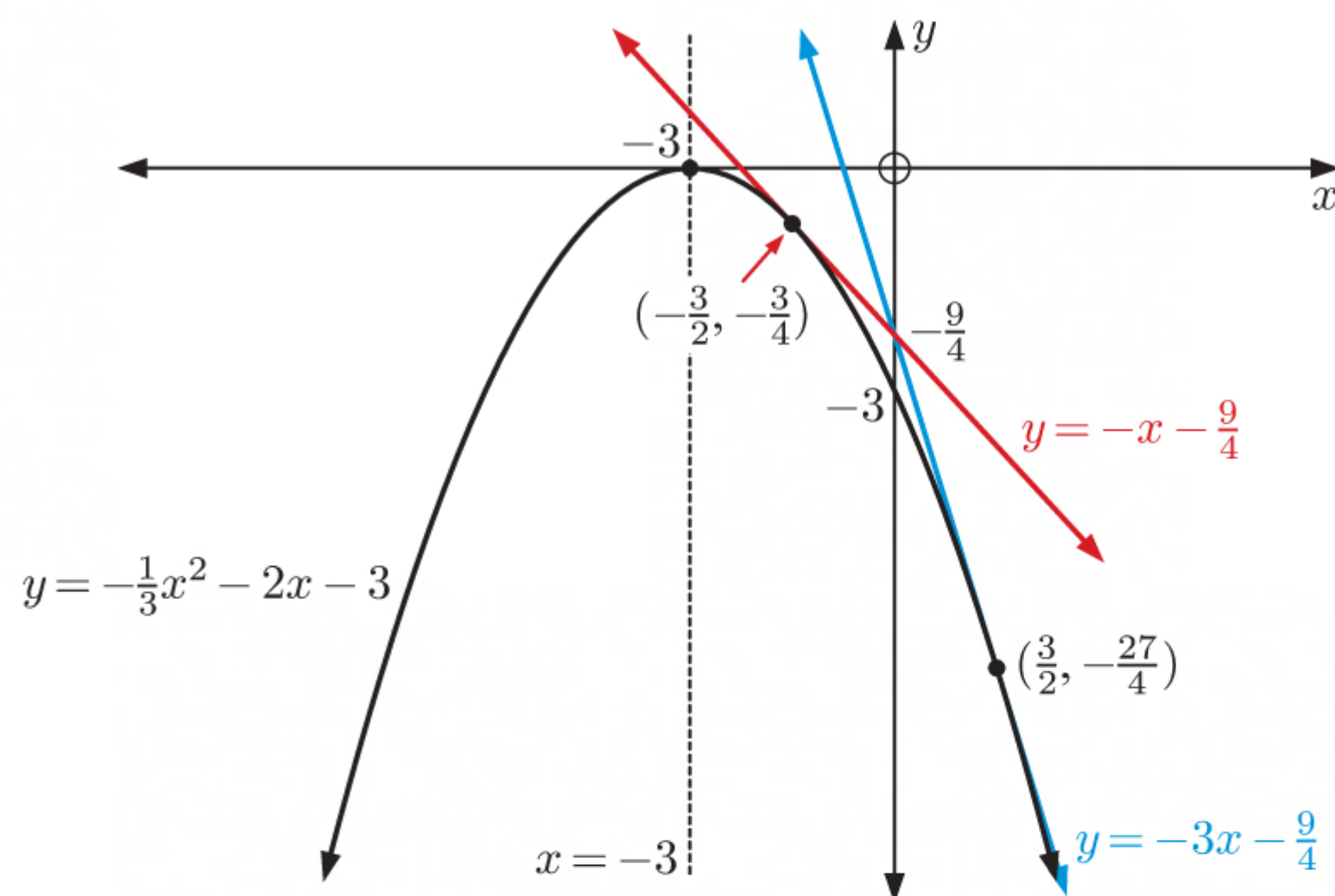
$$\therefore 4x^2 + 12x + 9 = 0$$

$$\therefore (2x + 3)^2 = 0$$

$$\therefore x = -\frac{3}{2}$$

Substituting into $y = -x - \frac{9}{4}$, when $x = -\frac{3}{2}$, $y = -\frac{3}{4}$.

\therefore the tangent $y = -x - \frac{9}{4}$ meets the curve at $(-\frac{3}{2}, -\frac{3}{4})$.



21 Let the quadratic function be $y = ax^2 + bx + c$.

The y -intercept is 4. $\therefore c = 4$

Now, $y = ax^2 + bx + 4$ meets $y = x - 5$ where $ax^2 + bx + 4 = x - 5$

$$\therefore ax^2 + (b-1)x + 9 = 0$$

The graphs touch when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (b-1)^2 - 4(a)(9) = 0$$

$$\therefore (b-1)^2 - 36a = 0$$

$$\therefore 36a = (b-1)^2$$

$$\therefore a = \frac{1}{36}(b-1)^2 \quad \dots (*)$$

Also, $y = ax^2 + bx + 4$ meets $y = -2x$ where $ax^2 + bx + 4 = -2x$

$$\therefore ax^2 + (b+2)x + 4 = 0$$

The graphs touch when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (b+2)^2 - 4(a)(4) = 0$$

$$\therefore (b+2)^2 - 16a = 0$$

$$\therefore (b+2)^2 = 16a$$

$$\therefore (b+2)^2 = \frac{16}{36}(b-1)^2 \quad \{\text{using (*)}\}$$

$$\therefore b^2 + 4b + 4 = \frac{4}{9}(b^2 - 2b + 1)$$

$$\therefore 9(b^2 + 4b + 4) = 4(b^2 - 2b + 1)$$

$$\therefore 9b^2 + 36b + 36 = 4b^2 - 8b + 4$$

$$\therefore 5b^2 + 44b + 32 = 0$$

$$\therefore (5b+4)(b+8) = 0$$

$$\therefore b = -\frac{4}{5} \text{ or } -8$$


Substituting into (*), when $b = -\frac{4}{5}$, $a = \frac{1}{36}\left(-\frac{4}{5} - 1\right)^2 = \frac{9}{100}$,

and when $b = -8$, $a = \frac{1}{36}(-8 - 1)^2 = \frac{9}{4}$.

\therefore the quadratic function is $y = \frac{9}{100}x^2 - \frac{4}{5}x + 4$ or $y = \frac{9}{4}x^2 - 8x + 4$.

22 a Total profit $P = x\left[\left(42 - \frac{x}{15}\right) - \left(26 + \frac{10}{x}\right)\right]$
 $= x\left[-\frac{x}{15} + 16 - \frac{10}{x}\right]$
 $= -\frac{1}{15}x^2 + 16x - 10$ euros

b $P = -\frac{1}{15}x^2 + 16x - 10$ has $a = -\frac{1}{15}$, $b = 16$, and $c = -10$

Since $a < 0$, the shape is 

The maximum profit occurs when $x = \frac{-b}{2a} = \frac{-16}{2\left(-\frac{1}{15}\right)} = 120$

So, 120 radios should be made per day to maximise profit.

c When $x = 120$, $P = -\frac{1}{15}(120)^2 + 16(120) - 10$
 $= 950$

So, the maximum profit is €950.

23 a Let the height of the equilateral triangle ends be h cm.

$$\therefore \left(\frac{x}{2}\right)^2 + h^2 = x^2 \quad \{\text{Pythagoras}\}$$

$$\therefore \frac{x^2}{4} + h^2 = x^2$$

$$\therefore h^2 = \frac{3x^2}{4}$$

$$\therefore h = \frac{\sqrt{3}}{2}x \quad \{h > 0\}$$

So, area of end $= \frac{1}{2} \times x \times h$
 $= \frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x$
 $= \frac{\sqrt{3}}{4}x^2 \text{ cm}^2$

b The sum of all side lengths of the prism must be 1.8 m or 180 cm.

$$\therefore 6x + 3y = 180$$

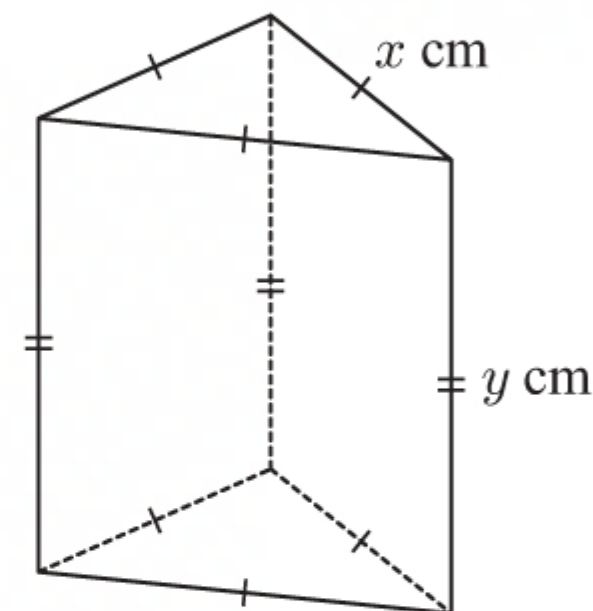
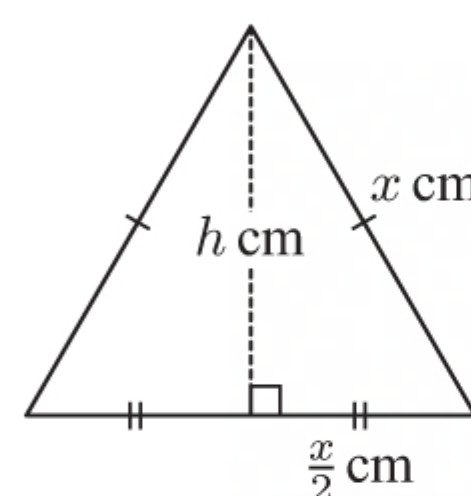
$$\therefore 3y = 180 - 6x$$

$$\therefore y = 60 - 2x$$

So, area of rectangular face $= x \times y$

$$= x(60 - 2x)$$


$$= 60x - 2x^2 \text{ cm}^2$$



\therefore the total surface area $A = 2 \times \text{area of end} + 3 \times \text{area of rectangular face}$

$$\begin{aligned} &= 2\left(\frac{\sqrt{3}}{4}x^2\right) + 3(60x - 2x^2) \quad \{\text{using a}\} \\ &= \frac{\sqrt{3}}{2}x^2 + 180x - 6x^2 \\ &= \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x \text{ cm}^2 \end{aligned}$$

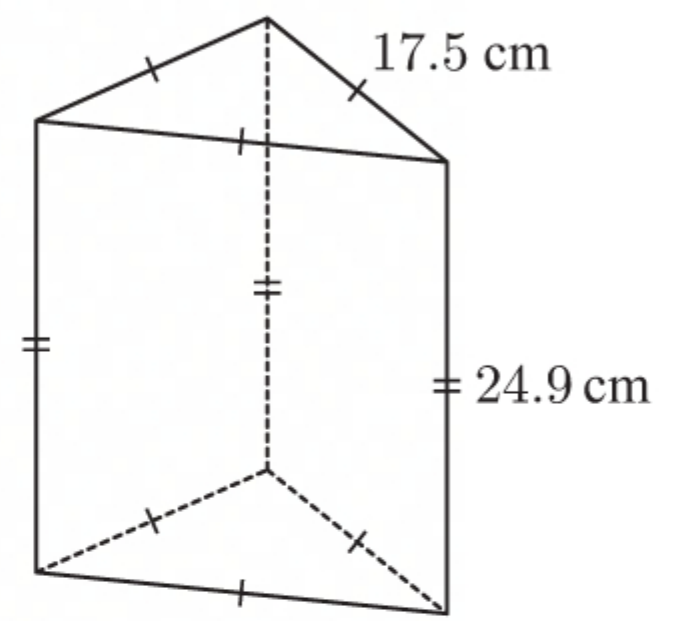
c $A = \left(\frac{\sqrt{3}}{2} - 6\right)x^2 + 180x$ has $a = \frac{\sqrt{3}}{2} - 6$, $b = 180$, and $c = 0$

Since $a < 0$, the shape is 

The maximum surface area occurs when $x = \frac{-b}{2a} = \frac{-180}{2\left(\frac{\sqrt{3}}{2} - 6\right)} \approx 17.5$

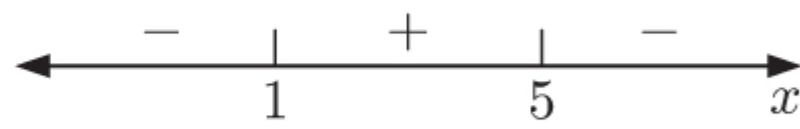
and $y = 60 - 2x \approx 24.9$

So, the dimensions Andreas should choose for the aquarium are shown alongside:



24 a $(x - 1)(5 - x) \leq 0$

Sign diagram of LHS is

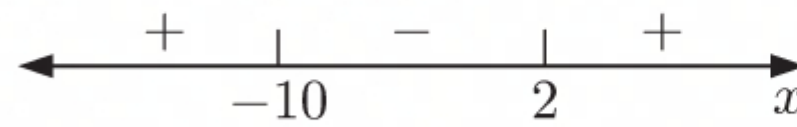


$\therefore x \leq 1$ or $x \geq 5$

b $x^2 + 8x - 20 < 0$

$\therefore (x + 10)(x - 2) < 0$

Sign diagram of LHS is



$\therefore -10 < x < 2$

c $-9x^2 + 4x + 5 \geq 0$

$\therefore 9x^2 - 4x - 5 \leq 0$

$\therefore (9x + 5)(x - 1) \leq 0$

Sign diagram of LHS is



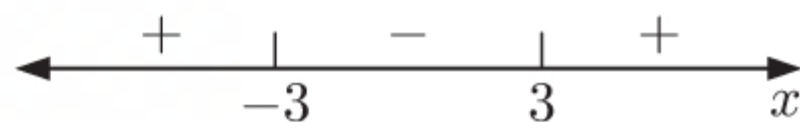
$\therefore -\frac{5}{9} \leq x \leq 1$

25 a $x^2 > 9$

$\therefore x^2 - 9 > 0$

$\therefore (x + 3)(x - 3) > 0$

Sign diagram of LHS is



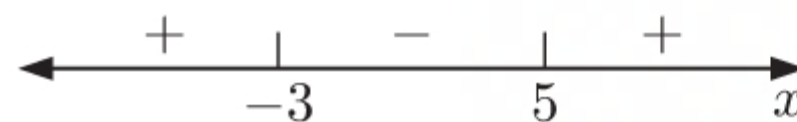
$\therefore x < -3$ or $x > 3$

b $x^2 - 15 \leq 2x$

$\therefore x^2 - 2x - 15 \leq 0$

$\therefore (x - 5)(x + 3) \leq 0$

Sign diagram of LHS is



$\therefore -3 \leq x \leq 5$

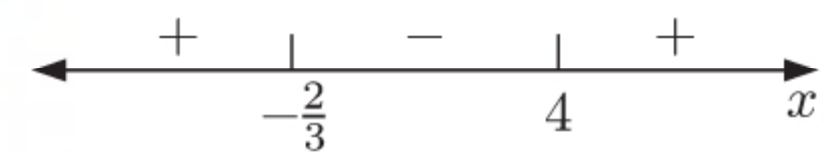
c $3x^2 < 2(5x + 4)$

$\therefore 3x^2 < 10x + 8$

$\therefore 3x^2 - 10x - 8 < 0$

$\therefore (3x + 2)(x - 4) < 0$

Sign diagram of LHS is



$\therefore -\frac{2}{3} < x < 4$

26 $y = kx^2 - (k - 6)x + (k - 6)$ has $a = k$, $b = -(k - 6)$, and $c = k - 6$

$\therefore \Delta = b^2 - 4ac$

$= [-(k - 6)]^2 - 4(k)(k - 6)$

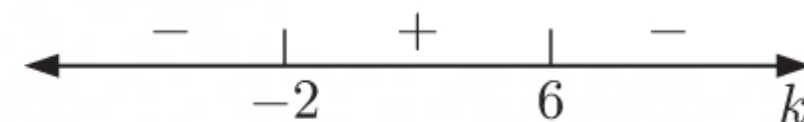
$= k^2 - 12k + 36 - 4k^2 + 24k$

$= -3k^2 + 12k + 36$

$= -3(k^2 - 4k - 12)$

$= -3(k - 6)(k + 2)$

So, Δ has sign diagram



a The graph cuts the x -axis twice if $\Delta > 0$.

$\therefore -2 < k < 6$, $k \neq 0$

b The graph touches the x -axis if $\Delta = 0$.

$\therefore k = -2$ or $k = 6$

c The graph misses the x -axis if $\Delta < 0$.

$\therefore k < -2$ or $k > 6$

27 $W(t) = 1000 - 0.5t$ litres

a $W(0) = 1000$

The initial amount of water in the tank was 1000 L.

c The tank is empty when $W(t) = 0$.

This occurs when $0.5t = 1000$

$\therefore t = 2000$

It will take 2000 hours, or 83 days, 8 hours, for the tank to empty.

b $W(t) = 700$, so $700 = 1000 - 0.5t$

$\therefore 0.5t = 300$

$\therefore t = 600$

After 600 hours, or 25 days, the amount of water in the tank is 700 L.

28 $f(x) = \frac{x-2}{x-3}$

a $f(-x) = \frac{-x-2}{-x-3}$
 $= \frac{-(x+2)}{-(x+3)}$
 $= \frac{x+2}{x+3}$

b $f(x+2) = \frac{(x+2)-2}{(x+2)-3}$
 $= \frac{x+2-2}{x+2-3}$
 $= \frac{x}{x-1}$

c $f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-2}{\frac{1}{x}-3} \times \frac{x}{x}$
 $= \frac{1-2x}{1-3x}$

29 From the graph: Domain is $\{x \mid -6 \leq x \leq 6 \text{ and } x \neq 3\}$

Range is $\{y \mid 0 \leq y \leq 5\}$

a $x = 0$ satisfies $-6 \leq x \leq 6$ and $x \neq 3$.

\therefore “0 is in the domain of f ” is true.

b $y = 0$ satisfies $0 \leq y \leq 5$.

\therefore “0 is in the range of f ” is true.

c $y = 6$ does not satisfy $0 \leq y \leq 5$.

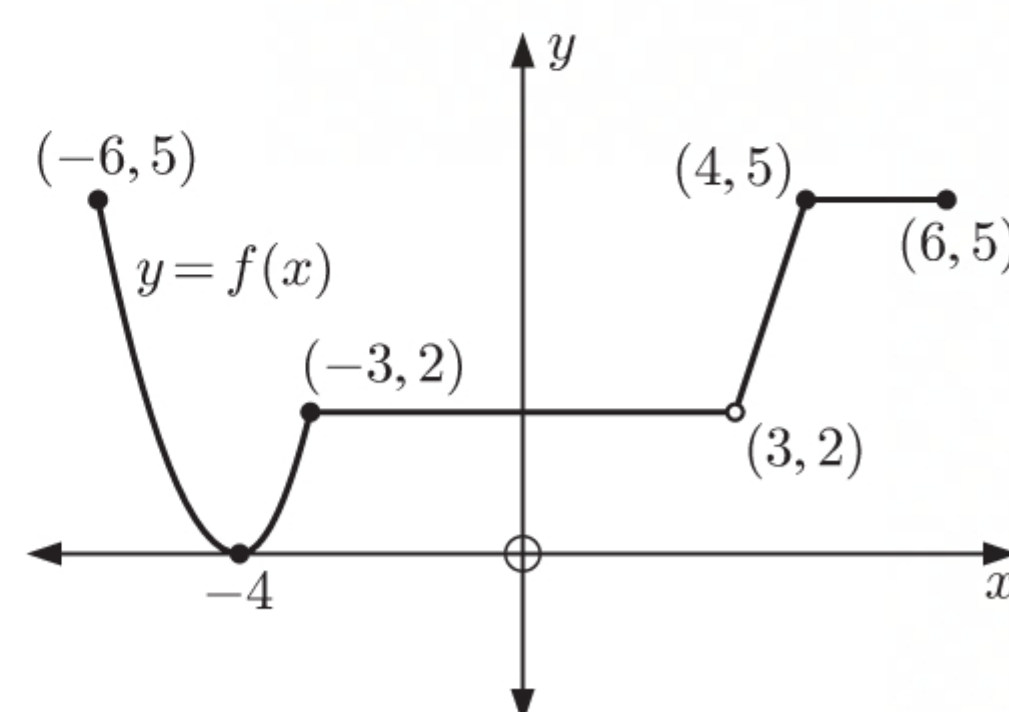
\therefore “6 is in the range of f ” is false.

d $x = 3$ does not satisfy $x \neq 3$.

\therefore “3 is in the domain of f ” is false.

e $y = 2$ satisfies $0 \leq y \leq 5$.

\therefore “2 is in the range of f ” is true.



30 a $\sqrt{3-2x}$ is defined when $3-2x \geq 0$

$\therefore 2x \leq 3$

$\therefore x \leq \frac{3}{2}$

\therefore the domain is $\{x \mid x \leq \frac{3}{2}\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.

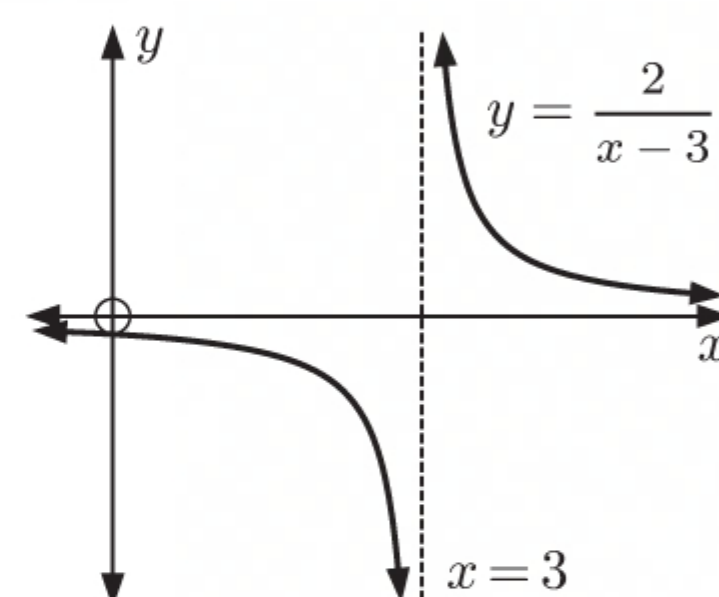
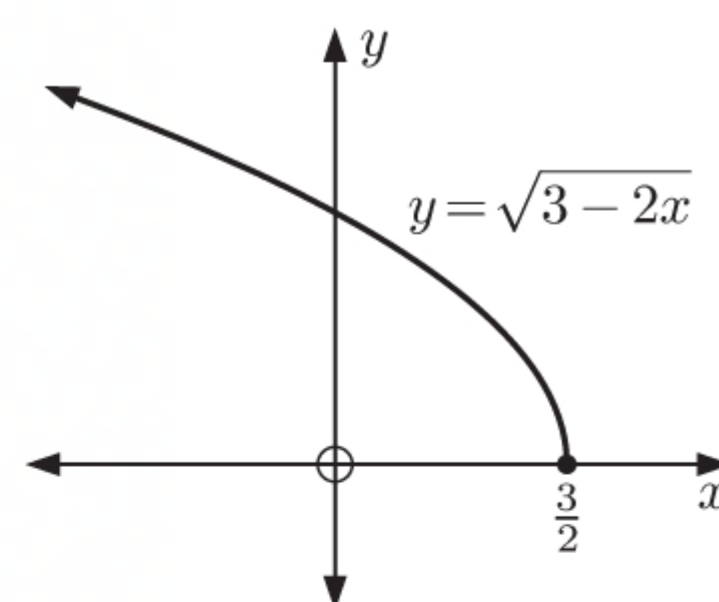
b $\frac{2}{x-3}$ is defined when $x-3 \neq 0$

$\therefore x \neq 3$

\therefore the domain is $\{x \mid x \neq 3\}$.

No matter how large or small x is, $y = f(x)$ is never zero.

\therefore the range is $\{y \mid y \neq 0\}$.



c $\frac{1}{\sqrt{x}-2}$ is defined when $x \geq 0$ and $\sqrt{x}-2 \neq 0$

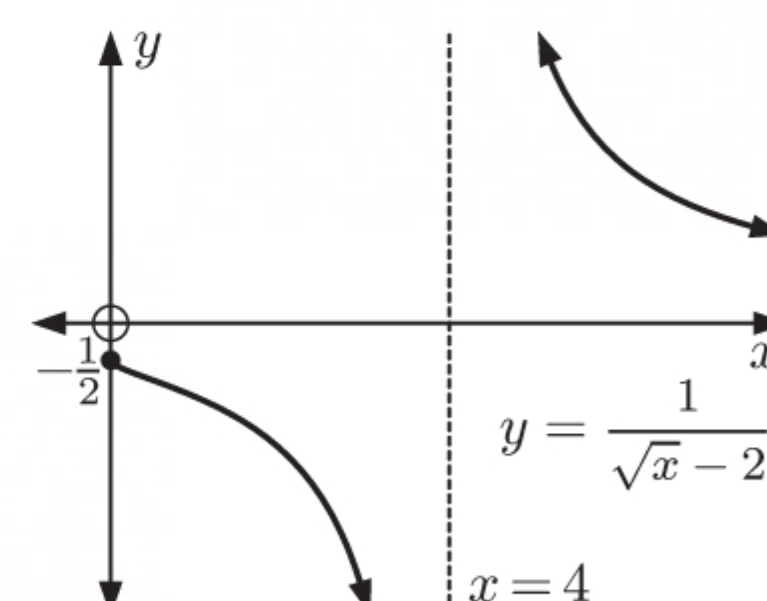
$\therefore x \geq 0$ and $\sqrt{x} \neq 2$

$\therefore x \geq 0$ and $x \neq 4$

\therefore the domain is $\{x \mid x \geq 0 \text{ and } x \neq 4\}$.

For $0 \leq x < 4$, $y \leq -\frac{1}{2}$, and for $x > 4$, $y > 0$.

\therefore the range is $\{y \mid y \leq -\frac{1}{2} \text{ or } y > 0\}$.



31 $k(t) = 2t - 4$ for $0 \leq t < 4$, $t \in \mathbb{Z}$

a The function is defined for t such that $0 \leq t < 4$ and $t \in \mathbb{Z}$.

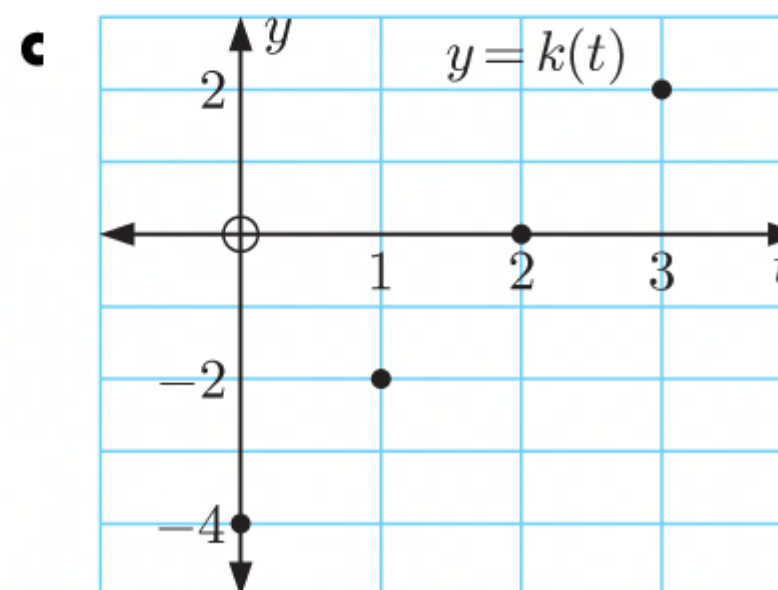
\therefore the domain is $\{0, 1, 2, 3\}$.

b $k(0) = -4$

$k(1) = -2$

$k(2) = 0$

$k(3) = 2 \quad \therefore$ the range is $\{-4, -2, 0, 2\}$.



32 a $f(x) = \sqrt{kx^2 + (2k+1)x + (k+2)}$ has natural domain $x \in \mathbb{R}$

$\therefore kx^2 + (2k+1)x + (k+2) \geq 0$ for all $x \in \mathbb{R}$ (*)

This quadratic has discriminant $\Delta = (2k+1)^2 - 4(k)(k+2)$
 $= 4k^2 + 4k + 1 - 4k^2 - 8k$
 $= -4k + 1$

Now (*) is true whenever $\Delta \leq 0$

$\therefore -4k + 1 \leq 0$

$\therefore -4k \leq -1$

$\therefore k \geq \frac{1}{4}$

b $f(x)$ is non-negative for all x and its minimum value occurs where the quadratic $kx^2 + (2k+1)x + (k+2)$ is minimised.

Now $k \geq \frac{1}{4} > 0$ {from **a**}

\therefore the vertex of the quadratic is a minimum turning point.

$\therefore f(x)$ is minimised at $x = -\frac{(2k+1)}{2k} = \frac{-2k-1}{2k}$.

\therefore minimum value is $f\left(\frac{-2k-1}{2k}\right) = \sqrt{k\left(\frac{-2k-1}{2k}\right)^2 + (2k+1)\left(\frac{-2k-1}{2k}\right) + (k+2)}$
 $= \sqrt{\frac{4k^2 + 4k + 1}{4k} - \left(\frac{4k^2 + 4k + 1}{2k}\right) + k + 2}$
 $= \sqrt{\frac{4k^2 + 4k + 1 - 8k^2 - 8k - 2 + 4k^2 + 8k}{4k}}$
 $= \sqrt{\frac{4k - 1}{4k}}$
 $= \sqrt{1 - \frac{1}{4k}}$

\therefore the range of $f(x)$ is $\left\{y \mid y \geq \sqrt{1 - \frac{1}{4k}}\right\}$.

33 $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$

a The domain is $\{x \mid x \neq 1\}$.

The range is $\{y \mid y \neq 1\}$.

c $f(0) = \frac{2}{-1} = -2$, so the y -intercept is -2 .

$f(x) = 0$ when $x + 2 = 0$

$\therefore x = -2$

\therefore the x -intercept is -2 .

e As $x \rightarrow 1^-$, $f(x) \rightarrow -\infty$

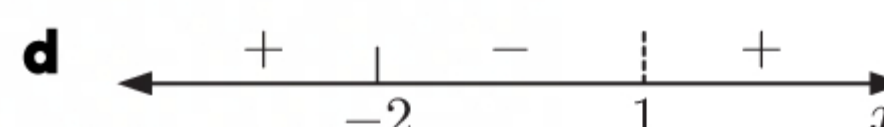
As $x \rightarrow 1^+$, $f(x) \rightarrow \infty$

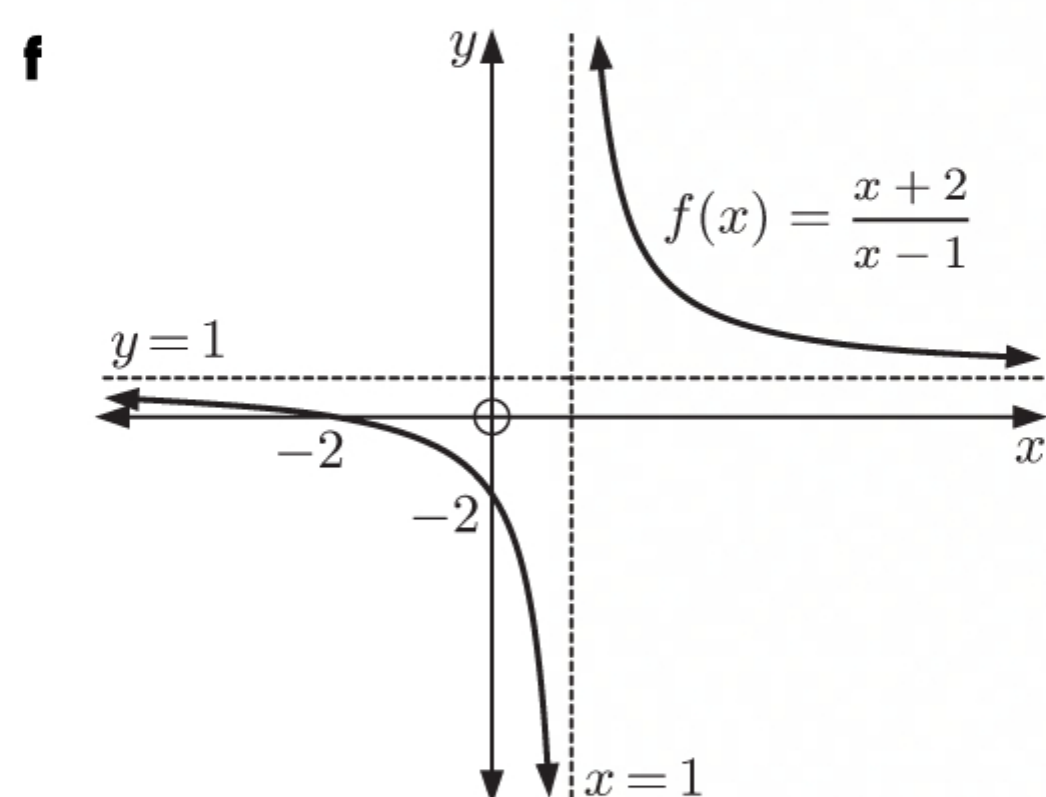
As $x \rightarrow -\infty$, $f(x) \rightarrow 1^-$

As $x \rightarrow \infty$, $f(x) \rightarrow 1^+$

b The vertical asymptote is $x = 1$.

The horizontal asymptote is $y = 1$.





34 $f(x) = 2 + \frac{4}{x+1}$

a $f(0) = 2 + \frac{4}{1} = 6$, so the y -intercept is 6.

$f(x) = 0$ when $2 + \frac{4}{x+1} = 0$

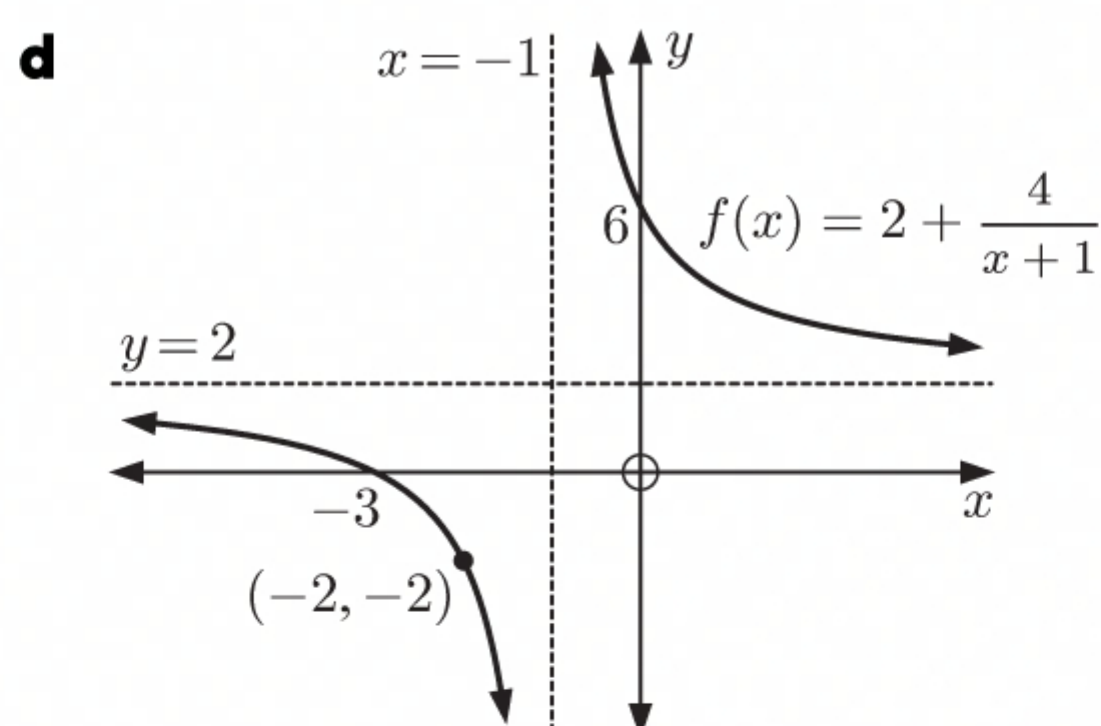
$\therefore 2(x+1) + 4 = 0$

$\therefore 2(x+1) = -4$

$\therefore x+1 = -2$

$\therefore x = -3$

\therefore the x -intercept is -3 .



b $f(-2) = 2 + \frac{4}{-2+1}$

$= 2 + \frac{4}{-1}$

$= -2$

c i The horizontal asymptote is $y = 2$.

ii The vertical asymptote is $x = -1$.

35 a i The horizontal asymptote is $y = 0$.

The function is undefined when $x^2 + 4x - 21 = 0$

$\therefore (x+7)(x-3) = 0$

$\therefore x = -7$ or 3

\therefore the vertical asymptotes are $x = -7$ and $x = 3$.

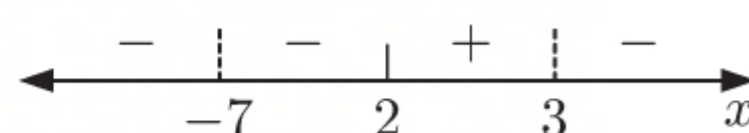
ii When $x = 0$, $y = -\frac{2}{21}$, so the y -intercept is $-\frac{2}{21}$.

When $y = 0$, $2 - x = 0$

$\therefore x = 2$

\therefore the x -intercept is 2.

iii $y = \frac{2-x}{x^2+4x-21}$ has sign diagram



iv As $x \rightarrow -7^-$, $y \rightarrow \infty$

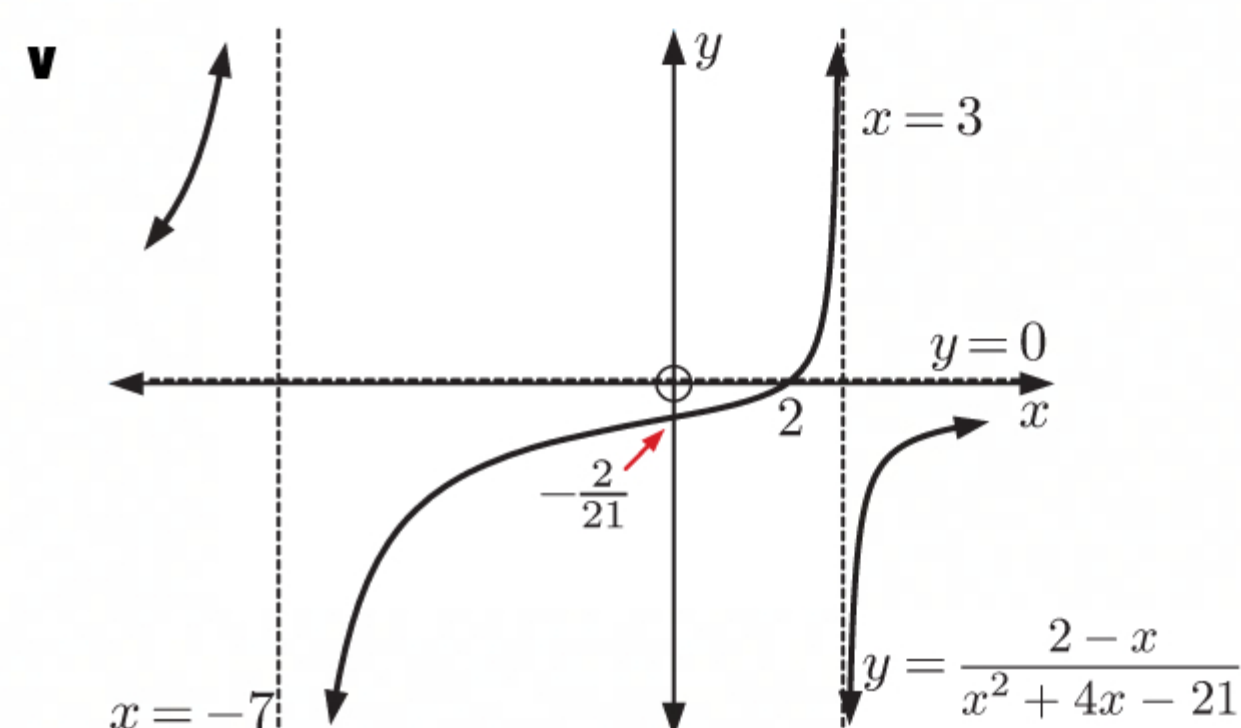
As $x \rightarrow -7^+$, $y \rightarrow -\infty$

As $x \rightarrow 3^-$, $y \rightarrow \infty$

As $x \rightarrow 3^+$, $y \rightarrow -\infty$

As $x \rightarrow -\infty$, $y \rightarrow 0^+$

As $x \rightarrow \infty$, $y \rightarrow 0^-$



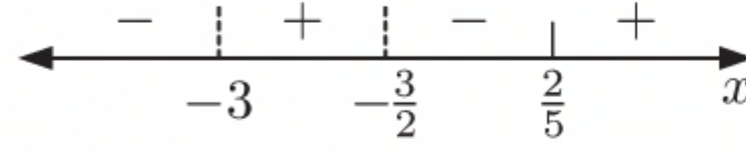
- b i** The horizontal asymptote is $y = 0$.

The function is undefined when

$$\begin{aligned} 2x^2 + 9x + 9 &= 0 \\ \therefore 2x^2 + 6x + 3x + 9 &= 0 \\ \therefore 2x(x + 3) + 3(x + 3) &= 0 \\ \therefore (x + 3)(2x + 3) &= 0 \\ \therefore x &= -3 \text{ or } -\frac{3}{2} \end{aligned}$$

\therefore the vertical asymptotes are $x = -3$ and $x = -\frac{3}{2}$.

- iii** $y = \frac{5x-2}{2x^2+9x+9}$ has sign diagram



- iv** As $x \rightarrow -3^-$, $y \rightarrow -\infty$

As $x \rightarrow -3^+$, $y \rightarrow \infty$

As $x \rightarrow -\frac{3}{2}^-$, $y \rightarrow \infty$

As $x \rightarrow -\frac{3}{2}^+$, $y \rightarrow -\infty$

As $x \rightarrow -\infty$, $y \rightarrow 0^-$

As $x \rightarrow \infty$, $y \rightarrow 0^+$

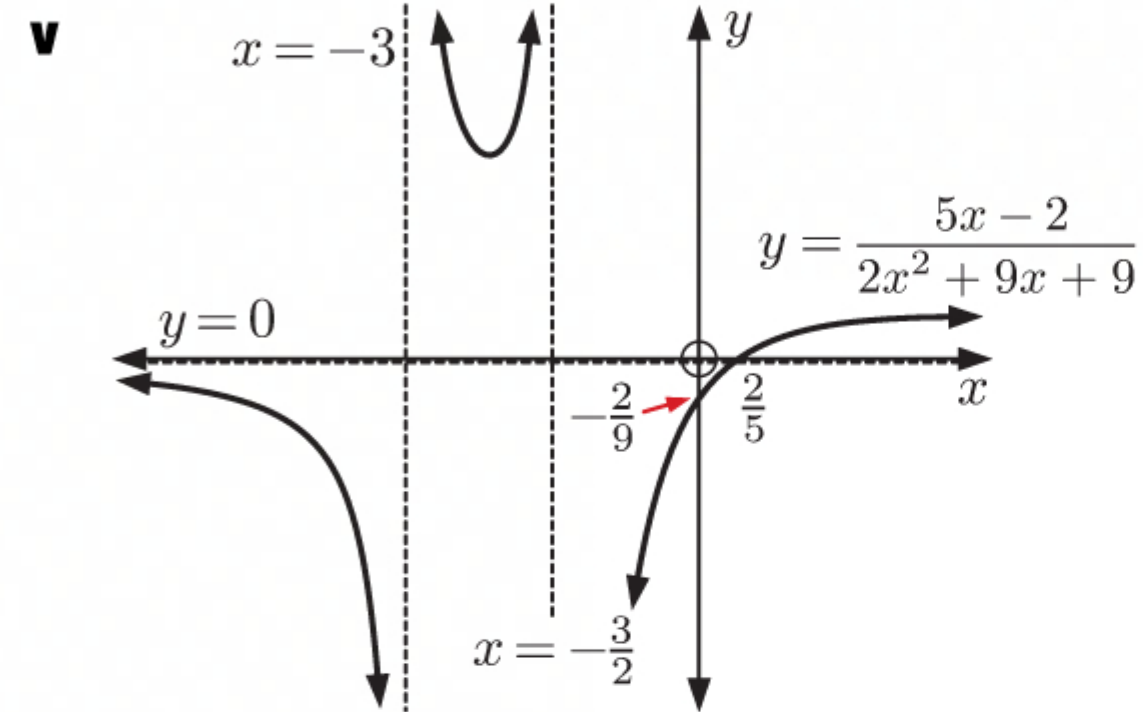
- ii** When $x = 0$, $y = -\frac{2}{9}$, so the y -intercept is $-\frac{2}{9}$.

When $y = 0$, $5x - 2 = 0$

$\therefore 5x = 2$

$\therefore x = \frac{2}{5}$

\therefore the x -intercept is $\frac{2}{5}$.



36 $f(x) = \frac{x+2}{x^2+bx+3}$

- a** $f(0) = \frac{2}{3}$, so the y -intercept is $\frac{2}{3}$.

$f(x) = 0$ when $x + 2 = 0$

$\therefore x = -2$

\therefore the x -intercept is -2 .

- b** $x^2 + bx + 3 = 0 \dots (*)$ has discriminant $\Delta = b^2 - 4(1)(3)$
 $= b^2 - 12$
 $= (b - \sqrt{12})(b + \sqrt{12})$

Δ has sign diagram

- i** $f(x)$ has no vertical asymptotes when $(*)$ has no real solutions.

$\therefore \Delta < 0$

$\therefore -\sqrt{12} < b < \sqrt{12}$

- ii** $f(x)$ has one vertical asymptote when $(*)$ has 1 real solution.

$\therefore \Delta = 0$

$\therefore b = \pm\sqrt{12}$

- iii** $f(x)$ has two vertical asymptotes when $(*)$ has 2 real solutions.

$\therefore \Delta > 0$

$\therefore b < -\sqrt{12} \text{ or } b > \sqrt{12}$

37 a $y = \frac{x^2 - 2x - 8}{x - 3}$

- i** The vertical asymptote is $x = 3$.

- ii** When $x = 0$, $y = \frac{-8}{-3} = \frac{8}{3}$, so the y -intercept is $\frac{8}{3}$.

When $y = 0$, $x^2 - 2x - 8 = 0$

$\therefore (x + 2)(x - 4) = 0$

$\therefore x = -2 \text{ or } 4$

\therefore the x -intercepts are -2 and 4 .

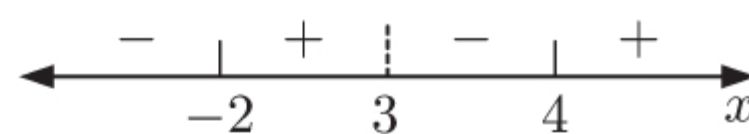
iii $y = \frac{x^2 - 2x - 8}{x - 3}$

$$= x + 1 - \frac{5}{x - 3}$$

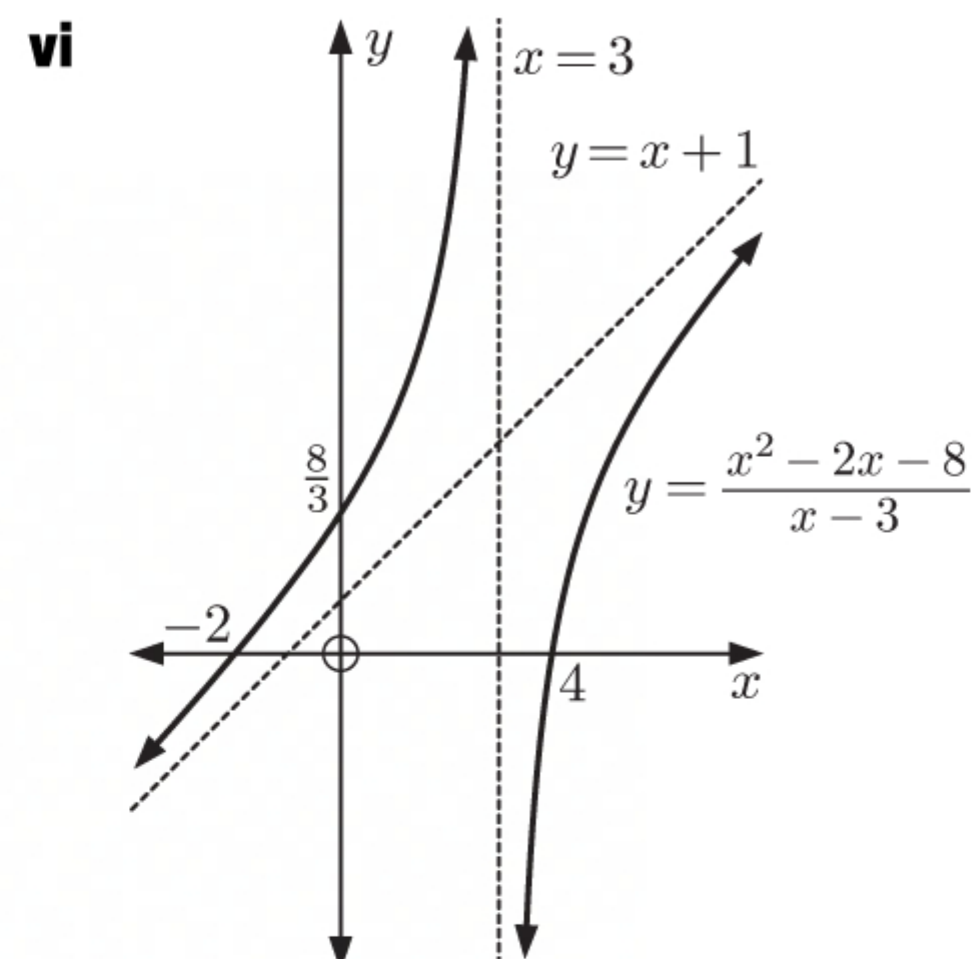
$$\begin{array}{r} x + 1 \\ x - 3 \overline{) x^2 - 2x - 8} \\ \underline{-(x^2 - 3x)} \\ x - 8 \\ \underline{-(x - 3)} \\ -5 \end{array}$$

\therefore the oblique asymptote is $y = x + 1$.

iv $y = \frac{x^2 - 2x - 8}{x - 3}$ has sign diagram



- v** As $x \rightarrow 3^-$, $y \rightarrow \infty$
 As $x \rightarrow 3^+$, $y \rightarrow -\infty$
 As $x \rightarrow -\infty$, $y \rightarrow x + 1^+$
 As $x \rightarrow \infty$, $y \rightarrow x + 1^-$



b $y = \frac{6x^2 - 7x - 5}{3x + 2}$

- i** The vertical asymptote is $x = -\frac{2}{3}$.
ii When $x = 0$, $y = -\frac{5}{2}$, so the y -intercept is $-\frac{5}{2}$.

When $y = 0$, $6x^2 - 7x - 5 = 0$
 $\therefore 6x^2 + 3x - 10x - 5 = 0$
 $\therefore 3x(2x + 1) - 5(2x + 1) = 0$
 $\therefore (2x + 1)(3x - 5) = 0$
 $\therefore x = -\frac{1}{2} \text{ or } \frac{5}{3}$

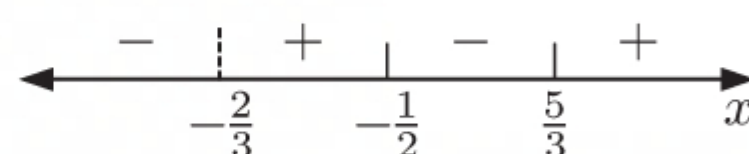
\therefore the x -intercepts are $-\frac{1}{2}$ and $\frac{5}{3}$.

iii $y = \frac{6x^2 - 7x - 5}{3x + 2}$
 $= 2x - \frac{11}{3} + \frac{7}{3(3x + 2)}$

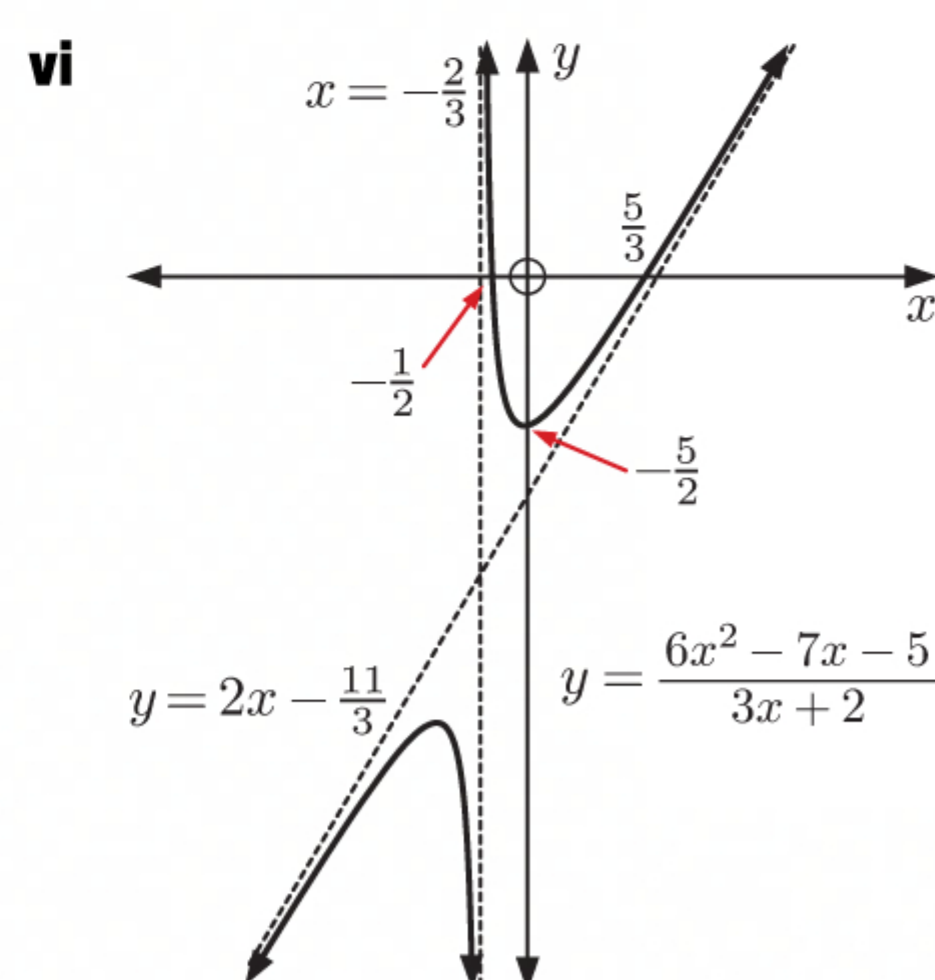
$$\begin{array}{r} 2x - \frac{11}{3} \\ 3x + 2 \overline{) 6x^2 - 7x - 5} \\ \underline{-(6x^2 + 4x)} \\ -11x - 5 \\ \underline{-(-11x - \frac{22}{3})} \\ \frac{7}{3} \end{array}$$

\therefore the oblique asymptote is $y = 2x - \frac{11}{3}$.

iv $y = \frac{6x^2 - 7x - 5}{3x + 2}$ has sign diagram



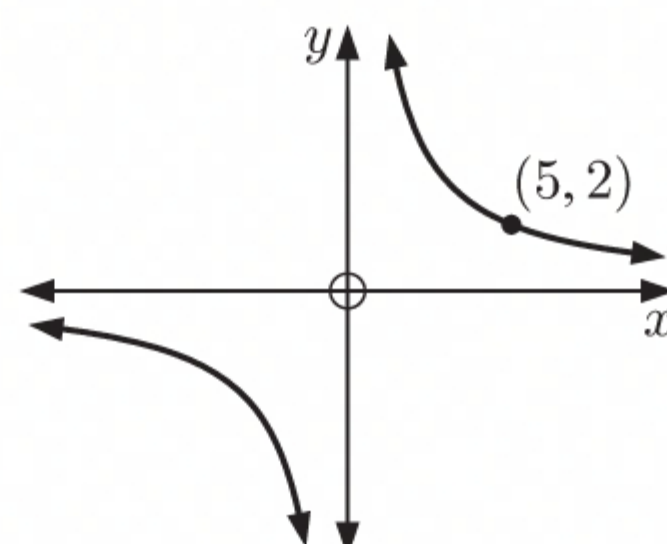
- v** As $x \rightarrow -\frac{2}{3}^-$, $y \rightarrow -\infty$
 As $x \rightarrow -\frac{2}{3}^+$, $y \rightarrow \infty$
 As $x \rightarrow -\infty$, $y \rightarrow 2x - \frac{11}{3}^-$
 As $x \rightarrow \infty$, $y \rightarrow 2x - \frac{11}{3}^+$



38 $f(x) = \frac{k}{x}$

- a** When $x = 5$, $y = 2$
 $\therefore 2 = \frac{k}{5}$
 $\therefore k = 10$

- b** The domain is $\{x \mid x \neq 0\}$.
 The range is $\{y \mid y \neq 0\}$.



$$\text{c } f\left(-\frac{1}{2}\right) = \frac{10}{-\frac{1}{2}} = -20$$

$$\text{d } f \text{ is } y = \frac{10}{x}$$

$$\therefore f^{-1} \text{ is } x = \frac{10}{y}$$

$$\therefore y = \frac{10}{x}$$

$$\text{So, } f^{-1}(x) = \frac{10}{x} = f(x).$$

$\therefore f$ is self-inverse.

$$39 \quad f(x) = \frac{1}{x-1} + \sqrt{x+1}, \quad g(x) = x^2$$

$$\text{a } \frac{1}{x-1} \text{ is defined when } x-1 \neq 0$$

$$\therefore x \neq 1$$

$$\sqrt{x+1} \text{ is defined when } x+1 \geq 0$$

$$\therefore x \geq -1$$

\therefore the domain of f is $\{x \mid x \geq -1 \text{ and } x \neq 1\}$.

c The domain of g is $\{x \mid x \in \mathbb{R}\}$.

$$\text{Now } \frac{1}{x^2-1} \text{ is defined when } x^2-1 \neq 0$$

$$\therefore (x+1)(x-1) \neq 0$$

$$\therefore x \neq -1 \text{ or } 1$$

and $x^2+1 \geq 0$, so $\sqrt{x^2+1}$ is defined for every value of x .

\therefore the domain of $(f \circ g)$ is $\{x \mid x \neq -1 \text{ and } x \neq 1\}$.

This is different to the domain of f and g since $(f \circ g)$ is defined using g whose range is $\{y \mid y \geq 0\}$.

$$40 \quad f: x \mapsto e^{x+1}, \quad g: x \mapsto \ln x - 1$$

$$\text{a } (f \circ g)(x) = f(g(x))$$

$$= f(\ln x - 1)$$

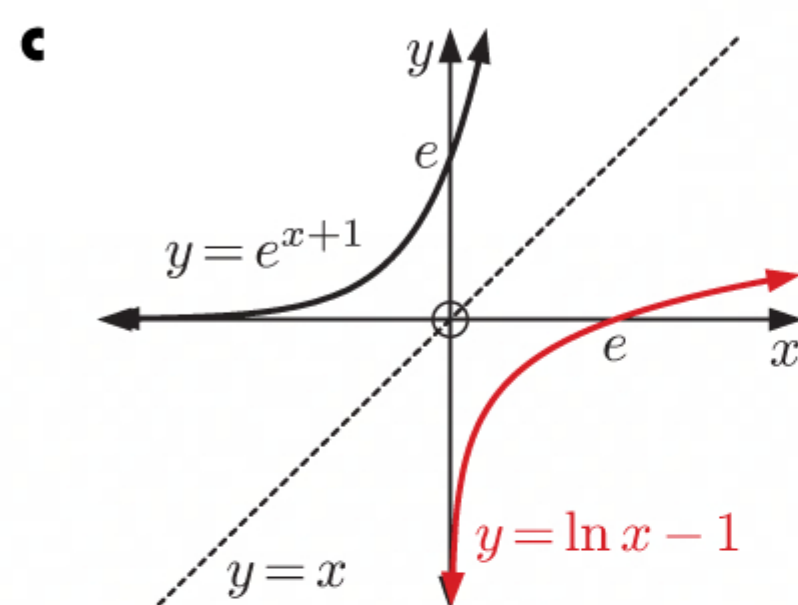
$$= e^{\ln x - 1 + 1}$$

$$= e^{\ln x}$$

$$= x$$

The domain is $\{x \mid x > 0\}$.

The range is $\{y \mid y > 0\}$.



$$\text{b } (g \circ f)(x) = g(f(x))$$

$$= g(e^{x+1})$$

$$= \ln(e^{x+1}) - 1$$

$$= x + 1 - 1$$

$$= x$$

The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \in \mathbb{R}\}$.

d The graph of $y = g(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$.

$\therefore f$ and g are inverses of one another.

$$41 \quad f(x) = \sqrt[3]{x}$$

$$f \text{ is } y = \sqrt[3]{x}$$

$$\therefore f^{-1} \text{ is } x = \sqrt[3]{y}$$

$$\therefore y = x^3$$

$$\therefore f^{-1}(x) = x^3$$

$$\text{a } \text{If } (f \circ g)(x) = 2x - 1,$$

$$\text{then } f^{-1}(f(g(x))) = f^{-1}(2x - 1)$$

$$\therefore (f^{-1} \circ f)(g(x)) = (2x - 1)^3$$

$$\therefore g(x) = (2x - 1)^3$$

$$\text{b } \text{If } (g \circ f)(x) = 2x - 1,$$

$$\text{then } (g \circ f)(f^{-1}(x)) = 2f^{-1}(x) - 1$$

$$\therefore g((f \circ f^{-1})(x)) = 2x^3 - 1$$

$$\therefore g(x) = 2x^3 - 1$$

$$42 \quad f(x) = \sqrt{x+4}, \quad g(x) = x^2 - 3$$

$$\begin{aligned} \mathbf{a} \quad (f \circ g)(x) &= f(g(x)) \\ &= f(x^2 - 3) \\ &= \sqrt{x^2 - 3 + 4} \\ &= \sqrt{x^2 + 1} \end{aligned}$$

$x^2 + 1 \geq 1$, so $\sqrt{x^2 + 1}$ is defined for every value of x .

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$.

$$x^2 + 1 \geq 1$$

$$\therefore \sqrt{x^2 + 1} \geq 1$$

\therefore the range is $\{y \mid y \geq 1\}$.

$$\begin{aligned} \mathbf{b} \quad (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt{x+4}) \\ &= (\sqrt{x+4})^2 - 3 \\ &= x + 1 \end{aligned}$$

$\sqrt{x+4}$ is defined when $x + 4 \geq 0$
 $\therefore x \geq -4$

\therefore the domain is $\{x \mid x \geq -4\}$.

$$x \geq -4$$

$$\therefore x + 1 \geq -3$$

\therefore the range is $\{y \mid y \geq -3\}$.

$$43 \quad f: x \mapsto 3x + 1, \quad g: x \mapsto 4 - x$$

$$\begin{aligned} \mathbf{a} \quad f(g(x)) &= f(4 - x) \\ &= 3(4 - x) + 1 \\ &= 12 - 3x + 1 \\ &= 13 - 3x \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (g \circ f)(-4) &= g(f(-4)) \\ &= g(3(-4) + 1) \\ &= g(-11) \\ &= 4 - (-11) \\ &= 15 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad f \text{ is } y &= 3x + 1 \\ \therefore f^{-1} \text{ is } x &= 3y + 1 \\ \therefore 3y &= x - 1 \\ \therefore y &= \frac{1}{3}x - \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{So, } f^{-1}(x) &= \frac{1}{3}x - \frac{1}{3} \\ \therefore f^{-1}\left(\frac{1}{2}\right) &= \frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{3} \\ &= \frac{1}{6} - \frac{1}{3} \\ &= -\frac{1}{6} \end{aligned}$$

$$44 \quad f: x \mapsto \ln x, \quad g: x \mapsto 3 + x$$

$$\begin{aligned} \mathbf{a} \quad f \text{ is } y &= \ln x & g \text{ is } y &= 3 + x \\ \therefore f^{-1} \text{ is } x &= \ln y & \therefore g^{-1} \text{ is } x &= 3 + y \\ \therefore y &= e^x & \therefore y &= x - 3 \\ \text{So, } f^{-1}(x) &= e^x & \text{So, } g^{-1}(x) &= x - 3 \end{aligned}$$

$$\therefore f^{-1}(2) \times g^{-1}(2) = e^2(2 - 3) = -e^2$$

$$\begin{aligned} \mathbf{b} \quad (f \circ g)(x) &= f(g(x)) & f \circ g \text{ is } y &= \ln(3 + x) \\ &= f(3 + x) & \therefore (f \circ g)^{-1} \text{ is } x &= \ln(3 + y) \\ &= \ln(3 + x) & \therefore e^x &= 3 + y \\ & & \therefore y &= e^x - 3 \\ & & \text{So, } (f \circ g)^{-1}(x) &= e^x - 3 \end{aligned}$$

$$\therefore (f \circ g)^{-1}(2) = e^2 - 3$$

$$\begin{aligned} \mathbf{c} \quad (g \circ f)(x) &= g(f(x)) & g \circ f \text{ is } y &= 3 + \ln x \\ &= g(\ln x) & \therefore (g \circ f)^{-1} \text{ is } x &= 3 + \ln y \\ &= 3 + \ln x & \therefore \ln y &= x - 3 \\ & & \therefore y &= e^{x-3} \\ & & \text{So, } (g \circ f)^{-1}(x) &= e^{x-3} \end{aligned}$$

$$\text{Now } (g \circ f)^{-1}(a) = \sqrt{e}$$

$$\therefore e^{a-3} = e^{\frac{1}{2}}$$

$$\therefore a - 3 = \frac{1}{2} \quad \{\text{equating indices}\}$$

$$\therefore a = \frac{7}{2}$$

45 $f: x \mapsto x + 5, \quad g: x \mapsto 7 - 3x$

a i f is $y = x + 5$
 $\therefore f^{-1}$ is $x = y + 5$
 $\therefore y = x - 5$
 So, $f^{-1}(x) = x - 5$

ii g is $y = 7 - 3x$
 $\therefore g^{-1}$ is $x = 7 - 3y$
 $\therefore 3y = 7 - x$
 $\therefore y = \frac{7}{3} - \frac{1}{3}x$
 So, $g^{-1}(x) = \frac{7}{3} - \frac{1}{3}x$

iii $(f \circ g)(x) = f(g(x))$
 $= f(7 - 3x)$
 $= (7 - 3x) + 5$
 $= 12 - 3x$

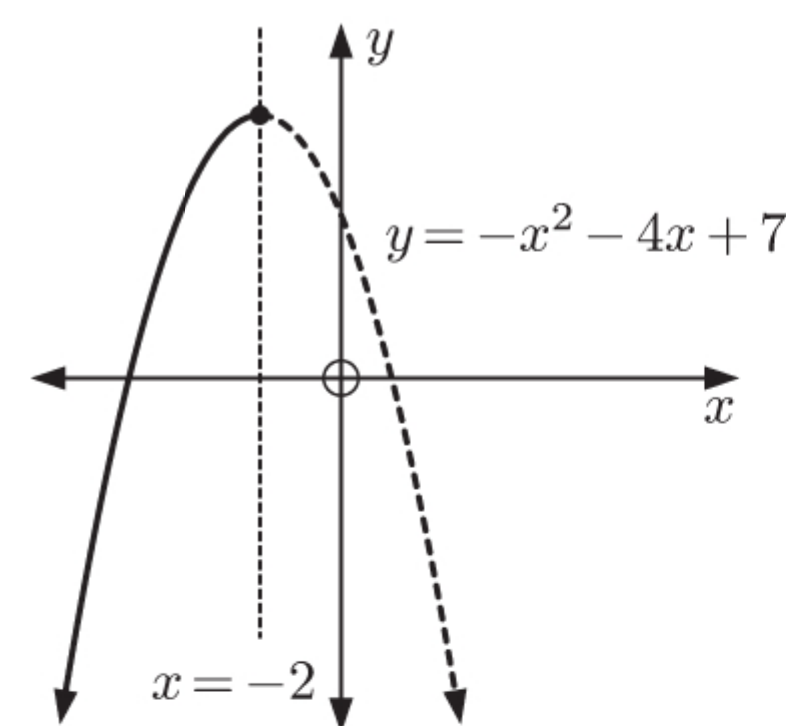
b $f \circ g$ is $y = 12 - 3x$ {using **a iii**}
 $\therefore (f \circ g)^{-1}$ is $x = 12 - 3y$
 $\therefore 3y = 12 - x$
 $\therefore y = 4 - \frac{1}{3}x$
 So, $(f \circ g)^{-1}(x) = 4 - \frac{1}{3}x$

$(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$
 $= g^{-1}(x - 5)$ {using **a i**}
 $= \frac{7}{3} - \frac{1}{3}(x - 5)$ {using **a ii**}
 $= \frac{7}{3} - \frac{1}{3}x + \frac{5}{3}$
 $= 4 - \frac{1}{3}x$
 $= (f \circ g)^{-1}(x)$

46 $f(x) = -x^2 - 4x + 7, \quad x \leq k$

a The largest value of k such that $f^{-1}(x)$ exists corresponds to the axis of symmetry of $y = -x^2 - 4x + 7$.

$\therefore k = \frac{-(-4)}{2(-1)} = \frac{4}{-2} = -2$



b i f is $y = -x^2 - 4x + 7, \quad x \leq -2$
 $\therefore f^{-1}$ is $x = -y^2 - 4y + 7, \quad y \leq -2$
 $\therefore x = -(y^2 + 4y - 7)$
 $\therefore x = -[(y + 2)^2 - 11]$
 $\therefore x = -(y + 2)^2 + 11$
 $\therefore x - 11 = -(y + 2)^2$
 $\therefore 11 - x = (y + 2)^2$
 $\therefore -\sqrt{11 - x} = y + 2 \quad \{y \leq -2\}$
 $\therefore -\sqrt{11 - x} - 2 = y$
 $\therefore f^{-1}(x) = -\sqrt{11 - x} - 2$

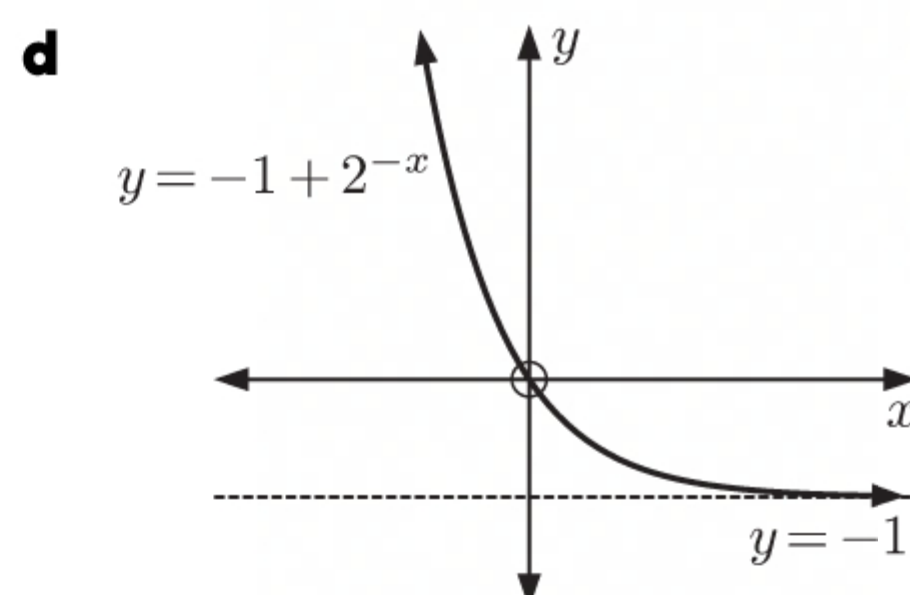
ii Domain = $\{x \mid x \leq 11\}$
 Range = $\{y \mid y \leq -2\}$

47 $y = -1 + 2^{-x}$

a When $x = 0, y = -1 + 1 = 0$
 \therefore the y -intercept is 0, and the x -intercept is 0.

c The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y > -1\}$.

b The horizontal asymptote is $y = -1$.



48 $y = a \times 2^x + b$

x	0	1	2	3
y	20	p	35	q

a When $x = 0$, $y = 20$
 $\therefore 20 = a \times 2^0 + b$
 $\therefore a + b = 20 \quad \dots (1)$

When $x = 2$, $y = 35$
 $\therefore 35 = a \times 2^2 + b$
 $\therefore 4a + b = 35 \quad \dots (2)$

b Using (1), $b = 20 - a \quad \dots (3)$

Substituting $b = 20 - a$ into (2) gives

$$4a + 20 - a = 35$$

$$\therefore 3a = 15$$

$$\therefore a = 5$$

Substituting $a = 5$ into (3) gives $b = 20 - 5 = 15$

$$\therefore a = 5 \text{ and } b = 15$$

c Using **b**, $y = 5 \times 2^x + 15$

When $x = 1$, $y = p$
 $\therefore p = 5 \times 2^1 + 15$
 $\therefore p = 25$

When $x = 3$, $y = q$
 $\therefore q = 5 \times 2^3 + 15$
 $\therefore q = 55$

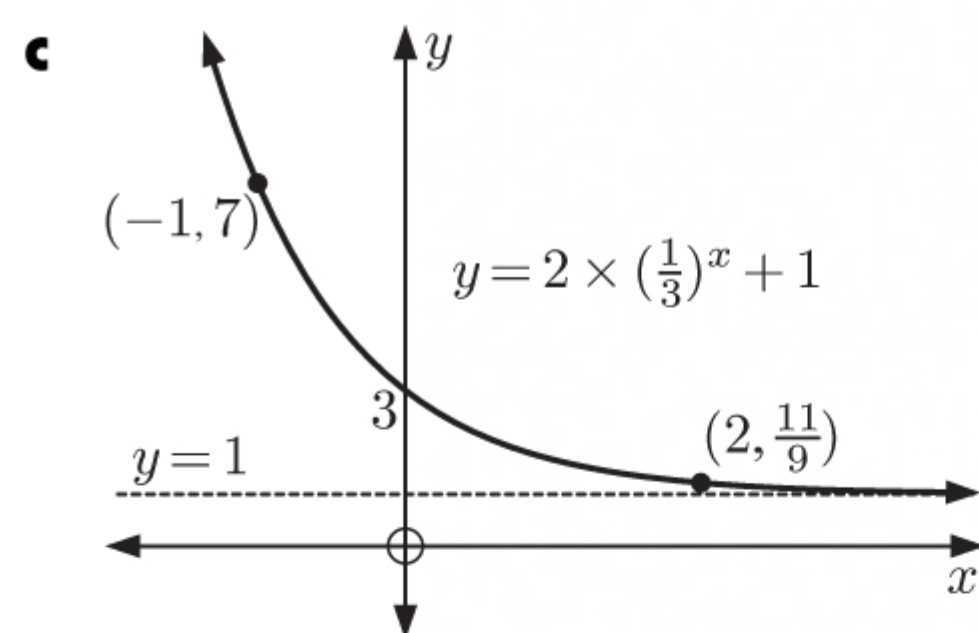
49 $f(x) = 2 \times \left(\frac{1}{3}\right)^x + 1$

a i $f(0) = 2 + 1$
 $= 3$

ii $f(2) = 2 \times \left(\frac{1}{3}\right)^2 + 1$
 $= \frac{2}{9} + 1$
 $= \frac{11}{9}$

iii $f(-1) = 2 \times \left(\frac{1}{3}\right)^{-1} + 1$
 $= 2 \times 3 + 1$
 $= 7$

b The horizontal asymptote is $y = 1$.



d The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y > 1\}$.

50 a i When $t = 4$, $P \approx 60$

\therefore there is about 60% of Carbon-14 remaining after 4000 years.

ii When $P = 50$, $t \approx 5.5$

\therefore it will take approximately 5500 years for the percentage of Carbon-14 to fall to 50%.

b $P = 100 \times (1.1318)^{-t}$, $t \geq 0$

i When $t = 8$, $P = 100 \times (1.1318)^{-8}$
 ≈ 37.1

\therefore there is about 37.1% of Carbon-14 remaining after 8000 years.

ii When $P = 15$, $15 = 100 \times (1.1318)^{-t}$

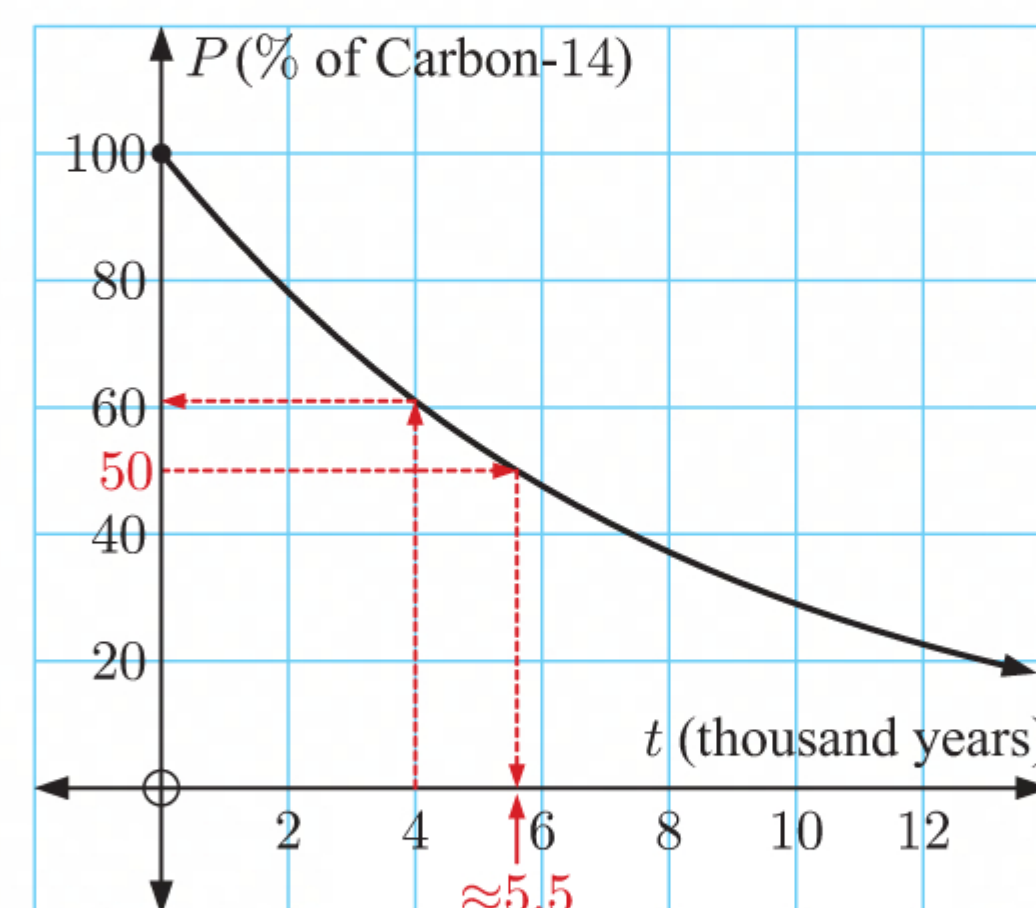
$$\therefore (1.1318)^{-t} = 0.15$$

$$\therefore -t \ln(1.1318) = \ln(0.15)$$

$$\therefore t = \frac{\ln(0.15)}{-\ln(1.1318)}$$

$$\approx 15.3$$

\therefore it will take approximately 15 300 years for the percentage of Carbon-14 to fall to 15%.



51 $N = 120 \times (1.04)^t$

a When $t = 0$, $N = 120$

\therefore there were 120 people who started the settlement.

b When $t = 4$,

$$N = 120 \times (1.04)^4 \\ \approx 140$$

\therefore there were about 140 people on the island after 4 years.

c When $N = 120 \times 2 = 240$,

$$240 = 120 \times (1.04)^t$$

$$\therefore (1.04)^t = 2$$

$$\therefore t \ln(1.04) = \ln 2$$

$$\therefore t = \frac{\ln 2}{\ln(1.04)}$$

$$\approx 17.7$$

\therefore it will take about 17.7 years for the number of people to double.

52 $T(t) = A \times B^{-t} + 3$

a i The initial internal temperature of the refrigerator was 27°C .

So, $T(0) = 27$

$$\therefore 27 = A + 3$$

$$\therefore A = 24$$

ii After 3 hours, the internal temperature was 6°C .

So, $T(3) = 6$

$$\therefore 6 = 24 \times B^{-3} + 3 \quad \{\text{using i}\}$$

$$\therefore 24 \times B^{-3} = 3$$

$$\therefore B^{-3} = \frac{1}{8}$$

$$\therefore B^3 = 8$$

$$\therefore B = 2$$

b $T(t) = 24 \times 2^{-t} + 3$ {using a}

$$\therefore T(5) = 24 \times 2^{-5} + 3 \\ = 3.75$$

\therefore the internal temperature is 3.75°C after 5 hours.

c As $t \rightarrow \infty$, $2^{-t} \rightarrow 0$

$$\therefore T(t) \rightarrow 24 \times 0 + 3 = 3$$

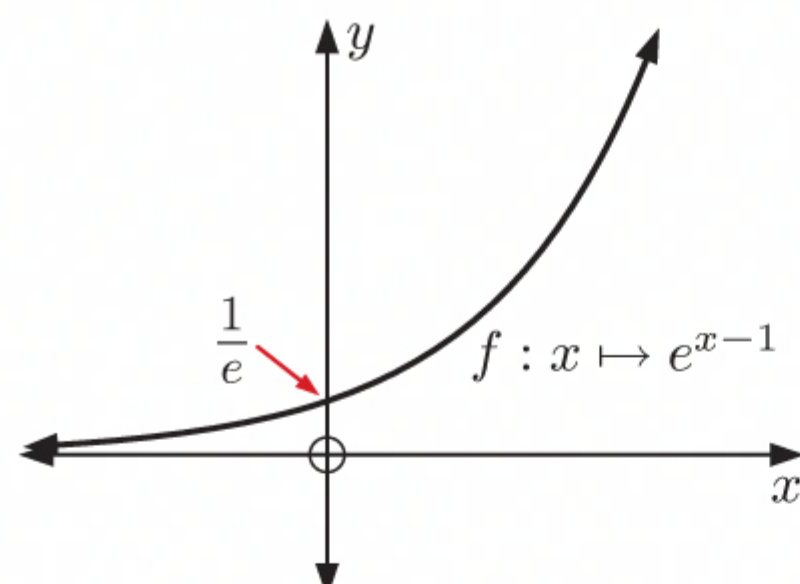
\therefore the minimum temperature that the refrigerator could be expected to reach is 3°C .

53 $f: x \mapsto e^{x-1}$

a $f(0) = e^{-1} = \frac{1}{e}$

$$f(1) = e^0 = 1$$

$$f(-1) = e^{-2} = \frac{1}{e^2}$$



b The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y > 0\}$.

54 $P = 1000 + ae^{kn}$

The initial population was 2000, so when $n = 0$, $P = 2000$

$$\therefore 2000 = 1000 + ae^0$$

$$\therefore a = 1000$$

$$\therefore P = 1000 + 1000e^{kn}$$

After 1 year, the population was 4000, so when $n = 1 \times 12 = 12$, $P = 4000$

$$\therefore 4000 = 1000 + 1000e^{12k}$$

$$\therefore 1000e^{12k} = 3000$$

$$\therefore e^{12k} = 3$$

$$\therefore e^k = 3^{\frac{1}{12}}$$

$$\therefore P = 1000 + 1000 \times 3^{\frac{n}{12}}$$

Now, when $P = 10\,000$, $10\,000 = 1000 + 1000 \times 3^{\frac{n}{12}}$

$$\therefore 9000 = 1000 \times 3^{\frac{n}{12}}$$

$$\therefore 3^{\frac{n}{12}} = 9$$

$$\therefore 3^{\frac{n}{12}} = 3^2$$

$$\therefore \frac{n}{12} = 2 \quad \{\text{equating indices}\}$$

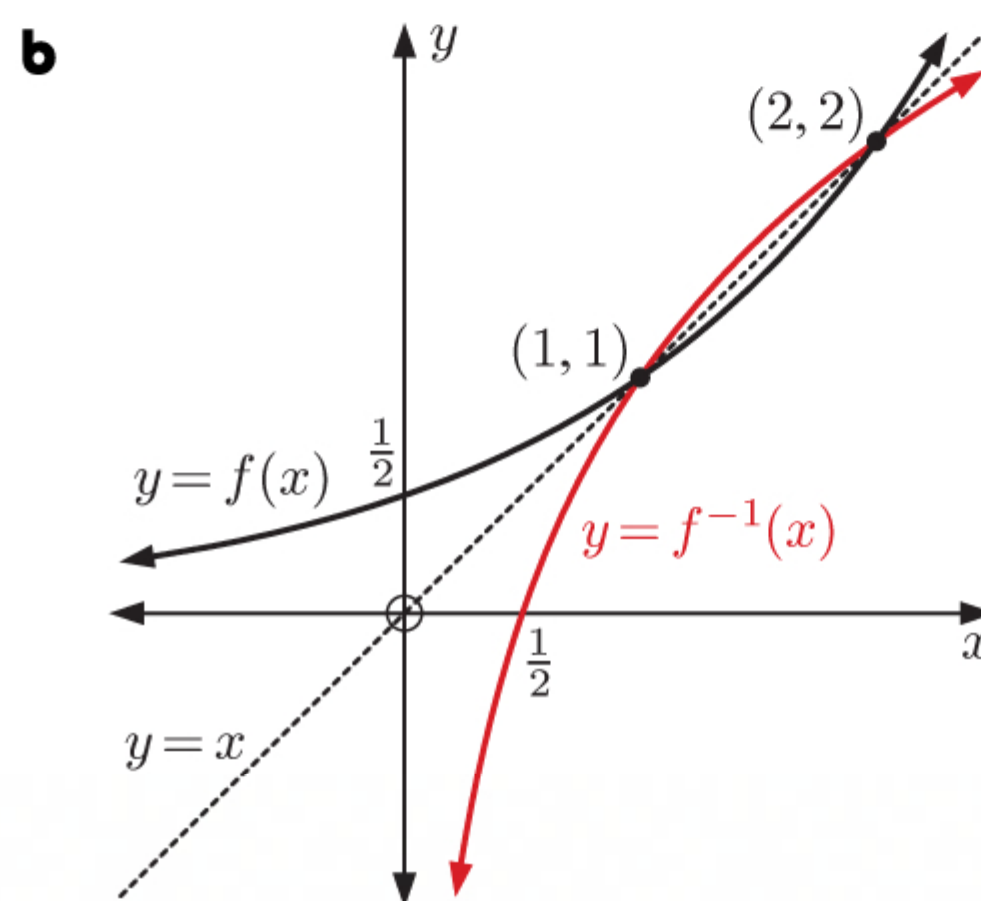
$$\therefore n = 24$$

\therefore it will take 24 months, or 2 years, for the population to reach 10 000.

$$\begin{aligned}
 55 \quad \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 &= \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} \\
 &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
 &= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} \\
 &= \frac{4}{4} \\
 &= 1
 \end{aligned}$$

$$56 \quad f(x) = 2^{x-1}$$

$$\begin{aligned}
 \mathbf{a} \quad f(1) &= 2^0 = 1 \\
 f(2) &= 2^1 = 2
 \end{aligned}$$



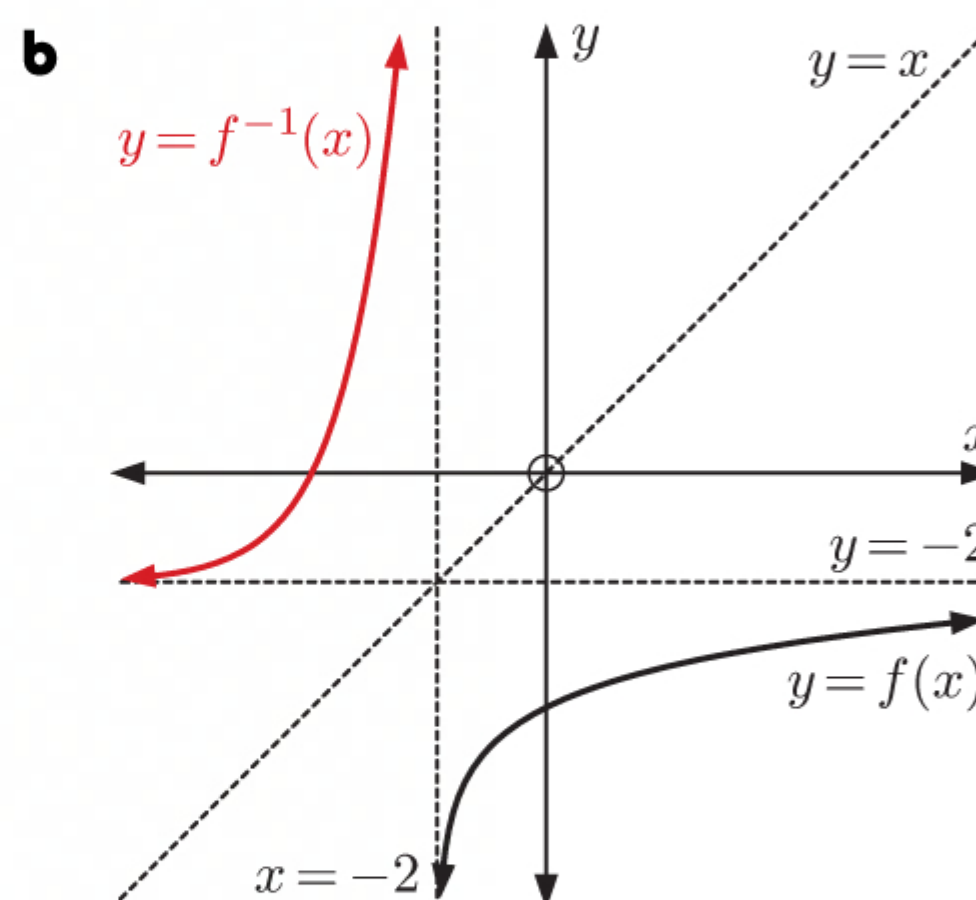
The graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$.

$$\begin{aligned}
 \mathbf{c} \quad f \text{ is } y &= 2^{x-1} \\
 \therefore f^{-1} \text{ is } x &= 2^{y-1} \\
 \therefore y - 1 &= \log_2 x \\
 \therefore y &= \log_2 x + 1 \\
 \text{So, } f^{-1}(x) &= \log_2 x + 1
 \end{aligned}$$

\mathbf{d} The domain of $f(x)$ is $\{x \mid x \in \mathbb{R}\}$.
 The range of $f(x)$ is $\{y \mid y > 0\}$.
 The domain of $f^{-1}(x)$ is $\{x \mid x > 0\}$.
 The range of $f^{-1}(x)$ is $\{y \mid y \in \mathbb{R}\}$.

$$57 \quad f: x \mapsto \ln(x+2) - 5, \quad x > -2$$

$$\begin{aligned}
 \mathbf{a} \quad f \text{ is } y &= \ln(x+2) - 5 \\
 \therefore f^{-1} \text{ is } x &= \ln(y+2) - 5 \\
 \therefore x + 5 &= \ln(y+2) \\
 \therefore y + 2 &= e^{x+5} \\
 \therefore y &= e^{x+5} - 2 \\
 \text{So, } f^{-1}(x) &= e^{x+5} - 2
 \end{aligned}$$



The graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$.

\mathbf{c} The domain of f^{-1} is $\{x \mid x \in \mathbb{R}\}$.
 The range of f^{-1} is $\{y \mid y > -2\}$.

$$58 \quad \mathbf{a} \quad f(x) = x^2 - 5x + 6$$

The graph of $y = g(x)$ is found by translating $y = f(x)$ 8 units upwards.

$$\begin{aligned}
 \therefore g(x) &= f(x) + 8 \\
 \therefore g(x) &= (x^2 - 5x + 6) + 8 \\
 \therefore g(x) &= x^2 - 5x + 14
 \end{aligned}$$

$$\mathbf{b} \quad f(x) = -2x^2 + x + 3$$

The graph of $y = g(x)$ is found by translating $y = f(x)$ 1 unit to the right.

$$\begin{aligned}
 \therefore g(x) &= f(x-1) \\
 \therefore g(x) &= -2(x-1)^2 + (x-1) + 3 \\
 \therefore g(x) &= -2(x^2 - 2x + 1) + x - 1 + 3 \\
 \therefore g(x) &= -2x^2 + 4x - 2 + x - 1 + 3 \\
 \therefore g(x) &= -2x^2 + 5x
 \end{aligned}$$

59 $f(x) = \frac{5}{x+2}$

The graph of $y = g(x)$ is found by translating $y = f(x)$ through $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$.

$$\therefore g(x) = f(x+4) + 6$$

$$\therefore g(x) = \frac{5}{(x+4)+2} + 6$$

$$\therefore g(x) = \frac{5}{x+6} + \frac{6(x+6)}{x+6}$$

$$\therefore g(x) = \frac{5+6x+36}{x+6}$$

$$\therefore g(x) = \frac{6x+41}{x+6}$$

60 a $f(x) = \frac{1}{\sqrt{x-4}} + 3$ is defined when $\sqrt{x-4} > 0$

$$\therefore x-4 > 0$$

$$\therefore x > 4$$

\therefore the domain is $\{x \mid x > 4\}$.

Now, $\frac{1}{\sqrt{x-4}} > 0$ {for $x > 4$ }

$$\therefore \frac{1}{\sqrt{x-4}} + 3 > 3$$

$$\therefore f(x) > 3$$

\therefore the range is $\{y \mid y > 3\}$.

b A translation through $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ maps $y = \frac{1}{\sqrt{x}}$ onto f .

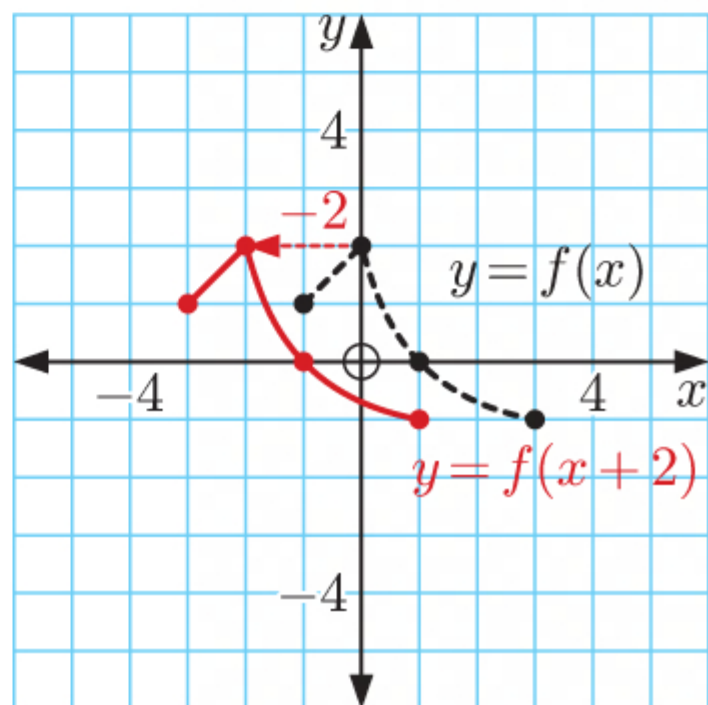
c $y = \frac{1}{\sqrt{x}}$ has vertical asymptote $x = 0$

and horizontal asymptote $y = 0$

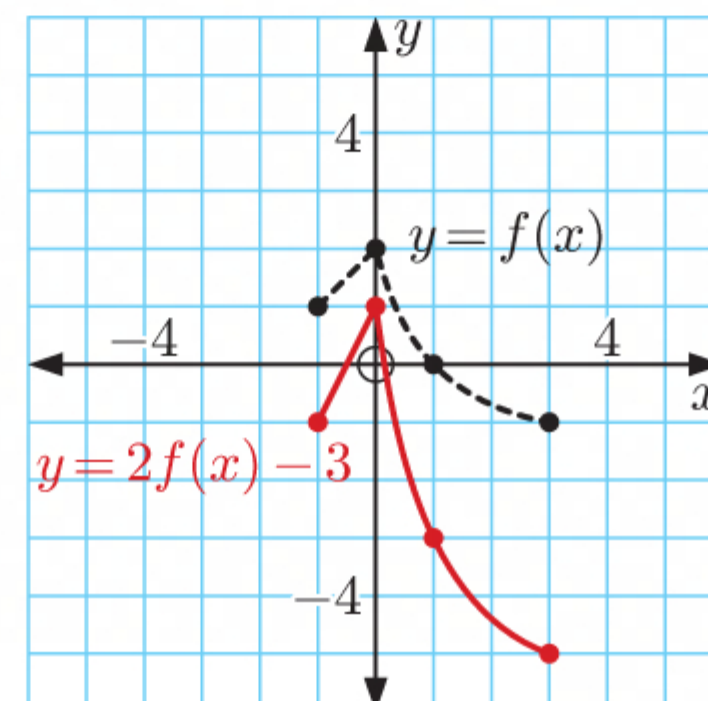
$\therefore y = f(x)$ has vertical asymptote $x = 4$

and horizontal asymptote $y = 3$.

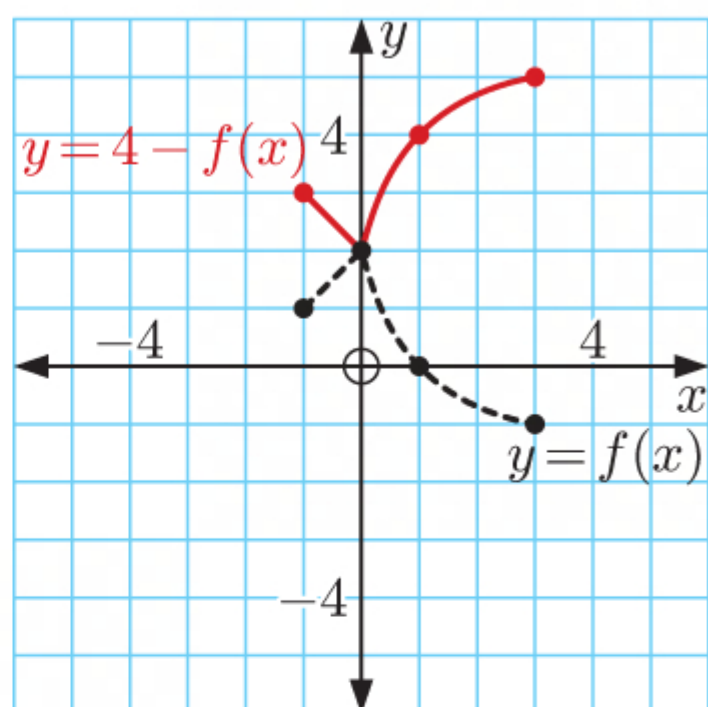
61 a The graph of $y = f(x+2)$ is found by translating $y = f(x)$ 2 units to the left.



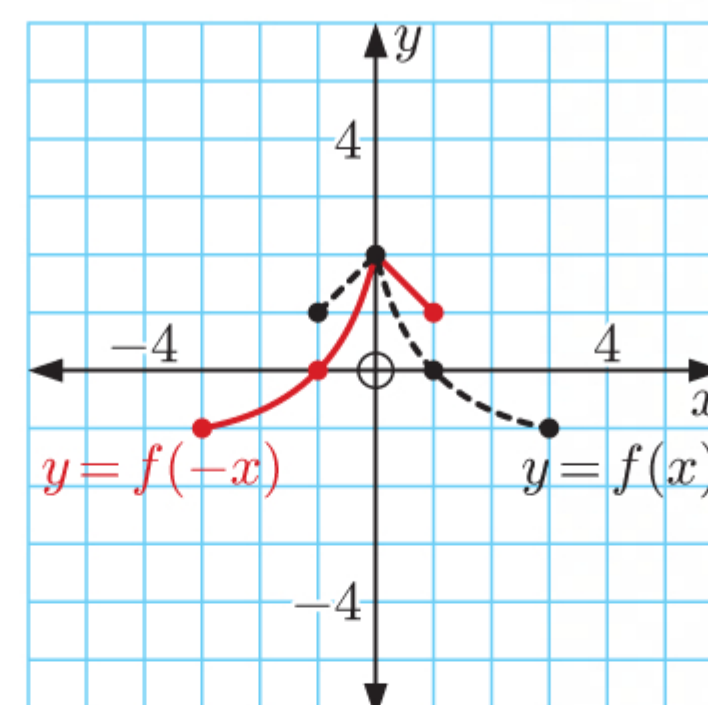
b The graph of $y = 2f(x) - 3$ is a vertical stretch of $y = f(x)$ with scale factor 2, followed by a translation 3 units downwards.



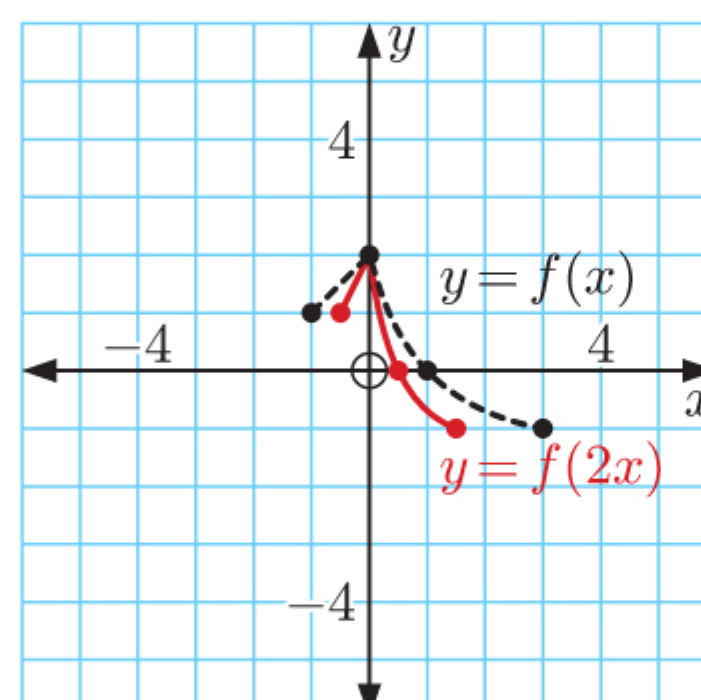
c The graph of $y = 4 - f(x)$ is found by reflecting $y = f(x)$ in the x -axis, then translating 4 units upwards.



d The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.



e The graph of $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.



62 $g: x \mapsto 4 - \ln(x - 2)$

a $\ln(x - 2)$ is defined when $x - 2 > 0$
 $\therefore x > 2$

So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

b As $x \rightarrow 2^+$, $y \rightarrow \infty$, so the vertical asymptote is $x = 2$.

As $x \rightarrow \infty$, $y \rightarrow -\infty$, so there is no horizontal asymptote.

c The graph of $y = h(x)$ is a horizontal stretch of $y = g(x)$ with scale factor $\frac{1}{2}$.

$\therefore h(x) = g(2x)$

$\therefore h(x) = 4 - \ln(2x - 2)$

d $y = g(x)$ has vertical asymptote $x = 2$ {using **b**}

$\therefore y = h(x)$ has vertical asymptote $x = \frac{1}{2}(2) = 1$.

63 a $f(x) \xrightarrow[\text{scale factor 2}]{\text{horizontal stretch}} f\left(\frac{1}{2}x\right) \xrightarrow[\text{scale factor 3}]{\text{vertical stretch}} 3f\left(\frac{1}{2}x\right) = g(x)$

A horizontal stretch with scale factor 2, then a vertical stretch with scale factor 3 maps $y = f(x)$ onto $y = g(x)$.

b Each point on $y = g(x)$ is 2 times their distance that $y = f(x)$ is from the y -axis, and 3 times their distance that $y = f(x)$ is from the x -axis.

The point $(-6, 3)$ on $y = f(x)$ is 6 units from the y -axis, and 3 units from the x -axis. The corresponding point on $y = g(x)$, which is $2 \times 6 = 12$ units from the y -axis and $3 \times 3 = 9$ units from the x -axis, is $(-12, 9)$.

c Each point on $y = f(x)$ is $\frac{1}{2}$ times their distance that $y = g(x)$ is from the y -axis, and $\frac{1}{3}$ times their distance that $y = g(x)$ is from the x -axis.

The point $(4, -9)$ on $y = g(x)$ is 4 units from the y -axis, and 9 units from the x -axis. The corresponding point on $y = f(x)$, which is $\frac{1}{2} \times 4 = 2$ units from the y -axis and $\frac{1}{3} \times 9 = 3$ units from the x -axis, is $(2, -3)$.

64 Let $y = f(x) = \frac{2}{x}$

a The image when $y = f(x)$ is reflected in the y -axis has equation $y = f(-x)$

$\therefore y = \frac{2}{-x} = -\frac{2}{x}$

b The image when $y = f(x)$ is translated through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ has equation $y = f(x + 1) + 2$

$\therefore y = \frac{2}{x+1} + 2$

c The image when $y = f(x)$ is stretched horizontally with scale factor 3 has equation $y = f\left(\frac{1}{3}x\right)$

$\therefore y = \frac{2}{\frac{1}{3}x}$

$\therefore y = \frac{6}{x}$

65 $f(x)$ has domain $\{x \mid x < 0, x \geq 2\}$ and range $\{y \mid y \geq 3\}$.

a $g(x) = f(x - 3) + 2$ translates every point on $y = f(x)$ 3 units to the right and 2 units upwards.

$\therefore g(x)$ has domain $\{x \mid x < 3, x \geq 5\}$ and range $\{y \mid y \geq 5\}$.

b $g(x) = 4 - \frac{1}{2}f(5x)$ transforms every point by a horizontal stretch with scale factor $\frac{1}{5}$, a vertical stretch with scale factor $\frac{1}{2}$, a reflection in the x -axis, then a translation 4 units upwards.

$\therefore g(x)$ has domain $\{x \mid x < 0, x \geq \frac{2}{5}\}$ and range $\{y \mid y \leq \frac{5}{2}\}$.

66 T_A is a horizontal translation 4 units to the right.

T_B is a reflection in the x -axis.

T_C is a translation through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a $f(x) \xrightarrow{T_A} f(x - 4) \xrightarrow{T_B} -f(x - 4)$

The resulting function is $-f(x - 4)$.

b $f(x) \xrightarrow{T_B} -f(x) \xrightarrow{T_A} -f(x - 4)$

The resulting function is $-f(x - 4)$.

c $f(x) \xrightarrow{T_B} -f(x) \xrightarrow{T_C} -f(x - 1) + 1$

The resulting function is $-f(x - 1) + 1$.

d $f(x) \xrightarrow{T_C} f(x - 1) + 1 \xrightarrow{T_B} -f(x - 1) - 1$

The resulting function is $-f(x - 1) - 1$.

67 $f : x \mapsto \ln(x - 2)$

a $\ln(x - 2)$ is defined when $x - 2 > 0$
 $\therefore x > 2$

\therefore the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

b As $x \rightarrow 2^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 2$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

c $f(x) \xrightarrow[\text{vertical stretch}]{\text{scale factor } 3} 3f(x) \xrightarrow[\text{reflection in the}]{y\text{-axis}} 3f(-x)$

The resulting function has equation $y = 3f(-x)$
 which is $y = 3\ln(-x - 2)$.

68 $f(x) = 3x^2 - 12x + 5$

$$= 3(x^2 - 4x + 4) + 5 - 12$$

$$= 3(x - 2)^2 - 7$$

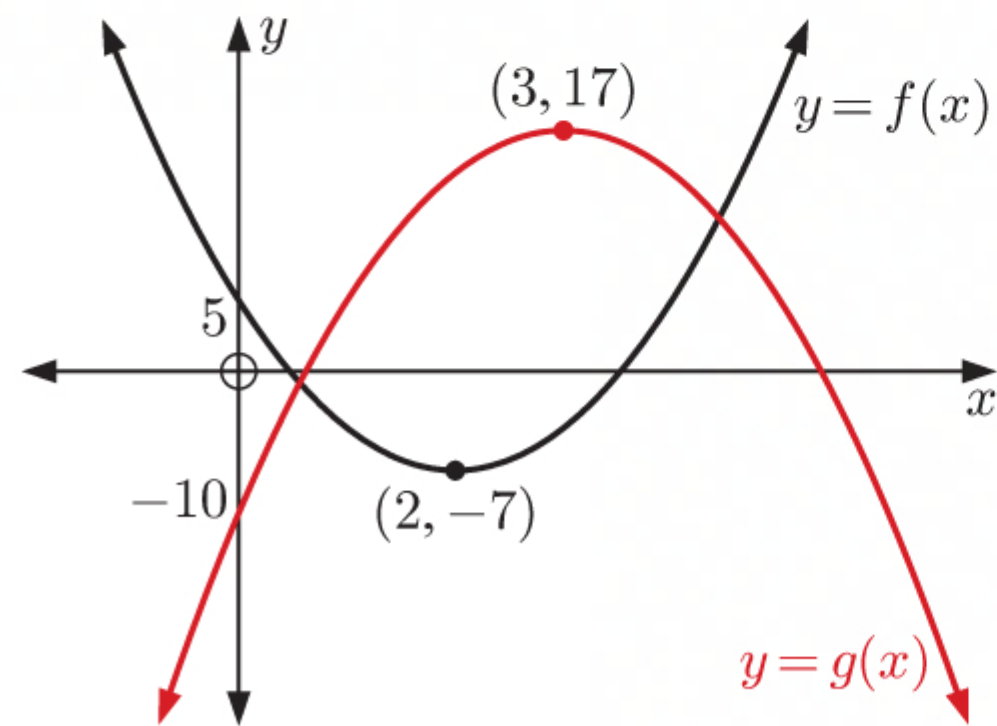
$g(x) = -3x^2 + 18x - 10$

$$= -3(x^2 - 6x + 9) - 10 + 27$$

$$= -3(x - 3)^2 + 17$$

$$= -(3(x - 1 - 2)^2 - 7 - 10)$$

$$= -(f(x - 1) - 10)$$



So, we translate $y = f(x)$ through $\begin{pmatrix} 1 \\ -10 \end{pmatrix}$ and then reflect the result in the x -axis.

69 a $f(x) \xrightarrow[\text{translation}]{\begin{pmatrix} 3 \\ -1 \end{pmatrix}} f(x - 3) - 1 \xrightarrow[\text{reflection in}]{x\text{-axis}} -f(x - 3) + 1$

The resulting function is $-f(x - 3) + 1$.

b $f(x) \xrightarrow[\text{reflection in}]{y\text{-axis}} f(-x) \xrightarrow[\text{translation}]{\begin{pmatrix} -2 \\ 7 \end{pmatrix}} f(-(x + 2)) + 7$

The resulting function is $f(-(x + 2)) + 7$.

c $f(x) \xrightarrow[\text{translation}]{\begin{pmatrix} 4 \\ 3 \end{pmatrix}} f(x - 4) + 3 \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{2}} f(2x - 4) + 3$

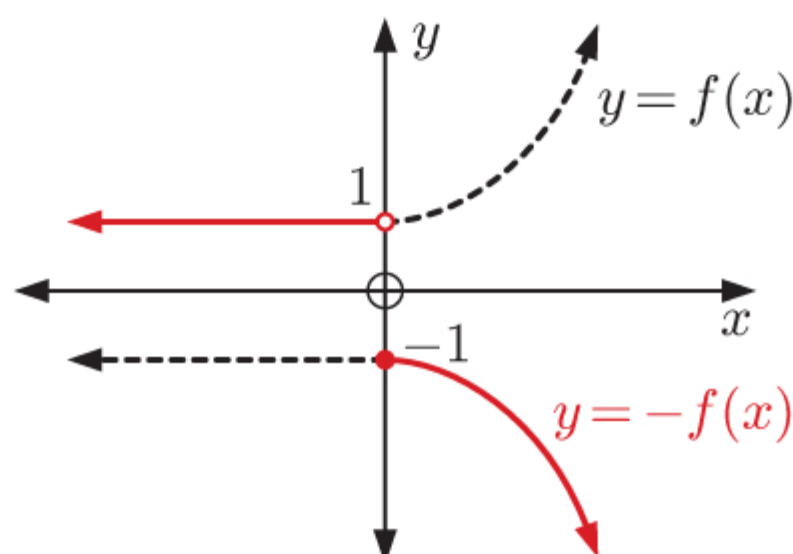
The resulting function is $f(2x - 4) + 3$.

70 a $f(x) \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{2}} f(2x) \xrightarrow[\text{translation}]{\begin{pmatrix} 1 \\ 3 \end{pmatrix}} f(2(x - 1)) + 3$

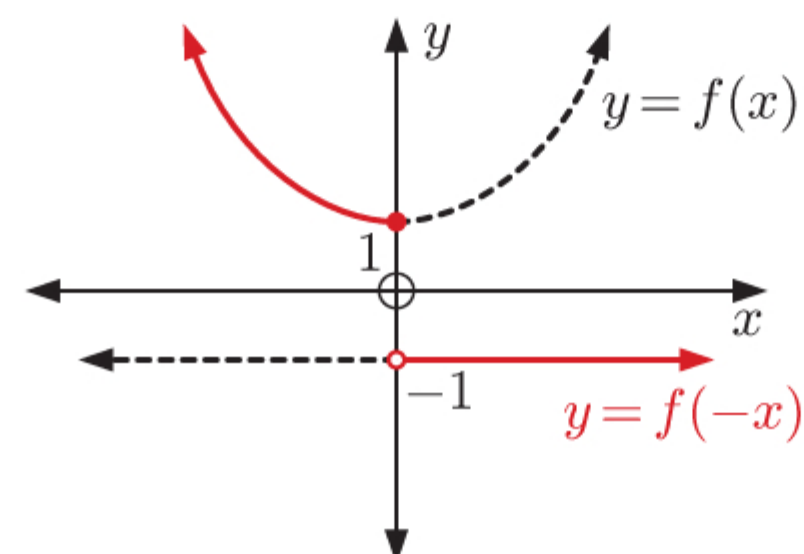
b $f(x) \xrightarrow[\text{horizontal stretch}]{\text{scale factor } 4} f(\frac{1}{4}x) \xrightarrow[\text{vertical stretch}]{\text{scale factor } 2} 2f(\frac{1}{4}x) \xrightarrow[\text{reflection in}]{x\text{-axis}} -2f(\frac{1}{4}x) \xrightarrow[\text{translation}]{\begin{pmatrix} 0 \\ 5 \end{pmatrix}} 5 - 2f(\frac{1}{4}x)$

c $f(x) \xrightarrow[\text{translation}]{\begin{pmatrix} 2 \\ 0 \end{pmatrix}} f(x - 2) \xrightarrow[\text{horizontal stretch}]{\text{scale factor } 3} f(\frac{1}{3}x - 2) \xrightarrow[\text{vertical stretch}]{\text{scale factor } 6} 6f(\frac{1}{3}x - 2) \xrightarrow[\text{translation}]{\begin{pmatrix} 0 \\ 4 \end{pmatrix}} 6f(\frac{1}{3}x - 2) + 4$

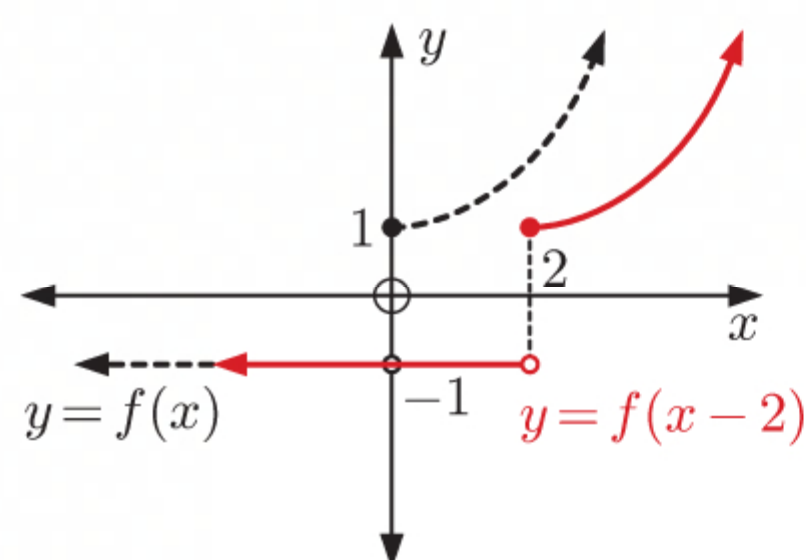
71 a The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.



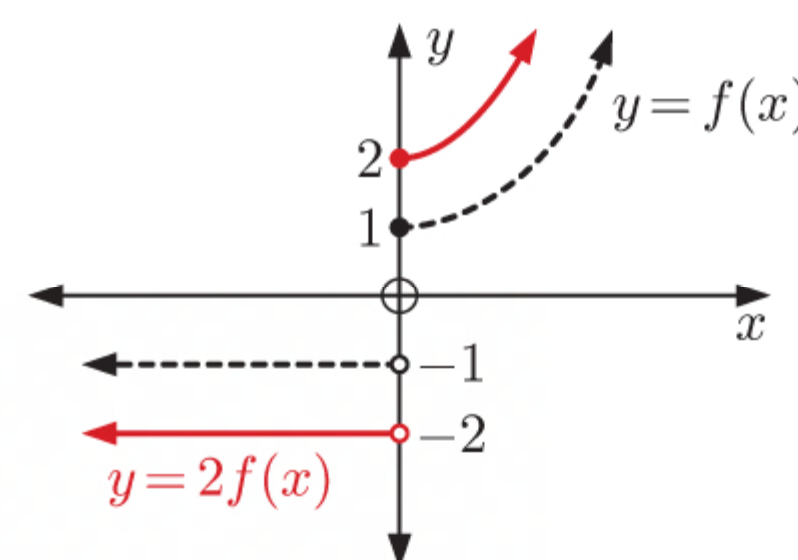
b The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.



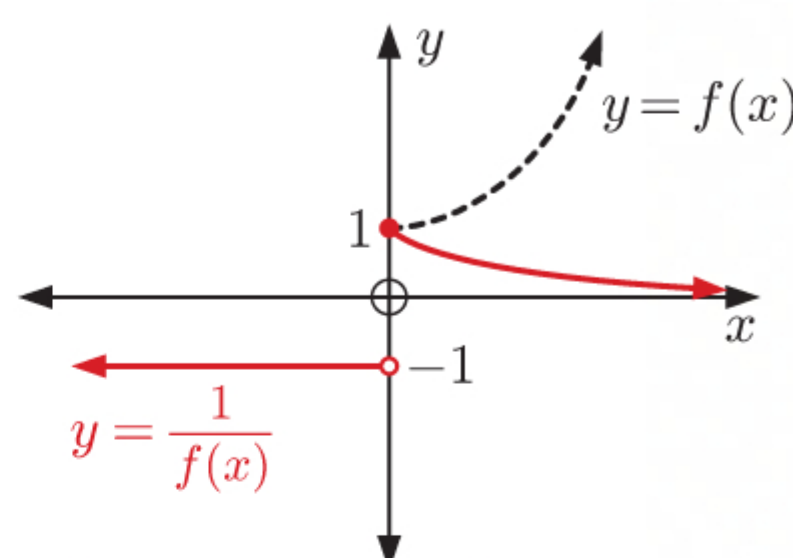
- c** The graph of $y = f(x - 2)$ is found by translating $y = f(x)$ 2 units to the right.



- d** The graph of $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.

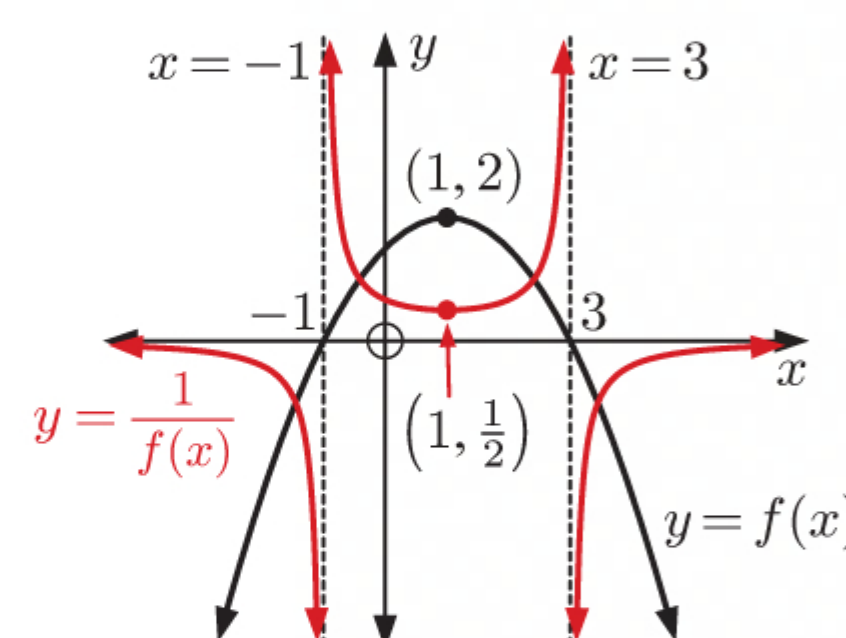


- e** $f(x) = -1$ for $x < 0$, so $\frac{1}{f(x)} = -1$ for $x < 0$.
 $f(x) \geq 1$ for $x \geq 0$, so $0 < \frac{1}{f(x)} \leq 1$ for $x \geq 0$.



- 72 a** $y = f(x)$ has x -intercepts -1 and 3 , so $y = \frac{1}{f(x)}$ has vertical asymptotes $x = -1$ and $x = 3$.

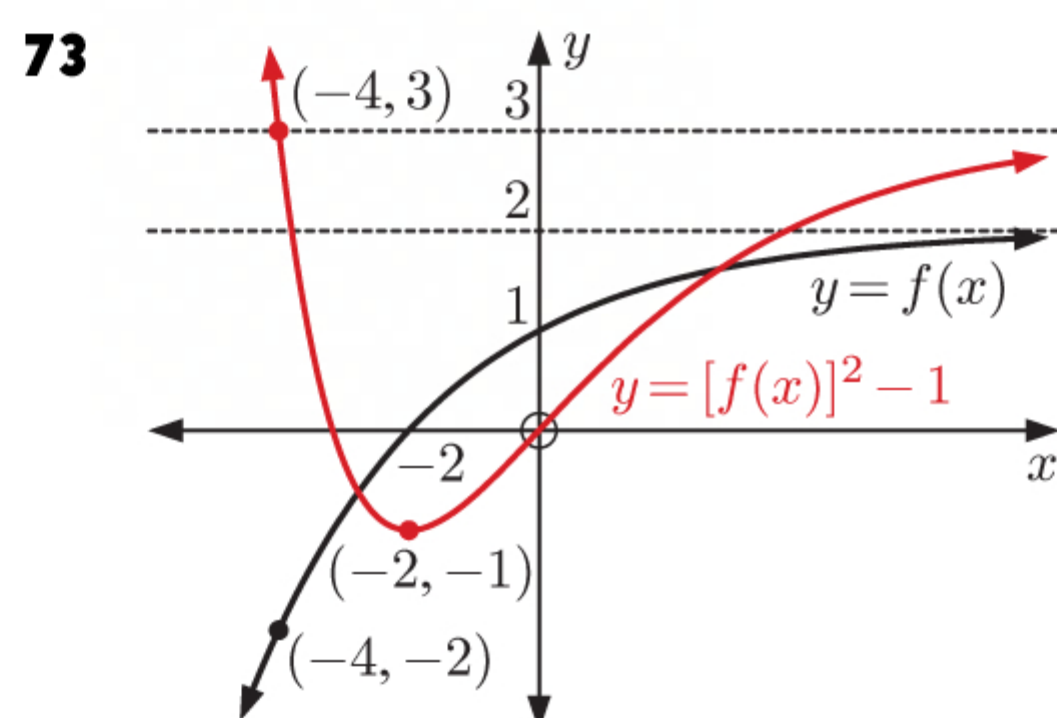
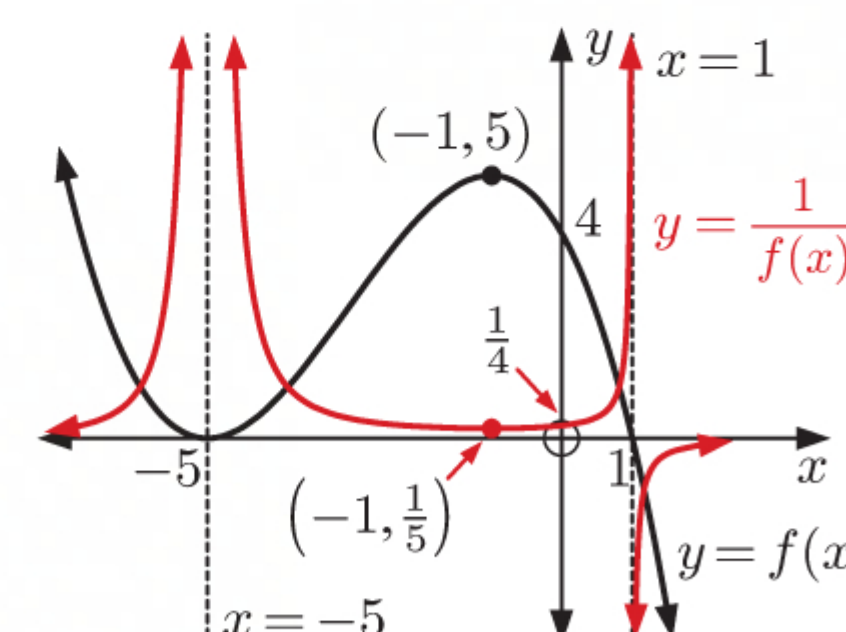
$y = f(x)$ has a local maximum at $(1, 2)$, so $y = \frac{1}{f(x)}$ has local minimum at $(1, \frac{1}{2})$.



- b** $y = f(x)$ has x -intercepts -5 and 1 , so $y = \frac{1}{f(x)}$ has vertical asymptotes $x = -5$ and $x = 1$.

$y = f(x)$ has local maximum $(-1, 5)$, so $y = \frac{1}{f(x)}$ has local minimum $(-1, \frac{1}{5})$.

$y = f(x)$ has y -intercept 4 , so $y = \frac{1}{f(x)}$ has y -intercept $\frac{1}{4}$.



74 $f(x) = \frac{6 - 2x}{x + 3}$

- a** $f(0) = \frac{6}{3} = 2$, so the y -intercept is 2 .

$$\begin{aligned} f(x) = 0 \text{ when } 6 - 2x &= 0 \\ \therefore 2x &= 6 \\ \therefore x &= 3 \end{aligned}$$

\therefore the x -intercept is 3 .

The vertical asymptote is $x = -3$.

$$\begin{aligned} f(x) &= \frac{6 - 2x}{x + 3} \\ &= \frac{6 - 2(x + 3) + 6}{x + 3} \\ &= \frac{12}{x + 3} - 2 \end{aligned}$$

\therefore the horizontal asymptote is $y = -2$.

- b** The y -intercept of $y = [f(x)]^2$ is $2^2 = 4$.

The x -intercept of $y = [f(x)]^2$ is 3 .

The vertical asymptote of $y = [f(x)]^2$ is $x = -3$.

The horizontal asymptote of $y = [f(x)]^2$ is $y = (-2)^2 = 4$.

c Invariant points occur where

$$f(x) = 0 \text{ or } f(x) = 1$$

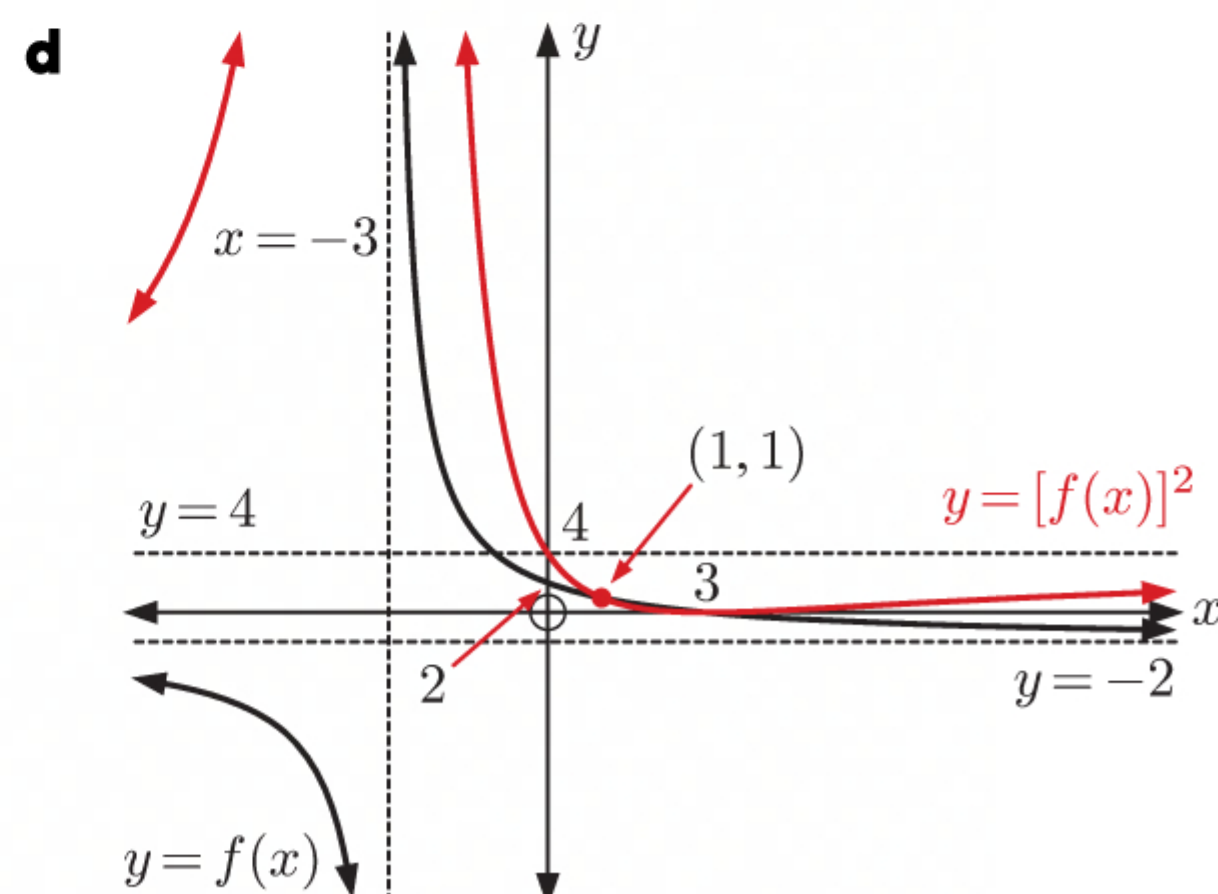
$$\therefore \frac{6-2x}{x+3} = 1$$

$$\therefore 6-2x = x+3$$

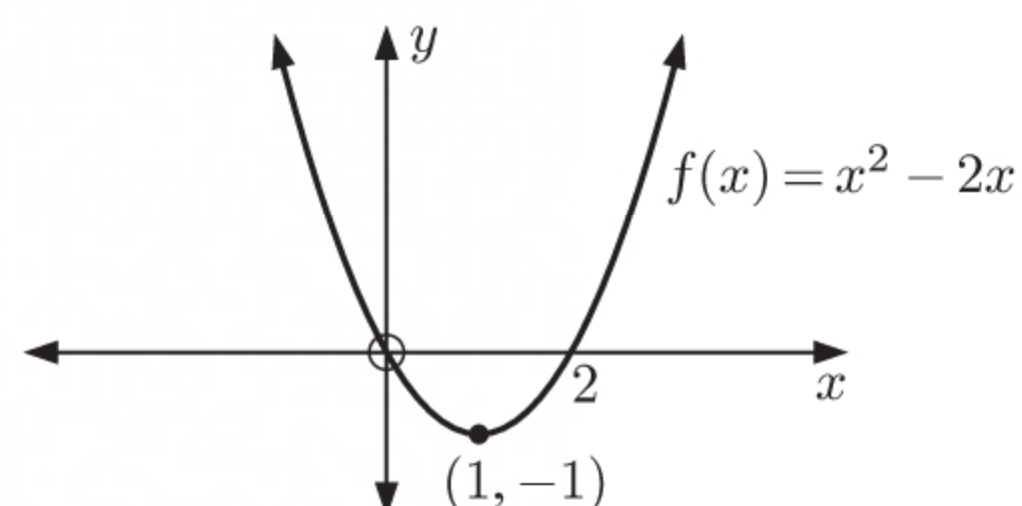
$$\therefore -3x = -3$$

$$\therefore x = 1$$

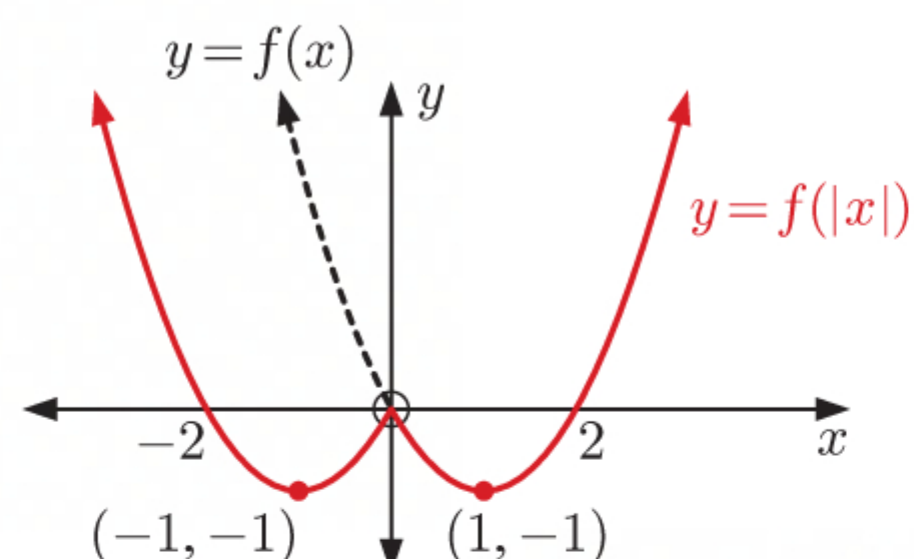
\therefore the invariant points are $(1, 1)$ and $(3, 0)$.



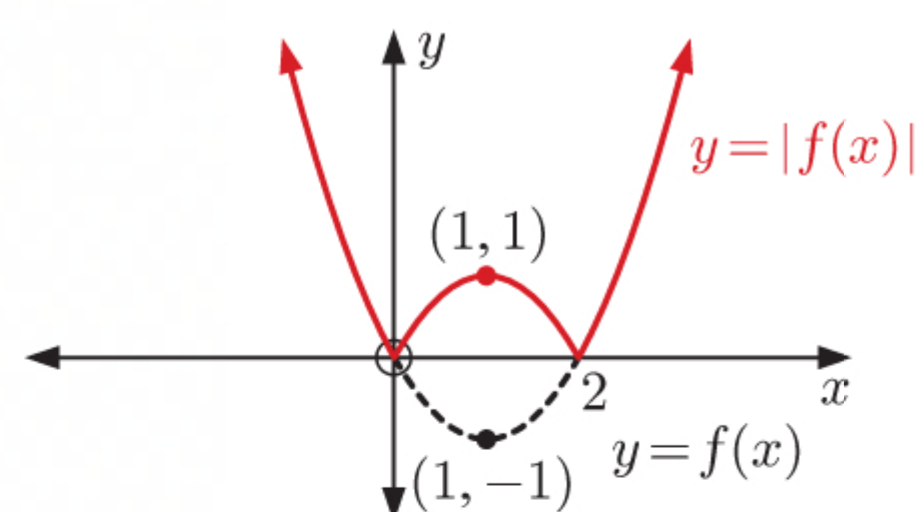
75 a



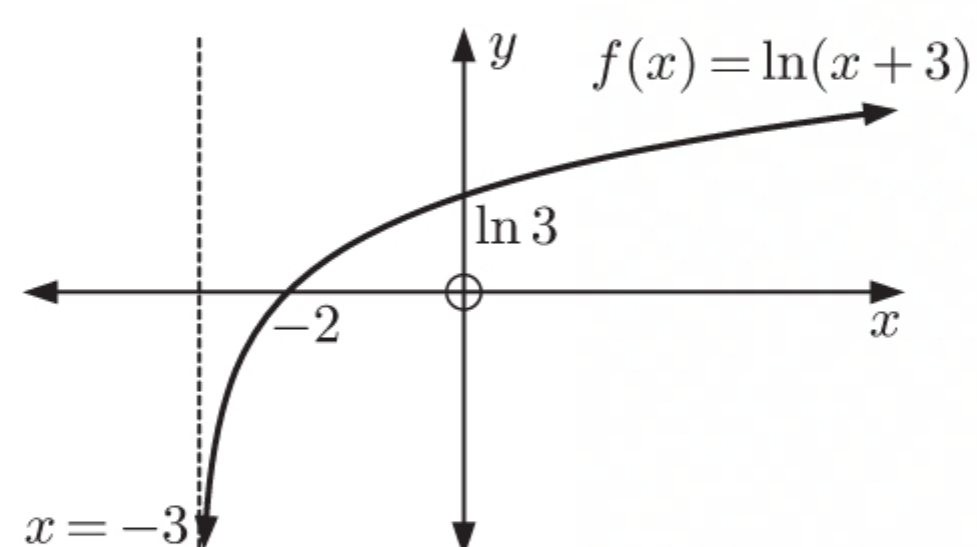
b i



ii



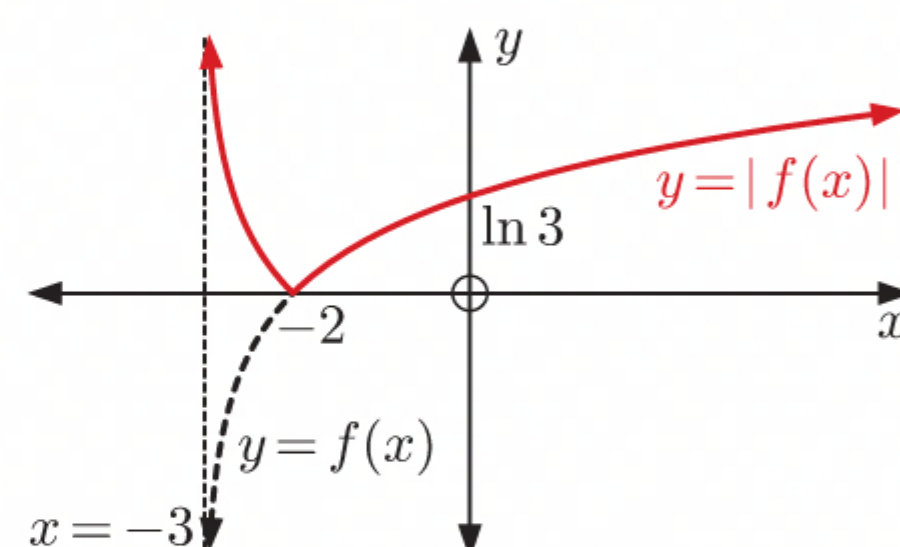
76 a



$$\text{Domain} = \{x \mid x > -3\}$$

$$\text{Range} = \{y \mid y \in \mathbb{R}\}$$

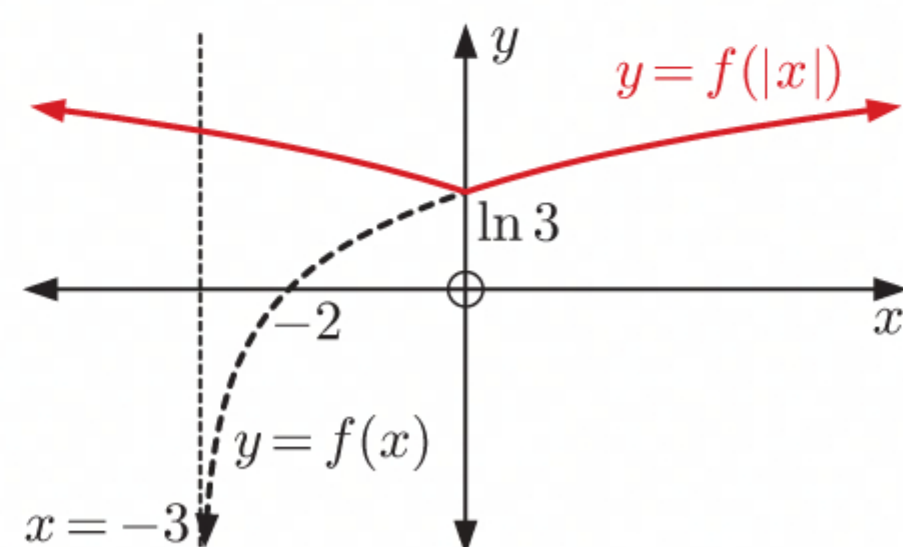
b



$$\text{Domain} = \{x \mid x > -3\}$$

$$\text{Range} = \{y \mid y \geq 0\}$$

c



$$\text{Domain} = \{x \mid x \in \mathbb{R}\}$$

$$\text{Range} = \{y \mid y \geq \ln 3\}$$

77 a If $|3-2x| = 5$, then $3-2x = \pm 5$

$$\therefore 3-2x = 5 \quad \text{or} \quad 3-2x = -5$$

$$\therefore -2x = 2 \quad \text{or} \quad -2x = -8$$

$$\therefore x = -1 \quad \text{or} \quad x = 4$$

So, $x = -1$ or 4 .

b If $\left| \frac{2x+5}{3-x} \right| = 2$, then $\frac{2x+5}{3-x} = \pm 2$

$$\therefore \frac{2x+5}{3-x} = 2 \quad \text{or} \quad \frac{2x+5}{3-x} = -2$$

$$\therefore 2x+5 = 6-2x \quad \text{or} \quad 2x+5 = -6+2x$$

$$\therefore 4x = 1 \quad \text{or} \quad 5 = -6 \quad \times$$

$$\therefore x = \frac{1}{4}$$

So, $x = \frac{1}{4}$.

c If $|2 - x| = 3|x + 4|$, then $2 - x = \pm 3(x + 4)$
 $\therefore 2 - x = 3(x + 4)$ or $2 - x = -3(x + 4)$
 $\therefore 2 - x = 3x + 12$ or $2 - x = -3x - 12$
 $\therefore -4x = 10$ or $2x = -14$
 $\therefore x = -\frac{5}{2}$ or $x = -7$

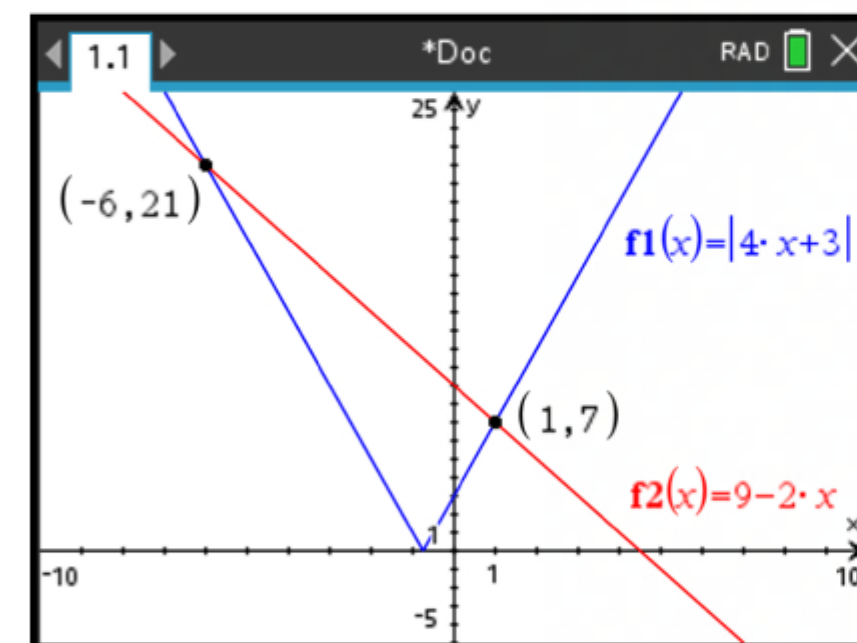
So, $x = -\frac{5}{2}$ or -7 .

78 a $|4x + 3| = 9 - 2x$

We graph $y = |4x + 3|$ and $y = 9 - 2x$ on the same set of axes.

The graphs intersect at $(-6, 21)$ and $(1, 7)$.

\therefore the solutions are $x = -6$ or 1 .

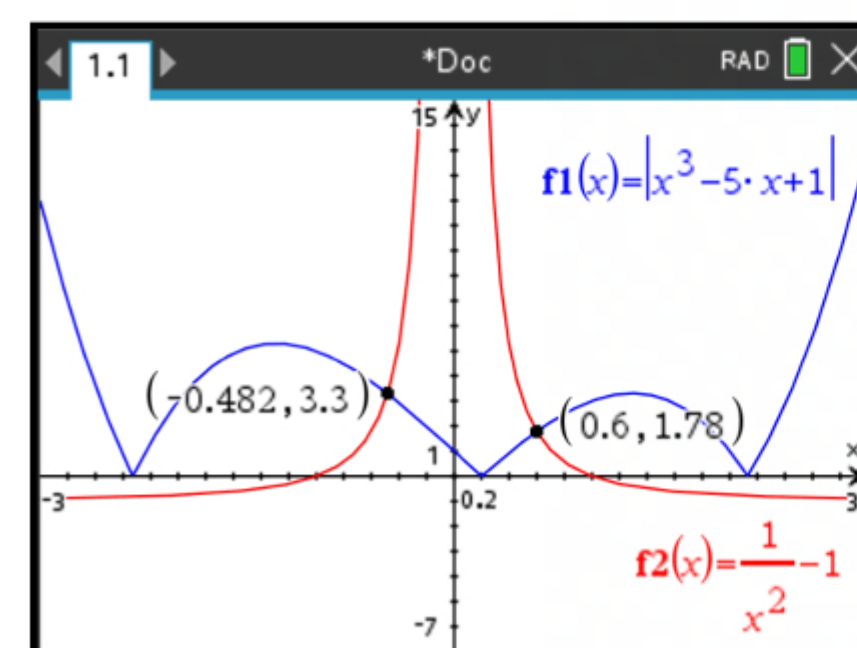


b $|x^3 - 5x + 1| = \frac{1}{x^2} - 1$

We graph $y = |x^3 - 5x + 1|$ and $y = \frac{1}{x^2} - 1$ on the same set of axes.

The graphs intersect at about $(-0.482, 3.30)$ and $(0.600, 1.78)$.

\therefore the solutions are $x \approx -0.482$ or ≈ 0.600 .



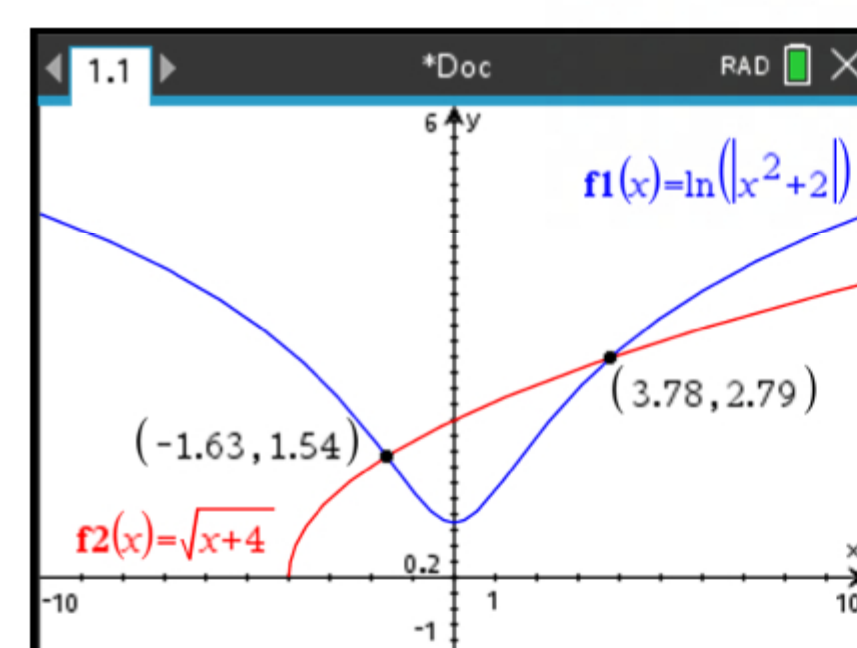
c $\ln|x^2 + 2| > \sqrt{x + 4}$

We graph $y = \ln|x^2 + 2|$ and $y = \sqrt{x + 4}$ on the same set of axes.

The graphs intersect at $x \approx -1.63$ and $x \approx 3.78$.

$y = \sqrt{x + 4}$ is undefined for $x < -4$.

$\therefore \ln|x^2 + 2| > \sqrt{x + 4}$ when $-4 \leq x < -1.63$ and $x > 3.78$.



79 a

$$|1 - 4x| > \frac{1}{3}|2x - 1|$$

$$\therefore 3|1 - 4x| > |2x - 1|$$

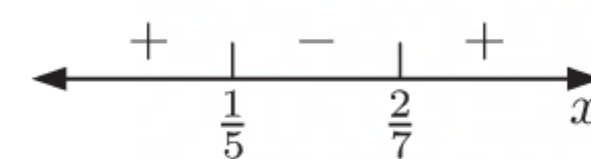
$$\therefore 9(1 - 4x)^2 > (2x - 1)^2$$

$$\therefore 9(1 - 4x)^2 - (2x - 1)^2 > 0$$

$$\therefore [3(1 - 4x) + (2x - 1)][3(1 - 4x) - (2x - 1)] > 0$$

$$\therefore (-10x + 2)(-14x + 4) > 0$$

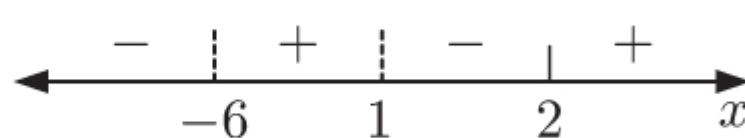
$$\therefore x < \frac{1}{5} \text{ or } x > \frac{2}{7}$$



b $\frac{x - 2}{6 - 5x - x^2} \leq 0$

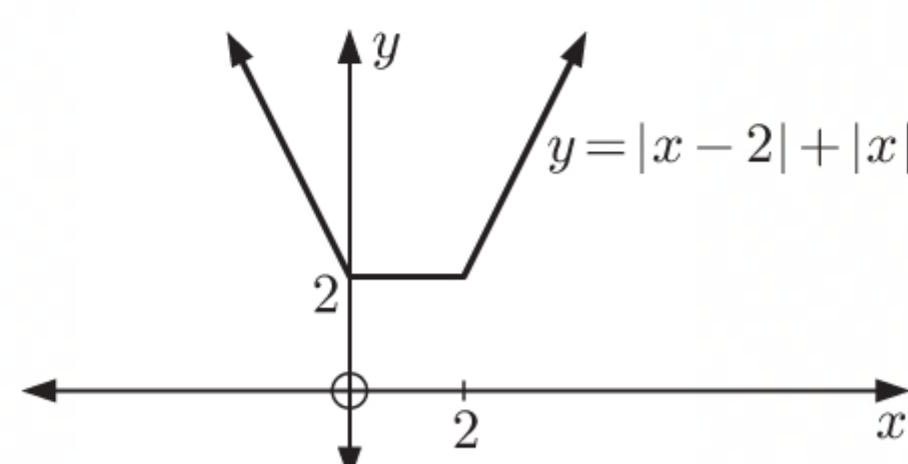
$$\therefore \frac{x - 2}{x^2 + 5x - 6} \geq 0$$

$$\therefore \frac{x - 2}{(x - 1)(x + 6)} \geq 0$$



$$\therefore -6 < x < 1 \text{ or } x \geq 2$$

80 a $|x - 2| + |x| = \begin{cases} 2 - 2x, & x < 0 \\ 2, & 0 \leq x < 2 \\ 2x - 2, & x \geq 2 \end{cases}$



b i $|x - 2| + |x| = 2$

$$\therefore 0 \leq x \leq 2$$

ii $|x - 2| + |x| \geq 3$ when $2 - 2x \geq 3$ or $2x - 2 \geq 3$

$$\therefore x \leq -\frac{1}{2} \text{ or } x \geq \frac{5}{2}$$

81 a If $f(x) = x - \frac{1}{x}$
 then $f(-x) = -x - \frac{1}{(-x)}$

$$= -\left(x - \frac{1}{x}\right)$$

$$= -f(x)$$

 $\therefore y = x - \frac{1}{x}$ is an odd function.

b If $f(x) = \sec 2x$
 then $f(-x) = \sec(2(-x))$

$$= \sec(-2x)$$

$$= \frac{1}{\cos(-2x)}$$

$$= \frac{1}{\cos 2x}$$

$$= \sec 2x$$

$$= f(x)$$

 $\therefore y = \sec 2x$ is an even function.

c If $f(x) = (x^2 + 1)(x^3 + x)$
 then $f(-x) = ((-x)^2 + 1)((-x)^3 + (-x))$

$$= (x^2 + 1)(-x^3 - x)$$

$$= -(x^2 + 1)(x^3 + x)$$

$$= -f(x)$$

 $\therefore y = (x^2 + 1)(x^3 + x)$ is an odd function.

82 $f(x) = (5x - 2)\left(\frac{a}{x} + 3\right)$ is an odd function
 $\therefore f(-x) = -f(x)$
 $\therefore (5(-x) - 2)\left(\frac{a}{-x} + 3\right) = -(5x - 2)\left(\frac{a}{x} + 3\right)$
 $\therefore -(5x + 2)\left(3 - \frac{a}{x}\right) = -(5x - 2)\left(\frac{a}{x} + 3\right)$
 $\therefore (5x + 2)\left(3 - \frac{a}{x}\right) = (5x - 2)\left(\frac{a}{x} + 3\right)$
 $\therefore (5x + 2)(3x - a) = (5x - 2)(a + 3x), \quad x \neq 0$
 $\therefore \cancel{15x^2} - 5ax + 6x - \cancel{2a} = 5ax + \cancel{15x^2} - \cancel{2a} - 6x$
 $\therefore (12 - 10a)x = 0$
 $\therefore 12 - 10a = 0 \quad \{x \neq 0\}$
 $\therefore 10a = 12$
 $\therefore a = \frac{6}{5}$

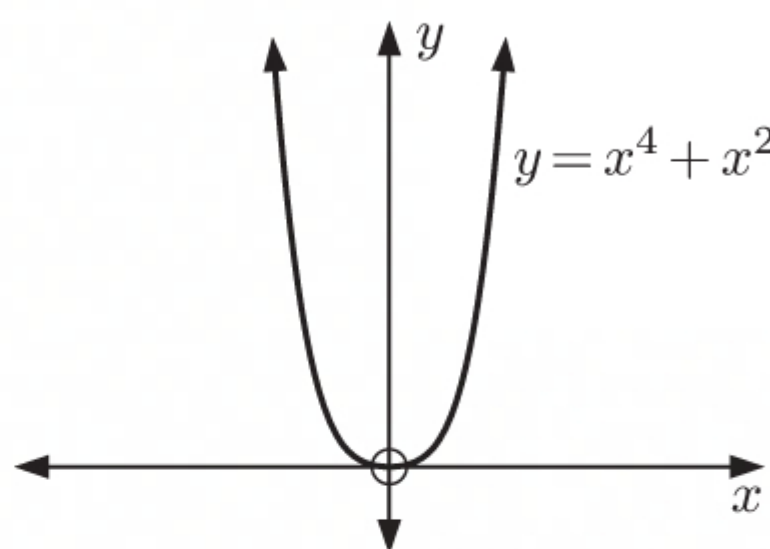
- 83 a** A function is even if $f(-x) = f(x)$ for all x in the domain of the function.
 \therefore for any value in the range of f , there exists at least two corresponding values of x in the domain (except in the trivial case $f(x) = k$ with domain $x = 0$)
 \therefore the function fails the horizontal line test, and does not have an inverse.

b If $f(x) = x^4 + x^2$
 then $f(-x) = (-x)^4 + (-x)^2$

$$= x^4 + x^2$$

$$= f(x)$$

 $\therefore f(x)$ is even.



From the graph, we observe that the function is strictly increasing for $x \geq 0$.

\therefore we can choose the domain restriction $x \geq 0$ for the function to have an inverse.

84 The zeros $2 \pm i\sqrt{3}$ have sum $= 2 + i\sqrt{3} + 2 - i\sqrt{3} = 4$ and
 product $= (2 + i\sqrt{3})(2 - i\sqrt{3}) = 4 + 3 = 7$

\therefore they come from the quadratic factor $x^2 - 4x + 7$.

1 comes from the linear factor $x - 1$.

-3 comes from the linear factor $x + 3$.

$\therefore P(x) = a(x - 1)(x + 3)(x^2 - 4x + 7), \quad a \neq 0.$

$$\begin{aligned}
 85 \quad 9x^4 + 4 &= (3x^2)^2 - (2i)^2 \\
 &= (3x^2 + 2i)(3x^2 - 2i) \quad \{\text{difference of two squares}\}
 \end{aligned}$$

$$\begin{aligned}
 86 \quad \frac{x^2 + 3}{x^2 - 1} &= k \\
 \therefore x^2 + 3 &= k(x^2 - 1) \\
 \therefore x^2 + 3 &= kx^2 - k \\
 \therefore x^2(1 - k) &= -k - 3 \\
 \therefore x^2 &= \frac{-k - 3}{1 - k} = \frac{k + 3}{k - 1}
 \end{aligned}$$

\therefore the equation has imaginary roots if $\frac{k + 3}{k - 1} < 0$

Sign diagram: $\begin{array}{ccccccc} & + & & - & & + & \\ & | & & | & & | & \\ \leftarrow & -3 & & 1 & & k & \rightarrow \end{array}$

So, the equation has imaginary roots for $-3 < k < 1$.

$$\begin{aligned}
 87 \quad \mathbf{a} \quad x^4 + 7x^3 + 13x^2 - 9x - 40 &= (x^2 + ax - 5)(x^2 + bx + 8) \\
 &= x^4 + bx^3 + 8x^2 + ax^3 + abx^2 + 8ax - 5x^2 - 5bx - 40 \\
 &= x^4 + (a + b)x^3 + (3 + ab)x^2 + (8a - 5b)x - 40
 \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} a + b = 7 & \dots (1) & \{x^3 \text{ s}\} \\ 3 + ab = 13 & \dots (2) & \{x^2 \text{ s}\} \\ 8a - 5b = -9 & \dots (3) & \{x \text{ s}\} \end{cases}$$

$$\begin{aligned}
 \text{Now } 8a - 5b &= -9 \quad \{(3)\} \\
 \frac{5a + 5b}{13a} &= \frac{35}{26} \quad \{(1) \times 5\} \\
 \text{Adding, } 13a &= 26 \\
 \therefore a &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } a = 2 \text{ into (1) gives } 2 + b &= 7 \\
 \therefore b &= 5
 \end{aligned}$$

$$\therefore x^4 + 7x^3 + 13x^2 - 9x - 40 = (x^2 + 2x - 5)(x^2 + 5x + 8)$$

$$\mathbf{b} \quad x^4 + 7x^3 + 13x^2 = 9x + 40$$

$$\therefore x^4 + 7x^3 + 13x^2 - 9x - 40 = 0$$

$$\therefore (x^2 + 2x - 5)(x^2 + 5x + 8) = 0 \quad \{\text{using a}\}$$

$$\therefore x^2 + 2x - 5 = 0$$

$$\text{or } x^2 + 5x + 8 = 0$$

$$\begin{aligned}
 \therefore x &= \frac{-2 \pm \sqrt{4 - 4(1)(-5)}}{2} \\
 &= \frac{-2 \pm \sqrt{24}}{2} \\
 &= \frac{-2 \pm 2\sqrt{6}}{2} \\
 &= -1 \pm \sqrt{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{or } x &= \frac{-5 \pm \sqrt{25 - 4(1)(8)}}{2} \\
 &= \frac{-5 \pm \sqrt{-7}}{2} \\
 &= -\frac{5}{2} \pm \frac{\sqrt{7}}{2}i
 \end{aligned}$$

$$\therefore x = -1 \pm \sqrt{6} \quad \{x \in \mathbb{R}\}$$

$$\begin{aligned}
 88 \quad \text{Let } x^3 - 3x^2 - 24x + c &= (x - a)^2(x - b) \\
 &= (x^2 - 2ax + a^2)(x - b) \\
 &= x^3 - bx^2 - 2ax^2 + 2abx + a^2x - a^2b \\
 &= x^3 - (b + 2a)x^2 + (2ab + a^2)x - a^2b
 \end{aligned}$$

$$\text{Equating coefficients } \begin{cases} b + 2a = 3 & \dots (1) & \{x^2 \text{ s}\} \\ 2ab + a^2 = -24 & \dots (2) & \{x \text{ s}\} \\ c = -a^2b & \dots (3) & \{\text{constants}\} \end{cases}$$

$$\text{From (1), } b = 3 - 2a \quad \dots (4).$$

Substituting (4) into (2) gives $2a(3 - 2a) + a^2 = -24$

$$\therefore 6a - 4a^2 + a^2 = -24$$

$$\therefore -3a^2 + 6a + 24 = 0$$

$$\therefore a^2 - 2a - 8 = 0$$

$$\therefore (a + 2)(a - 4) = 0$$

$$\therefore a = -2 \text{ or } 4$$

If $a = -2$, then $b = 3 - 2(-2)$ and $c = -(-2)^2(7)$

$$= 3 + 4 = -(4)(7)$$

$$= 7 = -28$$

If $a = 4$, then $b = 3 - 2(4)$ and $c = -(4)^2(-5)$

$$= 3 - 8 = -16(-5)$$

$$= -5 = 80$$

So if $c = -28$, the zeros are -2 and 7 , and

if $c = 80$, the zeros are 4 and -5 .

$$\begin{array}{r} \text{89 a} \quad \begin{array}{r} x^3 - 2x^2 + 7x - 13 \\ x + 2 \overline{) x^4 + 0x^3 + 3x^2 + x + 0} \\ \underline{-(x^4 + 2x^3)} \\ -2x^3 + 3x^2 \\ \underline{-(-2x^3 - 4x^2)} \\ 7x^2 + x \\ \underline{-(7x^2 + 14x)} \\ -13x + 0 \\ \underline{-(-13x - 26)} \\ +26 \end{array} \end{array}$$

\therefore the quotient is $x^3 - 2x^2 + 7x - 13$ and the remainder is 26 .

$$\text{b} \quad (x - 1)^2 = x^2 - 2x + 1$$

$$\begin{array}{r} x^2 + 2x + 6 \\ x^2 - 2x + 1 \overline{) x^4 + 0x^3 + 3x^2 + x + 0} \\ \underline{-(x^4 - 2x^3 + x^2)} \\ 2x^3 + 2x^2 + x \\ \underline{-(2x^3 - 4x^2 + 2x)} \\ 6x^2 - x + 0 \\ \underline{-(6x^2 - 12x + 6)} \\ 11x - 6 \end{array}$$

\therefore the quotient is $x^2 + 2x + 6$ and the remainder is $11x - 6$.

$$\begin{array}{r} \text{90 a} \quad \begin{array}{r} x + 1 \\ x - 1 \overline{) x^2 + 0x + 2} \\ \underline{-(x^2 - x)} \\ x + 2 \\ \underline{-(x - 1)} \\ 3 \end{array} \end{array}$$

$$\therefore \frac{x^2 + 2}{x - 1} = x + 1 + \frac{3}{x - 1}$$

$$\begin{array}{r} \text{b} \quad \begin{array}{r} x - 1 \\ x + 4 \overline{) x^2 + 3x + 5} \\ \underline{-(x^2 + 4x)} \\ -x + 5 \\ \underline{-(-x - 4)} \\ 9 \end{array} \end{array}$$

$$\therefore \frac{x^2 + 3x + 5}{x + 4} = x - 1 + \frac{9}{x + 4}$$

$$\begin{array}{r} \text{c} \quad \begin{array}{r} 3x + 8 \\ x - 2 \overline{) 3x^2 + 2x - 4} \\ \underline{-(3x^2 - 6x)} \\ 8x - 4 \\ \underline{-(8x - 16)} \\ 12 \end{array} \end{array}$$

$$\therefore \frac{3x^2 + 2x - 4}{x - 2} = 3x + 8 + \frac{12}{x - 2}$$

91 Let $P(x) = x^3 + mx + m$

Now $P(m) = m$ {Remainder theorem}

$$\therefore m^3 + m^2 + m = m$$

$$\therefore m^3 + m^2 = 0$$

$$\therefore m^2(m + 1) = 0$$

$$\therefore m = 0 \text{ or } -1$$

92 $P(x)$ divided by $(x - 1)(x - 2)$ gives a remainder of $2x + 3$.

$$\therefore P(x) = (x - 1)(x - 2)Q(x) + 2x + 3 \text{ for some polynomial } Q(x).$$

$$\therefore P(1) = (0)(-1)Q(1) + 2(1) + 3$$

$$= 5$$

\therefore the remainder when $P(x)$ is divided by $x - 1$ is 5 . {Remainder theorem}

$$\begin{array}{ll}
 \text{93 a} & \text{By the Remainder theorem, } f(1) = 1 \quad \text{and} \quad f(-2) = 1 \\
 & \therefore 3(1)^3 + a(1)^2 - 5(1) + b = 1 \quad \text{and} \quad 3(-2)^3 + a(-2)^2 - 5(-2) + b = 1 \\
 & \therefore 3 + a - 5 + b = 1 \quad \text{and} \quad -24 + 4a + 10 + b = 1 \\
 & \therefore a + b = 3 \quad \dots (1) \quad \text{and} \quad 4a + b = 15 \quad \dots (2)
 \end{array}$$

$$\begin{array}{l}
 \text{Solving simultaneously:} \quad a + b = 3 \quad \{(1)\} \\
 \quad \quad \quad -4a - b = -15 \quad \{-(2)\} \\
 \text{Adding,} \quad \quad \quad \underline{-3a = -12} \\
 \quad \quad \quad \therefore a = 4
 \end{array}$$

$$\begin{array}{l}
 \text{Substituting } a = 4 \text{ into (1), } 4 + b = 3 \\
 \therefore b = -1
 \end{array}$$

$$\text{b Using a, } f(x) = 3x^3 + 4x^2 - 5x - 1$$

$$\begin{array}{l}
 \text{Now } f(3) = 3(3)^3 + 4(3)^2 - 5(3) - 1 \\
 \quad = 81 + 36 - 15 - 1 \\
 \quad = 101
 \end{array}$$

$$\therefore \text{ the remainder when } f(x) \text{ is divided by } x - 3 \text{ is } 101. \quad \{\text{Remainder theorem}\}$$

$$\text{94 Let } \frac{P(x)}{(x-a)^2} = Q(x) + \frac{bx+c}{(x-a)^2} \text{ for some polynomial } Q(x)$$

$$\therefore P(x) = Q(x)(x-a)^2 + bx + c$$

$$\therefore P(a) = Q(a) \times 0 + ab + c = ab + c \quad \dots (1)$$

$$\text{Also, } P'(x) = Q'(x)(x-a)^2 + Q(x)2(x-a) + b$$

$$\therefore P'(a) = 0 + 0 + b = b \quad \dots (2)$$

$$\begin{array}{l}
 \text{So, the remainder is } bx + c = bx + (ab + c) - ab \\
 \quad = P'(a)x + P(a) - aP'(a) \quad \{\text{using (1) and (2)}\} \\
 \quad = P'(a)(x-a) + P(a)
 \end{array}$$

$$\text{95 Let } P(x) = 6x^2 + ax + b$$

$$\begin{array}{ll}
 \text{By the Remainder theorem, } P(-3) = 22 & \text{and} \quad P(-\frac{1}{2}) = 2 \\
 \therefore 6(-3)^2 + a(-3) + b = 22 & \text{and} \quad 6(-\frac{1}{2})^2 + a(-\frac{1}{2}) + b = 2 \\
 \therefore 54 - 3a + b = 22 & \text{and} \quad \frac{3}{2} - \frac{1}{2}a + b = 2 \\
 \therefore -3a + b = -32 \quad \dots (1) & \text{and} \quad -a + 2b = 1 \quad \dots (2)
 \end{array}$$

$$\begin{array}{l}
 \text{Solving simultaneously:} \quad -3a + b = -32 \quad \{(1)\} \\
 \quad \quad \quad 3a - 6b = -3 \quad \{-3 \times (2)\} \\
 \text{Adding,} \quad \quad \quad \underline{-5b = -35} \\
 \quad \quad \quad \therefore b = 7
 \end{array}$$

$$\begin{array}{l}
 \text{Substituting } b = 7 \text{ into (2), } -a + 14 = 1 \\
 \therefore -a = -13 \\
 \therefore a = 13
 \end{array}$$

$$\text{96 Let } P(x) = 3x^3 + ax^2 + bx + 6$$

$$\begin{array}{ll}
 \text{By the Factor theorem, } P(-3) = 0 & \text{and} \quad P(2) = 0 \\
 \therefore 3(-3)^3 + a(-3)^2 + b(-3) + 6 = 0 & \text{and} \quad 3(2)^3 + a(2)^2 + b(2) + 6 = 0 \\
 \therefore -81 + 9a - 3b + 6 = 0 & \text{and} \quad 24 + 4a + 2b + 6 = 0 \\
 \therefore 9a - 3b = 75 & \text{and} \quad 4a + 2b = -30 \\
 \therefore 3a - b = 25 \quad \dots (1) & \text{and} \quad 2a + b = -15 \quad \dots (2)
 \end{array}$$

$$\begin{array}{l}
 \text{Solving simultaneously:} \quad 3a - b = 25 \quad \{(1)\} \\
 \quad \quad \quad 2a + b = -15 \quad \{(2)\} \\
 \text{Adding,} \quad \quad \quad \underline{5a = 10} \\
 \quad \quad \quad \therefore a = 2
 \end{array}$$

$$\begin{array}{l}
 \text{Substituting } a = 2 \text{ into (2), } 4 + b = -15 \\
 \therefore b = -19
 \end{array}$$

97 $P(x) = 2x^3 + ax^2 + bx - 30$

a $P(3) = 0$ {Factor theorem}

$$\therefore 2(3)^3 + a(3)^2 + b(3) - 30 = 0$$

$$\therefore 54 + 9a + 3b - 30 = 0$$

$$\therefore 9a + 3b = -24$$

$$\therefore 3a + b = -8 \quad \dots (1)$$

$$P(-1) = -12 \quad \{\text{Remainder theorem}\}$$

$$\therefore 2(-1)^3 + a(-1)^2 + b(-1) - 30 = -12$$

$$\therefore -2 + a - b - 30 = -12$$

$$\therefore a - b = 20 \quad \dots (2)$$

Solving simultaneously: $3a + b = -8$ {(1)}

$$a - b = 20 \quad \{(2)\}$$

Adding, $4a = 12$

$$\therefore a = 3$$

Substituting $a = 3$ into (1), $9 + b = -8$

$$\therefore b = -17$$

b From **a**, $P(x) = 2x^3 + 3x^2 - 17x - 30$

$$\therefore P(-3) = 2(-3)^3 + 3(-3)^2 - 17(-3) - 30$$

$$= -54 + 27 + 51 - 30$$

$$= -6$$

\therefore the remainder when $P(x)$ is divided by $x + 3$ is -6 . {Remainder theorem}

c

$$\begin{array}{r} 2x^2 + 9x + 10 \\ x - 3 \overline{) 2x^3 + 3x^2 - 17x - 30} \\ \underline{-(2x^3 - 6x^2)} \downarrow \\ 9x^2 - 17x \downarrow \\ \underline{-(9x^2 - 27x)} \downarrow \\ 10x - 30 \downarrow \\ \underline{-(10x - 30)} \\ 0 \end{array}$$

$$\therefore P(x) = (x - 3)(2x^2 + 9x + 10)$$

d

$$2x^2 + 9x + 10 = 0$$

$$\therefore 2x^2 + 4x + 5x + 10 = 0$$

$$\therefore 2x(x + 2) + 5(x + 2) = 0$$

$$\therefore (x + 2)(2x + 5) = 0$$

$$\therefore x = -2 \text{ or } -\frac{5}{2}$$

\therefore the zeros of $P(x)$ are 3, -2 , and $-\frac{5}{2}$.

98 If $a + 2i$ is a root, so is its complex conjugate $a - 2i$ $\{a, b \in \mathbb{R}\}$.

$$\therefore \text{product of roots} = (a + 2i)(a - 2i)$$

$$= a^2 + 4$$

$$\therefore a^2 + 4 = a + 6$$

$$\therefore a^2 - a - 2 = 0$$

$$\therefore (a - 2)(a + 1) = 0$$

$$\therefore a = -1 \text{ or } 2$$

Also, sum of roots $= 2a$

$$\therefore b = -2a$$

$$\therefore a = -1, b = 2 \text{ or } a = 2, b = -4$$

99 $P(x)$ is a real polynomial, so $3 + 2i$ must also be a zero of $P(x)$.

$$(3 + 2i) + (3 - 2i) = 6 \quad \text{and} \quad (3 + 2i)(3 - 2i) = 9 + 4$$

$$= 13$$

So, $x^2 - 6x + 13$ is a factor of $P(x)$.

$$\therefore 2x^3 + mx^2 - (m + 1)x + (3 - 4m) = (x^2 - 6x + 13)(2x + b) \quad \text{for some } b$$

$$= 2x^3 + (b - 12)x^2 + (26 - 6b)x + 13b$$

Equating coefficients of x^2 : $m = b - 12$ (1)

Equating constants: $3 - 4m = 13b$ (2)

Substituting (1) into (2): $13b = 3 - 4(b - 12)$

$$\therefore 13b = 3 - 4b + 48$$

$$\therefore 17b = 51$$

$$\therefore b = 3$$

$$\therefore m = 3 - 12 = -9$$

$$\therefore P(x) = (x^2 - 6x + 13)(2x + 3)$$

\therefore the zeros of $P(x)$ are $3 \pm 2i$ and $-\frac{3}{2}$

100 Sum of roots $= \alpha + \beta = -m$

Product of roots $= \alpha\beta = 1$

$$\text{Now } \alpha\beta = \frac{1}{\alpha} + \frac{1}{\beta}$$

$$\therefore \alpha\beta = \frac{\alpha + \beta}{\alpha\beta}$$

$$\therefore 1 = \frac{-m}{1}$$

$$\therefore m = -1$$

101 a $x^2 + 3x - 5 = 0$

\therefore the sum of the roots $= -3$

The polynomial has degree 2.

\therefore the product of the roots $= (-1)^2(-5) = -5$

c $-3x^7 + 2x - 5 = 0$

\therefore the sum of the roots $= -\frac{0}{-3} = 0$

The polynomial has degree 7.

\therefore the product of the roots $= \frac{(-1)^7(-5)}{-3} = -\frac{5}{3}$

102 a The zeros of the cubic polynomial $P(x)$ are $2 \pm i\sqrt{3}$ and -1 .

$$\begin{aligned} \therefore \text{the sum of the zeros} &= (2 + i\sqrt{3}) + (2 - i\sqrt{3}) - 1 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{The product of the zeros} &= (2 + i\sqrt{3})(2 - i\sqrt{3})(-1) \\ &= (4 + 3)(-1) \\ &= -7 \end{aligned}$$

b Let $P(x) = 2x^3 + ax^2 + bx + c$ {leading coefficient is 2}.

$$\begin{aligned} \therefore \text{the sum of the zeros is } -\frac{a}{2} &= 3 \\ \therefore a &= -6 \end{aligned}$$

So, the coefficient of x^2 is -6 .

$$\begin{aligned} \text{c The product of the zeros is } \frac{(-1)^3 c}{2} &= -7 \\ \therefore -c &= -14 \\ \therefore c &= 14 \end{aligned}$$

So, the constant term is 14.

b $2x^3 + x^2 - 4x + 1 = 0$

\therefore the sum of the roots $= -\frac{1}{2}$

The polynomial has degree 3.

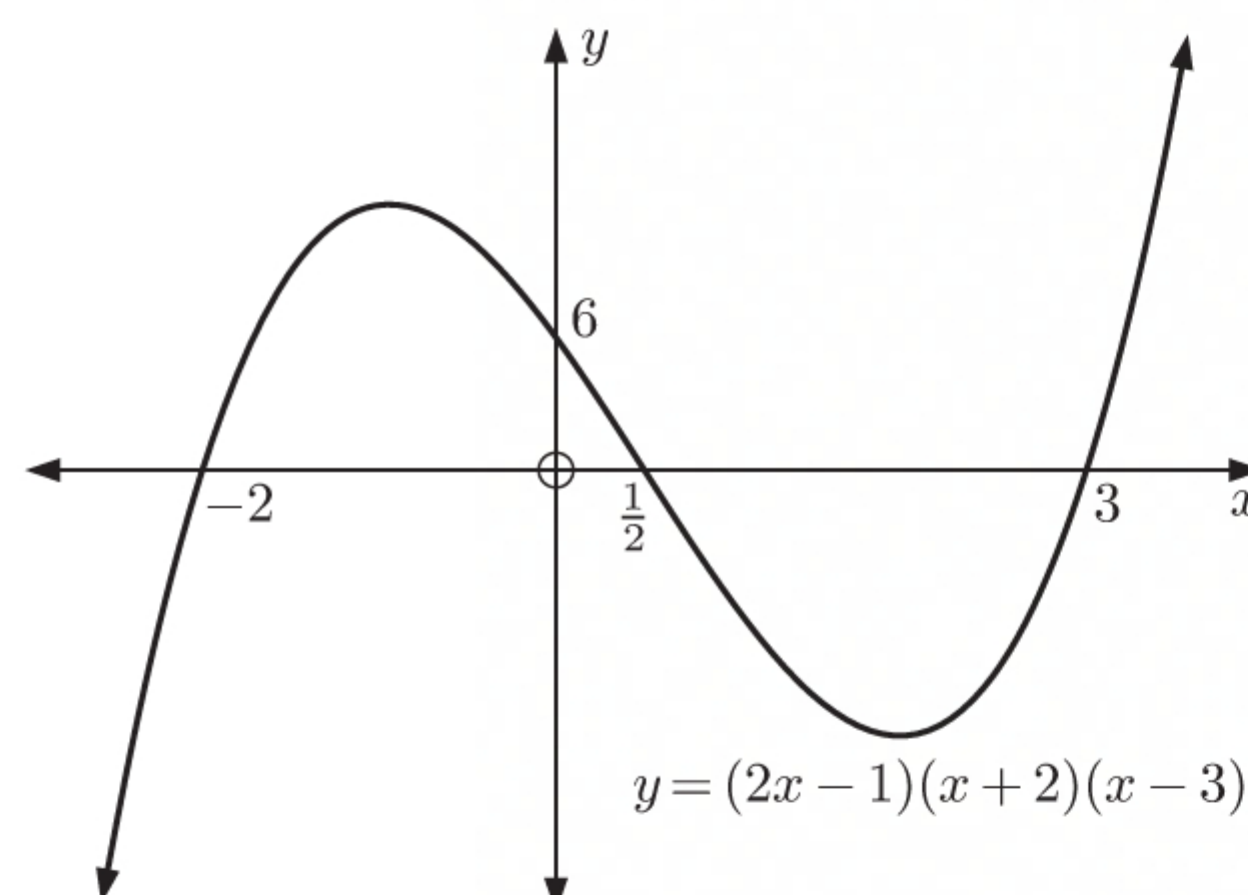
\therefore the product of the roots $= \frac{(-1)^3 1}{2} = -\frac{1}{2}$

103 a $y = (2x - 1)(x + 2)(x - 3)$

The graph cuts the x -axis at $\frac{1}{2}$, -2 , and 3 .

When $x = 0$, $y = (-1)(2)(-3) = 6$

\therefore the y -intercept is 6 .

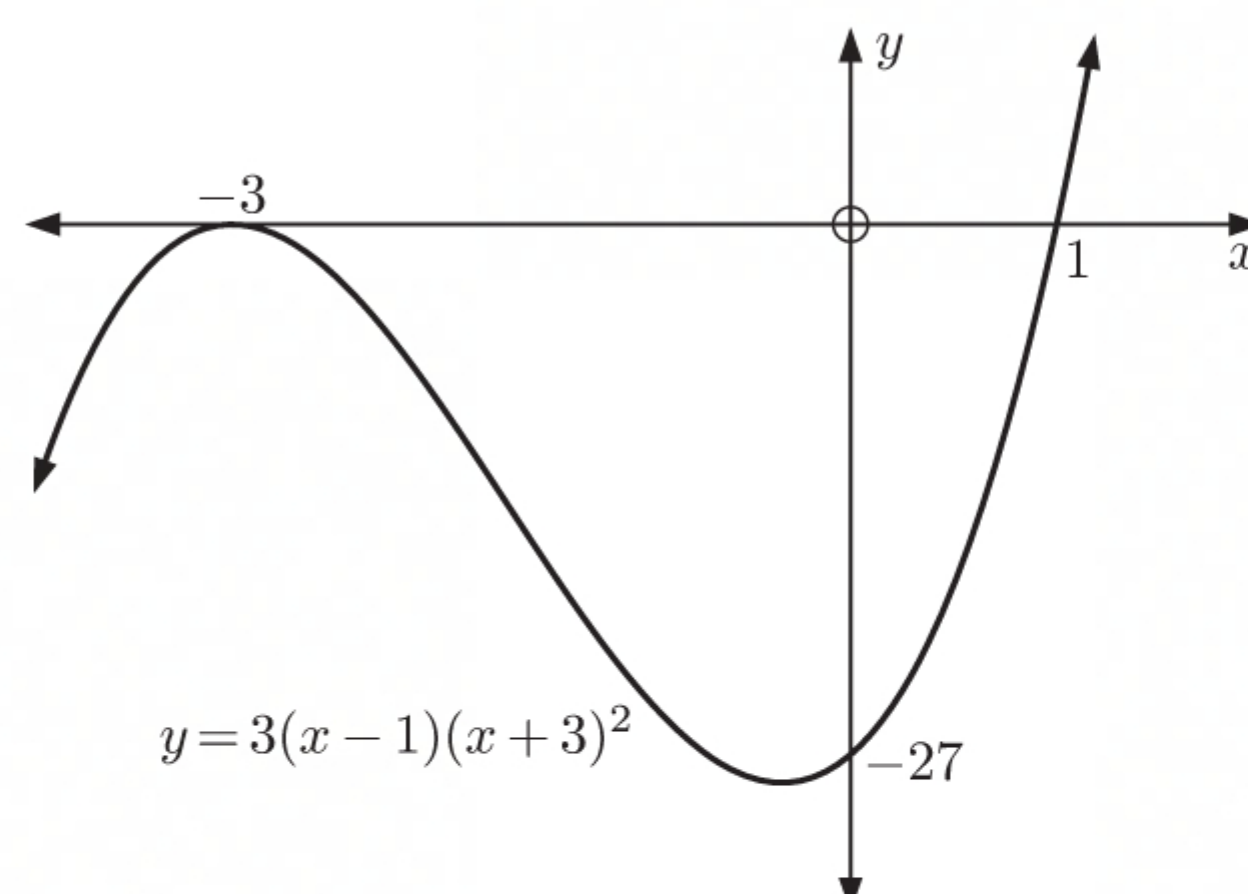


b $y = 3(x - 1)(x + 3)^2$

The graph cuts the x -axis at 1 and touches the x -axis at -3 .

When $x = 0$, $y = 3(-1)(3)^2$
 $= -27$

\therefore the y -intercept is -27 .



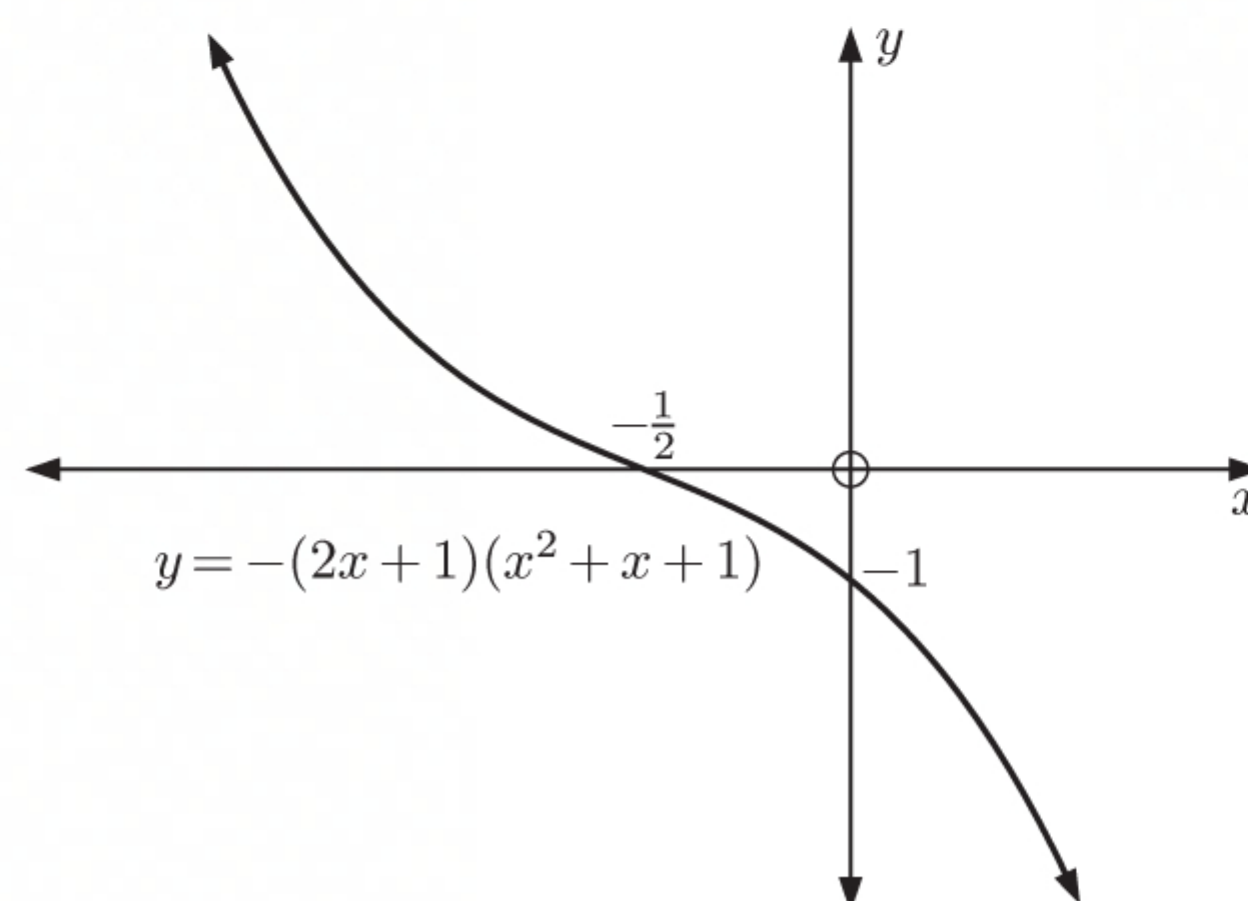
c $y = -(2x + 1)(x^2 + x + 1)$

The discriminant of the quadratic factor is $\Delta = 1^2 - 4(1)(1)$
 $= -3$
 < 0

So, the graph cuts the x -axis only at $-\frac{1}{2}$.

When $x = 0$, $y = -(1)(1) = -1$

\therefore the y -intercept is -1 .



104 a $y = a(x + 1)(2x + 1)(x - 2)$

But when $x = 1$, $y = -12$

$\therefore a(2)(3)(-1) = -12$

$\therefore -6a = -12$

$\therefore a = 2$

$\therefore y = 2(x + 1)(2x + 1)(x - 2)$

c $y = (x + 3)^2(ax + b)$

When $x = 1$, $y = -48$

$\therefore (4)^2(a + b) = -48$

$\therefore a + b = -3 \quad \dots (1)$

When $x = -2$, $y = -9$

$\therefore (1)^2(-2a + b) = -9$

$\therefore -2a + b = -9 \quad \dots (2)$

Solving simultaneously: $2a + 2b = -6 \quad \{2 \times (1)\}$

$-2a + b = -9 \quad \{(2)\}$

Adding, $3b = -15$

$\therefore b = -5$

Substituting $b = -5$ into (1) gives $a - 5 = -3$

$\therefore a = 2$

$\therefore y = (x + 3)^2(2x - 5)$

b $y = a(x - 4)^2(x + 2)$

But when $x = 7$, $y = -27$

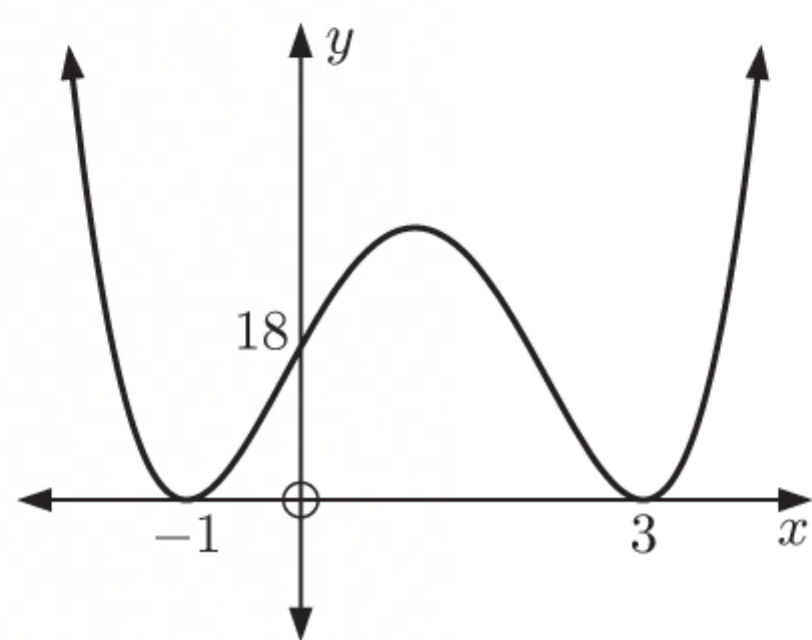
$\therefore a(3)^2(9) = -27$

$\therefore 81a = -27$

$\therefore a = -\frac{1}{3}$

$\therefore y = -\frac{1}{3}(x - 4)^2(x + 2)$

105 a



The graph touches the x -axis at -1 and 3 .

$$\therefore y = a(x+1)^2(x-3)^2$$

But when $x = 0$, $y = 18$

$$\therefore a(1)^2(-3)^2 = 18$$

$$\therefore 9a = 18$$

$$\therefore a = 2$$

$$\therefore y = 2(x+1)^2(x-3)^2$$

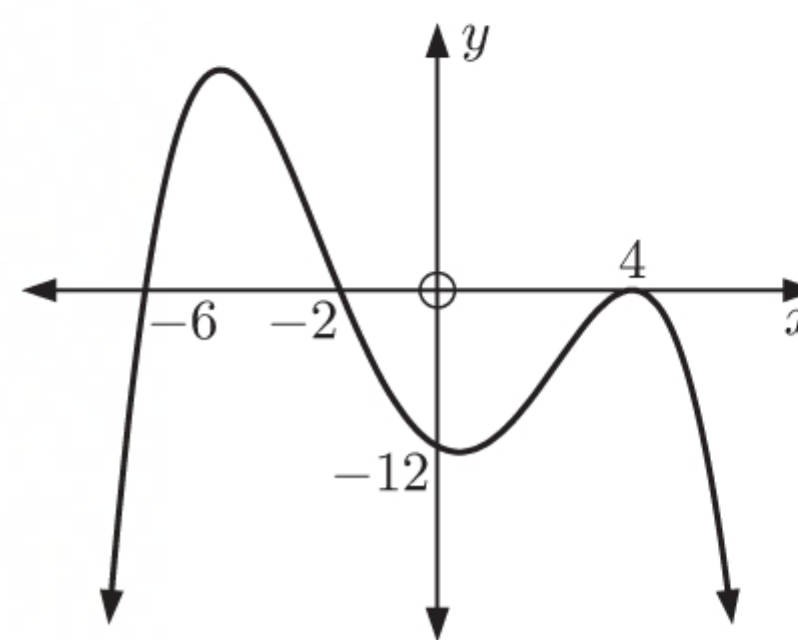
$$\therefore y = 2(x^2 + 2x + 1)(x^2 - 6x + 9)$$

$$\begin{aligned} \therefore y &= 2(x^4 - 6x^3 + 9x^2 \\ &\quad + 2x^3 - 12x^2 + 18x \\ &\quad + x^2 - 6x + 9) \end{aligned}$$

$$\therefore y = 2(x^4 - 4x^3 - 2x^2 + 12x + 9)$$

$$\therefore y = 2x^4 - 8x^3 - 4x^2 + 24x + 18$$

b



The graph touches the x -axis at 4 , and cuts the x -axis at -6 and -2 .

$$\therefore y = a(x-4)^2(x+6)(x+2)$$

But when $x = 0$, $y = -12$

$$\therefore a(-4)^2(6)(2) = -12$$

$$\therefore 192a = -12$$

$$\therefore a = -\frac{1}{16}$$

$$\therefore y = -\frac{1}{16}(x-4)^2(x+6)(x+2)$$

$$\therefore y = -\frac{1}{16}(x^2 - 8x + 16)(x^2 + 8x + 12)$$

$$\begin{aligned} \therefore y &= -\frac{1}{16}(x^4 + 8x^3 + 12x^2 \\ &\quad - 8x^3 - 64x^2 - 96x \\ &\quad + 16x^2 + 128x + 192) \end{aligned}$$

$$\therefore y = -\frac{1}{16}(x^4 - 36x^2 + 32x + 192)$$

$$\therefore y = -\frac{1}{16}x^4 + \frac{9}{4}x^2 - 2x - 12$$

106 a Using technology, $0.5 = \frac{1}{2}$ is a zero, so $(2x - 1)$ is a factor of the cubic.

$$\therefore 2x^3 + 7x^2 - 2 = (2x - 1)(x^2 + ax + 2) \text{ for some } a.$$

Equating coefficients of x^2 : $-1 + 2a = 7$

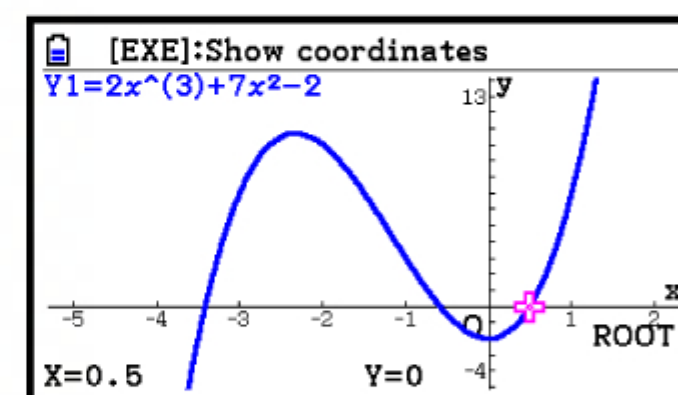
$$\therefore 2a = 8$$

$$\therefore a = 4$$

Equating coefficients of x : $-a + 4 = 0$ ✓

So the quadratic factor is $(x^2 + 4x + 2)$ which has zeros $\frac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2} = -2 \pm \sqrt{2}$

\therefore the zeros are $\frac{1}{2}$ and $-2 \pm \sqrt{2}$.



b Using technology, 2 is a repeated zero, so $(x - 2)^2$ is a factor of the quartic.

$$\therefore 3x^4 - 13x^3 + 17x^2 - 8x + 4$$

$$= (x - 2)^2(3x^2 + ax + 1) \text{ for some } a.$$

$$= (x^2 - 4x + 4)(3x^2 + ax + 1)$$

$$= 3x^4 + ax^3 + x^2$$

$$- 12x^3 - 4ax^2 - 4x$$

$$+ 12x^2 + 4ax + 4$$

$$= 3x^4 + (a - 12)x^3 + (13 - 4a)x^2 + (4a - 4)x + 4$$

Equating coefficients of x^3 : $a - 12 = -13$

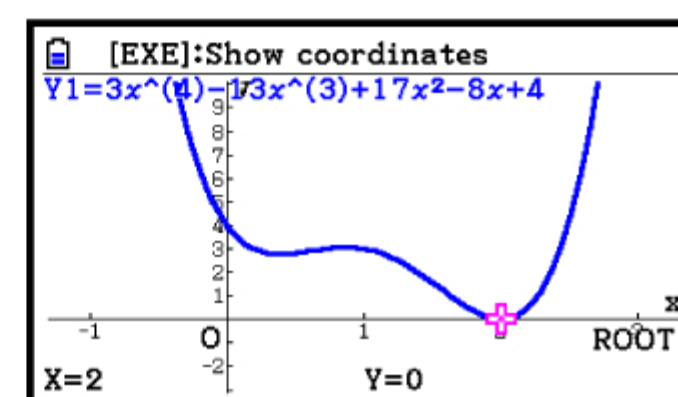
$$\therefore a = -1$$

Equating coefficients of x^2 : $13 - 4a = 13 + 4 = 17$ ✓

Equating coefficients of x : $4a - 4 = -4 - 4 = -8$ ✓

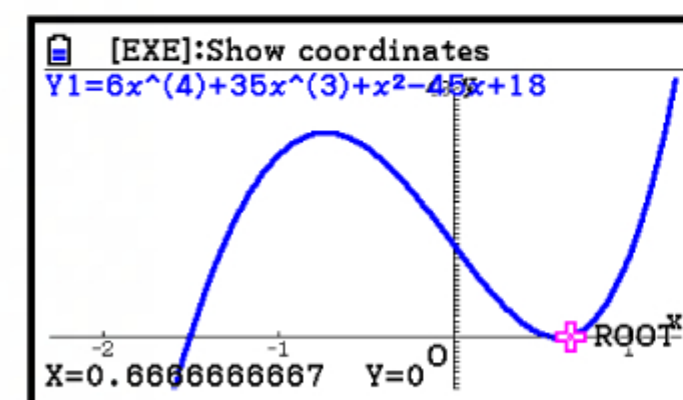
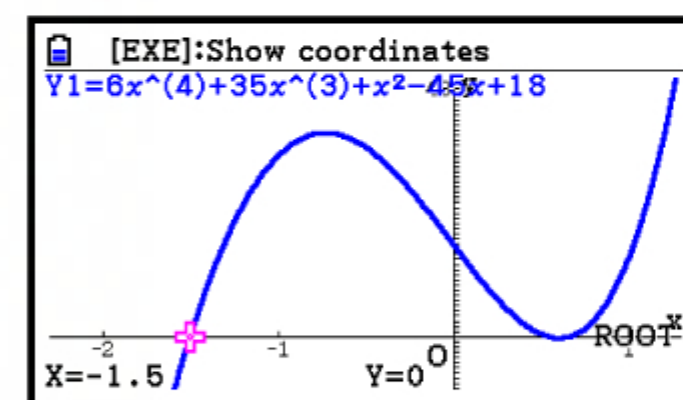
So the other quadratic factor is $(3x^2 - x + 1)$ which has zeros $\frac{1 \pm \sqrt{(-1)^2 - 4(3)(1)}}{2} = \frac{1}{2} \pm \frac{\sqrt{11}}{2}i$.

\therefore the zeros are 2 and $\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$.



- c** Using technology, $-1.5 = -\frac{3}{2}$ and $\approx 0.6667 \approx \frac{2}{3}$ are zeros, so $(2x + 3)$ and $(3x - 2)$ are factors of the quartic.

$$\begin{aligned}
 \therefore 6x^4 + 35x^3 + x^2 - 45x + 18 & \\
 &= (2x + 3)(3x - 2)(x^2 + ax - 3) \quad \text{for some } a. \\
 &= (6x^2 + 5x - 6)(x^2 + ax - 3) \\
 &= 6x^4 + 6ax^3 - 18x^2 \\
 &\quad + 5x^3 + 5ax^2 - 15x \\
 &\quad - 6x^2 - 6ax + 18 \\
 &= 6x^4 + (6a + 5)x^3 + (5a - 24)x^2 - (6a + 15)x + 18
 \end{aligned}$$



Equating coefficients of x^3 : $6a + 5 = 35$

$$\therefore 6a = 30$$

$$\therefore a = 5$$

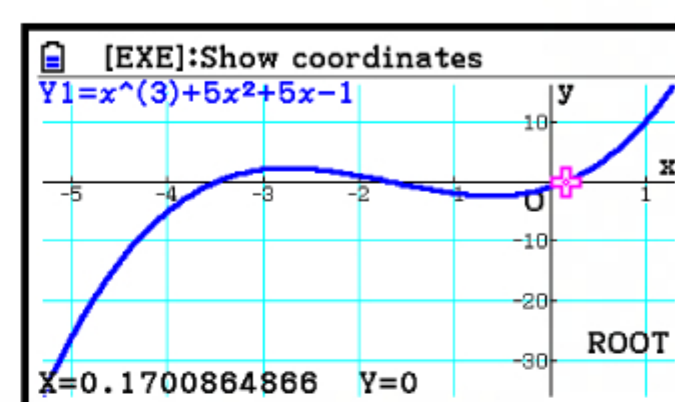
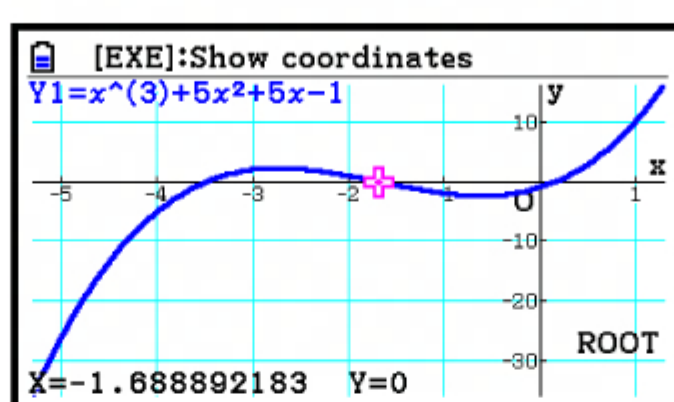
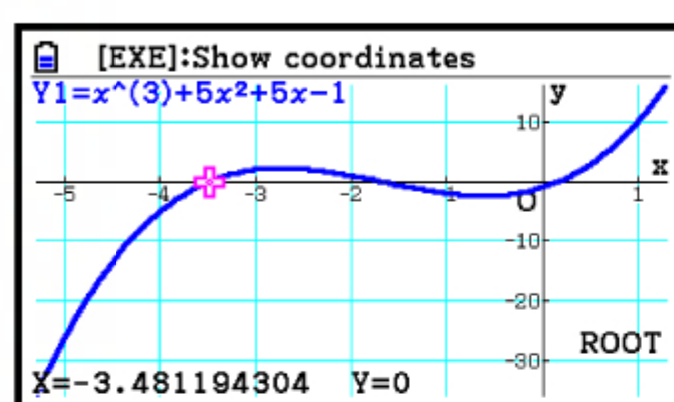
Equating coefficients of x^2 : $5a - 24 = 25 - 24 = 1$ ✓

Equating coefficients of x : $6a + 15 = 30 + 15 = 45$ ✓

So the other quadratic factor is $(x^2 + 5x - 3)$ which has zeros $\frac{-5 \pm \sqrt{5^2 - 4(1)(-3)}}{2} = \frac{-5 \pm \sqrt{37}}{2}$.

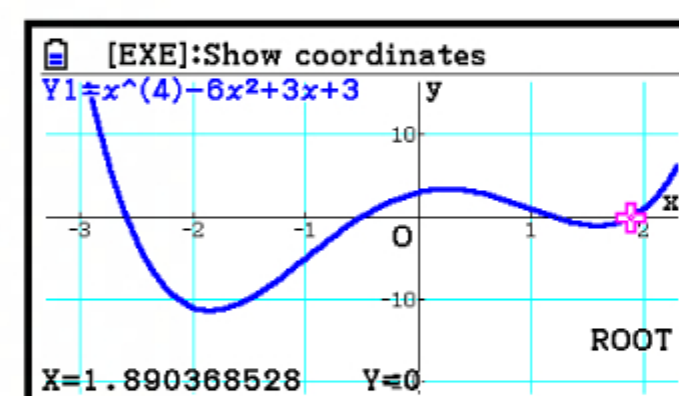
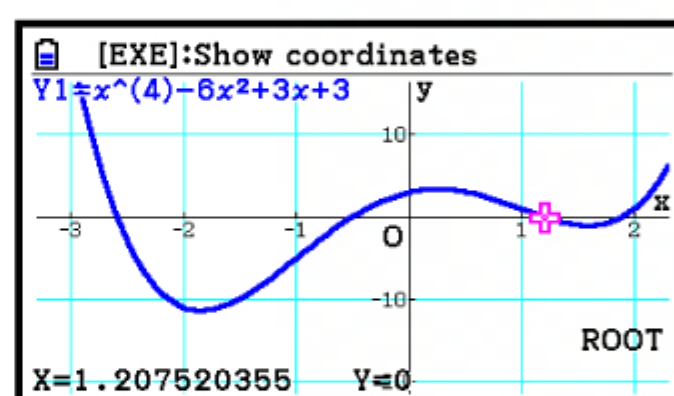
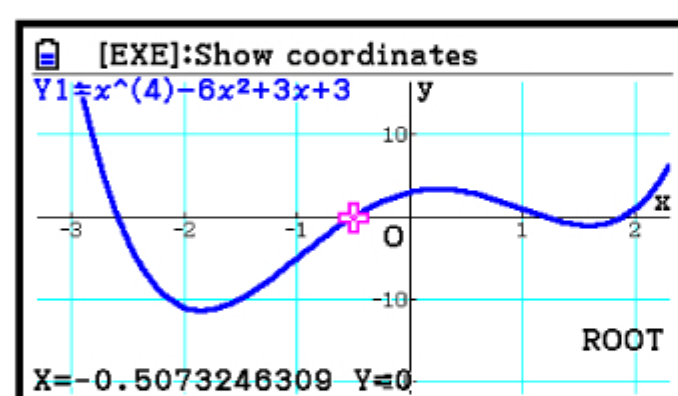
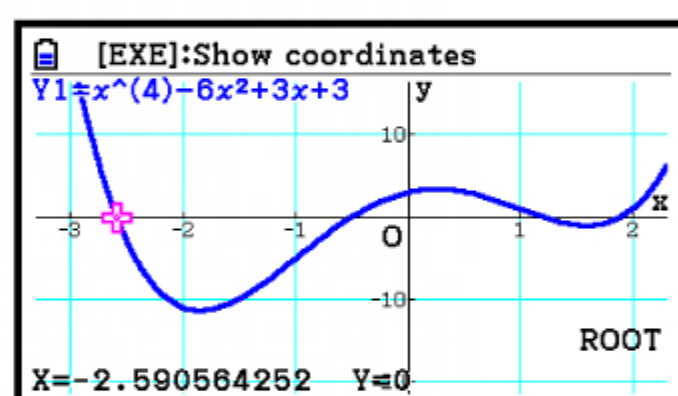
\therefore the zeros are $-\frac{3}{2}$, $\frac{2}{3}$, and $\frac{-5 \pm \sqrt{37}}{2}$.

107 a $x^3 + 5x^2 + 5x - 1 = 0$



$\therefore x \approx -3.48, -1.69, \text{ or } 0.170$

b $x^4 - 6x^2 + 3x + 3 = 0$



$\therefore x \approx -2.59, -0.507, 1.21, \text{ or } 1.89$

- 108 a** As a , b , and c are real, $P(z)$ is a real polynomial.


So, since $-3 + 2i$ is a zero, $-3 - 2i$ is also a zero.

These have sum -6 and product $9 + 4 = 13$.

$$\begin{aligned}
 \text{Thus } P(z) &= (z + 2)(z^2 + 6z + 13) \\
 &= z^3 + 8z^2 + 25z + 26
 \end{aligned}$$

$\therefore a = 8, b = 25, \text{ and } c = 26$

- b** If $P(z) \geq 0$ then $(z + 2)(z^2 + 6z + 13) \geq 0$.

Now $z^2 + 6z + 13 > 0$ for all z , since its roots are complex and it has shape 

$\therefore P(z) \geq 0$ provided $z + 2 \geq 0$

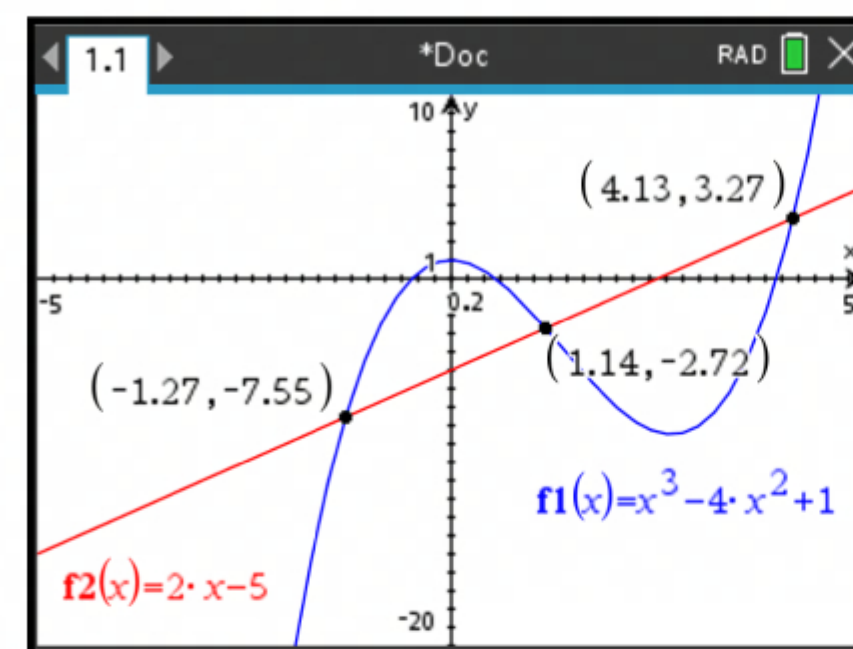
$$\therefore z \geq -2$$

- 109 a** We graph $y = x^3 - 4x^2 + 1$ and $y = 2x - 5$ on the same set of axes.

Using technology, the graphs intersect at $x \approx -1.27$, $x \approx 1.14$, and $x \approx 4.13$.

$x^3 - 4x^2 + 1 < 2x - 5$ whenever the graph of $y = x^3 - 4x^2 + 1$ is below the graph of $y = 2x - 5$.

This occurs when $x < -1.27$ or $1.14 < x < 4.13$.

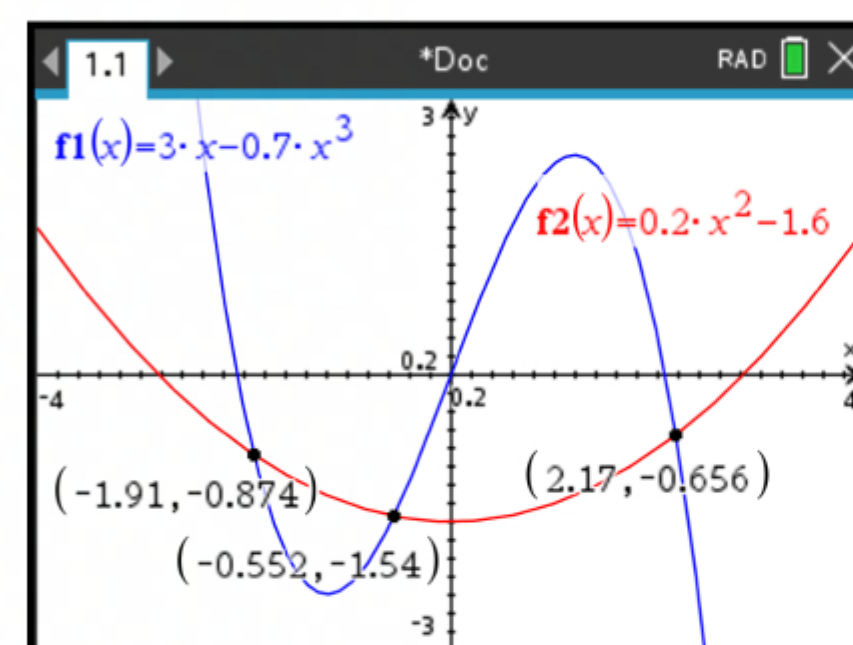


- b** We graph $y = 3x - 0.7x^3$ and $y = 0.2x^2 - 1.6$ on the same set of axes.

Using technology, the graphs intersect at $x \approx -1.91$, $x \approx -0.552$, and $x \approx 2.17$.

$3x - 0.7x^3 \geq 0.2x^2 - 1.6$ whenever the graph of $y = 3x - 0.7x^3$ is above the graph of $y = 0.2x^2 - 1.6$.

This occurs when $x \leq -1.91$ or $-0.552 \leq x \leq 2.17$.



TOPIC 3 SKILL BUILDER QUESTIONS

- 1 a** Let the height of the triangles be h cm.

$$\text{Now } h^2 = 20^2 + 5^2$$

$$\therefore h = \sqrt{20^2 + 5^2} = 5\sqrt{17}$$

Surface area

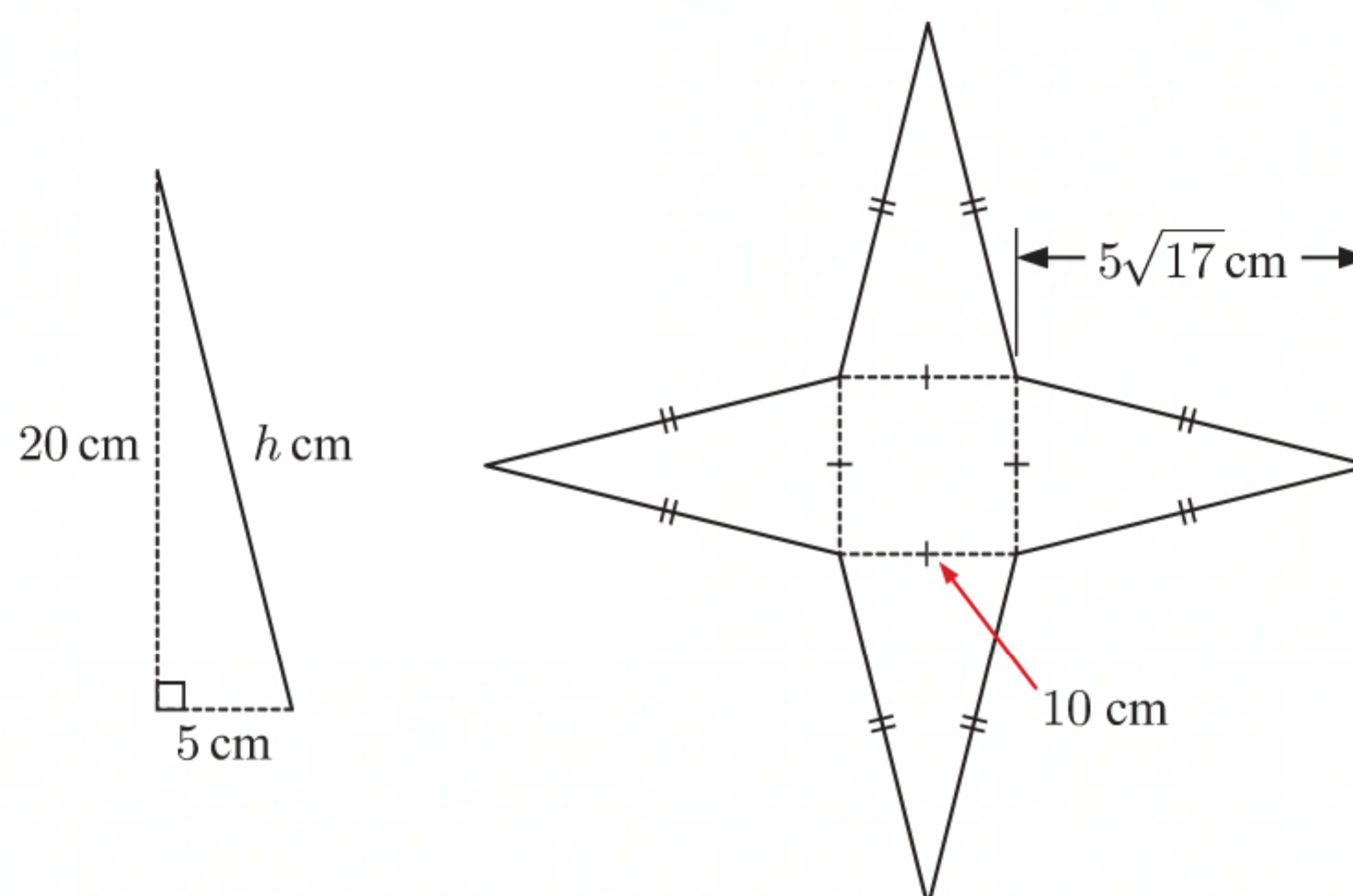
$$= \text{area of square} + 4 \times \text{area of triangle}$$

$$= 10^2 + 4\left(\frac{1}{2} \times 10 \times 5\sqrt{17}\right)$$

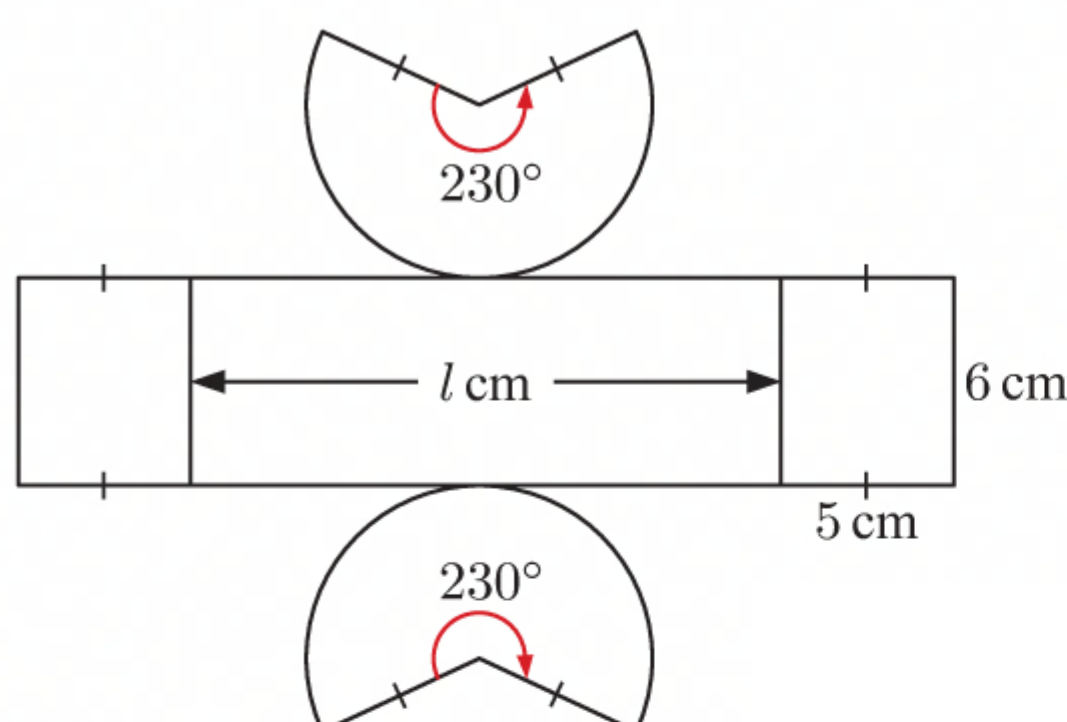
$$= 100 + 4(25\sqrt{17})$$

$$= 100 + 100\sqrt{17} \text{ cm}^2$$

$$\approx 512 \text{ cm}^2$$



b



$$l = 2\pi r \times \frac{230}{360}$$

$$= 2\pi(5) \times \frac{23}{36}$$

$$= \frac{115\pi}{18}$$

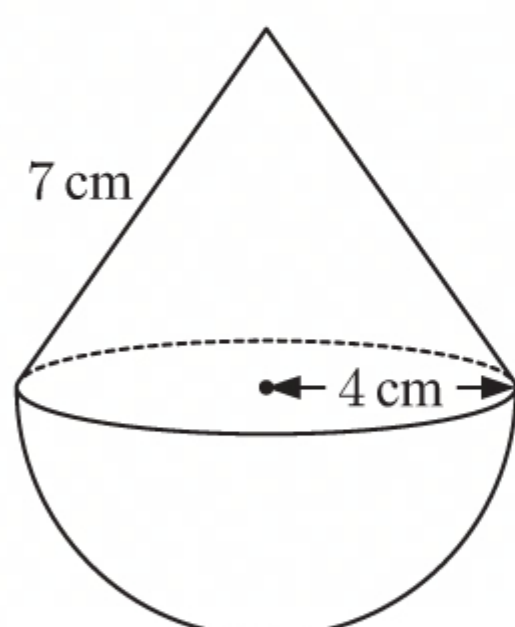
$$\text{Surface area} = 2 \times \text{area of sector} + \text{area of curved surface} + 2 \times \text{area of rectangle}$$

$$= 2 \times \frac{230}{360} \times \pi(5)^2 + \frac{115\pi}{18} \times 6 + 2 \times 6 \times 5$$

$$= \frac{575\pi}{18} + \frac{115\pi}{3} + 60$$

$$\approx 281 \text{ cm}^2$$

c



$$\text{Surface area} = \frac{1}{2}4\pi r^2 + \pi rs$$

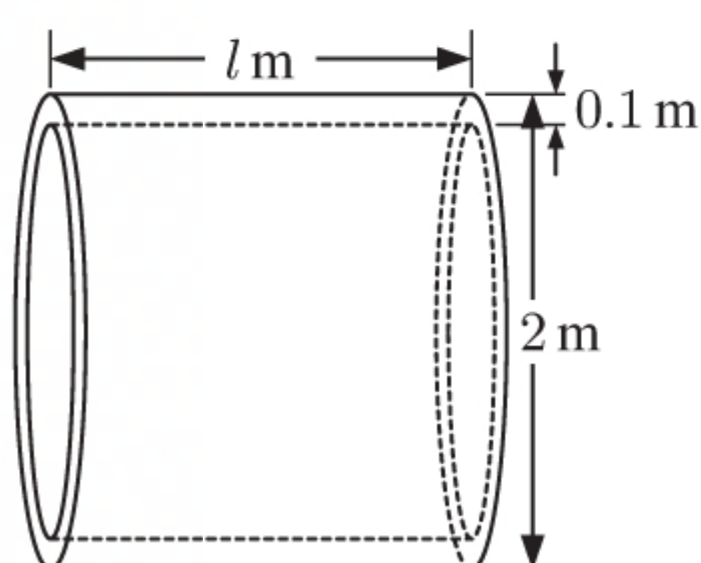
$$= \frac{1}{2} \times 4\pi(4)^2 + \pi(4)(7)$$

$$= 32\pi + 28\pi$$

$$= 60\pi \text{ cm}^2$$

$$\approx 188 \text{ cm}^2$$

2



Let the pipe have length l m.

Now volume of concrete = volume of whole cylinder – volume of hollow section

$$\therefore 3 = \pi(1)^2 \times l - \pi(0.9)^2 \times l$$

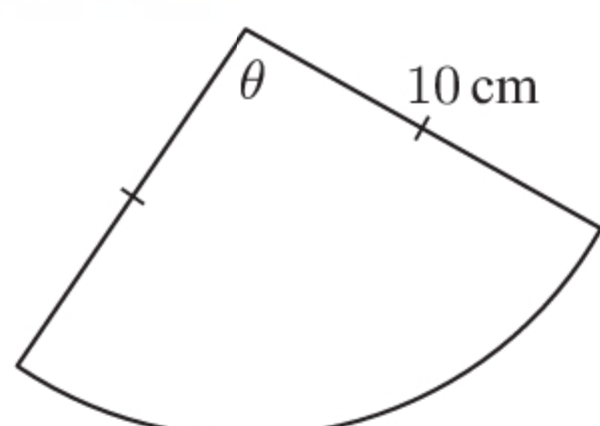
$$\therefore 3 = \pi l - 0.81\pi l$$

$$\therefore 3 = 0.19\pi l$$

$$\therefore l = \frac{3}{0.19\pi} \approx 5.03$$

\therefore the pipe is approximately 5.03 m long.

3



a Perimeter = $2r + \text{arc length}$

$$\therefore 40 = 2(10) + \text{arc length}$$

$$\therefore \text{arc length} = 20 \text{ cm}$$

b arc length = θr

$$\therefore 20 = 10\theta \quad \{\text{from a}\}$$

$$\therefore \theta = 2^\circ$$

$$\therefore \text{area} = \frac{1}{2}\theta r^2$$

$$= \frac{1}{2} \times 2 \times 10^2$$

$$= 100 \text{ cm}^2$$

- 4 a** Total height = hemisphere radius + cone height
 $\therefore 7 = \text{hemisphere radius} + 4$
 $\therefore \text{hemisphere radius} = 3 \text{ m}$
 $\therefore \text{cone radius} = 3 \text{ m}$ {hemisphere radius = cone radius}

b Volume = volume of hemisphere + volume of cone

$$= \frac{1}{2} \times \frac{4}{3} \pi r^3 + \frac{1}{3} \times \pi r^2 \times h$$

$$= \frac{1}{2} \times \frac{4}{3} \pi (3)^3 + \frac{1}{3} \times \pi (3)^2 \times 4$$

$$= 18\pi + 12\pi$$

$$= 30\pi \approx 94.2 \text{ m}^3$$

- c** Let the slant height of the cone be $s \text{ m}$.

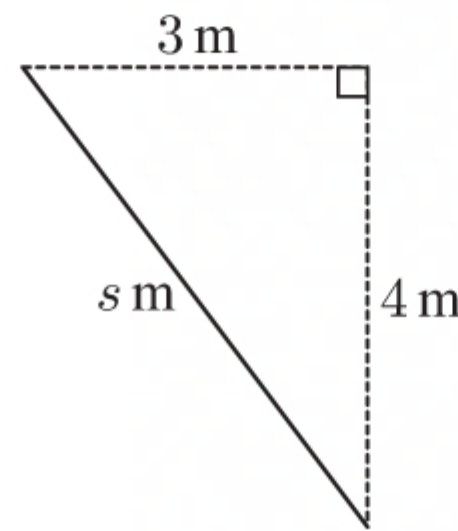
$$\therefore s^2 = 4^2 + 3^2 \quad \{\text{Pythagoras}\}$$

$$\therefore s^2 = 16 + 9$$

$$\therefore s^2 = 25$$

$$\therefore s = 5 \quad \{s > 0\}$$

\therefore the slant height of the cone is 5 m.



d Surface area = $\frac{1}{2} \times 4\pi r^2 + \pi r s$

$$= 2\pi(3)^2 + \pi(3)(5)$$

$$= 18\pi + 15\pi$$

$$= 33\pi \approx 104 \text{ m}^2$$

e Weight = surface area \times weight of polymer per m^2

$$= 33\pi \times 1.23$$

$$\approx 128 \text{ kg}$$

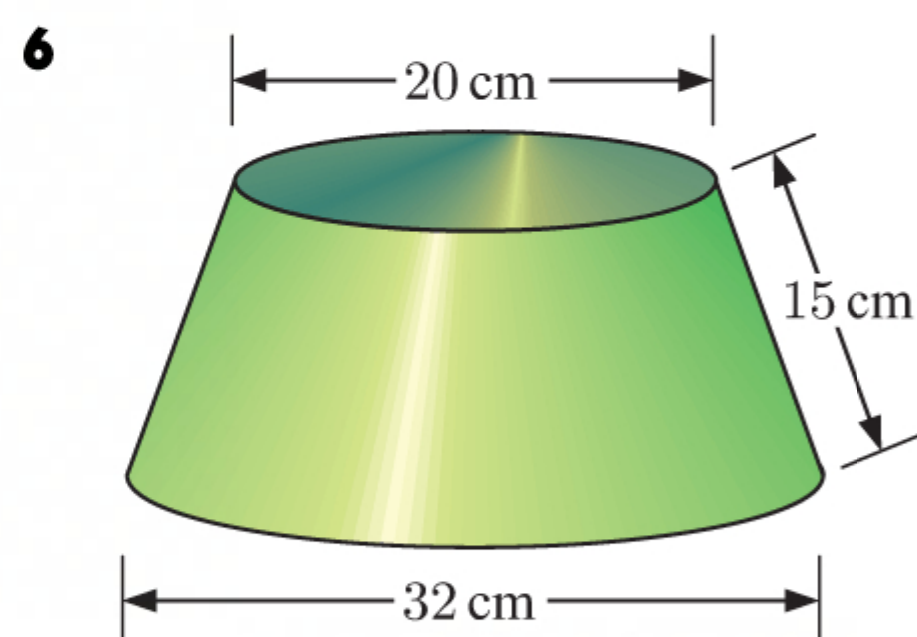
5 a Arc length = θr
 $\therefore \pi = \theta(3)$
 $\therefore \theta = \frac{\pi}{3}$

b Shaded area = $\pi r^2 - \frac{1}{2} \theta r^2$

$$= \pi \times 3^2 - \frac{1}{2} \left(\frac{\pi}{3} \right) \times 3^2$$

$$= 9\pi - \frac{3\pi}{2}$$

$$= \frac{15\pi}{2} \approx 23.6 \text{ cm}^2$$



The shorter arc length = $2\pi \left(\frac{20}{2} \right) = 20\pi \text{ cm}$

The longer arc length = $2\pi \left(\frac{32}{2} \right) = 32\pi \text{ cm}$

Now, the shorter arc length = θr

$$\therefore 20\pi = \theta r \quad \dots (*)$$

and the longer arc length = $\theta(r + 15)$

$$\therefore 32\pi = \theta r + 15\theta$$

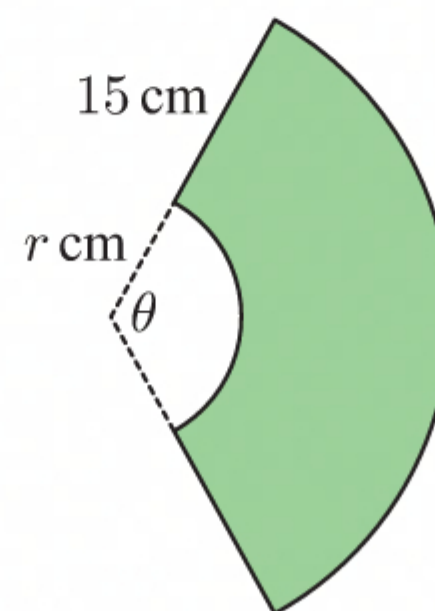
$$\therefore 32\pi = 20\pi + 15\theta \quad \{\text{using } (*)\}$$

$$\therefore 15\theta = 12\pi$$

$$\therefore \theta = \frac{4\pi}{5}$$

Substituting $\theta = \frac{4\pi}{5}$ into (*) gives $20\pi = \frac{4\pi}{5} r$

$$\therefore r = 25$$



7 $P(k, 6, -5)$ and $Q(2, -1, -8)$

$$\begin{aligned} PQ &= \sqrt{(2-k)^2 + (-1-6)^2 + (-8-(-5))^2} \\ &= \sqrt{(2-k)^2 + (-7)^2 + (-3)^2} \\ &= \sqrt{4 - 4k + k^2 + 49 + 9} \\ &= \sqrt{k^2 - 4k + 62} \end{aligned}$$

Now $PQ = 9$ units

$$\therefore \sqrt{k^2 - 4k + 62} = 9$$

$$\therefore k^2 - 4k + 62 = 81$$

$$\therefore k^2 - 4k - 19 = 0$$

$$\therefore k = \frac{4 \pm \sqrt{16 - 4(1)(-19)}}{2}$$

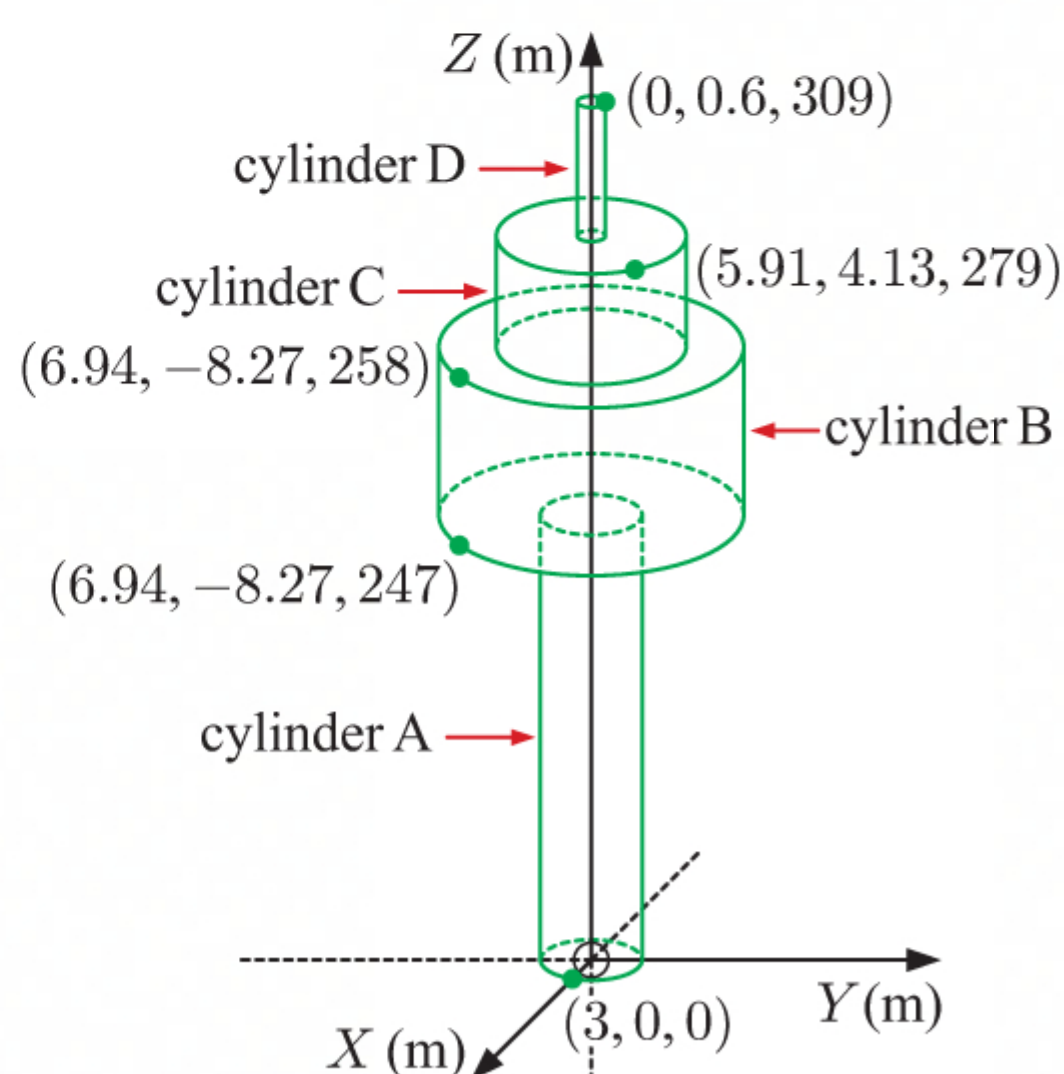
$$= \frac{4 \pm \sqrt{92}}{2}$$

$$= \frac{4 \pm 2\sqrt{23}}{2}$$

$$= 2 \pm \sqrt{23}$$

$$\therefore k = 2 - \sqrt{23} \quad \text{or} \quad 2 + \sqrt{23}$$

8



Height of cylinder A = 247 m

Radius of cylinder A = 3 m

Height of cylinder B = 258 - 247 = 11 m

$$\begin{aligned} \text{Radius of cylinder B} &= \sqrt{6.94^2 + (-8.27)^2} \\ &= \sqrt{116.5565} \text{ m} \end{aligned}$$

Height of cylinder C = 279 - 258 = 21 m

$$\begin{aligned} \text{Radius of cylinder C} &= \sqrt{5.91^2 + 4.13^2} \\ &= \sqrt{51.985} \text{ m} \end{aligned}$$

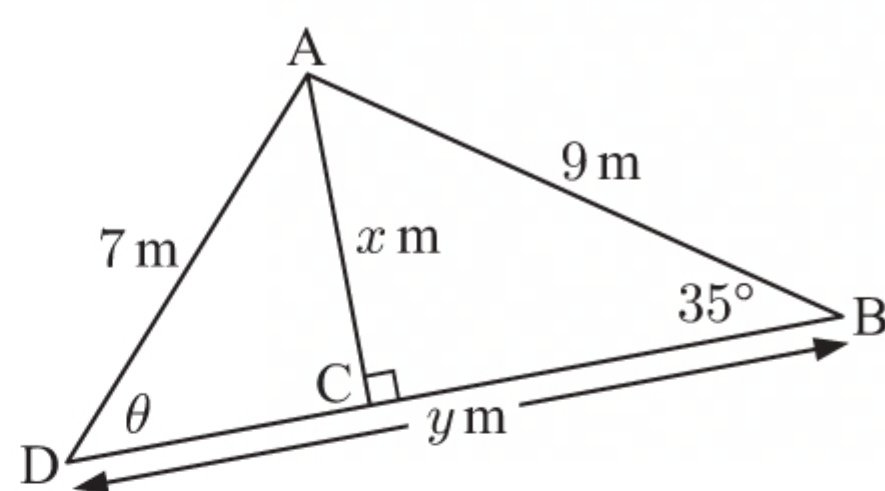
Height of cylinder D = 309 - 279 = 30 m

Radius of cylinder D = 0.6 m

Volume of tower = volume of cylinder A + volume of cylinder B + volume of cylinder C + volume of cylinder D

$$\begin{aligned} &= \pi(3)^2 \times 247 + \pi(\sqrt{116.5565})^2 \times 11 + \pi(\sqrt{51.985})^2 \times 21 + \pi(0.6)^2 \times 30 \\ &= 2223\pi + 1282.1215\pi + 1091.685\pi + 10.8\pi \\ &= 4607.6065\pi \\ &\approx 14\,475.22 \\ &\approx 14\,500 \text{ m}^3 \end{aligned}$$

9 a



$$\text{In triangle ABC, } \sin 35^\circ = \frac{x}{9}$$

$$\therefore x = 9 \sin 35^\circ \approx 5.16$$

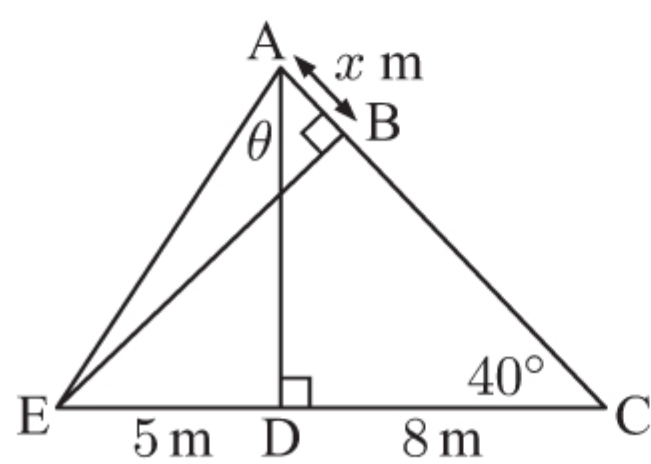
$$\text{In triangle ACD, } \sin \theta = \frac{x}{7}$$

$$= \frac{9 \sin 35^\circ}{7}$$

$$\therefore \theta = \sin^{-1}\left(\frac{9 \sin 35^\circ}{7}\right) \approx 47.5^\circ$$

$$\begin{aligned} \widehat{DAB} &\approx 180^\circ - 35^\circ - 47.5^\circ \quad \{\text{angles in triangle DAB}\} \\ &\approx 97.5^\circ \end{aligned}$$

$$\text{Using the cosine rule in triangle DAB, } y \approx \sqrt{7^2 + 9^2 - 2(7)(9) \cos 97.5^\circ} \approx 12.1$$

b


$$\text{In triangle ACD, } \tan 40^\circ = \frac{AD}{8}$$

$$\therefore AD = 8 \tan 40^\circ$$

$$\text{In triangle ADE, } \tan \theta = \frac{5}{AD}$$

$$= \frac{5}{8 \tan 40^\circ}$$

$$\therefore \theta = \tan^{-1}\left(\frac{5}{8 \tan 40^\circ}\right)$$

$$\approx 36.68^\circ$$

$$\approx 36.7^\circ$$

$$\text{and } \sin 36.68^\circ \approx \frac{5}{AE}$$

$$\therefore AE \approx \frac{5}{\sin 36.68^\circ}$$

$$\text{Now } \widehat{DAC} = 90^\circ - 40^\circ \quad \{\text{angles in triangle ACD}\}$$

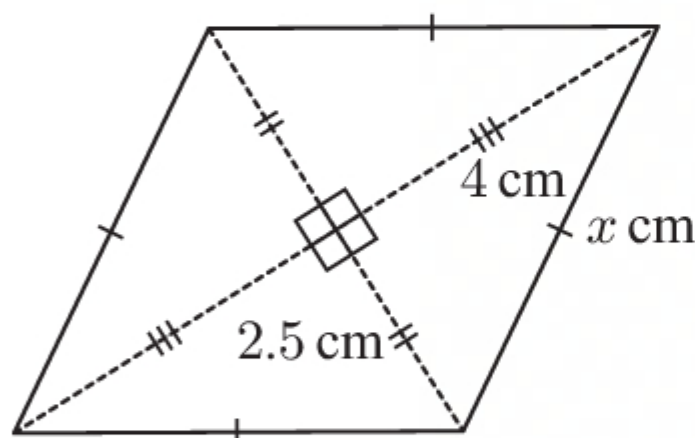
$$= 50^\circ$$

$$\therefore \widehat{EAB} \approx 50^\circ + 36.68^\circ \approx 86.68^\circ$$

$$\text{In triangle ABE, } \cos \widehat{EAB} = \frac{x}{AE}$$

$$\therefore \cos 86.68^\circ \approx \frac{x}{\left(\frac{5}{\sin 36.68^\circ}\right)}$$

$$\therefore x \approx \frac{5 \cos 86.68^\circ}{\sin 36.68^\circ} \approx 0.485$$

10 a

b Let x cm be the side length of the rhombus.

$$\therefore x^2 = 4^2 + (2.5)^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x^2 = 22.25$$

$$\therefore x = \sqrt{22.25} \quad \{x > 0\}$$

$$\therefore x \approx 4.72$$

 \therefore the length of the rhombus' sides are approximately 4.72 cm.

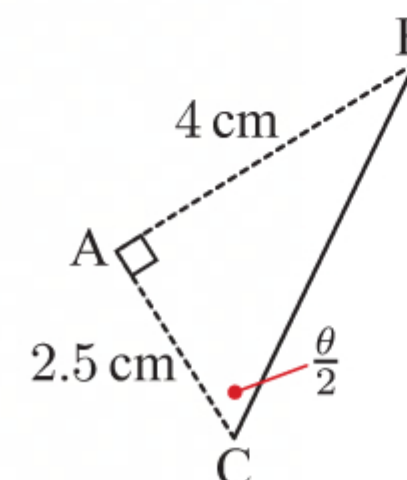
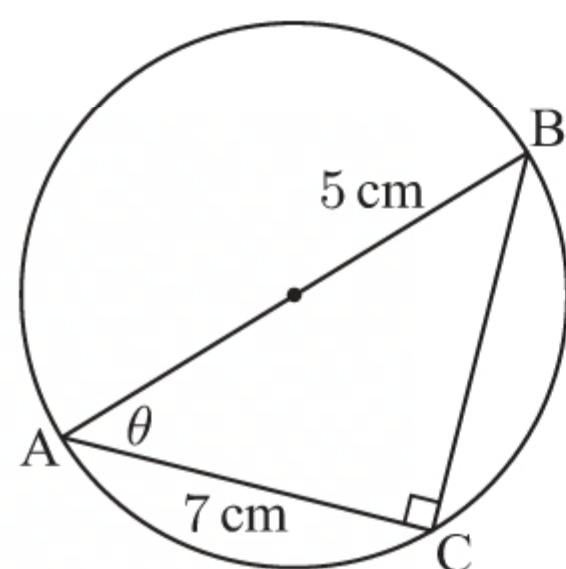
c Let θ be the larger angle in the rhombus.

$$\therefore \widehat{ACB} = \frac{\theta}{2} \quad \{\text{diagonals of a rhombus bisect its angles}\}$$

$$\therefore \tan \frac{\theta}{2} = \frac{4}{2.5}$$

$$\therefore \frac{\theta}{2} = \tan^{-1}\left(\frac{4}{2.5}\right)$$

$$\therefore \theta = 2 \tan^{-1}\left(\frac{4}{2.5}\right) \approx 116^\circ$$


11 a


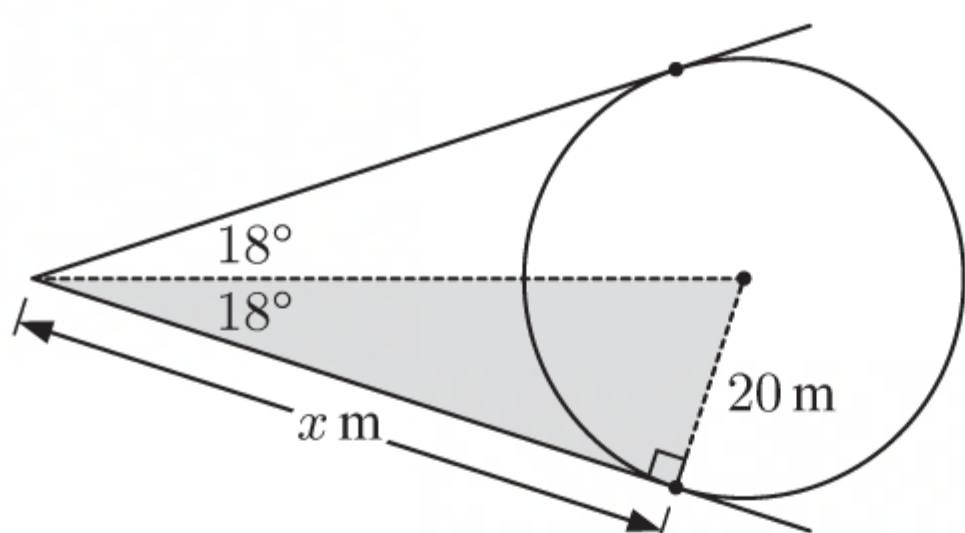
$$\widehat{ACB} = 90^\circ \quad \{\text{angle in a semi-circle}\}$$

$$\therefore \triangle ABC \text{ is right angled at C.}$$

$$\therefore \cos \theta = \frac{7}{AB}$$

$$\therefore \cos \theta = \frac{7}{10}$$

$$\therefore \theta = \cos^{-1}\left(\frac{7}{10}\right) \approx 45.6^\circ$$

b


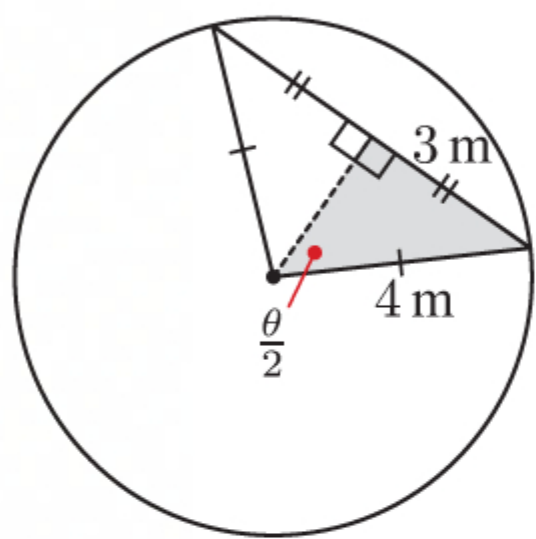
We construct the right angled triangle as shown.

$$\text{For the shaded triangle, } \tan 18^\circ = \frac{20}{x}$$

$$\therefore x = \frac{20}{\tan 18^\circ}$$

$$\therefore x \approx 61.6$$

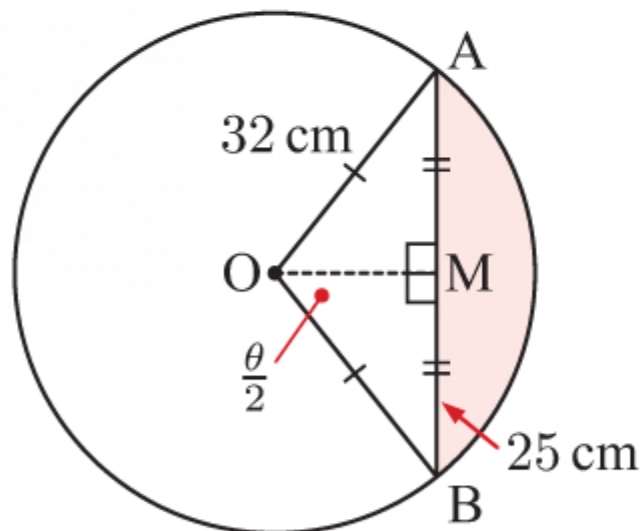
c



We construct the altitude as shown.

$$\begin{aligned}\text{For the shaded triangle, } \sin \frac{\theta}{2} &= \frac{3}{4} \\ \therefore \frac{\theta}{2} &= \sin^{-1}\left(\frac{3}{4}\right) \\ \therefore \theta &= 2 \sin^{-1}\left(\frac{3}{4}\right) \\ \therefore \theta &\approx 97.2^\circ\end{aligned}$$

12 a



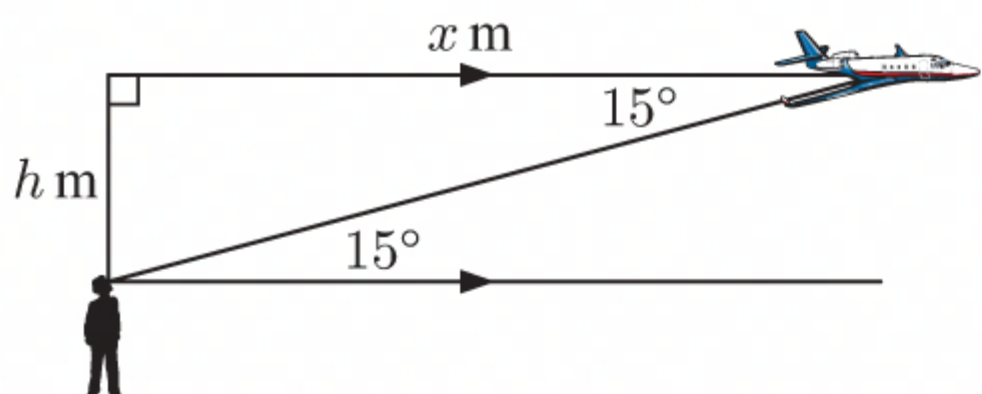
$$\text{Let } \widehat{AOB} = \theta \quad \therefore \widehat{BOM} = \frac{\theta}{2}$$

$$\begin{aligned}\text{In triangle OMB, } \sin \frac{\theta}{2} &= \frac{25}{32} \\ \therefore \frac{\theta}{2} &= \sin^{-1}\left(\frac{25}{32}\right) \\ \therefore \theta &= 2 \sin^{-1}\left(\frac{25}{32}\right) \\ \therefore \theta &\approx 1.79 \text{ radians}\end{aligned}$$

$$\begin{aligned}\text{b Sector area} &= \frac{1}{2}\theta r^2 & \text{Triangle area} &= \frac{1}{2}r^2 \sin \theta \\ &\approx \frac{1}{2}(1.79)(32)^2 & &\approx \frac{1}{2}(32)^2 \sin(1.79) \\ &\approx 918.19 \text{ cm}^2 & &\approx 499.37 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\therefore \text{area of shaded segment} &= \text{sector area} - \text{triangle area} \\ &\approx 918.19 - 499.37 \\ &\approx 418.81 \\ &\approx 419 \text{ cm}^2\end{aligned}$$

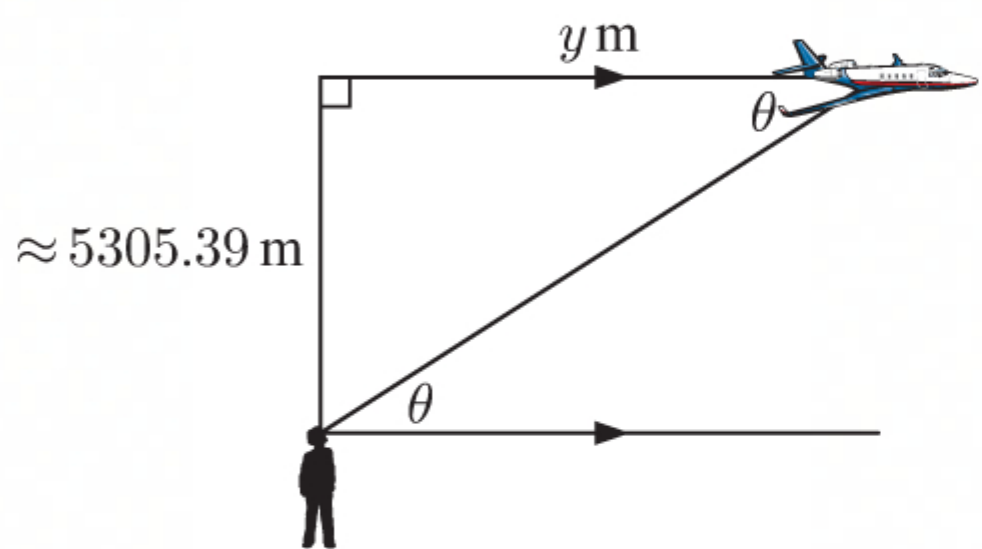
13 a



$$\begin{aligned}x &= \text{speed} \times \text{time} \\ &= 110 \times 180 \quad \{3 \text{ minutes} = 180 \text{ seconds}\} \\ &= 19800 \\ \therefore \tan 15^\circ &= \frac{h}{19800} \\ \therefore h &= 19800 \tan 15^\circ \\ \therefore h &\approx 5305.39 \approx 5310\end{aligned}$$

\therefore the plane is approximately 5310 m \approx 5.31 km above the ground.

b



$$\begin{aligned}y &= \text{speed} \times \text{time} \\ &= 110 \times 420 \quad \{7 \text{ minutes} = 420 \text{ seconds}\} \\ &= 46200 \\ \therefore \tan \theta &\approx \frac{5305.39}{46200} \\ \therefore \theta &\approx \tan^{-1}\left(\frac{5305.39}{46200}\right) \\ \therefore \theta &\approx 6.55^\circ\end{aligned}$$

\therefore the angle of elevation of the plane at 2:42 pm is approximately 6.55° .

14 Consider the diagram alongside.

Since pizzerias A and B are 10 km apart, $AB = 10$ km.

The circles centred at A and B represent the free delivery regions for pizzerias A and B, respectively.

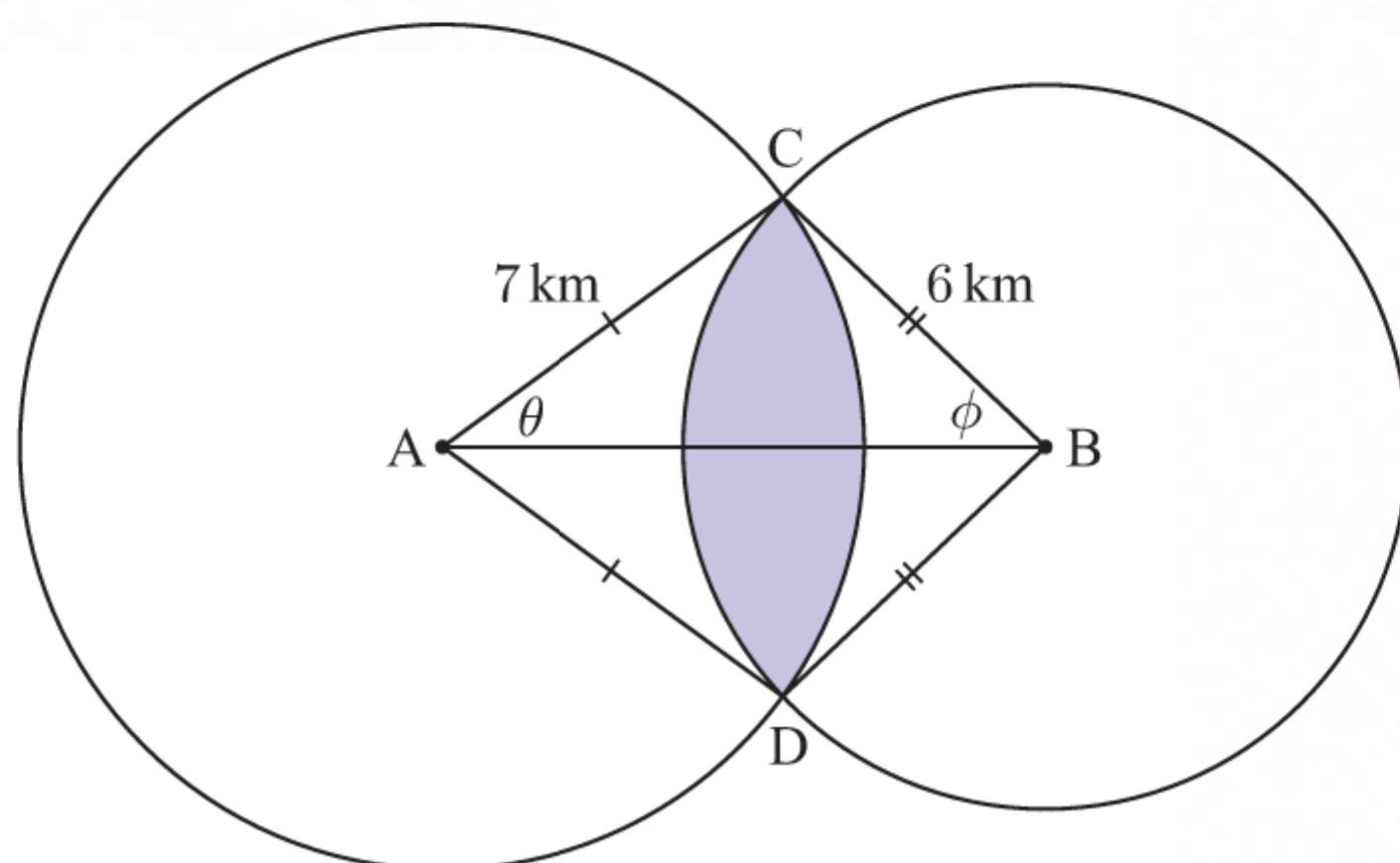
So, the shaded area represents the region which receives free delivery from *both* pizzerias.

Using the cosine rule in $\triangle ABC$:

$$\begin{aligned}\cos \theta &= \frac{7^2 + 10^2 - 6^2}{2 \times 7 \times 10} & \text{and } \cos \phi &= \frac{6^2 + 10^2 - 7^2}{2 \times 6 \times 10} \\ \therefore \theta &= \cos^{-1}\left(\frac{113}{140}\right) & \therefore \phi &= \cos^{-1}\left(\frac{29}{40}\right)\end{aligned}$$

Now $\triangle ABC$ and $\triangle ABD$ are congruent. {SSS}

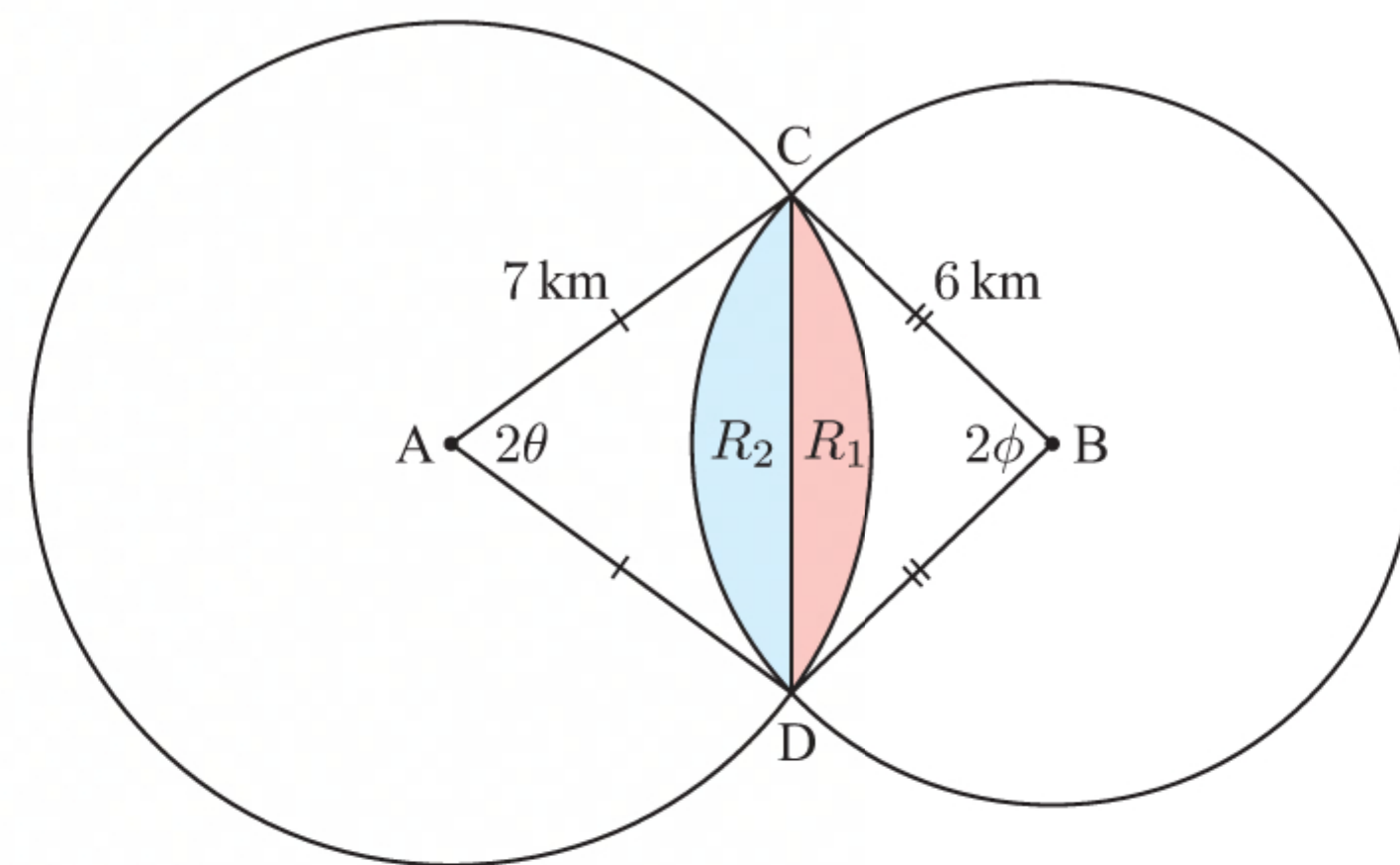
$$\therefore \widehat{CAD} = 2\theta \text{ and } \widehat{CBD} = 2\phi$$



We divide the shaded area into regions R_1 and R_2 .

$$\begin{aligned}\text{Now } R_1 &= \text{area of sector ACD} - \text{area of } \triangle ACD \\ &= \frac{1}{2}(2\theta)(7)^2 - \frac{1}{2}(7)(7) \sin 2\theta \\ &= 49\theta - \frac{49}{2} \sin 2\theta\end{aligned}$$

$$\begin{aligned}\text{and } R_2 &= \text{area of sector BCD} - \text{area of } \triangle BCD \\ &= \frac{1}{2}(2\phi)(6)^2 - \frac{1}{2}(6)(6) \sin 2\phi \\ &= 36\phi - 18 \sin 2\phi\end{aligned}$$



$$\begin{aligned}\text{So, area of shaded region} &= R_1 + R_2 \\ &= 49\theta - \frac{49}{2} \sin 2\theta + 36\phi - 18 \sin 2\phi \\ &= 49 \cos^{-1}\left(\frac{113}{140}\right) - \frac{49}{2} \sin\left(2 \cos^{-1}\left(\frac{113}{140}\right)\right) + 36 \cos^{-1}\left(\frac{29}{40}\right) - 18 \sin\left(2 \cos^{-1}\left(\frac{29}{40}\right)\right) \\ &\approx 17.0 \text{ km}^2\end{aligned}$$

15 a M is $(-5, 4, 6)$

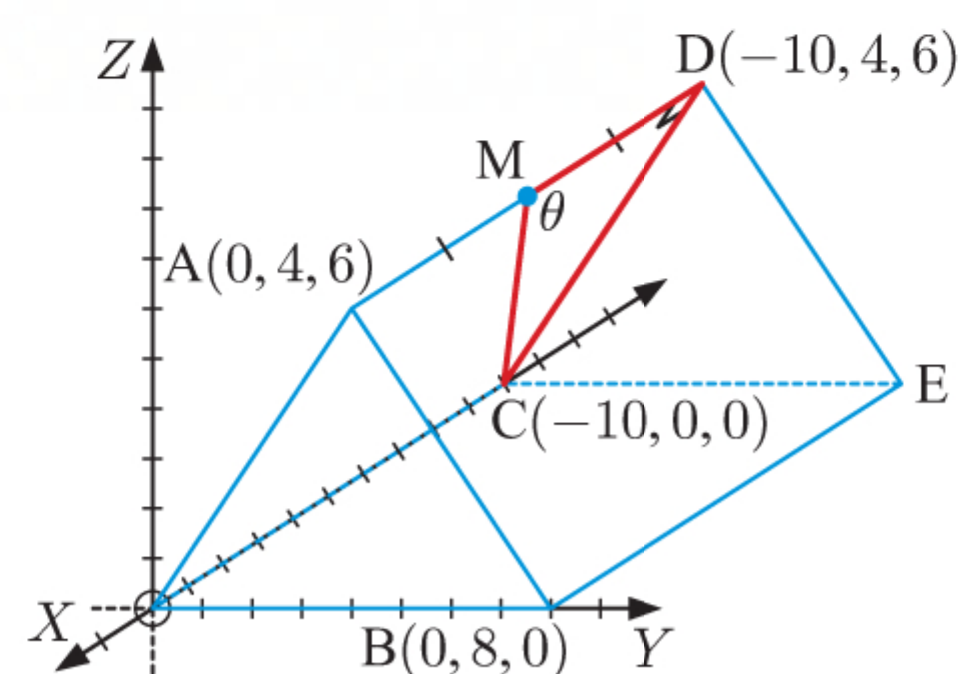
b Let \widehat{CMD} be θ .

$$\text{Now } DM = 5 \text{ units}$$

$$\begin{aligned}\text{and } CD &= \sqrt{(-10 - (-10))^2 + (4 - 0)^2 + (6 - 0)^2} \\ &= \sqrt{0^2 + 4^2 + 6^2} \\ &= \sqrt{52} \text{ units}\end{aligned}$$

$$\begin{aligned}\therefore \tan \theta &= \frac{\sqrt{52}}{5} \\ \therefore \theta &= \tan^{-1}\left(\frac{\sqrt{52}}{5}\right) \approx 55.3^\circ\end{aligned}$$

$$\therefore \widehat{CMD} \approx 55.3^\circ$$



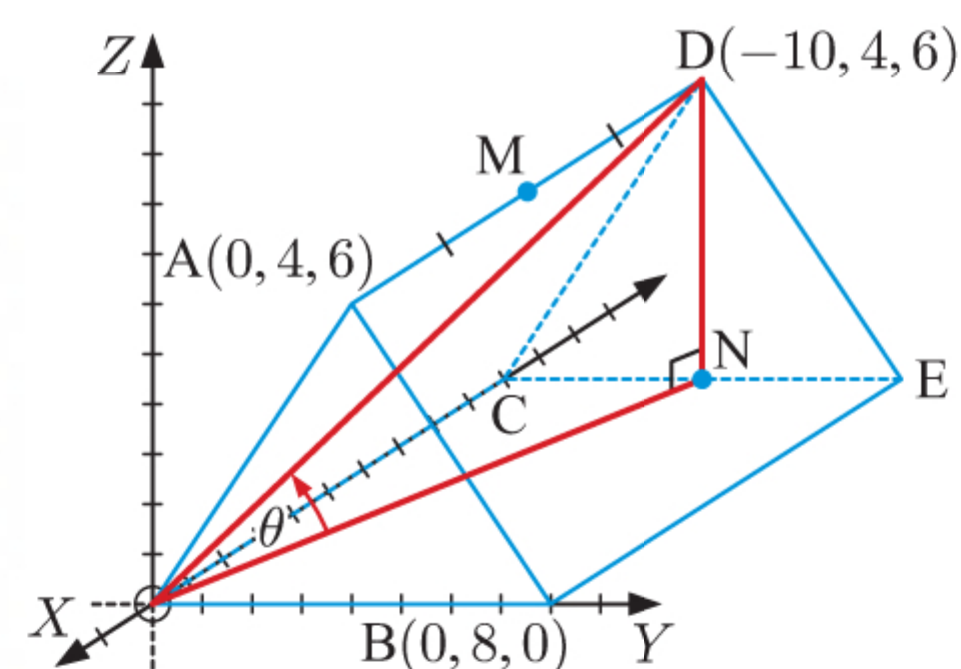
c i The required angle is \widehat{DON} , where N has coordinates $(-10, 4, 0)$.

$$\text{Now } DN = 6 \text{ units}$$

$$\begin{aligned}\text{and } NO &= \sqrt{(-10 - 0)^2 + (4 - 0)^2 + (0 - 0)^2} \\ &= \sqrt{(-10)^2 + 4^2 + 0^2} \\ &= \sqrt{116} \text{ units}\end{aligned}$$

$$\begin{aligned}\therefore \tan \theta &= \frac{6}{\sqrt{116}} \\ \therefore \theta &= \tan^{-1}\left(\frac{6}{\sqrt{116}}\right) \approx 29.1^\circ\end{aligned}$$

The angle is about 29.1° .



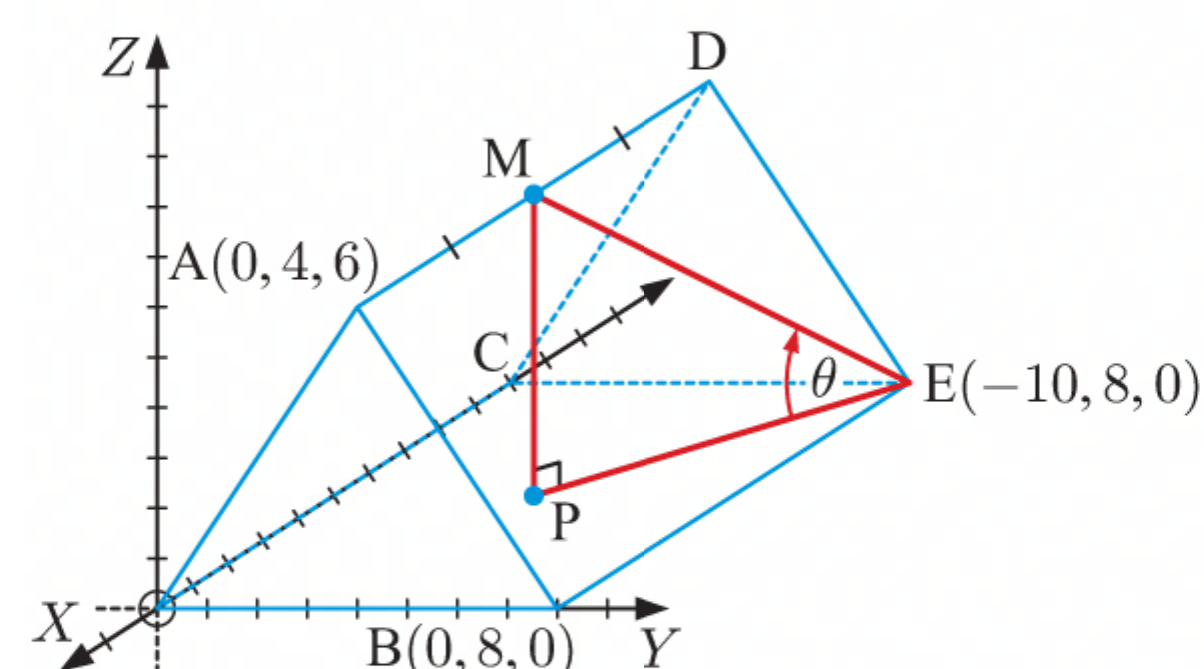
ii The required angle is \widehat{MEP} , where P has coordinates $(-5, 4, 0)$.

$$\text{Now } MP = 6 \text{ units}$$

$$\begin{aligned}\text{and } PE &= \sqrt{(-10 - (-5))^2 + (8 - 4)^2 + (0 - 0)^2} \\ &= \sqrt{(-5)^2 + 4^2 + 0^2} \\ &= \sqrt{41} \text{ units}\end{aligned}$$

$$\begin{aligned}\therefore \tan \theta &= \frac{6}{\sqrt{41}} \\ \therefore \theta &= \tan^{-1}\left(\frac{6}{\sqrt{41}}\right) \approx 43.1^\circ\end{aligned}$$

The angle is about 43.1° .



- 16** Let the equal sides of the pyramid be x .

The projection of $[AX]$ onto the base plane is $[AM]$, where M is the point directly below A on the shaded base plane.

\therefore the required angle is \widehat{AXM} .

Consider the triangular base of the pyramid.

The angle at the centre of the triangle is 360° . {angles at a point}

$$\therefore \widehat{BMX} = \frac{360^\circ}{3} = 120^\circ$$

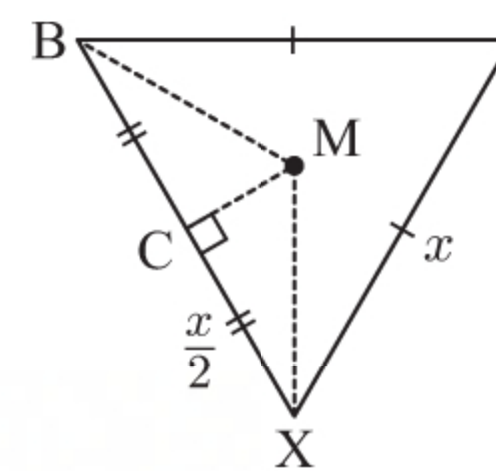
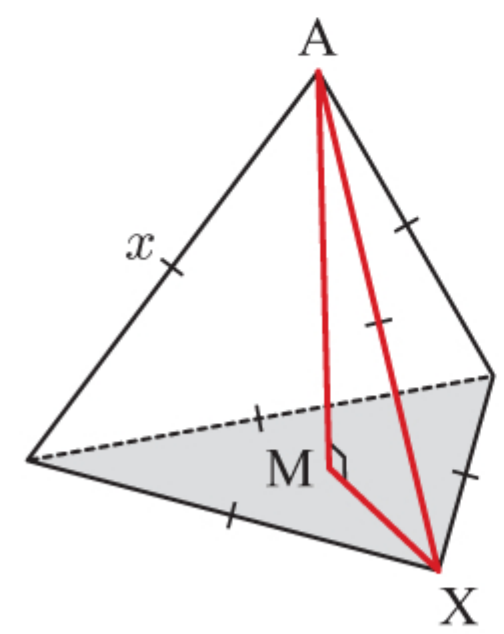
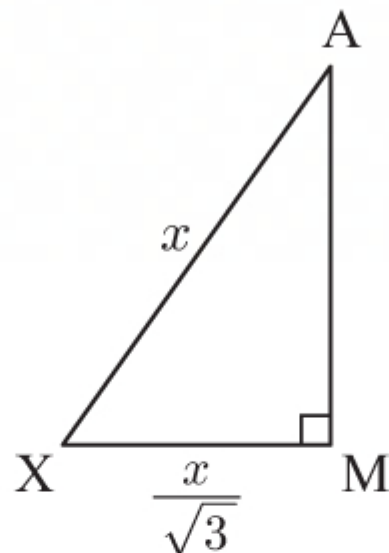
$$\therefore \widehat{CMX} = \frac{120^\circ}{2} = 60^\circ$$

$$\text{In } \triangle CMX, \quad \sin 60^\circ = \frac{\frac{x}{2}}{MX}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{x}{2MX}$$

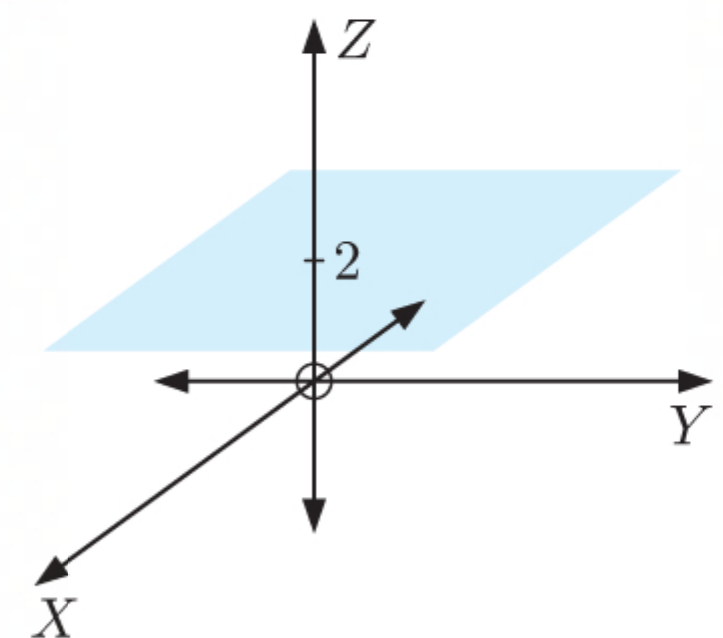
$$\therefore MX = \frac{x}{\sqrt{3}}$$

$$\begin{aligned} \text{In } \triangle AMX, \quad \cos \widehat{AXM} &= \frac{\frac{x}{\sqrt{3}}}{x} \\ &= \frac{1}{\sqrt{3}} \\ \therefore \widehat{AXM} &= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &\approx 54.7^\circ \end{aligned}$$

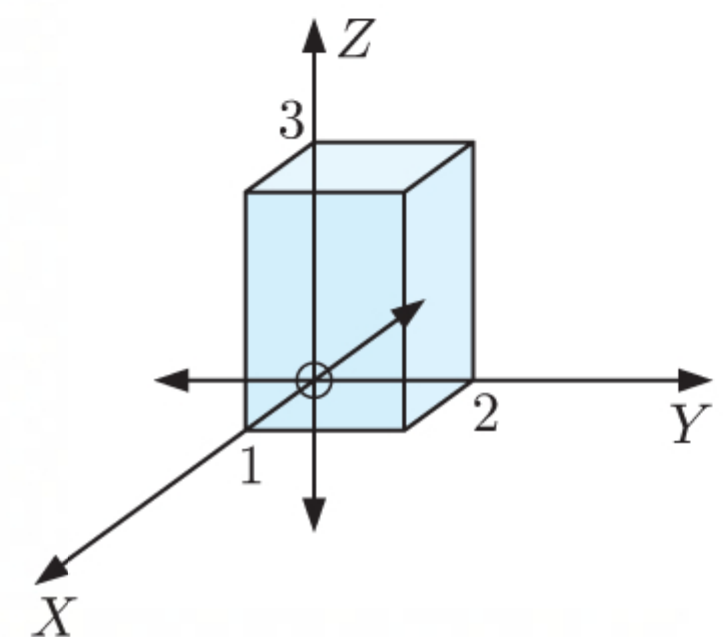


So, the angle between $[AX]$ and the shaded base plane is approximately 54.7° .

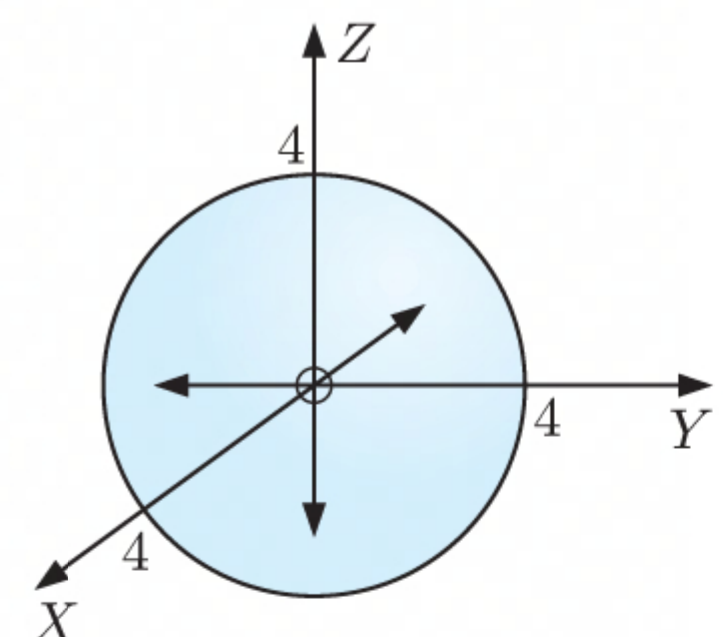
- 17 a** $\{(x, y, z) \mid z = 2\}$ is the plane parallel to the YZ -plane, passing through $(0, 0, 2)$.



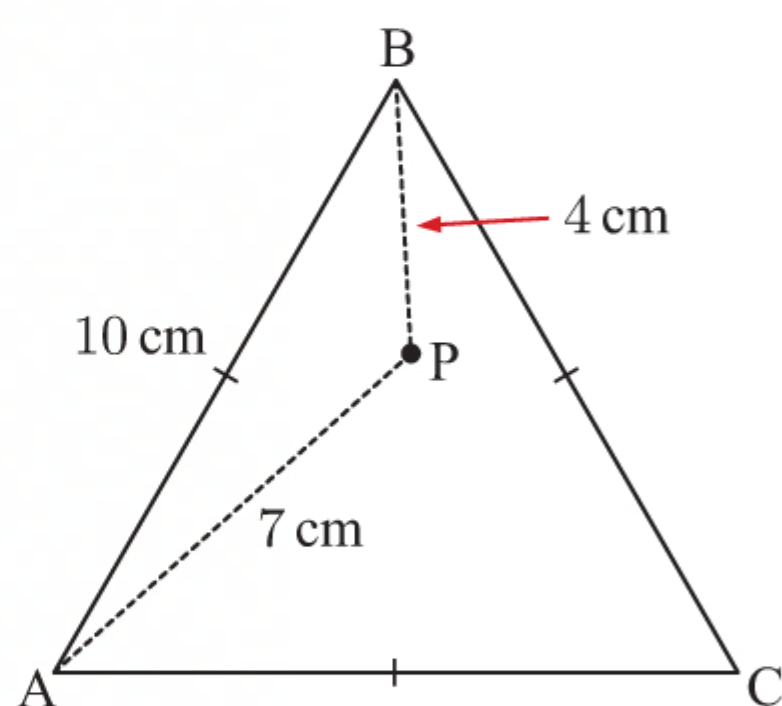
- b** $\{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3\}$ is the set of points on and within the $1 \times 2 \times 3$ rectangular prism (as shown).



- c** $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 16\}$ is the set of points on and within a sphere with centre $(0, 0, 0)$ and radius 4 units.



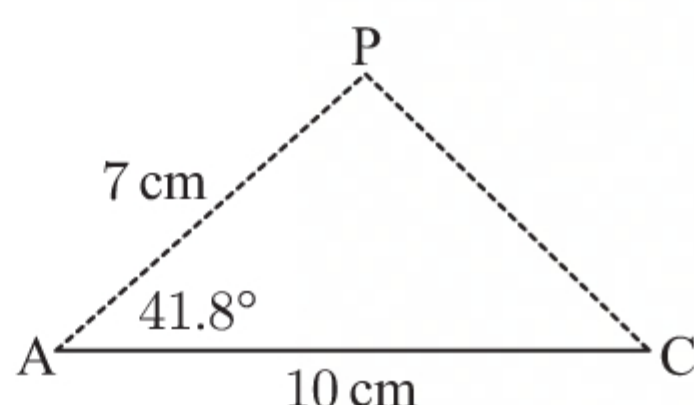
18 a


 b i By the cosine rule in $\triangle BAP$:

$$\begin{aligned}\cos \widehat{BAP} &= \frac{10^2 + 7^2 - 4^2}{2 \times 7 \times 10} \\ \therefore \cos \widehat{BAP} &= \frac{133}{140} \\ \therefore \widehat{BAP} &= \cos^{-1}\left(\frac{133}{140}\right) \approx 18.2^\circ\end{aligned}$$

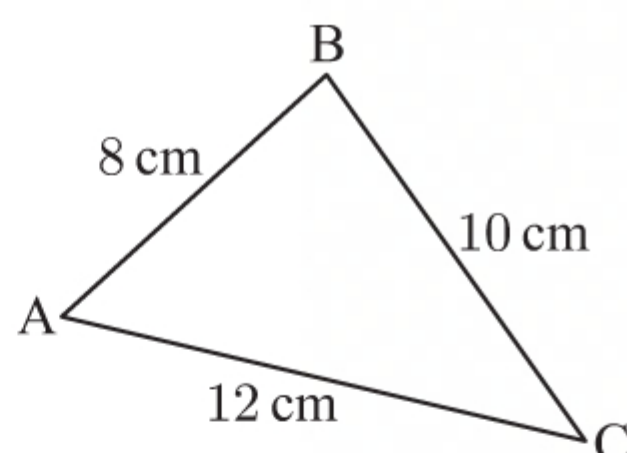
$$\begin{aligned}\text{ii} \quad \widehat{BAC} &= 60^\circ \quad \{\text{angles in an equilateral triangle}\} \\ \therefore \widehat{CAP} &= 60^\circ - \widehat{BAP} \\ &\approx 60^\circ - 18.2^\circ \\ &\approx 41.8^\circ\end{aligned}$$

c


 By the cosine rule in $\triangle APC$:

$$\begin{aligned}CP^2 &\approx 10^2 + 7^2 - 2(10)(7) \cos 41.8^\circ \\ \therefore CP &\approx \sqrt{10^2 + 7^2 - 2(10)(7) \cos 41.8^\circ} \\ \therefore CP &\approx 6.68 \text{ cm}\end{aligned}$$

19 a



b The smallest angle in triangle ABC is opposite the shortest side.

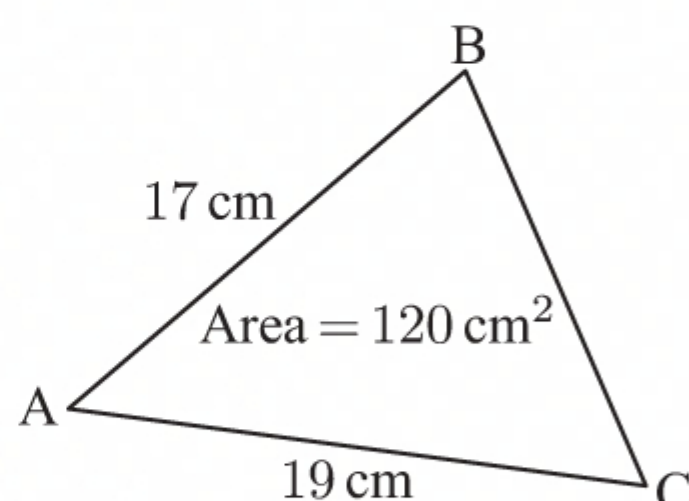
 $\therefore \widehat{BCA}$ is the smallest angle.

By the cosine rule:

$$\begin{aligned}\cos \widehat{BCA} &= \frac{12^2 + 10^2 - 8^2}{2 \times 12 \times 10} \\ \therefore \cos \widehat{BCA} &= \frac{180}{240} = \frac{3}{4} \\ \therefore \widehat{BCA} &= \cos^{-1}\left(\frac{3}{4}\right) \approx 41.4^\circ\end{aligned}$$

$$\begin{aligned}\text{c} \quad \text{Area} &= \frac{1}{2}ab \sin C \\ &\approx \frac{1}{2} \times 10 \times 12 \times \sin 41.4^\circ \\ &\approx 39.7 \text{ cm}^2\end{aligned}$$

20 a



b

$$\begin{aligned}\text{Area} &= 120 \text{ cm}^2 \\ \therefore \frac{1}{2} \times 17 \times 19 \times \sin \widehat{BAC} &= 120 \\ \therefore \sin \widehat{BAC} &= \frac{120 \times 2}{17 \times 19} \\ \therefore \sin \widehat{BAC} &= \frac{240}{323} \\ \therefore \widehat{BAC} &= \sin^{-1}\left(\frac{240}{323}\right) \approx 48.0^\circ\end{aligned}$$

c By the cosine rule:

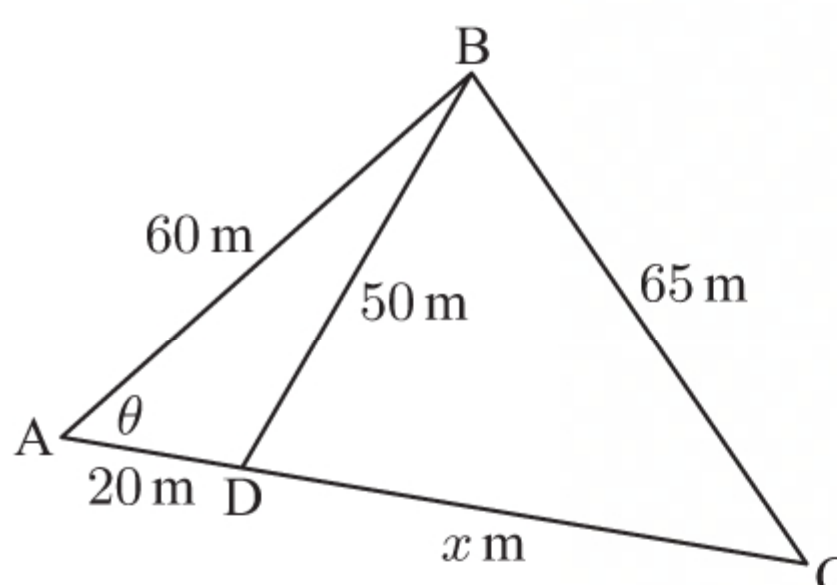
$$\begin{aligned}CP^2 &\approx 17^2 + 19^2 - 2(17)(19) \cos 48.0^\circ \\ \therefore CP &\approx \sqrt{17^2 + 19^2 - 2(17)(19) \cos 48.0^\circ} \\ \therefore CP &\approx 14.8 \text{ cm}\end{aligned}$$

 d Volume = cross-sectional area \times length

$$\begin{aligned}&= 120 \times 13.5 \\ &= 1620 \text{ cm}^3\end{aligned}$$

 21 a By the cosine rule in $\triangle BAD$:

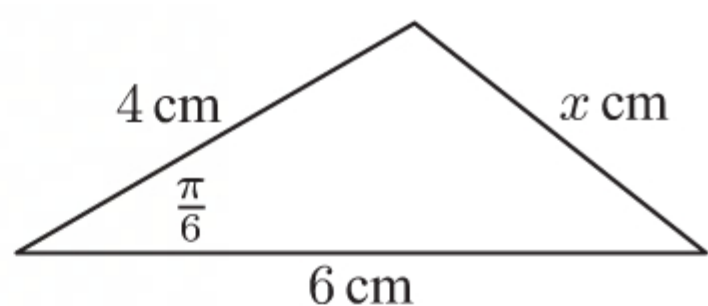
$$\begin{aligned}\cos \theta &= \frac{60^2 + 20^2 - 50^2}{2 \times 60 \times 20} \\ \therefore \cos \theta &= \frac{1500}{2400} \\ \therefore \cos \theta &= \frac{5}{8}\end{aligned}$$



b By the cosine rule in $\triangle ABC$:

$$\begin{aligned}
 65^2 &= 60^2 + (20 + x)^2 - 2(60)(20 + x) \cos \theta \\
 \therefore 65^2 &= 60^2 + (20 + x)^2 - 2(60)(20 + x) \left(\frac{5}{8}\right) \quad \{\text{using a}\} \\
 \therefore 4225 &= 3600 + 400 + 40x + x^2 - 1500 - 75x \\
 \therefore x^2 - 35x - 1725 &= 0 \\
 \therefore x &= \frac{35 \pm \sqrt{1225 - 4(1)(-1725)}}{2(1)} \quad \{\text{quadratic formula}\} \\
 \therefore x &= \frac{35 \pm \sqrt{8125}}{2} \\
 \therefore x &= \frac{35 \pm 25\sqrt{13}}{2} \\
 \therefore x &= \frac{35 + 25\sqrt{13}}{2} \quad \{x > 0\} \\
 \therefore x &\approx 62.6
 \end{aligned}$$

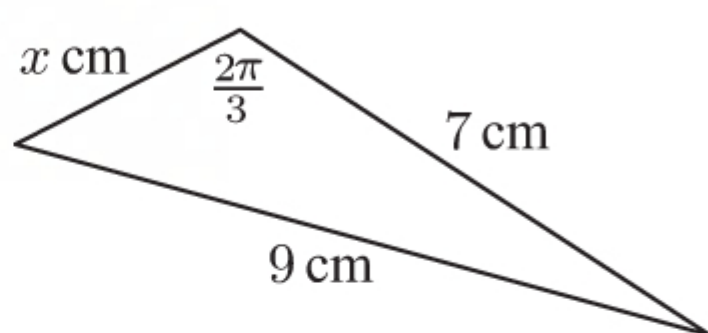
22 a



By the cosine rule:

$$\begin{aligned}
 x^2 &= 4^2 + 6^2 - 2(4)(6) \cos \frac{\pi}{6} \\
 \therefore x^2 &= 16 + 36 - 48 \left(\frac{\sqrt{3}}{2}\right) \\
 \therefore x^2 &= 52 - 24\sqrt{3} \\
 \therefore x &= \sqrt{52 - 24\sqrt{3}} \quad \{x > 0\} \\
 \therefore x &\approx 3.23
 \end{aligned}$$

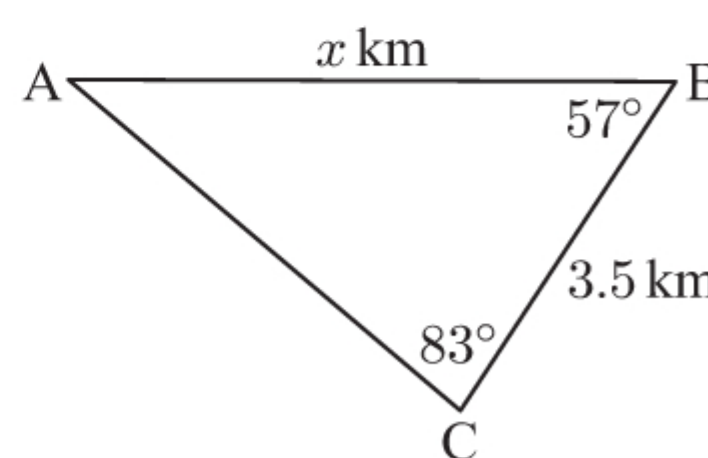
b



By the cosine rule:

$$\begin{aligned}
 9^2 &= x^2 + 7^2 - 2(x)(7) \cos \frac{2\pi}{3} \\
 \therefore 81 &= x^2 + 49 - 14\left(-\frac{1}{2}\right)x \\
 \therefore 81 &= x^2 + 49 + 7x \\
 \therefore x^2 + 7x - 32 &= 0 \\
 \therefore x &= \frac{-7 \pm \sqrt{49 - 4(1)(-32)}}{2(1)} \quad \{\text{quadratic formula}\} \\
 \therefore x &= \frac{-7 \pm \sqrt{177}}{2} \\
 \therefore x &= \frac{-7 + \sqrt{177}}{2} \quad \{x > 0\} \\
 \therefore x &\approx 3.15
 \end{aligned}$$

23 a

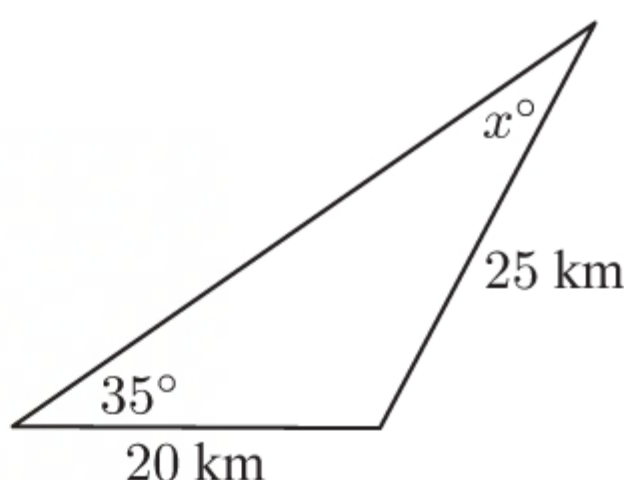


$$\begin{aligned}
 \widehat{BAC} &= 180^\circ - 57^\circ - 83^\circ \quad \{\text{angles in a triangle}\} \\
 &= 40^\circ
 \end{aligned}$$

Now by the sine rule:

$$\begin{aligned}
 \frac{x}{\sin 83^\circ} &= \frac{3.5}{\sin 40^\circ} \\
 \therefore x &= \frac{3.5 \sin 83^\circ}{\sin 40^\circ} \\
 \therefore x &\approx 5.40
 \end{aligned}$$

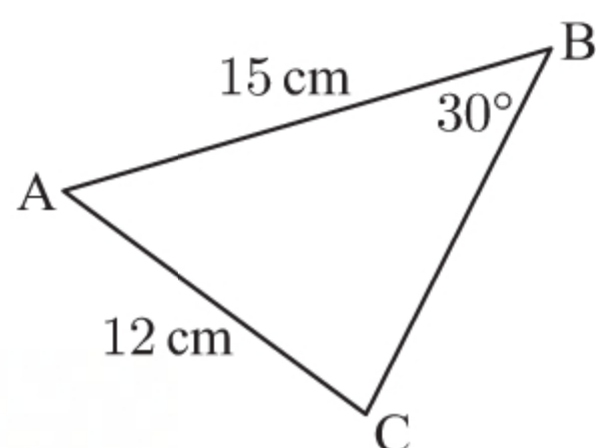
b



By the sine rule:

$$\begin{aligned}
 \frac{\sin x^\circ}{20} &= \frac{\sin 35^\circ}{25} \\
 \therefore \sin x^\circ &= \frac{20 \sin 35^\circ}{25} \\
 \therefore x &= \sin^{-1} \left(\frac{20 \sin 35^\circ}{25} \right) \\
 \therefore x &\approx 27.3
 \end{aligned}$$

24 a



$$\frac{\sin \hat{ACB}}{15} = \frac{\sin 30^\circ}{12} \quad \{\text{sine rule}\}$$

$$\therefore \sin \hat{ACB} = \frac{15 \sin 30^\circ}{12}$$

$$\therefore \hat{ACB} = \sin^{-1}\left(\frac{15 \sin 30^\circ}{12}\right) \quad \text{or its supplement}$$

$$\therefore \hat{ACB} \approx 38.7^\circ \quad \text{or } 180^\circ - 38.7^\circ$$

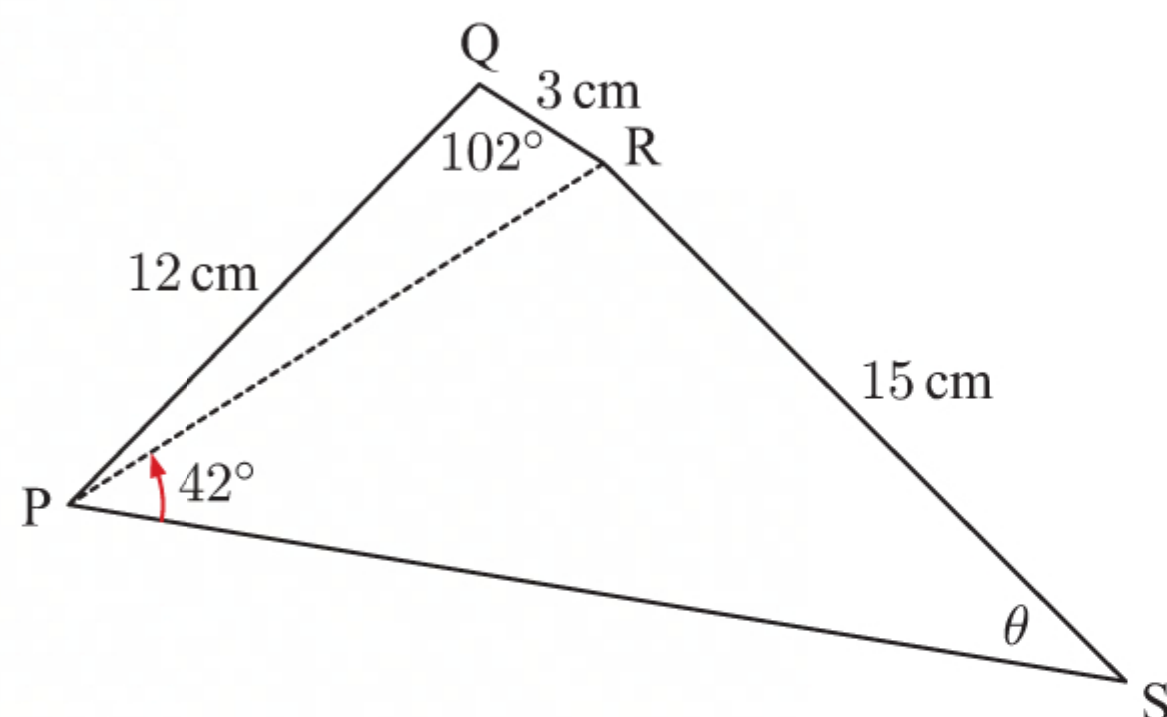
$$\therefore \hat{ACB} \approx 38.7^\circ \quad \text{or } 141.3^\circ$$

 b \hat{BAC} is acute if \hat{ACB} is obtuse.

$$\therefore \hat{BAC} \approx 180^\circ - 30^\circ - 141.3^\circ \quad \{\text{using a}\}$$

$$\approx 8.7^\circ$$

25 a


 By the cosine rule in $\triangle PQR$:

$$PR^2 = 3^2 + 12^2 - 2(3)(12) \cos 102^\circ$$

$$\therefore PR = \sqrt{3^2 + 12^2 - 2(3)(12) \cos 102^\circ}$$

$$\therefore PR \approx 12.96 \text{ cm} \approx 13.0 \text{ cm}$$

 b By the sine rule in $\triangle PRS$:

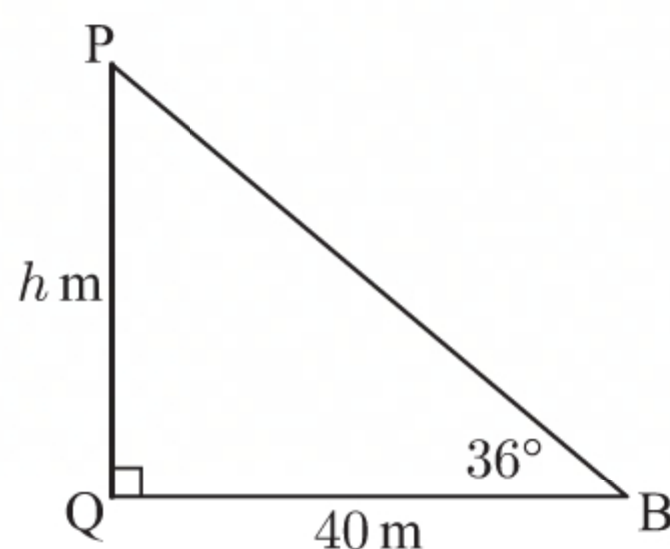
$$\frac{\sin \theta}{PR} = \frac{\sin 42^\circ}{15}$$

$$\therefore \sin \theta = \frac{PR \sin 42^\circ}{15}$$

$$\therefore \sin \theta \approx \frac{12.96 \sin 42^\circ}{15} \quad \{\text{from a}\}$$

$$\therefore \theta \approx \sin^{-1}\left(\frac{12.96 \sin 42^\circ}{15}\right) \approx 35.3^\circ$$

26 a

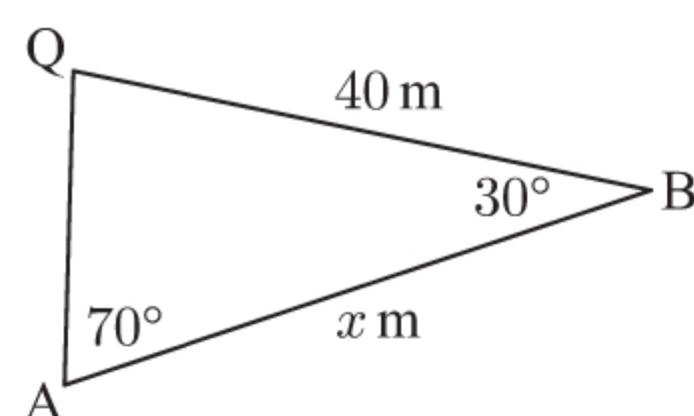

 Let PQ be h m.

$$\therefore \tan 36^\circ = \frac{h}{40}$$

$$\therefore h = 40 \tan 36^\circ \approx 29.1$$

The height of the pole is about 29.1 m.

b


 Let AB be x m.

$$\hat{AQB} = 180^\circ - 70^\circ - 30^\circ \quad \{\text{angles in a triangle}\}$$

$$= 80^\circ$$

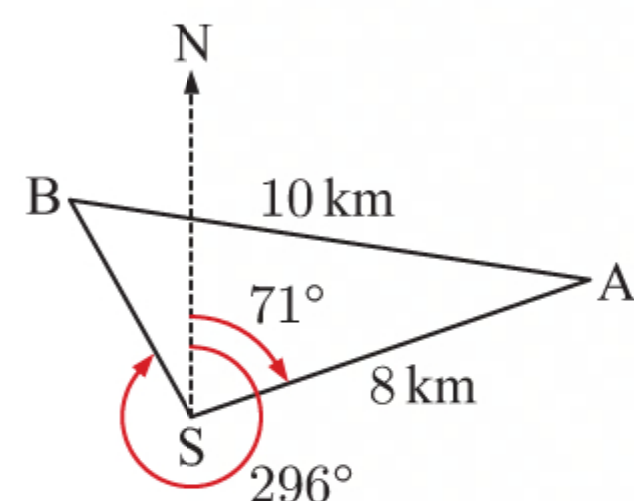
$$\therefore \frac{x}{\sin 80^\circ} = \frac{40}{\sin 70^\circ} \quad \{\text{sine rule}\}$$

$$\therefore x = \frac{40 \sin 80^\circ}{\sin 70^\circ}$$

$$\therefore x \approx 41.9$$

The distance between A and B is about 41.9 m.

27 a



$$\begin{aligned} \mathbf{b} \quad N_1\hat{S}B &= 360^\circ - 296^\circ \quad \{\text{angles at a point}\} \\ &= 64^\circ \end{aligned}$$

$$\therefore \hat{BSA} = 64^\circ + 71^\circ = 135^\circ$$

$$\therefore \frac{\sin \alpha}{8} = \frac{\sin 135^\circ}{10} \quad \{\text{sine rule}\}$$

$$\therefore \sin \alpha = \frac{8 \sin 135^\circ}{10}$$

$$\therefore \alpha = \sin^{-1}\left(\frac{8 \sin 135^\circ}{10}\right) \approx 34.4^\circ$$

$$\begin{aligned} \text{Now } \theta &= 180^\circ - 135^\circ - \alpha \quad \{\text{angles in a triangle}\} \\ &\approx 180^\circ - 135^\circ - 34.4^\circ \\ &\approx 10.6^\circ \end{aligned}$$

$$\begin{aligned} N_2\hat{AS} &= 180^\circ - 71^\circ \quad \{\text{co-interior angles}\} \\ &= 109^\circ \end{aligned}$$

$$\therefore \text{bearing of B from A} \approx 360^\circ - 109^\circ + 10.6^\circ \approx 262^\circ$$

c By the cosine rule:

$$BS^2 = 10^2 + 8^2 - 2(10)(8) \cos \theta$$

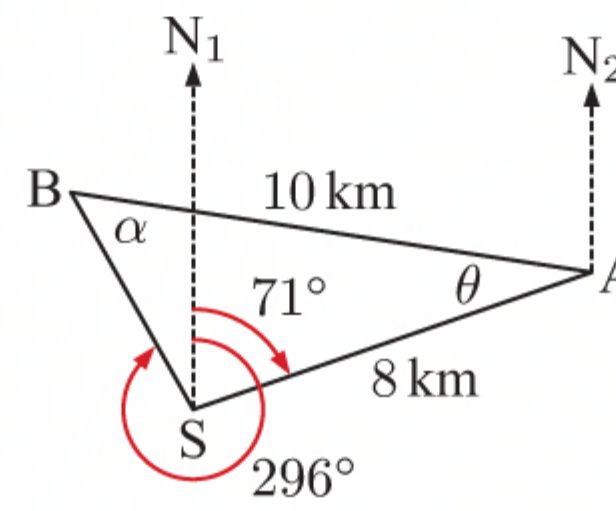
$$\therefore BS = \sqrt{10^2 + 8^2 - 2(10)(8) \cos \theta}$$

$$\therefore BS \approx \sqrt{10^2 + 8^2 - 2(10)(8) \cos 10.6^\circ} \quad \{\text{from b}\}$$

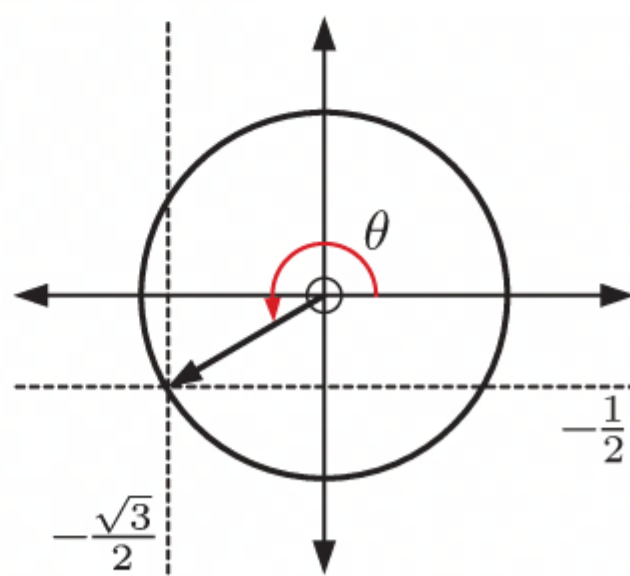
$$\therefore BS \approx 2.59 \text{ km} \approx 2590 \text{ m}$$

$$\begin{aligned} \text{Now time} &= \frac{\text{distance}}{\text{speed}} \\ &\approx \frac{2590}{7} \approx 370 \text{ seconds} \end{aligned}$$

It will take about 370 seconds or 6 minutes 10 seconds for train B to reach the train station.



28 a



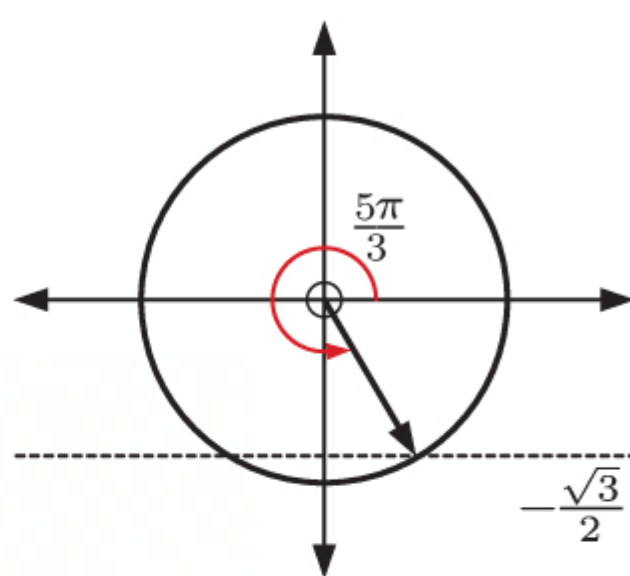
$$\sin \theta = -\frac{1}{2} \text{ and } \cos \theta = -\frac{\sqrt{3}}{2}$$

$$\therefore \theta = 210^\circ$$

$$\begin{aligned} \mathbf{b} \quad \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

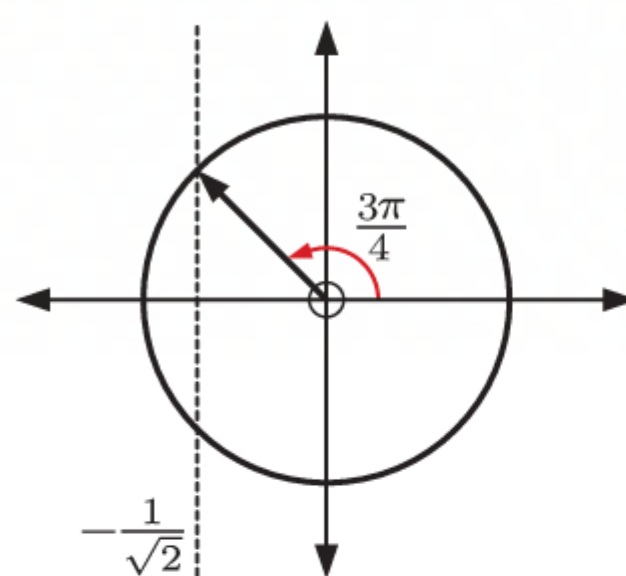
$$\begin{aligned} \mathbf{c} \quad \tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} \quad \{\text{double angle formulae}\} \\ &= \frac{2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right)^2} \\ &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \end{aligned}$$

29 a



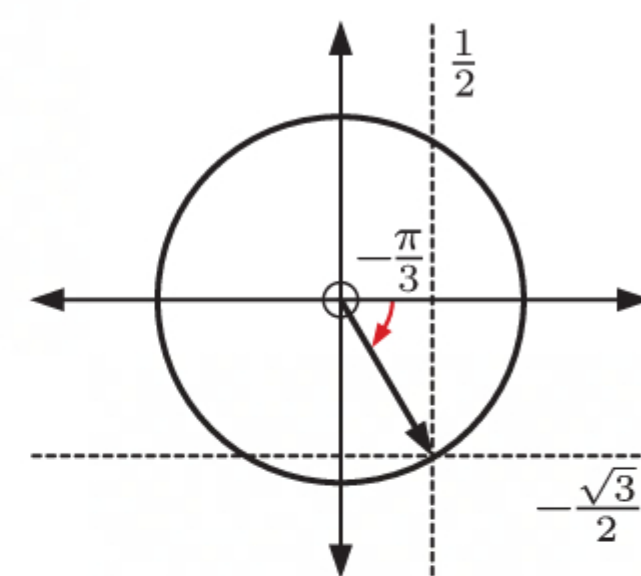
$$\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$$

b



$$\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

c



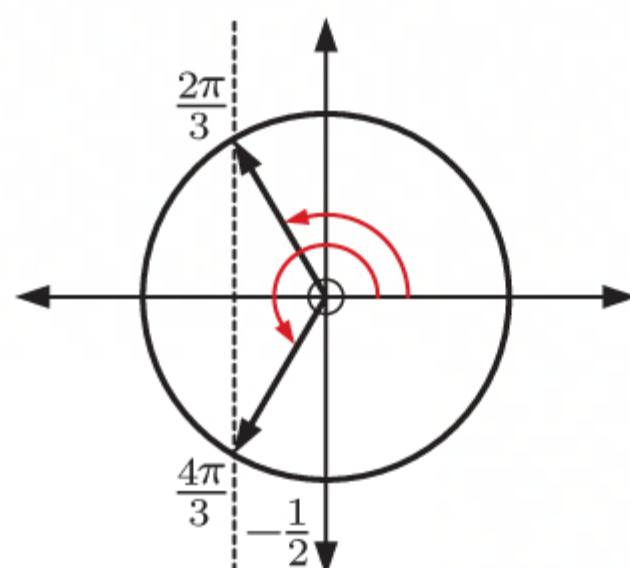
$$\begin{aligned} \tan\left(-\frac{\pi}{3}\right) &= \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} \\ &= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \end{aligned}$$

$$\begin{aligned}
 \text{30 a } \sin \frac{\pi}{3} \cos \frac{\pi}{4} &= \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) \\
 &= \frac{\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
 &= \frac{\sqrt{6}}{4}
 \end{aligned}$$

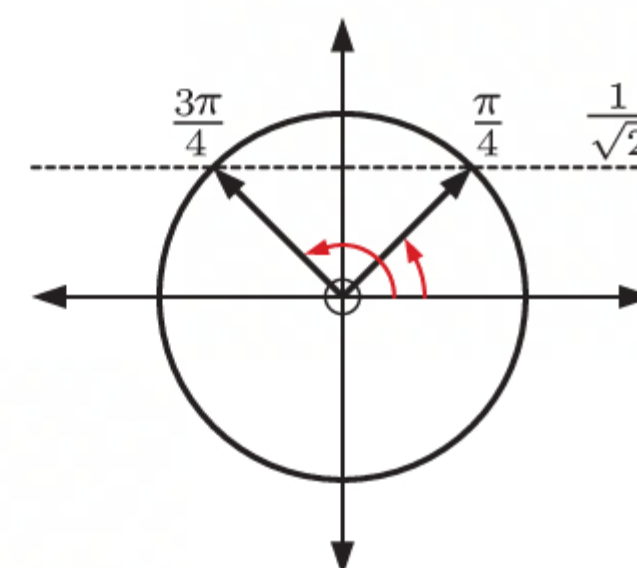
$$\begin{aligned}
 \text{b } 2 \tan^2 \left(\frac{2\pi}{3}\right) + 1 &= 2 \left(\frac{\sin \frac{2\pi}{3}}{\cos \frac{2\pi}{3}}\right)^2 + 1 \\
 &= 2 \left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)^2 + 1 \\
 &= 2(-\sqrt{3})^2 + 1 \\
 &= 2(3) + 1 \\
 &= 7
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \frac{\cos \frac{5\pi}{6} \tan^2 \left(\frac{3\pi}{4}\right)}{\sin \left(-\frac{\pi}{3}\right)} &= \frac{\cos \frac{5\pi}{6} \left(\frac{\sin \frac{3\pi}{4}}{\cos \frac{3\pi}{4}}\right)^2}{\sin \left(-\frac{\pi}{3}\right)} \\
 &= \frac{\left(-\frac{\sqrt{3}}{2}\right) \left(\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right)^2}{\left(-\frac{\sqrt{3}}{2}\right)} \\
 &= (-1)^2 \\
 &= 1
 \end{aligned}$$

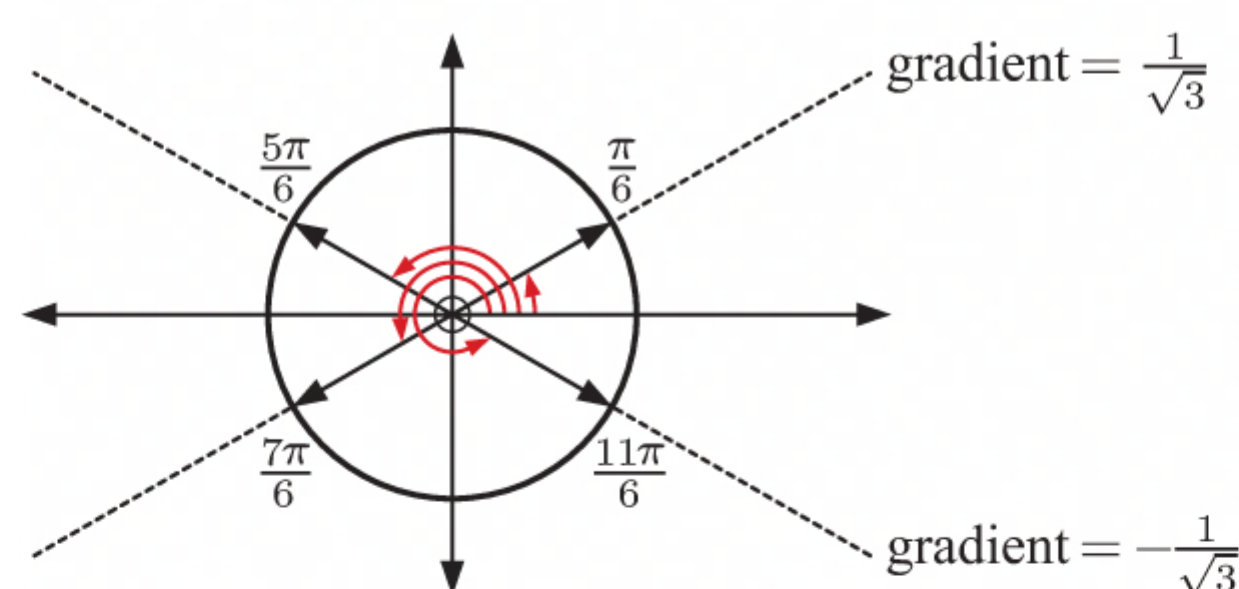
$$\begin{aligned}
 \text{31 a } \cos \theta &= -\frac{1}{2} \\
 \therefore \theta &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}
 \end{aligned}$$



$$\begin{aligned}
 \text{b } \sin \theta &= \frac{1}{\sqrt{2}} \\
 \therefore \theta &= \frac{\pi}{4} \text{ or } \frac{3\pi}{4}
 \end{aligned}$$



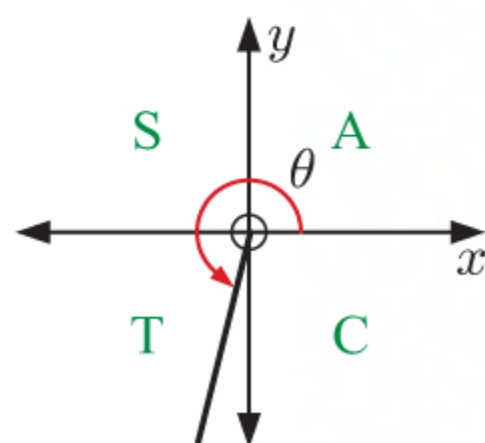
$$\begin{aligned}
 \text{c } \tan^2 \theta &= \frac{1}{3} \\
 \therefore \tan \theta &= \pm \frac{1}{\sqrt{3}} \\
 \therefore \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}
 \end{aligned}$$



$$\begin{aligned}
 \text{32 a } \sin \theta &= \frac{4}{5} \\
 \text{Now } \cos^2 \theta + \sin^2 \theta &= 1 \\
 \therefore \cos^2 \theta + \left(\frac{4}{5}\right)^2 &= 1 \\
 \therefore \cos^2 \theta + \frac{16}{25} &= 1 \\
 \therefore \cos^2 \theta &= \frac{9}{25} \\
 \therefore \cos \theta &= \pm \frac{3}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \cos \theta &= -\frac{2}{7} \\
 \text{Now } \cos^2 \theta + \sin^2 \theta &= 1 \\
 \therefore \left(-\frac{2}{7}\right)^2 + \sin^2 \theta &= 1 \\
 \therefore \frac{4}{49} + \sin^2 \theta &= 1 \\
 \therefore \sin^2 \theta &= \frac{45}{49} \\
 \therefore \sin \theta &= \pm \frac{\sqrt{45}}{7}
 \end{aligned}$$

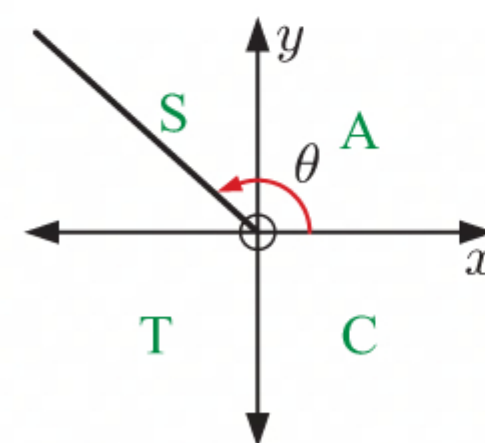
$$\text{33 a } \cos \theta = -\frac{1}{4} \text{ and } \pi < \theta < \frac{3\pi}{2}$$



θ is in quadrant 3, so $\sin \theta$ is negative.

$$\begin{aligned}
 \text{Now } \cos^2 \theta + \sin^2 \theta &= 1 \\
 \therefore \left(-\frac{1}{4}\right)^2 + \sin^2 \theta &= 1 \\
 \therefore \frac{1}{16} + \sin^2 \theta &= 1 \\
 \therefore \sin^2 \theta &= \frac{15}{16} \\
 \therefore \sin \theta &= -\frac{\sqrt{15}}{4} \quad \{\sin \theta < 0\}
 \end{aligned}$$

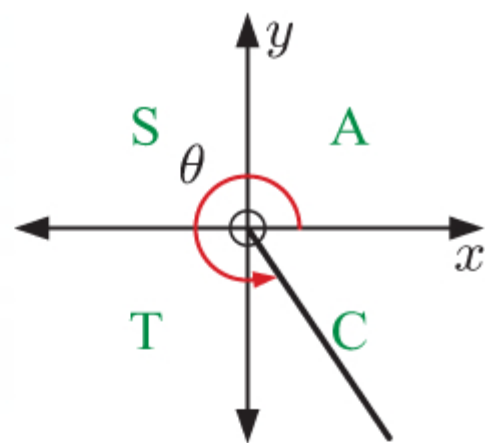
$$\text{b } \sin \theta = \frac{2}{3} \text{ and } \frac{\pi}{2} < \theta < \pi$$



θ is in quadrant 2, so $\cos \theta$ is negative.

$$\begin{aligned}
 \text{Now } \cos^2 \theta + \sin^2 \theta &= 1 \\
 \therefore \cos^2 \theta + \left(\frac{2}{3}\right)^2 &= 1 \\
 \therefore \cos^2 \theta + \frac{4}{9} &= 1 \\
 \therefore \cos^2 \theta &= \frac{5}{9} \\
 \therefore \cos \theta &= -\frac{\sqrt{5}}{3} \quad \{\cos \theta < 0\}
 \end{aligned}$$

c $\sin \theta = -\frac{5}{6}$ and $\frac{3\pi}{2} < \theta < 2\pi$



θ is in quadrant 4, so $\cos \theta$ is positive.

Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \left(-\frac{5}{6}\right)^2 = 1$$

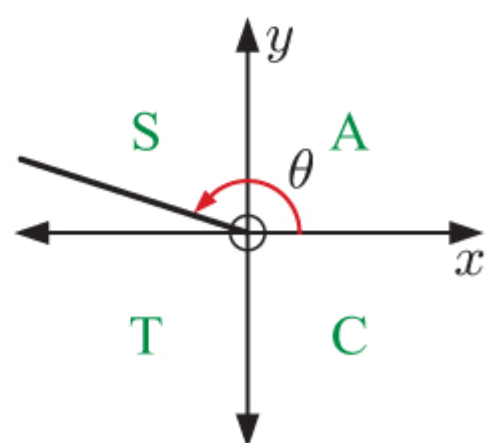
$$\therefore \cos^2 \theta + \frac{25}{36} = 1$$

$$\therefore \cos^2 \theta = \frac{11}{36}$$

$$\therefore \cos \theta = \frac{\sqrt{11}}{6} \quad \{\cos \theta > 0\}$$

$$\begin{aligned} \text{So, } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{-\frac{5}{6}}{\frac{\sqrt{11}}{6}} \\ &= -\frac{5}{\sqrt{11}} \end{aligned}$$

34 a $\tan \theta = -\frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$



θ is in quadrant 2, so $\sin \theta > 0$ and $\cos \theta < 0$.

Now $\tan \theta = -\frac{1}{3}$

$$\therefore \frac{\sin \theta}{\cos \theta} = -\frac{1}{3}$$

$$\therefore \cos \theta = -3 \sin \theta \quad \dots (*)$$

So, $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore 9 \sin^2 \theta + \sin^2 \theta = 1 \quad \{\text{using } (*)\}$$

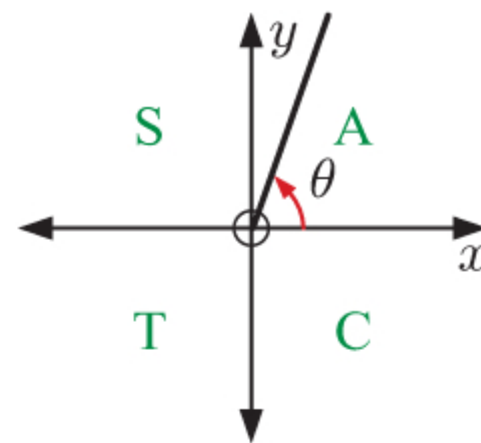
$$\therefore 10 \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{1}{10}$$

$$\therefore \sin \theta = \frac{1}{\sqrt{10}} \quad \{\sin \theta > 0\}$$

$$\therefore \cos \theta = -\frac{3}{\sqrt{10}}$$

d $\cos \theta = \frac{1}{3}$ and $0 < \theta < \frac{\pi}{2}$



θ is in quadrant 1, so $\sin \theta$ is positive.

Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(\frac{1}{3}\right)^2 + \sin^2 \theta = 1$$

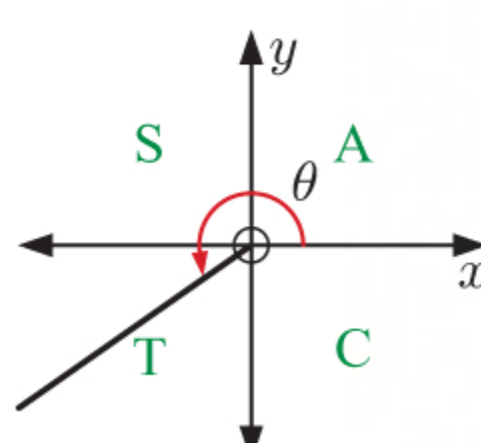
$$\therefore \frac{1}{9} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{8}{9}$$

$$\therefore \sin \theta = \frac{\sqrt{8}}{3} \quad \{\sin \theta > 0\}$$

$$\begin{aligned} \text{So, } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{\frac{\sqrt{8}}{3}}{\frac{1}{3}} \\ &= \sqrt{8} \end{aligned}$$

b $\tan \theta = \frac{1}{\sqrt{2}}$ and $\pi < \theta < \frac{3\pi}{2}$



θ is in quadrant 3, so $\sin \theta < 0$ and $\cos \theta < 0$.

Now $\tan \theta = \frac{1}{\sqrt{2}}$

$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{1}{\sqrt{2}}$$

$$\therefore \cos \theta = \sqrt{2} \sin \theta \quad \dots (*)$$

So, $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore 2 \sin^2 \theta + \sin^2 \theta = 1 \quad \{\text{using } (*)\}$$

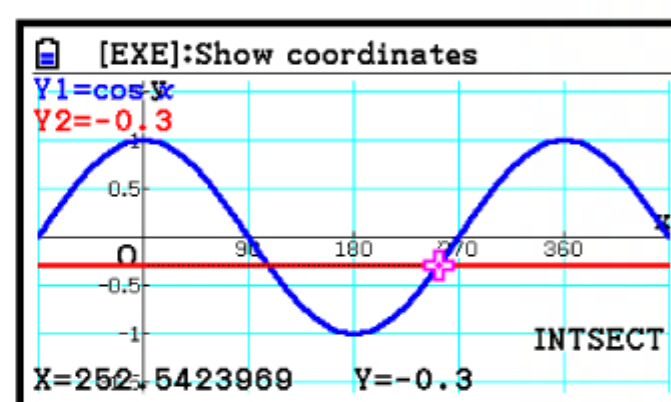
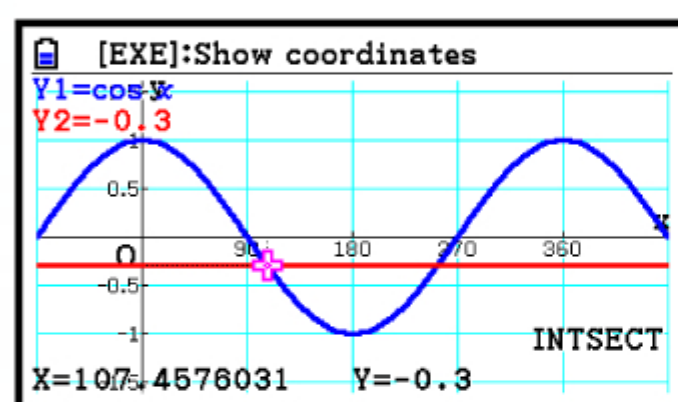
$$\therefore 3 \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{1}{3}$$

$$\therefore \sin \theta = -\frac{1}{\sqrt{3}} \quad \{\sin \theta < 0\}$$

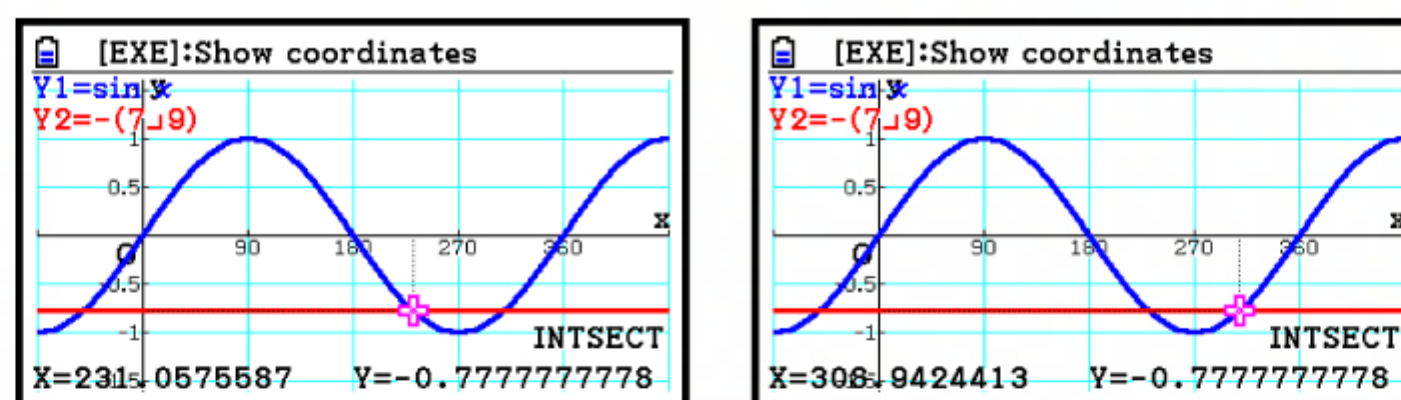
$$\therefore \cos \theta = -\sqrt{\frac{2}{3}}$$

35 a We graph the functions $Y_1 = \cos X$ and $Y_2 = -0.3$ on the same set of axes.



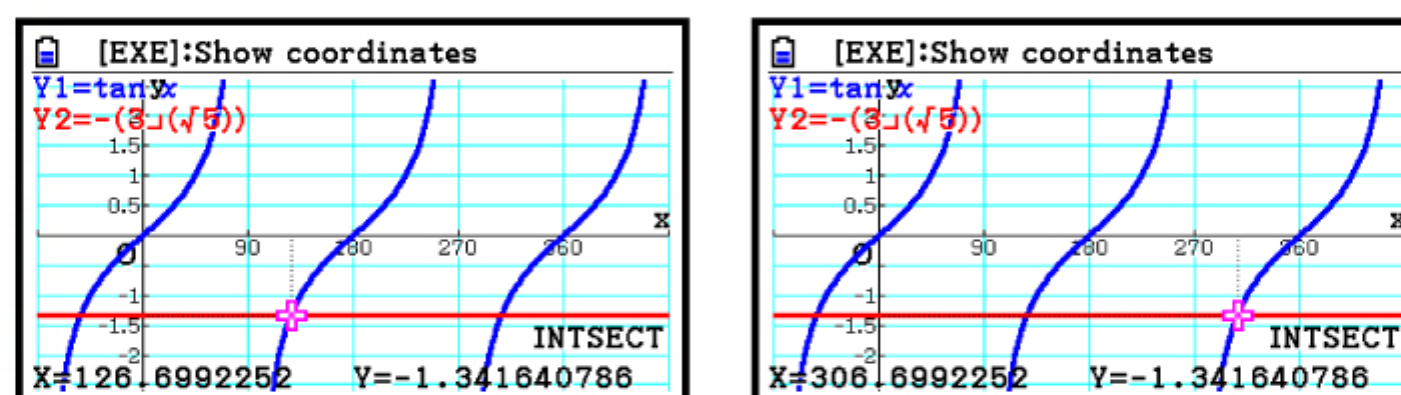
The solutions are $\theta \approx 107^\circ, 253^\circ$.

- b** We graph the functions $Y_1 = \sin X$ and $Y_2 = -\frac{7}{9}$ on the same set of axes.

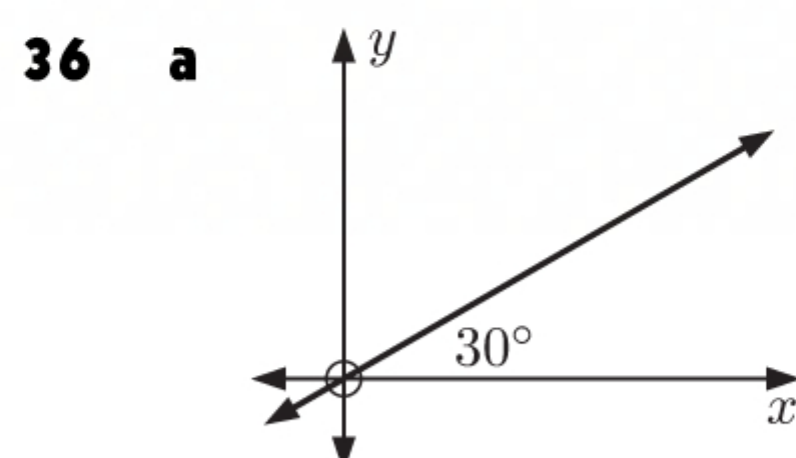


The solutions are $\theta \approx 231^\circ, 309^\circ$.

- c** We graph the functions $Y_1 = \tan X$ and $Y_2 = -\frac{3}{\sqrt{5}}$ on the same set of axes.

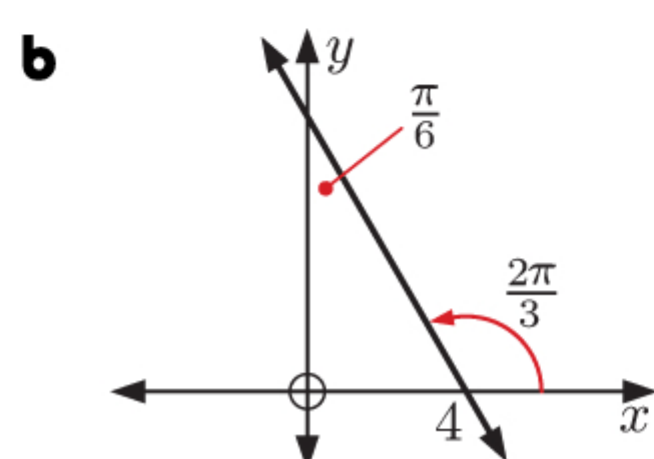


The solutions are $\theta \approx 127^\circ, 307^\circ$.



The line has gradient $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$ and y -intercept 0.

\therefore the line has equation $y = \frac{1}{\sqrt{3}}x$.



The line makes an angle of $\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ with the positive x -axis.
{exterior angle of a triangle theorem}

\therefore the line has gradient $m = \tan \frac{2\pi}{3} = -\sqrt{3}$.

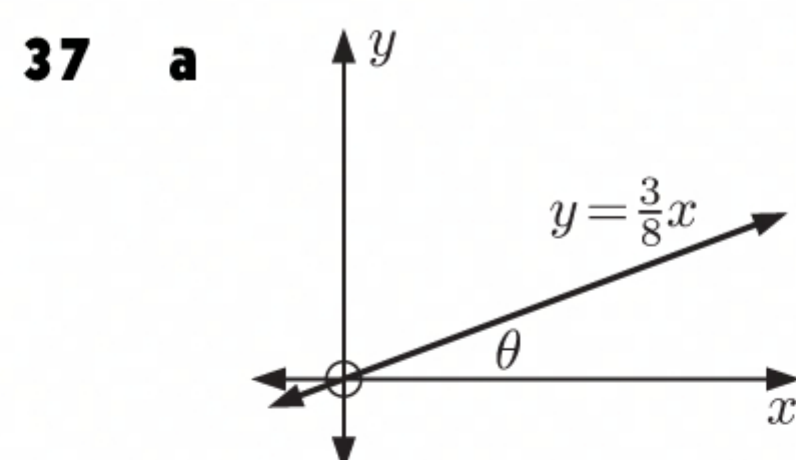
\therefore the line has equation $y = -\sqrt{3}x + c$.

When $x = 4$, $y = 0$, $\therefore 0 = -\sqrt{3}(4) + c$

$$\therefore 0 = -4\sqrt{3} + c$$

$$\therefore c = 4\sqrt{3}$$

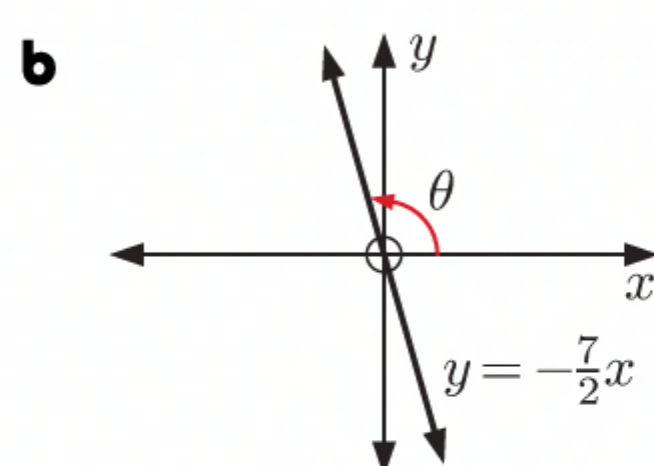
\therefore the equation of the line is $y = -\sqrt{3}x + 4\sqrt{3}$.



The line has gradient $\frac{3}{8}$, so $\tan \theta = \frac{3}{8}$.

Using technology, $\tan^{-1}\left(\frac{3}{8}\right) \approx 0.359$

$\therefore \theta \approx 0.359$



The line has gradient $-\frac{7}{2}$, so $\tan \theta = -\frac{7}{2}$.

Using technology, $\tan^{-1}\left(-\frac{7}{2}\right) \approx -1.29$

But $0 < \theta < \pi$, so $\theta \approx \pi - 1.29 \approx 1.85$

- 38 a** For the function $f(x) = \sin 4x$:

- the amplitude is $|a| = 1$
- the principal axis is $y = 0$
- the period is $\frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$.

- b** For the function $f(x) = -2 \sin \frac{x}{2} - 1$:

- the amplitude is $|a| = |-2| = 2$
- the principal axis is $y = -1$
- the period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$.

- 39 a**
- i** The amplitude is $|a| = 2$.
 - ii** The principal axis is $y = 0$.
 - iii** The period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$.

- b**
- i** The amplitude is $|a| = 1$.
 - ii** The principal axis is $y = 2$.
 - iii** The period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$.

- c**
- i** The amplitude is $|a| = 3$.
 - ii** The principal axis is $y = 0$.
 - iii** The period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

- d**
- i** The amplitude is $|a| = 1$.
 - ii** The principal axis is $y = -1$.
 - iii** The period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$.

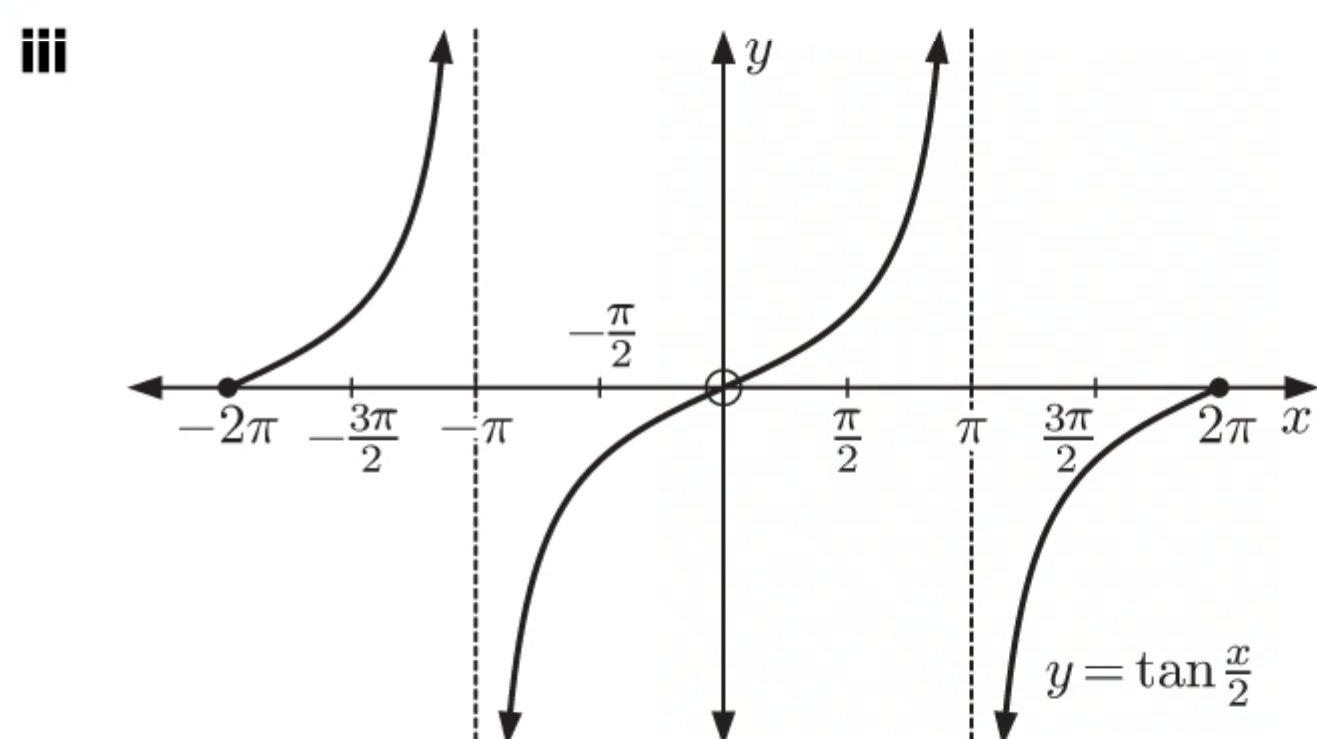
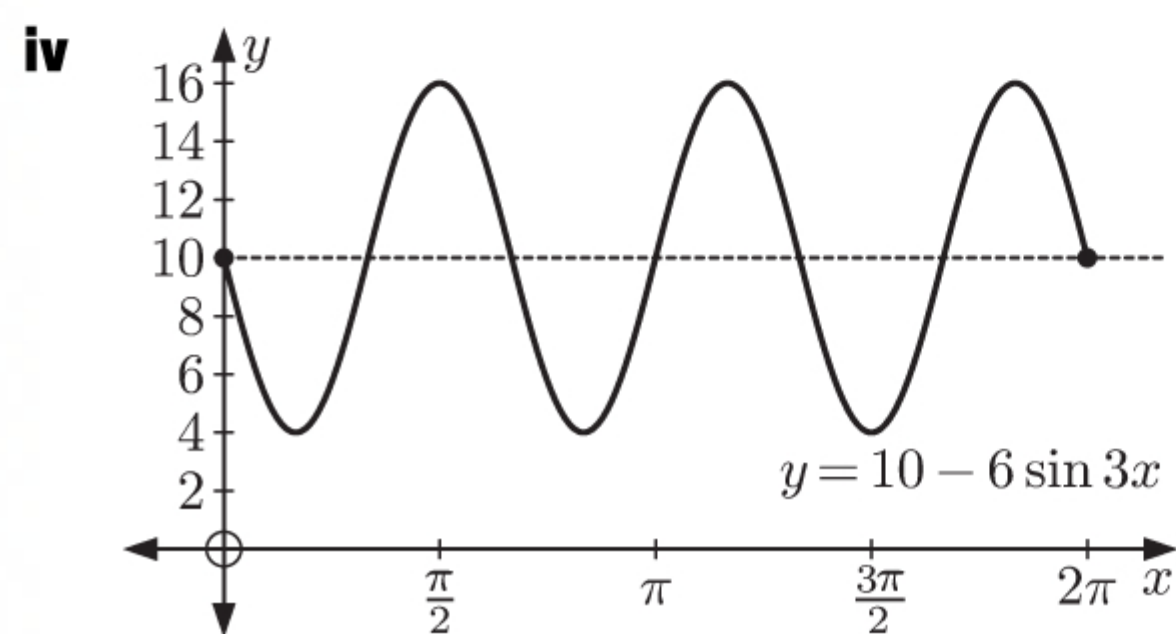
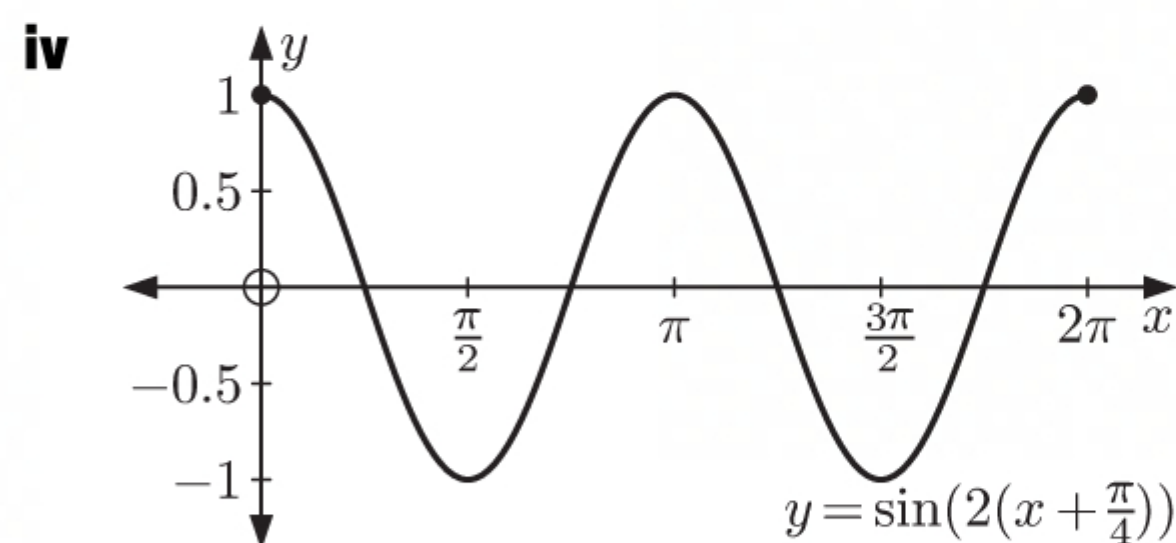
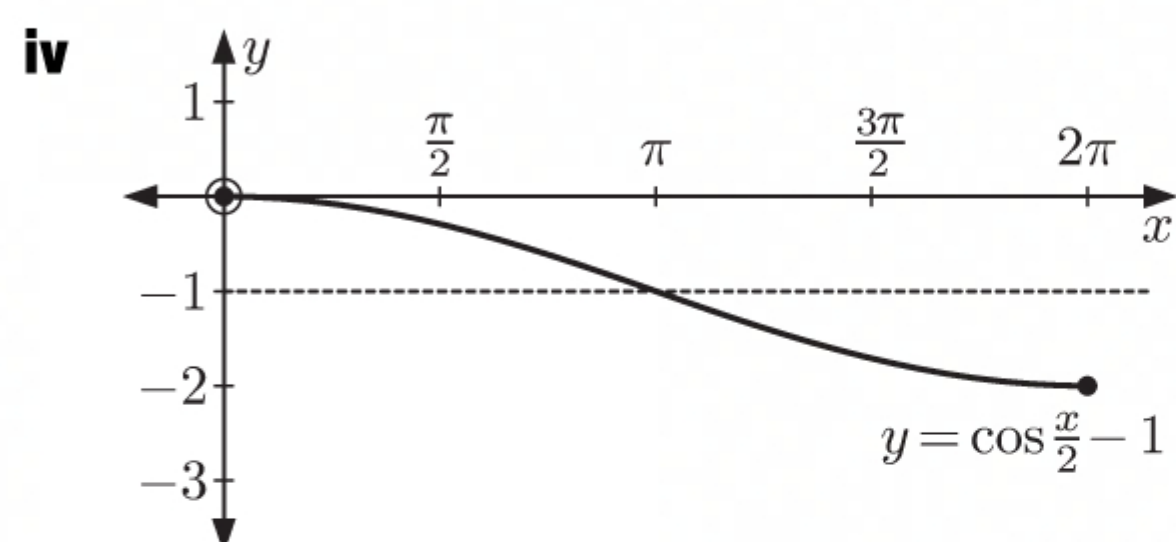
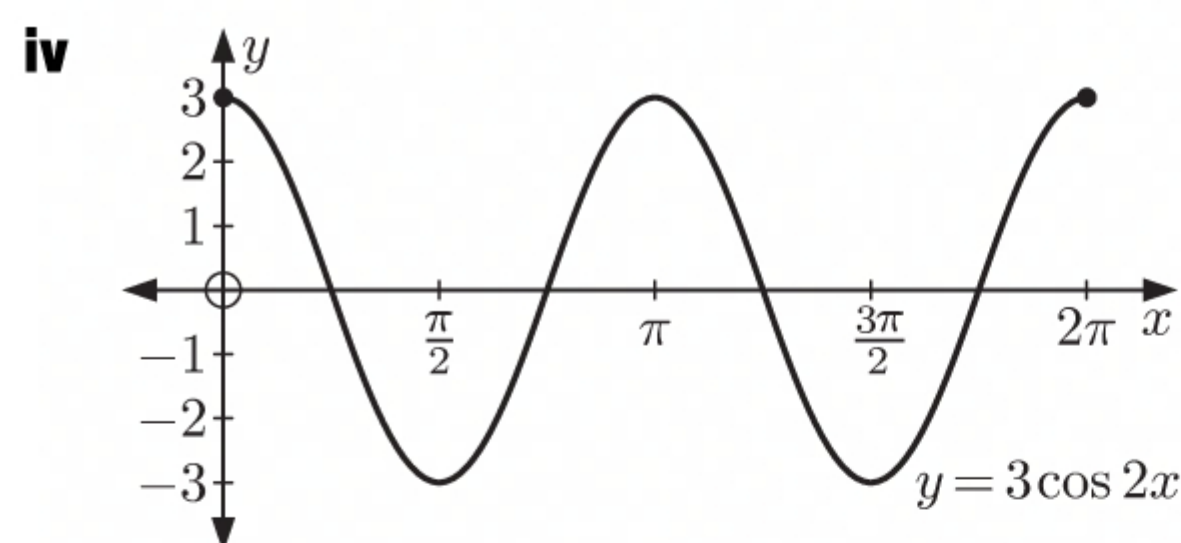
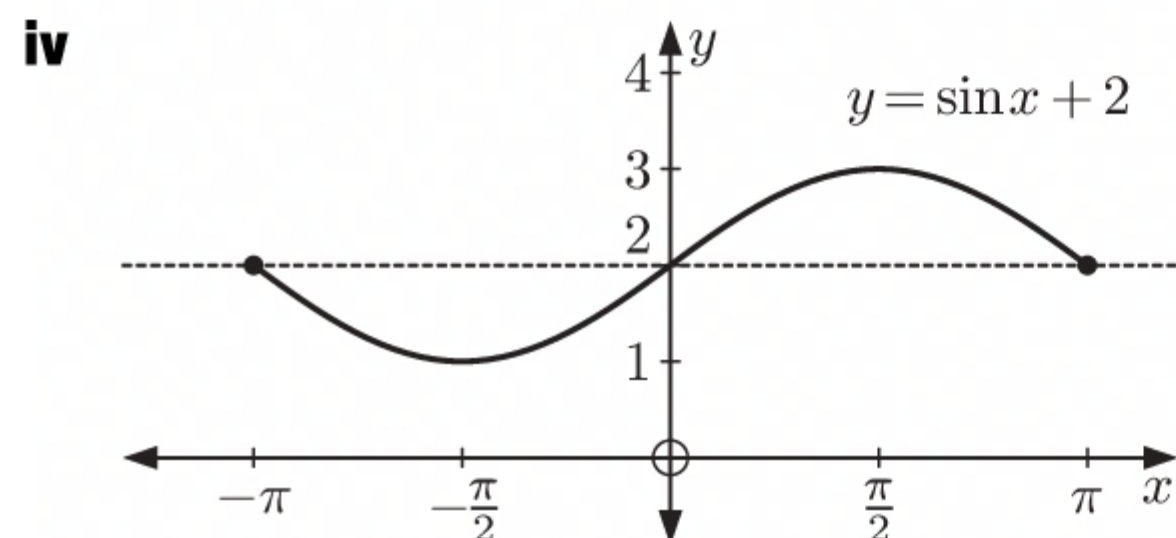
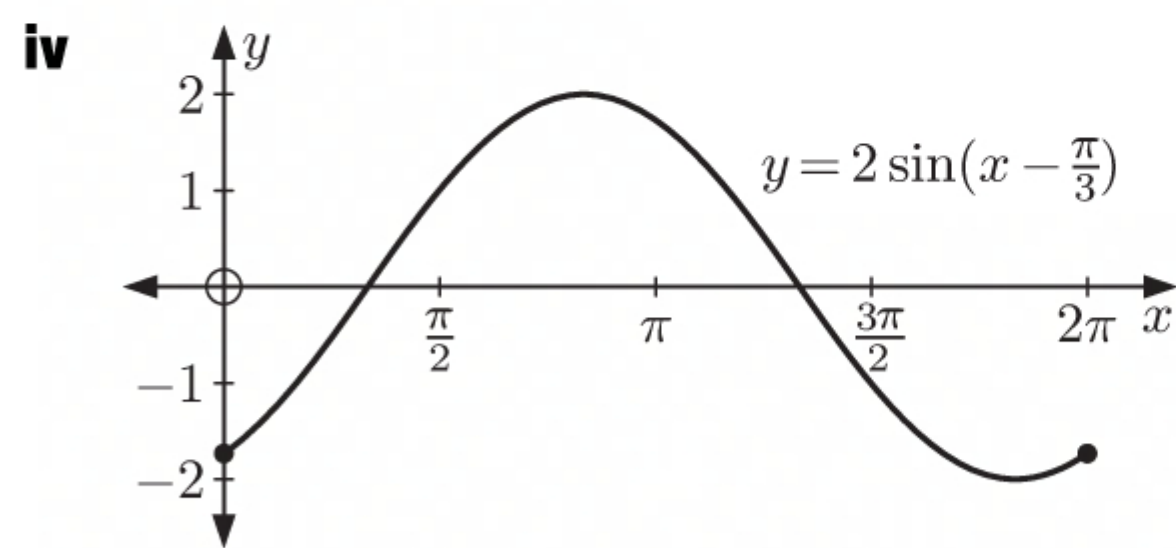
- e**
- i** The amplitude is $|a| = 1$.
 - ii** The principal axis is $y = 0$.
 - iii** The period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

- f**
- i** The amplitude is $|a| = |-6| = 6$.
 - ii** The principal axis is $y = 10$.
 - iii** The period is $\frac{2\pi}{b} = \frac{2\pi}{3}$.

- 40 a**
- i** $y = \tan \frac{x}{2}$ has period $\frac{\pi}{b} = \frac{\pi}{(\frac{1}{2})} = 2\pi$.
 - ii** $y = \tan \frac{x}{2}$ is undefined when

$$\frac{x}{2} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

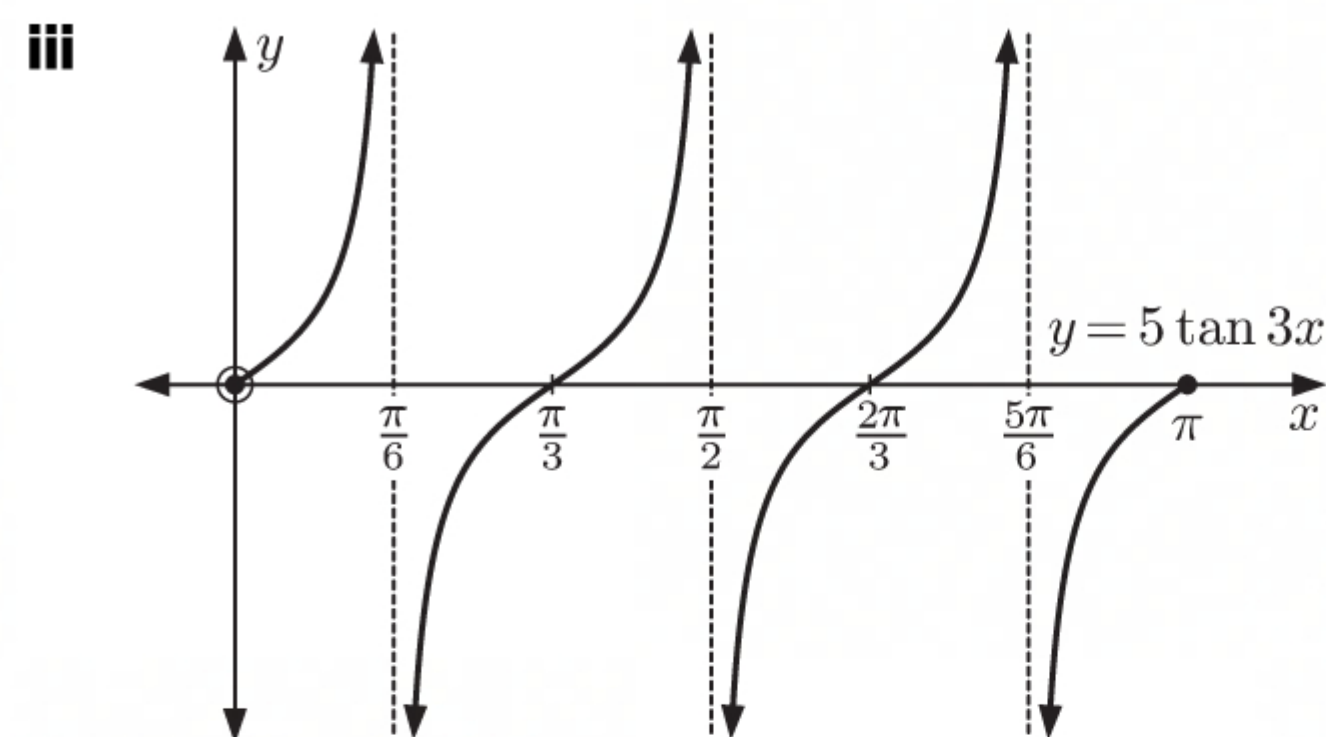


b i $y = 5 \tan 3x$ has period $\frac{\pi}{b} = \frac{\pi}{3}$.

ii $y = 5 \tan 3x$ is undefined when

$$3x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + \frac{k\pi}{3}, \quad k \in \mathbb{Z}$$

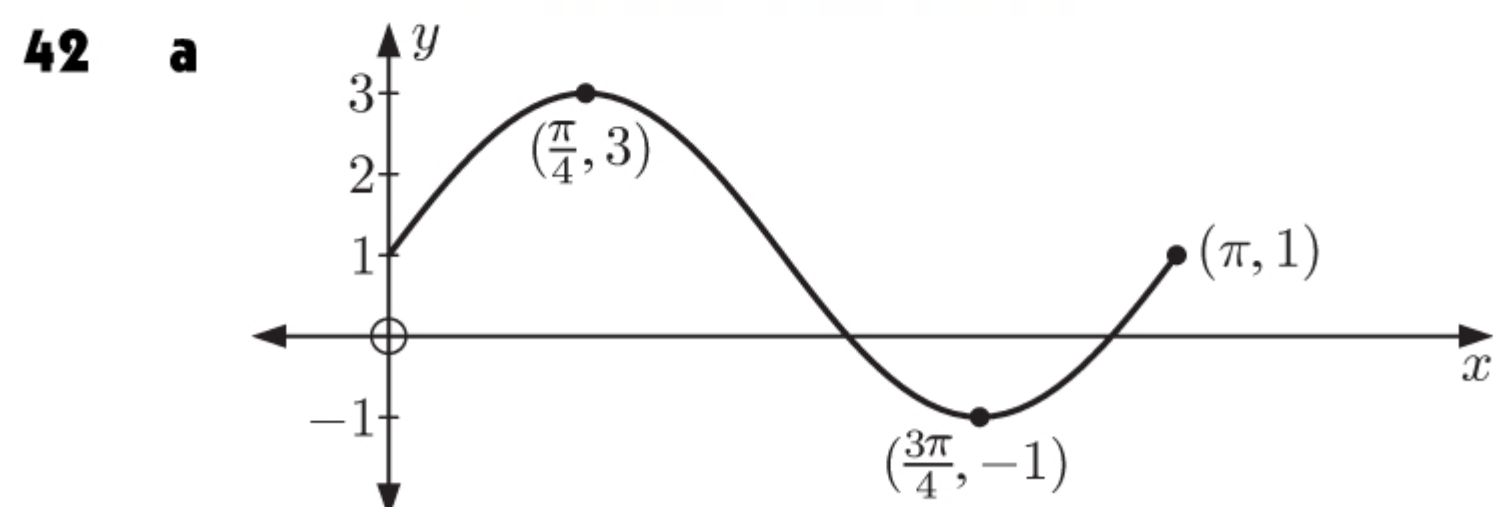


41 a $\sin x \xrightarrow[\text{scale factor 2}]{\text{vertical stretch}} 2 \sin x \xrightarrow[\text{scale factor 3}]{\text{horizontal stretch}} 2 \sin \frac{x}{3}$

A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 3 maps $y = \sin x$ onto $y = 2 \sin \frac{x}{3}$.

b $\sin x \xrightarrow[\text{translation } \begin{pmatrix} -\frac{\pi}{3} \\ -4 \end{pmatrix}]{\text{translation}} \sin\left(x + \frac{\pi}{3}\right) - 4$

A translation $\frac{\pi}{3}$ units left and 4 units downwards maps $y = \sin x$ onto $y = \sin\left(x + \frac{\pi}{3}\right) - 4$.



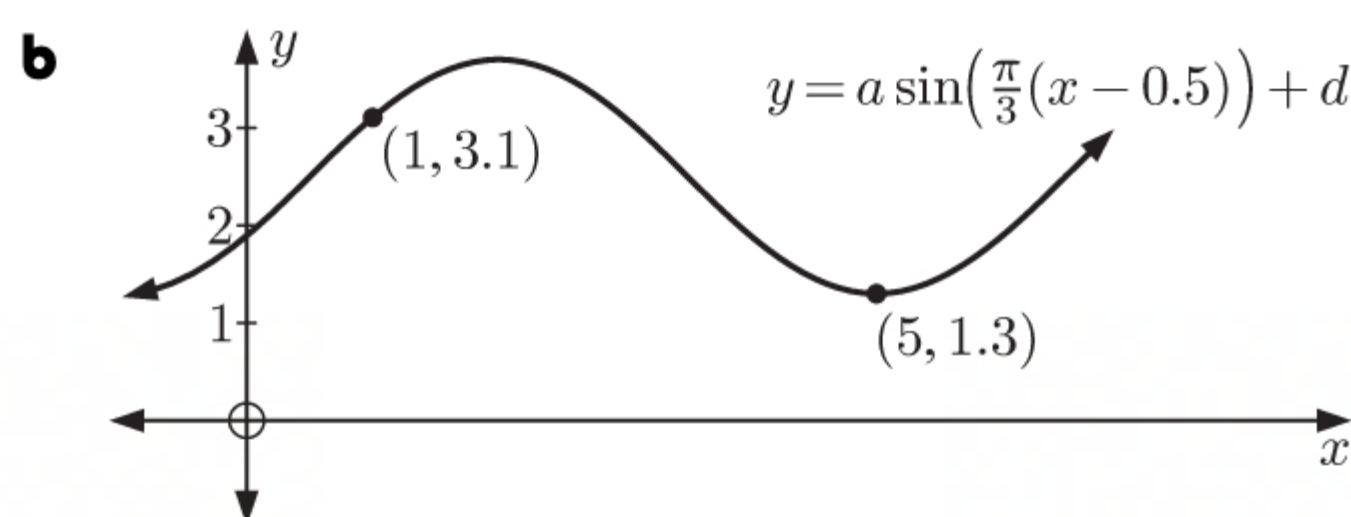
The amplitude is 2, so $a = 2$.

The period is π , so $\frac{2\pi}{b} = \pi$ and $\therefore b = 2$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 1$, so $d = 1$.

\therefore the equation of the function is $y = 2 \sin 2x + 1$.



When $x = 1$, $y = 3.1$

$$\therefore a \sin\left(\frac{\pi}{3}(1 - 0.5)\right) + d = 3.1$$

$$\therefore a \sin\left(\frac{\pi}{3} \times 0.5\right) + d = 3.1$$

$$\therefore a \sin \frac{\pi}{6} + d = 3.1$$

$$\therefore \frac{1}{2}a + d = 3.1 \quad \dots (1)$$

$$-\frac{1}{2}a - d = -3.1 \quad \{(1) \times -1\}$$

$$-a + d = 1.3 \quad \{(2)\}$$

Adding, $-\frac{3}{2}a = -1.8$

$$\therefore a = 1.2$$

and when $x = 5$, $y = 1.3$

$$\therefore a \sin\left(\frac{\pi}{3}(5 - 0.5)\right) + d = 1.3$$

$$\therefore a \sin\left(\frac{\pi}{3} \times 4.5\right) + d = 1.3$$

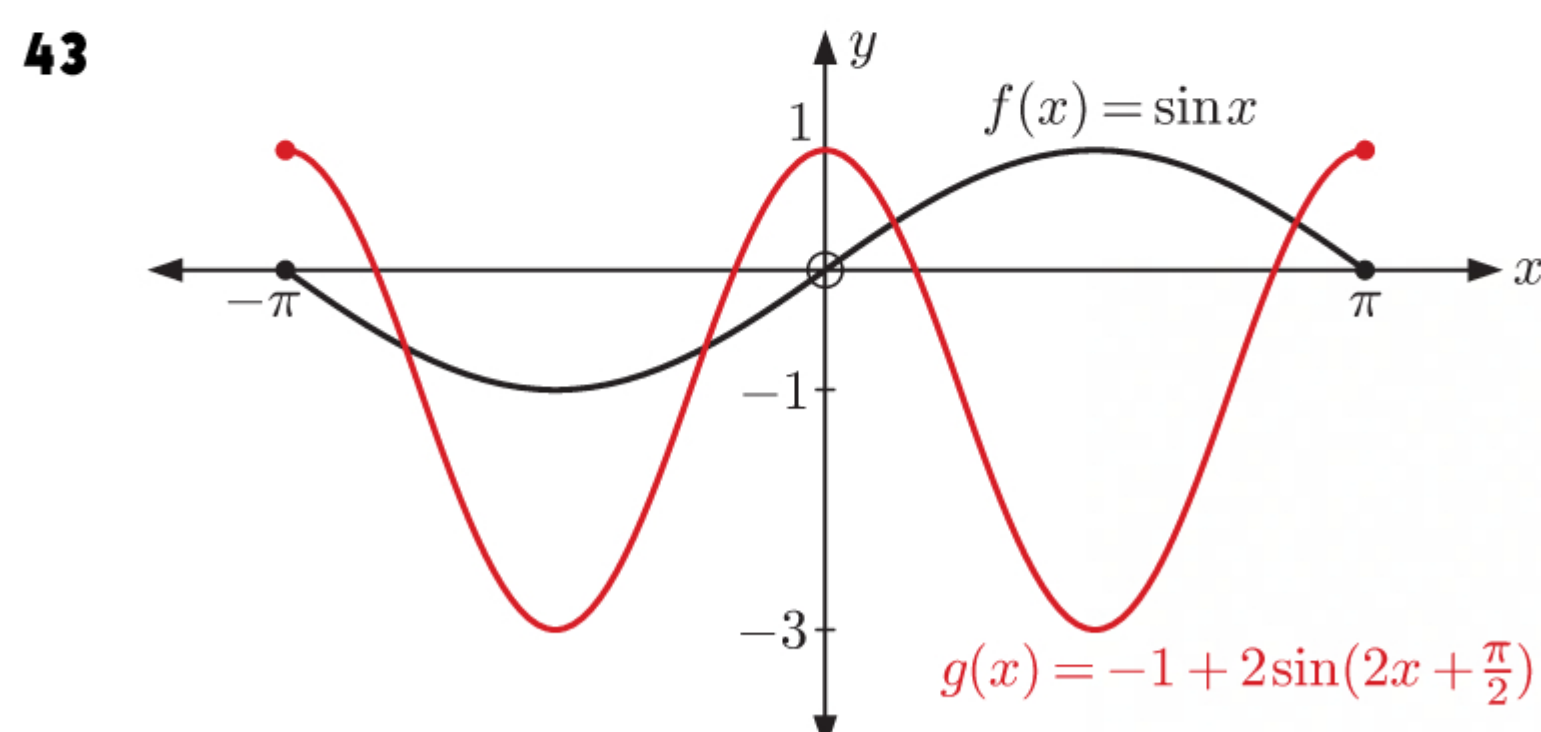
$$\therefore a \sin \frac{3\pi}{2} + d = 1.3$$

$$\therefore -a + d = 1.3 \quad \dots (2)$$

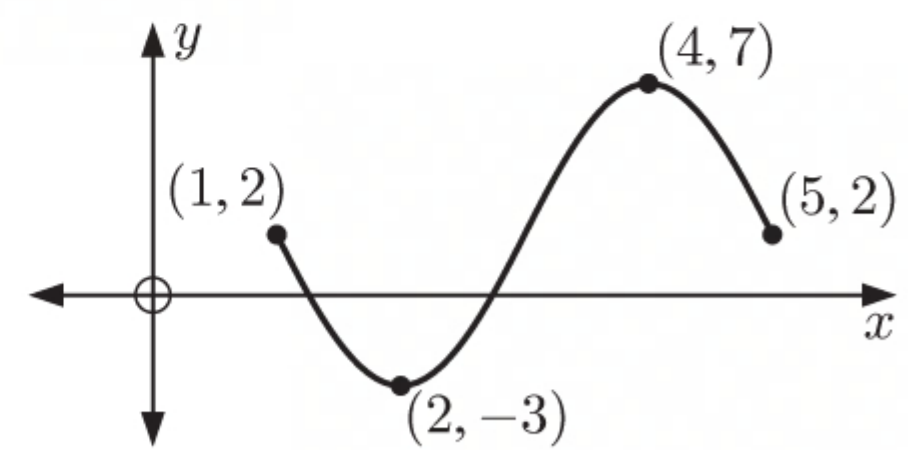
Substituting $a = 1.2$ into (2) gives $-1.2 + d = 1.3$

$$\therefore d = 2.5$$

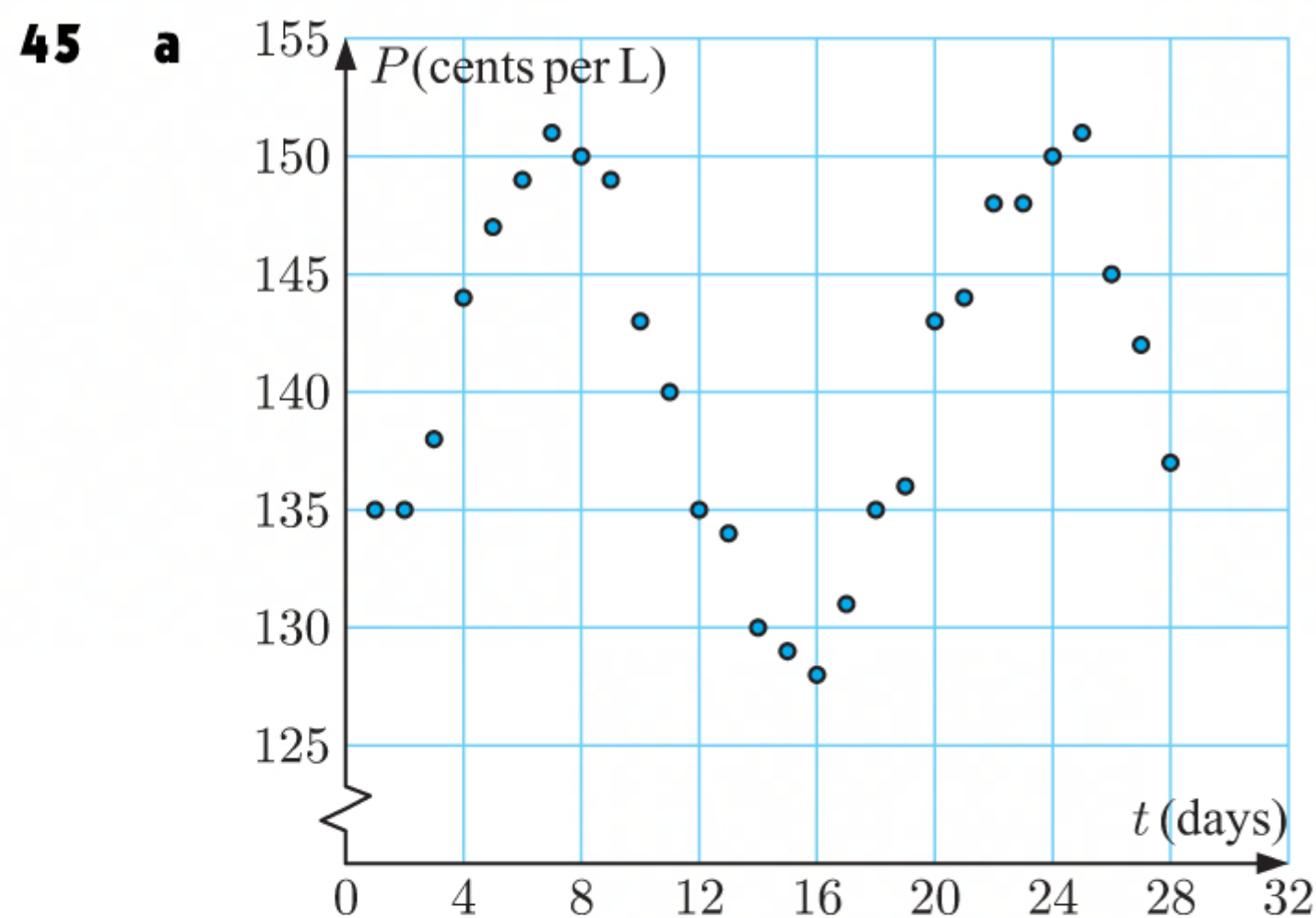
\therefore the equation of the sine function is $y = 1.2 \sin\left(\frac{\pi}{3}(x - 0.5)\right) + 2.5$.



- 44 a i** The amplitude is 5, so $a = 5$.
- ii** The period is 4, so $\frac{2\pi}{b} = 4$ and $b = \frac{\pi}{2}$.
- iii** The principal axis is $y = \frac{7 + (-3)}{2}$ which is $y = 2$, so $d = 2$.
- iv** The function would start its first period at $x = 4 - 4 = 0$.
- So, there is no horizontal translation and $c = 0$.



- b** From **a**, $y = 5 \cos \frac{\pi}{2}x + 2$
- $$\therefore y = 5 \sin\left(\frac{\pi}{2} - \frac{\pi}{2}x\right) + 2 \quad \left\{ \cos \theta = \sin\left(\frac{\pi}{2} - \theta\right) \right\}$$
- $$\therefore y = 5 \sin\left(\frac{\pi}{2}(1 - x)\right) + 2$$
- $$\therefore y = 5 \sin\left(-\frac{\pi}{2}(x - 1)\right) + 2$$
- $$\therefore y = -5 \sin\left(\frac{\pi}{2}(x - 1)\right) + 2 \quad \left\{ \sin(-\theta) = -\sin \theta \right\}$$



- b i** From the scatter diagram, the time between peaks (period) is about 16 days.

So, $\frac{2\pi}{b} \approx 16$ and $\therefore b \approx \frac{\pi}{8}$.

- ii** The amplitude $= \frac{\max - \min}{2} \approx \frac{151 - 128}{2} \approx 11.5$, so $a \approx 11.5$.

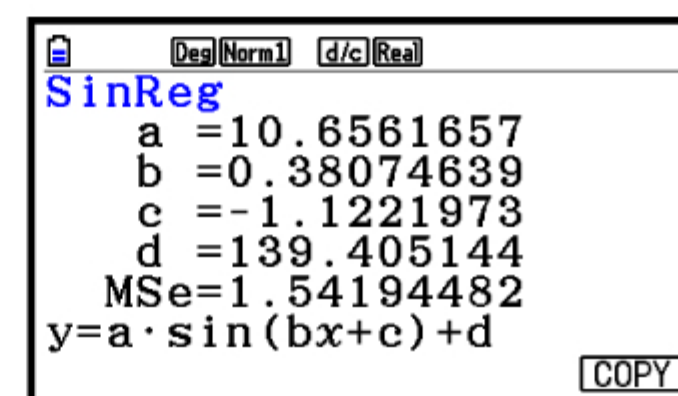
- iii** The principal axis is midway between the maximum and minimum, so $d \approx \frac{151 + 128}{2} \approx 139.5$.

- iv** The model is $P \approx 11.5 \sin\left(\frac{\pi}{8}(t - c)\right) + 139.5$ for some constant c .

From the scatter diagram, the first period starts somewhere between $t = 3$ and $t = 4$. So we estimate $c \approx 3.5$.

- c** From **b**, our model is $P \approx 11.5 \sin\left(\frac{\pi}{8}(t - 3.5)\right) + 139.5$
 $\approx 11.5 \sin(0.393t - 1.37) + 139.5$

Using technology, $P \approx 10.7 \sin(0.381t - 1.12) + 139.4$



46 $f(x) = 2 \tan(3(x - 1)) + 4, \quad -1 \leq x \leq 1$

- a** The period of $y = f(x)$ is $\frac{\pi}{b} = \frac{\pi}{3}$.

- b** $y = f(x)$ is undefined when $3(x - 1) = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

$$\therefore x - 1 = \frac{\pi}{6} + \frac{k\pi}{3}, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + \frac{k\pi}{3} + 1, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + \frac{k\pi}{3} + 1, \quad k = -2, -1 \quad \{-1 \leq x \leq 1\}$$

$$\therefore x = \frac{\pi}{6} - \frac{2\pi}{3} + 1, \quad \frac{\pi}{6} - \frac{\pi}{3} + 1$$

$$\therefore x = 1 - \frac{\pi}{2}, \quad 1 - \frac{\pi}{6}$$

$$\text{c } \tan x \xrightarrow[\text{vertical stretch}]{\text{scale factor } 2} 2 \tan x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{3}} 2 \tan 3x \xrightarrow[\text{translation } \begin{pmatrix} 1 \\ 4 \end{pmatrix}]{} 2 \tan(3(x-1)) + 4$$

So, a vertical stretch with scale factor 2, then a horizontal stretch with scale factor $\frac{1}{3}$, then a horizontal translation 1 unit to the right, then a vertical translation 4 units upwards will map $y = \tan x$ onto $y = f(x)$.

d Using **b**, the domain of $y = f(x)$ is $\{x \mid -1 \leq x \leq 1, x \neq 1 - \frac{\pi}{2}, x \neq 1 - \frac{\pi}{6}\}$.

$2 \tan(3(x-1)) + 4$ can take any real value for $1 - \frac{\pi}{2} < x < 1 - \frac{\pi}{6}$. So, the range of $y = f(x)$ is $\{y \mid y \in \mathbb{R}\}$.

47 a $f(x) = \operatorname{cosec} x, \quad -2\pi \leq x \leq 2\pi$
 $= \frac{1}{\sin x}$

The vertical asymptotes occur where $\sin x = 0$

$$\therefore x = -2\pi, -\pi, 0, \pi, \text{ or } 2\pi$$

\therefore the vertical asymptotes are $x = -2\pi, \pi, 0, \pi$, and 2π .

b $f(x) = \sec 2x, \quad -2\pi \leq x \leq 2\pi$
 $= \frac{1}{\cos 2x}$

The vertical asymptotes occur where $\cos 2x = 0$.

$$\text{Since } -2\pi \leq x \leq 2\pi$$

$$\therefore -4\pi \leq 2x \leq 4\pi$$

$$\text{So, } 2x = -\frac{7\pi}{2}, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ or } \frac{7\pi}{2}$$

$$\therefore x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$$

\therefore the vertical asymptotes are

$$x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4}.$$

c $g(x) = \cot \frac{x}{2}, \quad -2\pi \leq x \leq 2\pi$
 $= \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}$

The vertical asymptotes occur where $\sin(\frac{x}{2}) = 0$.

$$\text{Since } -2\pi \leq x \leq 2\pi$$

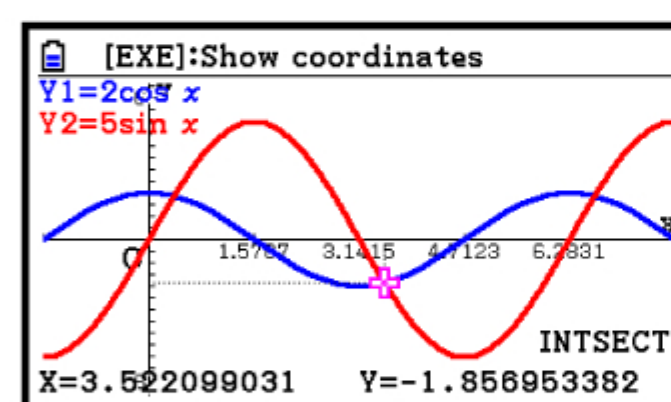
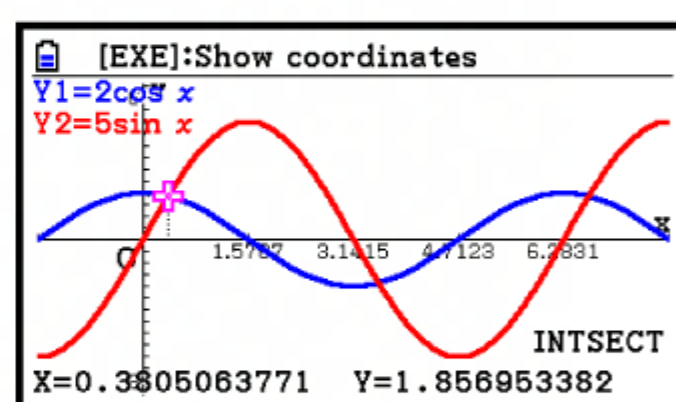
$$\therefore -\pi \leq \frac{x}{2} \leq \pi$$

$$\text{So, } \frac{x}{2} = -\pi, 0, \pi$$

$$\therefore x = -2\pi, 0, 2\pi$$

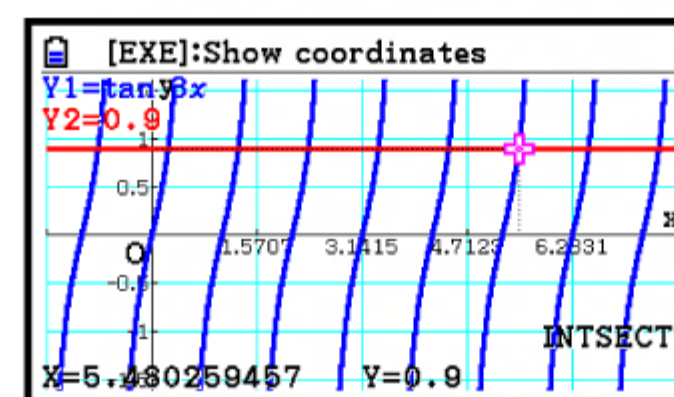
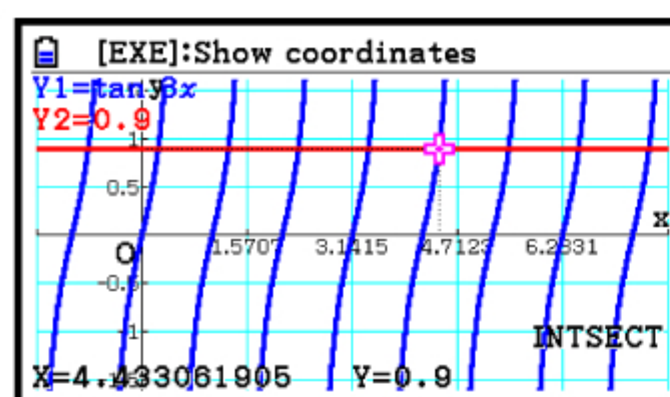
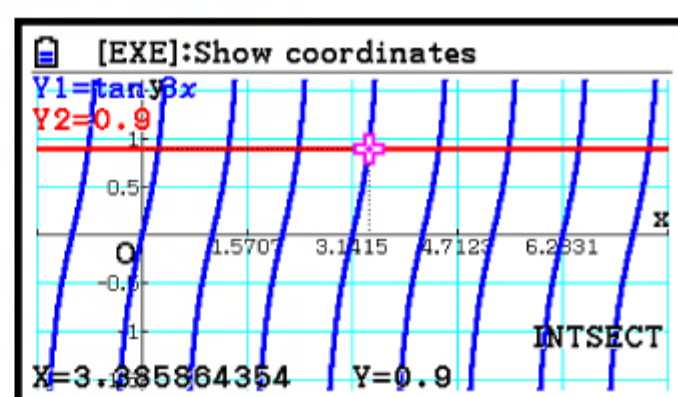
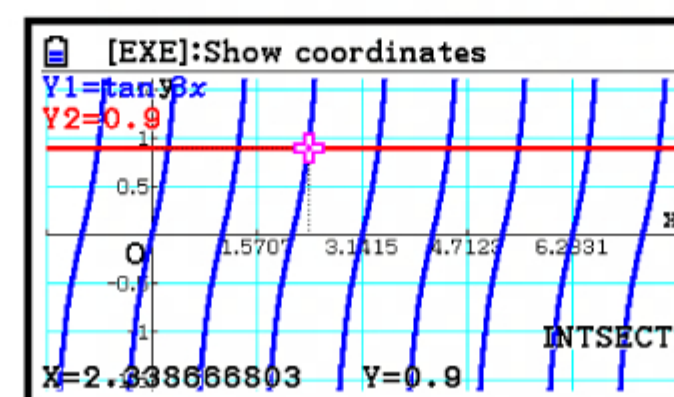
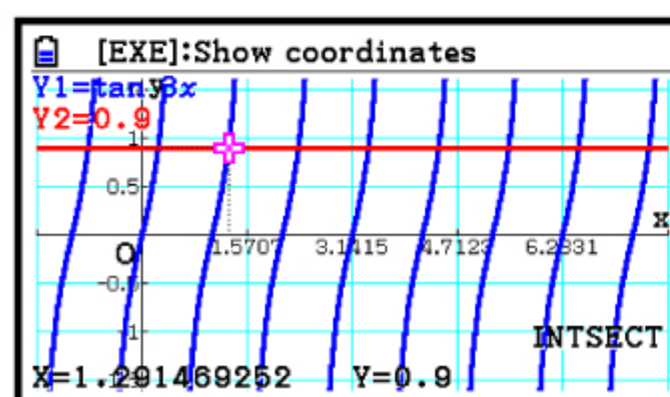
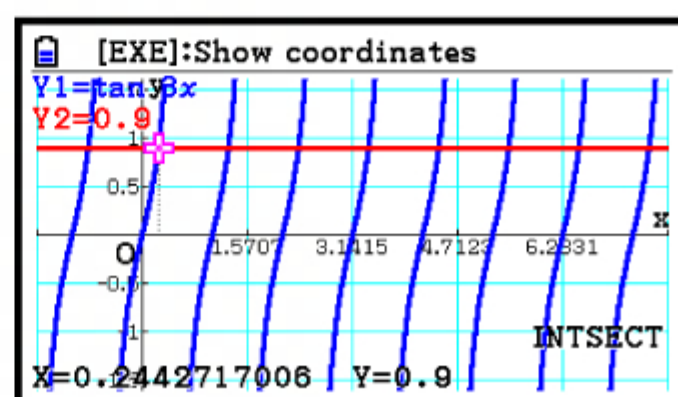
\therefore the vertical asymptotes are $x = -2\pi, 0$, and 2π .

48 a We graph the functions $Y_1 = 2 \cos X$ and $Y_2 = 5 \sin X$ on the same set of axes.



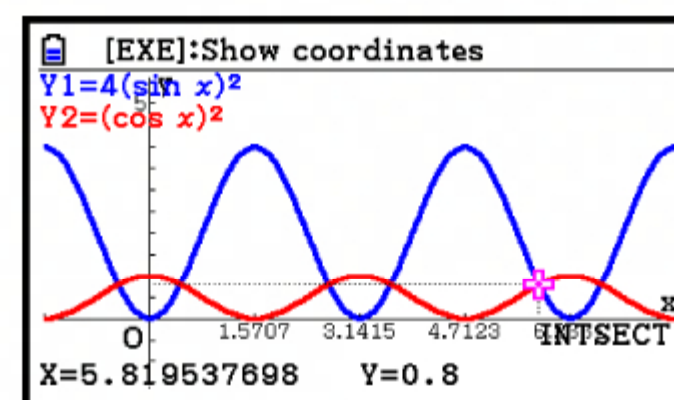
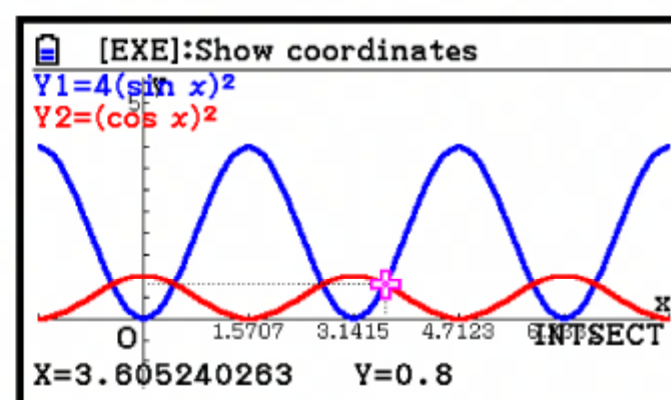
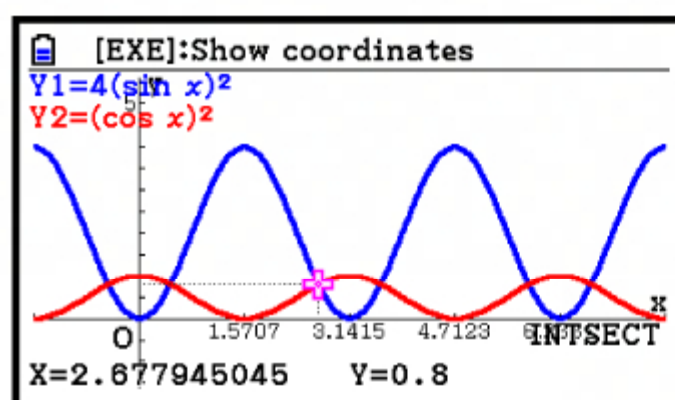
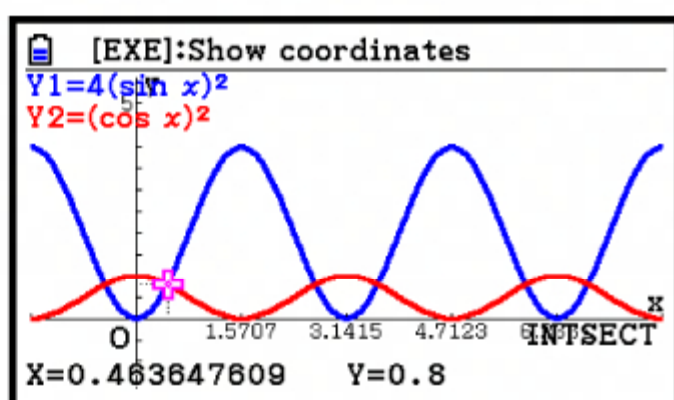
The solutions are $x \approx 0.381, 3.52$.

b We graph the functions $Y_1 = \tan 3X$ and $Y_2 = 0.9$ on the same set of axes.



The solutions are $x \approx 0.244, 1.29, 2.34, 3.39, 4.43, 5.48$.

- c** We graph the functions $Y_1 = 4\sin^2 X$ and $Y_2 = \cos^2 X$ on the same set of axes.



The solutions are $x \approx 0.464, 2.68, 3.61, 5.82$.

49 a $\sqrt{3} \tan \frac{x}{2} = -1$

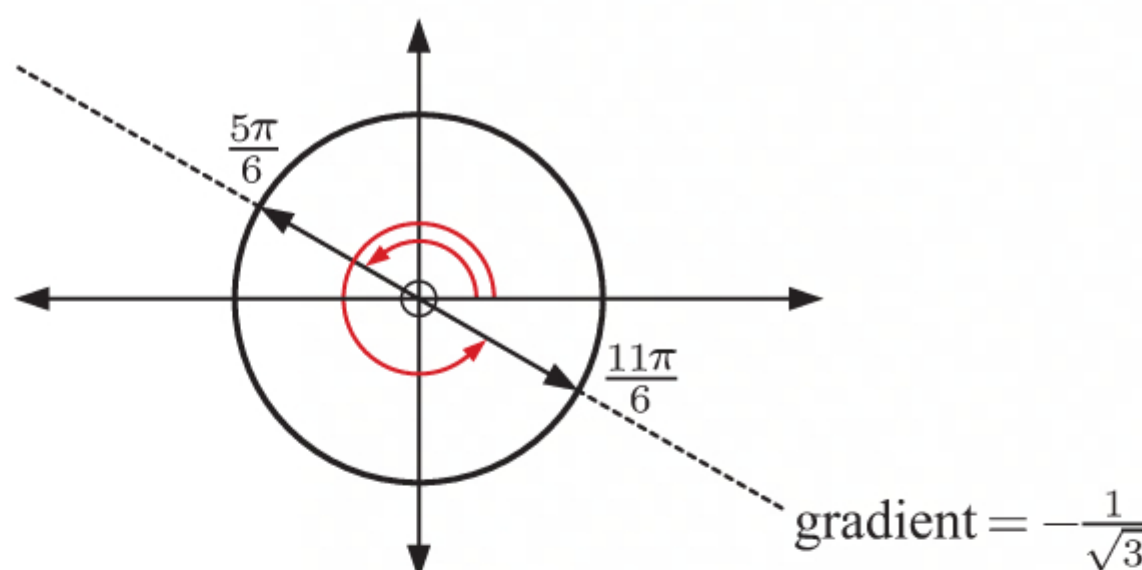
$$\therefore \tan \frac{x}{2} = -\frac{1}{\sqrt{3}}$$

Since $-\pi \leq x \leq 3\pi$,

$$-\frac{\pi}{2} \leq \frac{x}{2} \leq \frac{3\pi}{2}$$

So, $\frac{x}{2} = -\frac{\pi}{6}$ and $\frac{5\pi}{6}$

$$\therefore x = -\frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$



b $\sqrt{3} + 2\sin 2x = 0$

$$\therefore 2\sin 2x = -\sqrt{3}$$

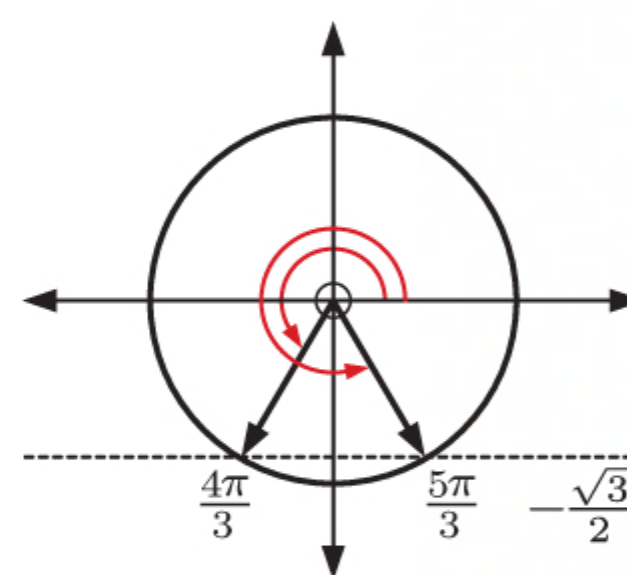
$$\therefore \sin 2x = -\frac{\sqrt{3}}{2}$$

Since $-\pi \leq x \leq 3\pi$,

$$-2\pi \leq 2x \leq 6\pi$$

So, $2x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \frac{17\pi}{3}$

$$\therefore x = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}, \frac{8\pi}{3}, \frac{17\pi}{6}$$



c $1 - \sqrt{2}\cos 3x = 0$

$$\therefore -\sqrt{2}\cos 3x = -1$$

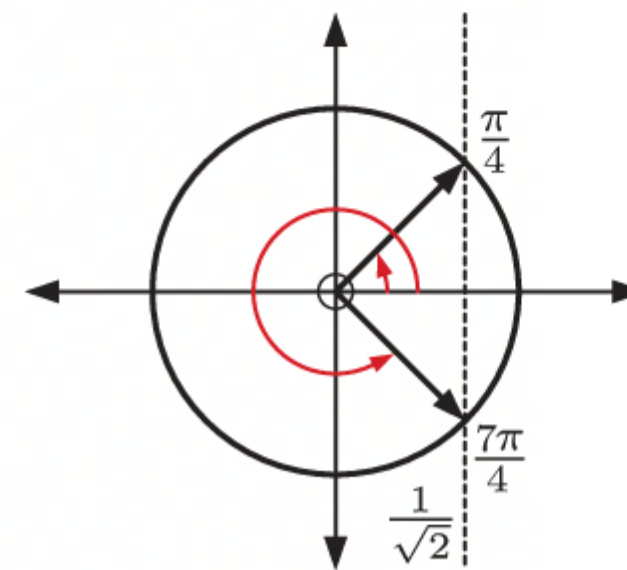
$$\therefore \cos 3x = \frac{1}{\sqrt{2}}$$

Since $-\pi \leq x \leq 3\pi$

$$-3\pi \leq 3x \leq 9\pi$$

So, $3x = -\frac{9\pi}{4}, -\frac{7\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}, \frac{25\pi}{4}, \frac{31\pi}{4}, \frac{33\pi}{4}$

$$\therefore x = -\frac{3\pi}{4}, -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}, \frac{25\pi}{12}, \frac{31\pi}{12}, \frac{11\pi}{4}$$



d $10\sin \frac{x}{3} = 5\sqrt{3}$

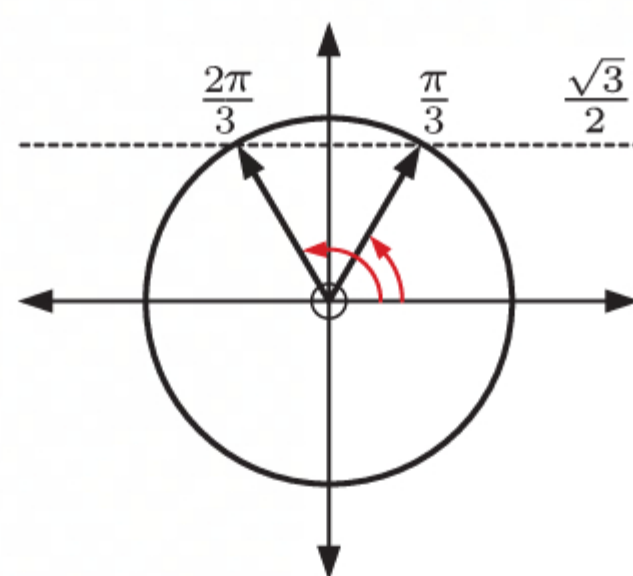
$$\therefore \sin \frac{x}{3} = \frac{\sqrt{3}}{2}$$

Since $-\pi \leq x \leq 3\pi$

$$\therefore -\frac{\pi}{3} \leq \frac{x}{3} \leq \pi$$

So, $\frac{x}{3} = \frac{\pi}{3}$ and $\frac{2\pi}{3}$

$$\therefore x = \pi \text{ and } 2\pi$$



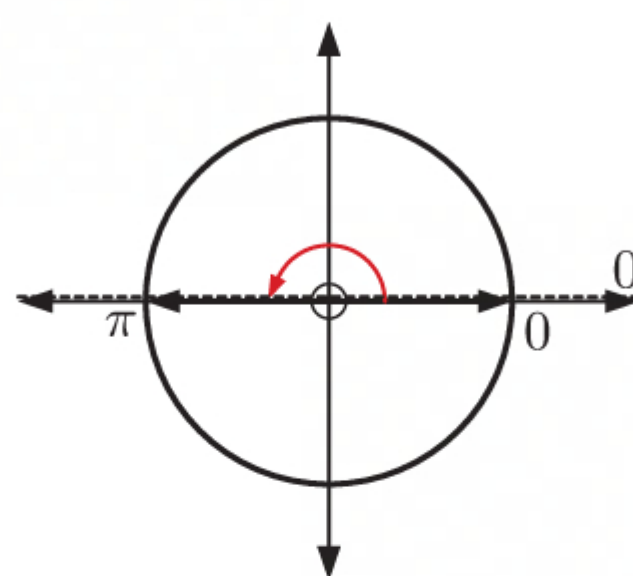
50 a $\sin(x - \frac{\pi}{3}) = 0$

Since $-2\pi \leq x \leq \pi$

$$-\frac{7\pi}{3} \leq x - \frac{\pi}{3} \leq \frac{2\pi}{3}$$

So, $x - \frac{\pi}{3} = -2\pi, -\pi, 0$

$$\therefore x = -\frac{5\pi}{3}, -\frac{2\pi}{3}, \frac{\pi}{3}$$



b $\cos(3x + \frac{\pi}{4}) = -\frac{1}{2}$

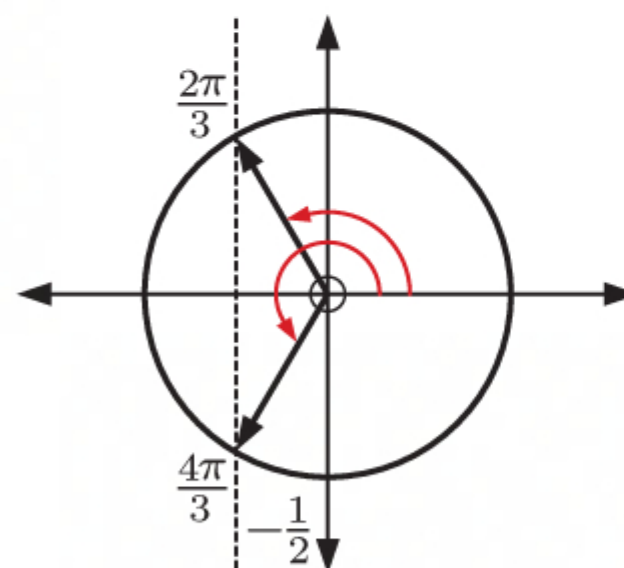
Since $0 \leq x \leq \pi$

$$\therefore \frac{\pi}{4} \leq 3x + \frac{\pi}{4} \leq \frac{13\pi}{4}$$

So, $3x + \frac{\pi}{4} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$$\therefore 3x = \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{29\pi}{12}$$

$$\therefore x = \frac{5\pi}{36}, \frac{13\pi}{36}, \frac{29\pi}{36}$$

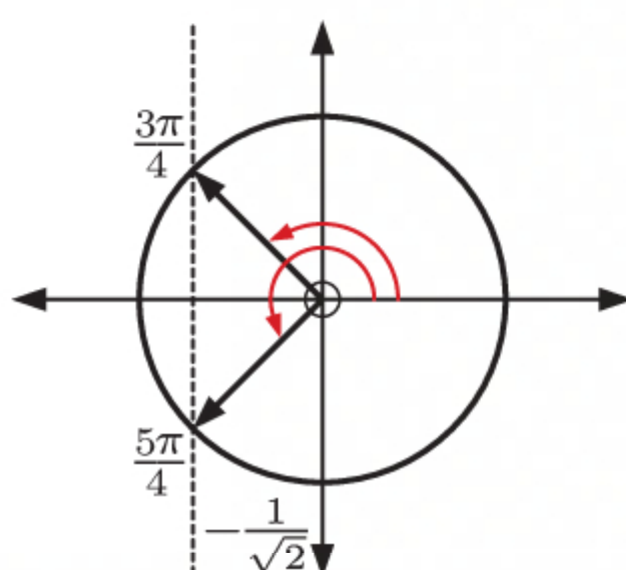


51 a $\sqrt{2} \cos x + 1 = 0$

$$\therefore \sqrt{2} \cos x = -1$$

$$\therefore \cos x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

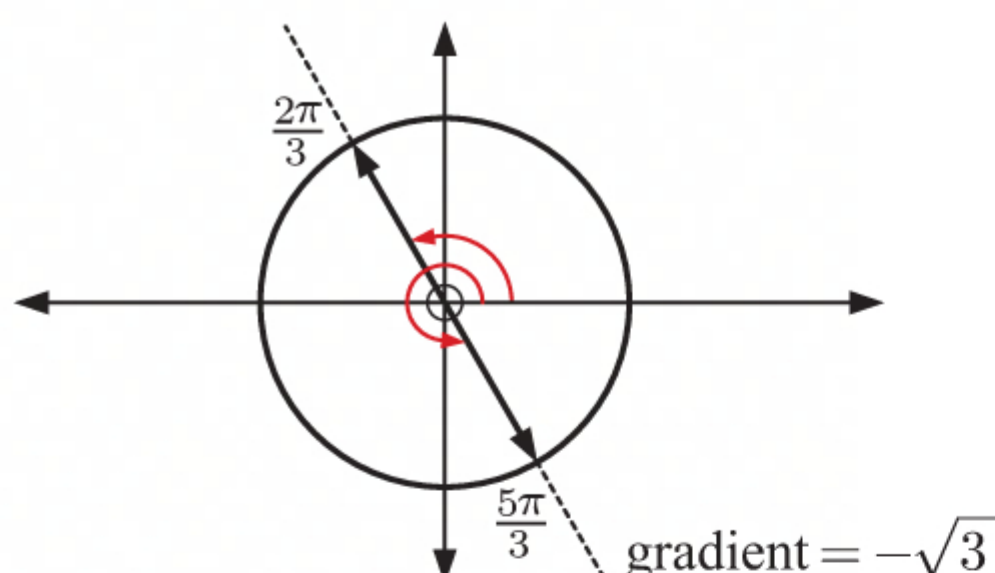


b $\sin x = -\sqrt{3} \cos x$

$$\therefore \frac{\sin x}{\cos x} = -\sqrt{3}$$

$$\therefore \tan x = -\sqrt{3}$$

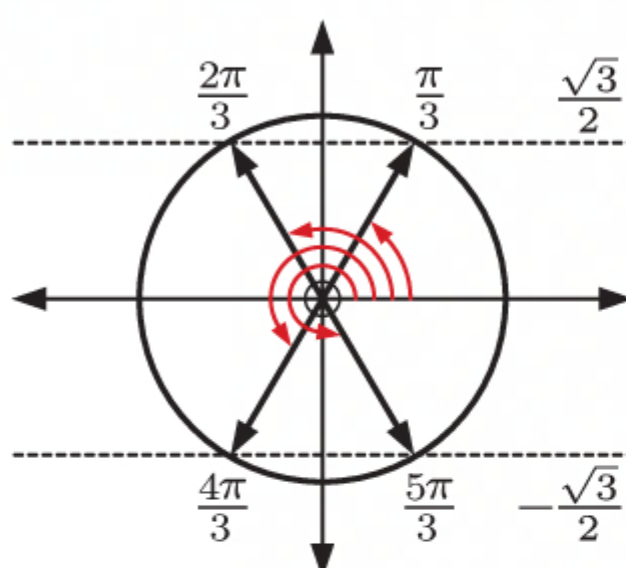
$$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}$$



c $\sin^2 x = \frac{3}{4}$

$$\therefore \sin x = \pm \frac{\sqrt{3}}{2}$$

$$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$



d $\tan^3 2x - 3 \tan 2x = 0$

$$\therefore \tan 2x (\tan^2 2x - 3) = 0$$

$$\therefore \tan 2x = 0 \quad \text{or} \quad \tan^2 2x = 3$$

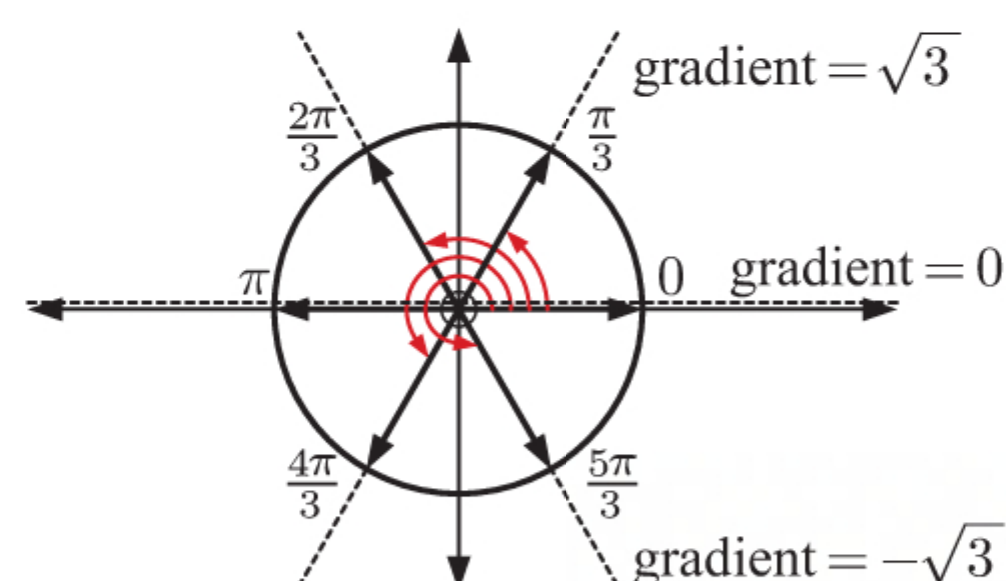
$$\therefore \tan 2x = \pm \sqrt{3}$$

Since $0 \leq x \leq 2\pi$,

$$0 \leq 2x \leq 4\pi$$

So, $2x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}, \frac{8\pi}{3}, 3\pi, \frac{10\pi}{3}, \frac{11\pi}{3}, 4\pi$

$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$$



e $4 \cos^2 x - 3 = 4 \cos x$

$$\therefore 4 \cos^2 x - 4 \cos x - 3 = 0$$

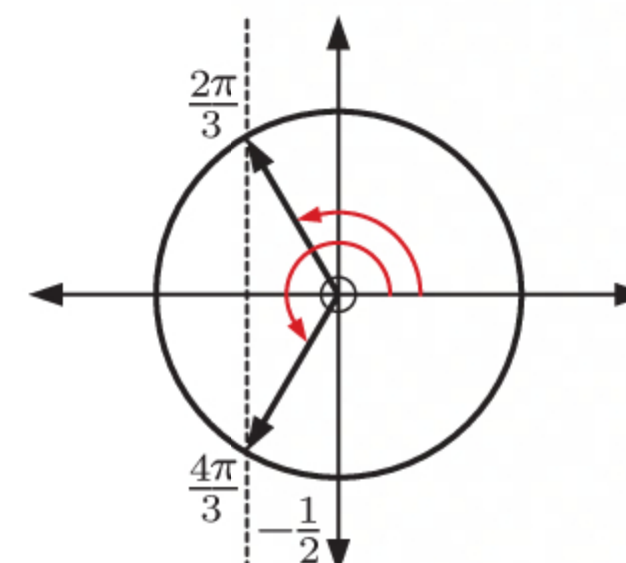
$$\therefore 4 \cos^2 x + 2 \cos x - 6 \cos x - 3 = 0$$

$$\therefore 2 \cos x (2 \cos x + 1) - 3(2 \cos x + 1) = 0$$

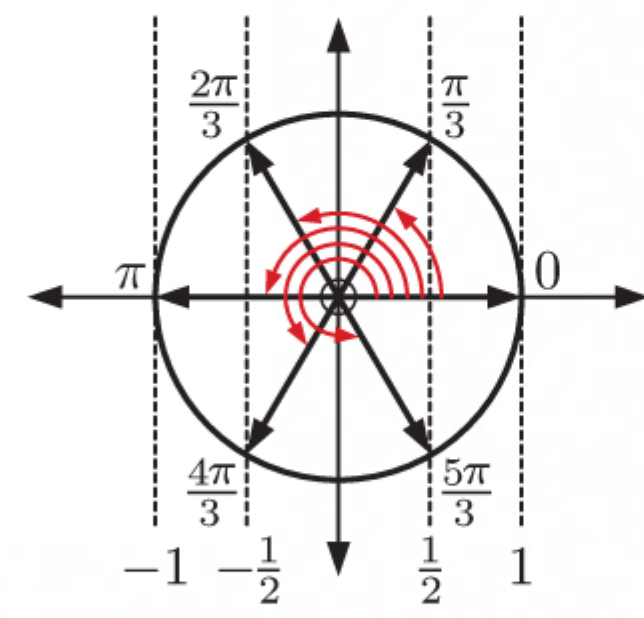
$$\therefore (2 \cos x + 1)(2 \cos x - 3) = 0$$

$$\therefore \cos x = -\frac{1}{2} \quad \{-1 \leq \cos x \leq 1 \text{ for all } x\}$$

$$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}$$



$$\begin{aligned}
 \mathbf{f} \quad & 4 \cos^4 x + 1 = 5 \cos^2 x \\
 & \therefore 4 \cos^4 x - 5 \cos^2 x + 1 = 0 \\
 & \therefore 4 \cos^4 x - 4 \cos^2 x - \cos^2 x + 1 = 0 \\
 & \therefore 4 \cos^2 x (\cos^2 x - 1) - (\cos^2 x - 1) = 0 \\
 & \therefore (\cos^2 x - 1)(4 \cos^2 x - 1) = 0 \\
 & \therefore \cos^2 x = 1 \quad \text{or} \quad \cos^2 x = \frac{1}{4} \\
 & \therefore \cos x = \pm 1 \quad \text{or} \quad \cos x = \pm \frac{1}{2} \\
 & \therefore x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi
 \end{aligned}$$



$$\mathbf{52} \quad P(t) = 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right), \quad 0 \leq t \leq 10$$

$$\begin{aligned}
 \mathbf{a} \quad \mathbf{i} \quad & P(0) = 4500 + 500 \sin\left(\frac{2\pi}{7}(0-3)\right) \\
 & = 4500 + 500 \sin\left(-\frac{6\pi}{7}\right) \\
 & \approx 4283.06 \\
 & \approx 4280 \text{ butterflies}
 \end{aligned}$$

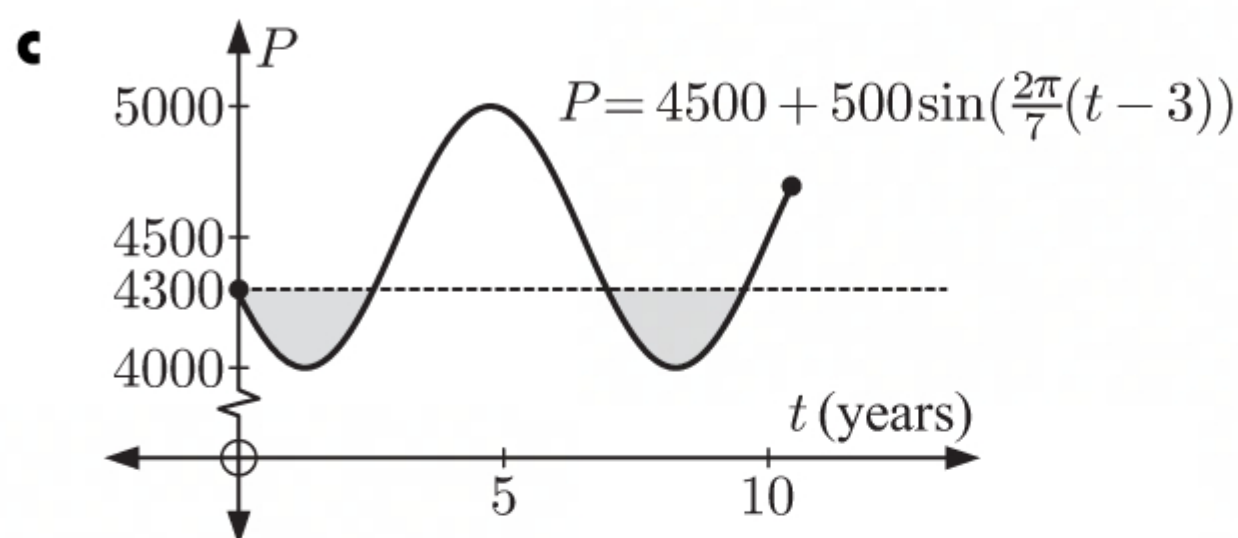
$$\begin{aligned}
 \mathbf{ii} \quad & P(3) = 4500 + 500 \sin\left(\frac{2\pi}{7}(3-3)\right) \\
 & = 4500 + 500 \sin 0 \\
 & = 4500 \text{ butterflies}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{i} \quad & \text{We need to solve } P(t) = 4200, \text{ so} \\
 & 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right) = 4200 \\
 & \text{Using technology, } t \approx 0.217, 2.28, 7.22, 9.28.
 \end{aligned}$$

So, the population is 4200 after about 0.217 years, 2.28 years, 7.22 years, and 9.28 years.

$$\begin{aligned}
 \mathbf{ii} \quad & \text{We need to solve } P(t) = 4900, \text{ so} \\
 & 4500 + 500 \sin\left(\frac{2\pi}{7}(t-3)\right) = 4900 \\
 & \text{Using technology, } t \approx 4.03, 5.47.
 \end{aligned}$$

So, the population is 4900 after about 4.03 years and 5.47 years.



$$\mathbf{53} \quad \mathbf{a} \quad 2 \sin^2 \theta + 3 \sin^2 \theta = 5 \sin^2 \theta$$

$$\begin{aligned}
 \mathbf{b} \quad & \cos x \tan x - 2 \sin x \\
 & = \cos x \times \frac{\sin x}{\cos x} - 2 \sin x \\
 & = \sin x - 2 \sin x \\
 & = -\sin x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{-\cos\left(\frac{\pi}{2} - \theta\right) \sin(-\theta)}{\cos(-\theta) \sin(\pi - \theta)} \\
 & = \frac{-\sin \theta (-\sin \theta)}{\cos \theta \sin \theta} \\
 & = \frac{\sin \theta}{\cos \theta} \\
 & = \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{54} \quad \mathbf{a} \quad & -3 \sin^2 \theta - 3 \cos^2 \theta \\
 & = -3(\sin^2 \theta + \cos^2 \theta) \\
 & = -3(1) \\
 & = -3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \sin \theta \cos^2 \theta + \sin^3 \theta \\
 & = \sin \theta (\cos^2 \theta + \sin^2 \theta) \\
 & = \sin \theta (1) \\
 & = \sin \theta
 \end{aligned}$$

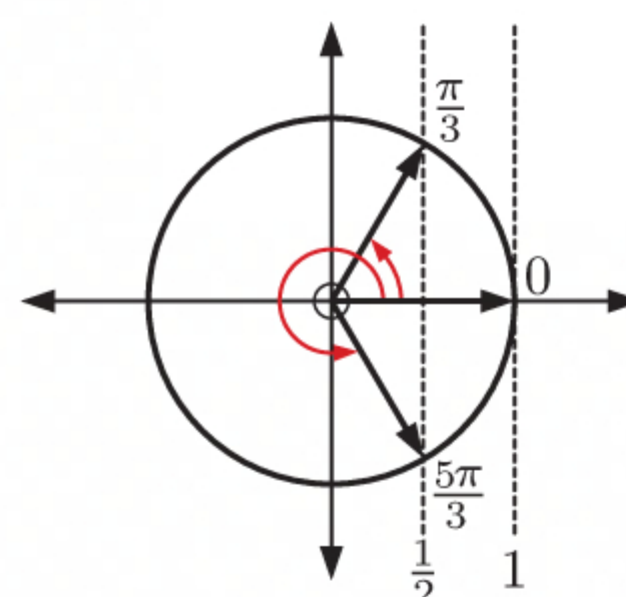
$$\begin{aligned}
 \mathbf{c} \quad & \frac{\sin^2 \theta - 1}{\cos \theta} \\
 & = \frac{-\cos^2 \theta}{\cos \theta} \\
 & = -\cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{55} \quad \mathbf{a} \quad & (\sin^2 x + 3)^2 \\
 & = (\sin^2 x)^2 + 6 \sin^2 x + 9 \\
 & = \sin^4 x + 6 \sin^2 x + 9
 \end{aligned}$$

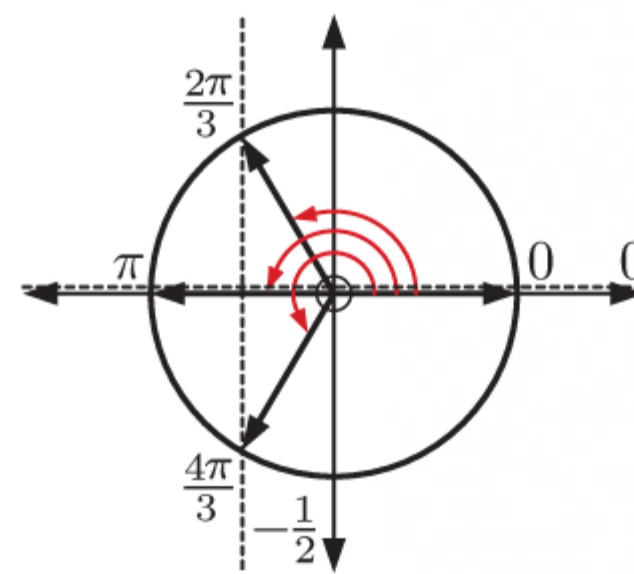
$$\begin{aligned}
 \mathbf{b} \quad & (\tan \alpha + 1)^2 \\
 & = \tan^2 \alpha + 2 \tan \alpha + 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (\sin x - 1)(\sin x + 1) \\
 & = \sin^2 x - 1 \\
 & = -\cos^2 x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{56} \quad \mathbf{a} \quad & 2 \sin^2 x + 3 \cos x = 3, \quad 0 \leq x \leq 2\pi \\
 & \therefore 2(1 - \cos^2 x) + 3 \cos x = 3 \\
 & \therefore 2 - 2 \cos^2 x + 3 \cos x = 3 \\
 & \therefore 2 \cos^2 x - 3 \cos x + 1 = 0 \\
 & \therefore 2 \cos^2 x - 2 \cos x - \cos x + 1 = 0 \\
 & \therefore 2 \cos x (\cos x - 1) - (\cos x - 1) = 0 \\
 & \therefore (\cos x - 1)(2 \cos x - 1) = 0 \\
 & \therefore \cos x = 1 \quad \text{or} \quad \cos x = \frac{1}{2} \\
 & \therefore x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad & \sin 2x + \sin x = 0 \\
 \therefore & 2 \sin x \cos x + \sin x = 0 \quad \{\text{double angle formula}\} \\
 \therefore & \sin x(2 \cos x + 1) = 0 \\
 \therefore & \sin x = 0 \text{ or } \cos x = -\frac{1}{2} \\
 \therefore & x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi
 \end{aligned}$$



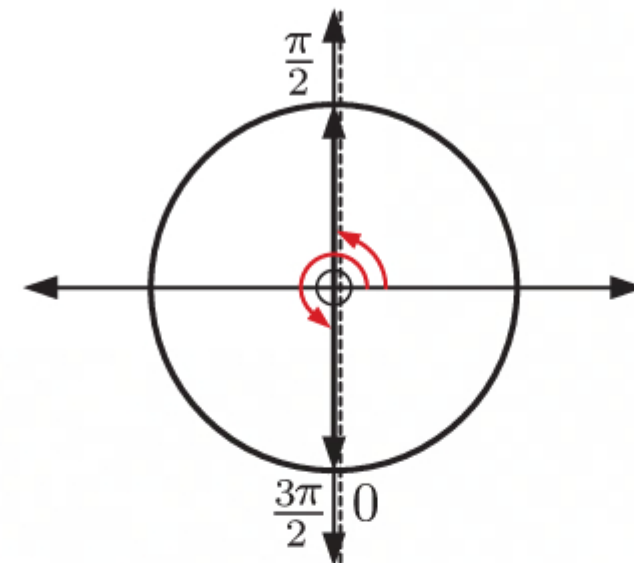
$$\begin{aligned}
 \text{c} \quad & \sin^2 x - \cos^2 x = 0, \quad 0 \leq x \leq 2\pi \\
 \therefore & \cos^2 x - \sin^2 x = 0 \\
 \therefore & \cos 2x = 0
 \end{aligned}$$

$$\text{Since } 0 \leq x \leq 2\pi,$$

$$0 \leq 2x \leq 4\pi$$

$$\text{So, } 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$



$$\text{57 a } 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$$

$$\begin{aligned}
 \text{b } 2 \cos^2 \alpha - 7 \cos \alpha \sin \alpha - 4 \sin^2 \alpha &= 2 \cos^2 \alpha + \cos \alpha \sin \alpha - 8 \cos \alpha \sin \alpha - 4 \sin^2 \alpha \\
 &= \cos \alpha(2 \cos \alpha + \sin \alpha) - 4 \sin \alpha(2 \cos \alpha + \sin \alpha) \\
 &= (2 \cos \alpha + \sin \alpha)(\cos \alpha - 4 \sin \alpha)
 \end{aligned}$$

$$\begin{aligned}
 \text{c } 2 \cos^2 \theta - 3 \sin \theta &= 2(1 - \sin^2 \theta) - 3 \sin \theta \\
 &= 2 - 2 \sin^2 \theta - 3 \sin \theta \\
 &= -2 \sin^2 \theta - 4 \sin \theta + \sin \theta + 2 \\
 &= -2 \sin \theta(\sin \theta + 2) + (\sin \theta + 2) \\
 &= (\sin \theta + 2)(1 - 2 \sin \theta)
 \end{aligned}$$

$$\text{58 a } \operatorname{cosec} x = \sqrt{2}, \quad 0 \leq x \leq 2\pi$$

$$\therefore \frac{1}{\sin x} = \sqrt{2}$$

$$\therefore \sin x = \frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\text{b The domain is } 0 \leq x \leq 2\pi$$

$$\therefore 0 \leq 2x \leq 4\pi$$

$$\sec(\pi - 2x) + 2 = 0$$

$$\therefore \sec(\pi - 2x) = -2$$

$$\therefore \frac{1}{\cos(\pi - 2x)} = -2$$

$$\therefore \cos(\pi - 2x) = -\frac{1}{2}$$

$$\therefore -\cos 2x = -\frac{1}{2} \quad \{\cos(\pi - \theta) = -\cos \theta\}$$

$$\therefore \cos 2x = \frac{1}{2}$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \text{ or } \frac{11\pi}{3}$$

$$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$$

$$\text{c } \cos^2 x + 5 \sin^2 x = \sec^2 x, \quad 0 \leq x \leq 2\pi$$

$$\therefore \cos^2 x + 5(1 - \cos^2 x) = \frac{1}{\cos^2 x}$$

$$\therefore \cos^2 x + 5 - 5 \cos^2 x = \frac{1}{\cos^2 x}$$

$$\therefore 5 - 4 \cos^2 x = \frac{1}{\cos^2 x}$$

$$\therefore 5 \cos^2 x - 4 \cos^4 x = 1$$

$$\therefore 4 \cos^4 x - 5 \cos^2 x + 1 = 0$$

$$\therefore (4 \cos^2 x - 1)(\cos^2 x - 1) = 0$$

$$\therefore 4 \cos^2 x = 1$$

$$\therefore \cos^2 x = \frac{1}{4}$$

$$\therefore \cos x = \pm \frac{1}{2}$$

$$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}$$

$$\therefore x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \text{ or } 2\pi$$

$$\text{or } \cos^2 x = 1$$

$$\therefore \cos x = \pm 1$$

$$\therefore x = 0, \pi, \text{ or } 2\pi$$

$$\begin{aligned}
 \text{59 a } \cot x \sec x &= \frac{\cos x}{\sin x} \times \frac{1}{\cos x} \\
 &= \frac{1}{\sin x} \\
 &= \operatorname{cosec} x
 \end{aligned}$$

$$\begin{aligned}
 \text{c } (2 \tan A + 3)^2 + (3 \tan A + 2)^2 \\
 &= 4 \tan^2 A + 12 \tan A + 9 + 9 \tan^2 A + 12 \tan A + 4 \\
 &= 13 \tan^2 A + 24 \tan A + 13 \\
 &= 13(\tan^2 A + 1) + 24 \tan A \\
 &= 13 \sec^2 A + 24 \tan A
 \end{aligned}$$

$$\begin{aligned}
 \text{60 a } \frac{1}{\tan \theta - \sec \theta} &= \frac{1}{\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta}} \\
 &= \frac{1}{\frac{\sin \theta - 1}{\cos \theta}} \\
 &= \left(\frac{\cos \theta}{\sin \theta - 1} \right) \times \left(\frac{\sin \theta + 1}{\sin \theta + 1} \right) \\
 &= \frac{\cos \theta (\sin \theta + 1)}{\sin^2 \theta - 1} \\
 &= \frac{\cos \theta (\sin \theta + 1)}{-\cos^2 \theta} \\
 &= -\left(\frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} \right) \\
 &= -(\tan \theta + \sec \theta)
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \frac{\sec x}{\sec x - 1} + \frac{\sec x}{\sec x + 1} &= \frac{\sec x(\sec x + 1) + \sec x(\sec x - 1)}{(\sec x - 1)(\sec x + 1)} \\
 &= \frac{\sec^2 x + \sec x + \sec^2 x - \sec x}{\sec^2 x - 1} \\
 &= \frac{2 \sec^2 x}{\tan^2 x} \\
 &= \frac{2 \times \frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} \\
 &= \frac{2}{\sin^2 x} \\
 &= 2 \operatorname{cosec}^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{61 a } \cot^2 \beta - \operatorname{cosec}^2 \beta &= \frac{\cos^2 \beta}{\sin^2 \beta} - \frac{1}{\sin^2 \beta} \\
 &= \frac{1}{\sin^2 \beta} (\cos^2 \beta - 1) \\
 &= -\frac{1}{\sin^2 \beta} (1 - \cos^2 \beta) \\
 &= -\frac{1}{\sin^2 \beta} \times \sin^2 \beta \\
 &= -1
 \end{aligned}$$

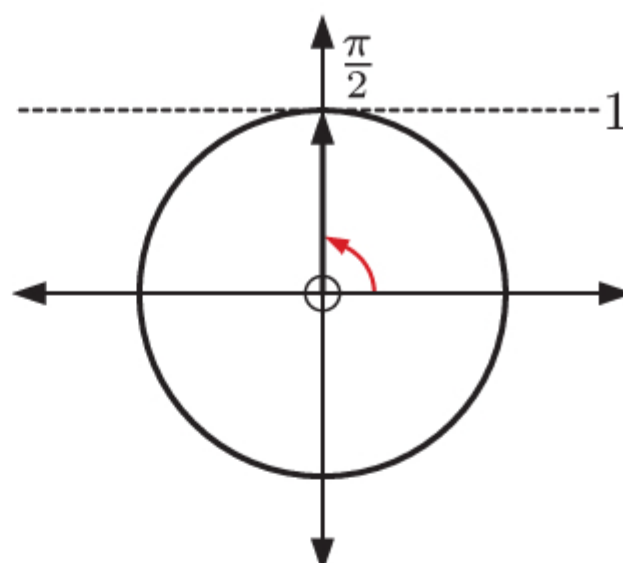
$$\begin{aligned}
 \text{62 } \cot \theta + \tan \theta &= 2, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
 \therefore \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} &= 2 \\
 \therefore \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} &= 2 \\
 \therefore 1 &= 2 \sin \theta \cos \theta \\
 \therefore 1 &= \sin 2\theta \\
 \therefore 2\theta &= \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z} \\
 \therefore \theta &= \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z} \\
 \therefore \theta &= \frac{\pi}{4} \quad \left\{ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right\}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{\operatorname{cosec}(\pi - \theta) \cos(\frac{\pi}{2} - \theta)}{\cot(\pi - \theta)} &= \frac{\frac{1}{\sin(\pi - \theta)} \times \cos(\frac{\pi}{2} - \theta)}{\frac{\cos(\pi - \theta)}{\sin(\pi - \theta)}} \\
 &= \frac{\frac{1}{\sin \theta} \times \sin \theta}{\frac{-\cos \theta}{\sin \theta}} \\
 &= -\frac{\sin \theta}{\cos \theta} = -\tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \frac{\cot^2 \theta (1 - \cos^2 \theta)}{1 + \cot^2 \theta} &= \frac{\frac{\cos^2 \theta}{\sin^2 \theta} \times \sin^2 \theta}{\operatorname{cosec}^2 \theta} \\
 &= \frac{\cos^2 \theta}{\frac{1}{\sin^2 \theta}} \\
 &= \sin^2 \theta \cos^2 \theta \\
 &= \frac{1}{4} (2 \sin \theta \cos \theta)^2 \\
 &= \frac{1}{4} \sin^2(2\theta)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (1 - \sin \theta)(1 + \operatorname{cosec} \theta) &= (1 - \sin \theta) \left(1 + \frac{1}{\sin \theta} \right) \\
 &= 1 + \frac{1}{\sin \theta} - \sin \theta - 1 \\
 &= \frac{1}{\sin \theta} - \frac{\sin^2 \theta}{\sin \theta} \\
 &= \frac{1 - \sin^2 \theta}{\sin \theta} \\
 &= \frac{\cos^2 \theta}{\sin \theta} \\
 &= \frac{\cos \theta}{\sin \theta} \times \cos \theta \\
 &= \cot \theta \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b } 3 \sec^2 \alpha + 7 \sec \alpha - 6 \\
 &= 3 \sec^2 \alpha + 9 \sec \alpha - 2 \sec \alpha - 6 \\
 &= 3 \sec \alpha (\sec \alpha + 3) - 2(\sec \alpha + 3) \\
 &= (3 \sec \alpha - 2)(\sec \alpha + 3)
 \end{aligned}$$



$$\begin{aligned} 63 \quad \arcsin\left(-\frac{1}{2}\right) + \arctan 1 + \arccos\left(-\frac{1}{2}\right) &= -\frac{\pi}{6} + \frac{\pi}{4} + \frac{2\pi}{3} \\ &= \frac{3\pi}{4} \end{aligned}$$

$$64 \quad \mathbf{a} \quad \text{The range of } y = \arcsin(2x - 3) \text{ is } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

$-\frac{\pi}{6}$ is within the range.

$$\therefore 2x - 3 = \sin\left(-\frac{\pi}{6}\right)$$

$$\therefore 2x - 3 = -\frac{1}{2}$$

$$\therefore 2x = \frac{5}{2}$$

$$\therefore x = \frac{5}{4}$$

$$\mathbf{c} \quad \text{The range of } y = \arctan(2 - x) \text{ is } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

$\frac{\pi}{4}$ is within the range.

$$\therefore 2 - x = \tan \frac{\pi}{4}$$

$$\therefore 2 - x = 1$$

$$\therefore x = 1$$

$$65 \quad \alpha \text{ is obtuse and } \sin \alpha = \frac{2}{3} \quad \therefore \cos \alpha \text{ is negative.}$$

$$\mathbf{a} \quad \sin^2 \alpha + \cos^2 \alpha = 1$$

$$\therefore \left(\frac{2}{3}\right)^2 + \cos^2 \alpha = 1$$

$$\therefore \frac{4}{9} + \cos^2 \alpha = 1$$

$$\therefore \cos^2 \alpha = \frac{5}{9}$$

$$\therefore \cos \alpha = -\frac{\sqrt{5}}{3} \quad \{\cos \alpha < 0\}$$

$$\mathbf{b} \quad \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$= \frac{5}{9} - \frac{4}{9} \quad \{\text{using } \mathbf{a}\}$$

$$= \frac{1}{9}$$

$$66 \quad \alpha \text{ is acute and } \cos 2\alpha = \frac{5}{13} \quad \therefore \cos \alpha \text{ and } \sin \alpha \text{ are positive.}$$

$$\mathbf{a} \quad \cos 2\alpha = 1 - 2\sin^2 \alpha$$

$$\therefore \frac{5}{13} = 1 - 2\sin^2 \alpha$$

$$\therefore -2\sin^2 \alpha = -\frac{8}{13}$$

$$\therefore \sin^2 \alpha = \frac{4}{13}$$

$$\therefore \sin \alpha = \frac{2}{\sqrt{13}} \quad \{\sin \alpha > 0\}$$

$$\mathbf{b} \quad \sin^2 \alpha + \cos^2 \alpha = 1$$

$$\therefore \frac{4}{13} + \cos^2 \alpha = 1 \quad \{\text{using } \mathbf{a}\}$$

$$\therefore \cos^2 \alpha = \frac{9}{13}$$

$$\therefore \cos \alpha = \frac{3}{\sqrt{13}} \quad \{\cos \alpha > 0\}$$

$$\mathbf{c} \quad \tan \alpha = \frac{\sin \alpha}{\cos \alpha}$$

$$= \frac{\frac{2}{\sqrt{13}}}{\frac{3}{\sqrt{13}}} \quad \{\text{using } \mathbf{a} \text{ and } \mathbf{b}\}$$

$$= \frac{2}{3}$$

$$67 \quad \cos 2x = \frac{5}{8}$$

$$\text{Now } \cos 2x = 1 - 2\sin^2 x$$

$$\therefore \frac{5}{8} = 1 - 2\sin^2 x$$

$$\therefore -2\sin^2 x = -\frac{3}{8}$$

$$\therefore \sin^2 x = \frac{3}{16}$$

$$\therefore \sin x = \pm \frac{\sqrt{3}}{4}$$

$$68 \quad \mathbf{a} \quad \sin 2x = \sin x, \quad -\pi \leq x \leq \pi$$

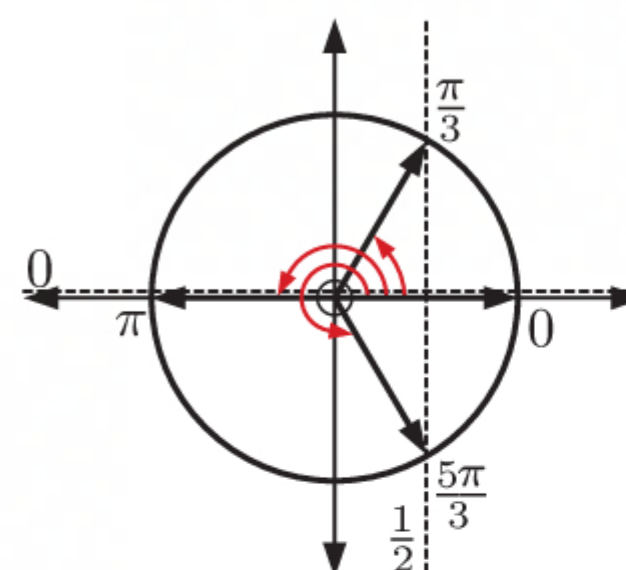
$$\therefore 2\sin x \cos x = \sin x$$

$$\therefore 2\sin x \cos x - \sin x = 0$$

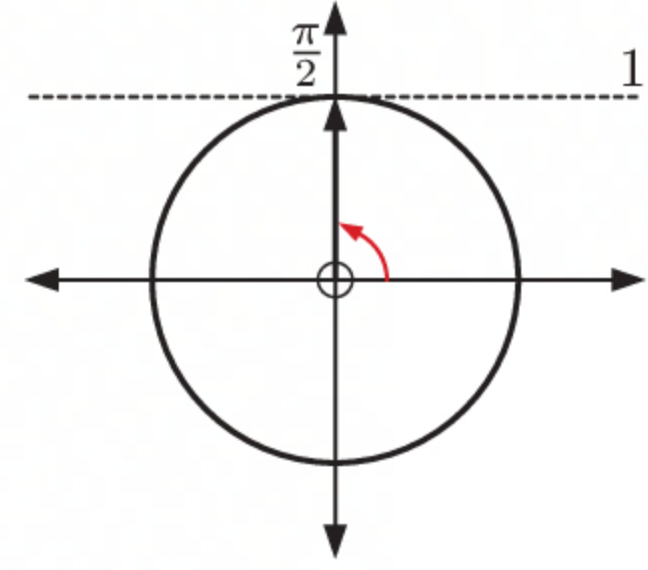
$$\therefore \sin x(2\cos x - 1) = 0$$

$$\therefore \sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

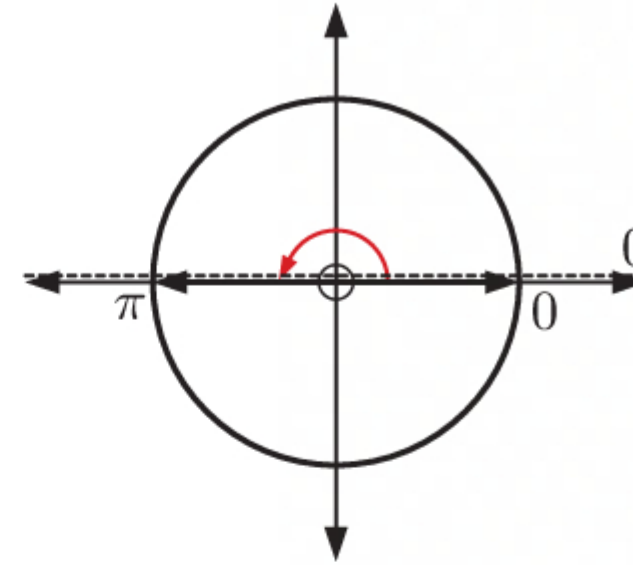
$$\therefore x = -\pi, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \pi$$



$$\begin{aligned}
 \text{b} \quad & -3 \cos 2x - 14 \sin x + 11 = 0, \quad -\pi \leq x \leq \pi \\
 \therefore & -3(1 - 2 \sin^2 x) - 14 \sin x + 11 = 0 \\
 \therefore & -3 + 6 \sin^2 x - 14 \sin x + 11 = 0 \\
 \therefore & 6 \sin^2 x - 14 \sin x + 8 = 0 \\
 \therefore & 3 \sin^2 x - 7 \sin x + 4 = 0 \\
 \therefore & 3 \sin^2 x - 3 \sin x - 4 \sin x + 4 = 0 \\
 \therefore & 3 \sin x(\sin x - 1) - 4(\sin x - 1) = 0 \\
 \therefore & (\sin x - 1)(3 \sin x - 4) = 0 \\
 \therefore & \sin x = 1 \quad \{-1 \leq \sin x \leq 1 \text{ for all } x\} \\
 \therefore & x = \frac{\pi}{2}
 \end{aligned}$$



$$\begin{aligned}
 \text{c} \quad & \sin x + \cos x = 1, \quad -\pi \leq x \leq \pi \\
 \therefore & (\sin x + \cos x)^2 = 1^2 \\
 \therefore & \sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 \\
 \therefore & 2 \sin x \cos x + 1 = 1 \\
 \therefore & 2 \sin x \cos x = 0 \\
 \therefore & \sin 2x = 0
 \end{aligned}$$



Since $-\pi \leq x \leq \pi$,
 $-2\pi \leq 2x \leq 2\pi$

So, $2x = -2\pi, -\pi, 0, \pi, 2\pi$
 $\therefore x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

Since we squared both sides of the equation, we must check that all the solutions satisfy the original equation.

Check: If $x = -\pi$, $\sin(-\pi) + \cos(-\pi) = 0 + (-1) = -1$ ✗
 If $x = -\frac{\pi}{2}$, $\sin(-\frac{\pi}{2}) + \cos(-\frac{\pi}{2}) = -1 + 0 = -1$ ✗
 If $x = 0$, $\sin 0 + \cos 0 = 0 + 1 = 1$ ✓
 If $x = \frac{\pi}{2}$, $\sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1 + 0 = 1$ ✓
 If $x = \pi$, $\sin \pi + \cos \pi = 0 + (-1) = -1$ ✗

So the solutions are $x = 0, \frac{\pi}{2}$.

$$\begin{aligned}
 \text{69} \quad & \tan 2A = \frac{3}{2} \\
 \therefore & \frac{2 \tan A}{1 - \tan^2 A} = \frac{3}{2} \\
 \therefore & 4 \tan A = 3 - 3 \tan^2 A \\
 \therefore & 3 \tan^2 A + 4 \tan A - 3 = 0 \\
 \therefore & \tan A = \frac{-4 \pm \sqrt{4^2 - 4(3)(-3)}}{2(3)} \\
 & = \frac{-4 \pm \sqrt{52}}{6} \\
 & = \frac{-4 \pm 2\sqrt{13}}{6} \\
 & = \frac{-2 \pm \sqrt{13}}{3}
 \end{aligned}$$

But A is acute $\therefore \tan A > 0$

$$\therefore \tan A = \frac{-2 + \sqrt{13}}{3}$$

$$\begin{aligned}
 \text{70} \quad & \cos\left(\frac{3\pi}{2} - \phi\right) \tan(\phi + \pi) = \left(\cos \frac{3\pi}{2} \cos \phi + \sin \frac{3\pi}{2} \sin \phi\right) \left(\frac{\tan \phi + \tan \pi}{1 - \tan \phi \tan \pi}\right) \\
 & = (-\sin \phi) \left(\frac{\tan \phi}{1}\right) \\
 & = -\sin \phi \tan \phi
 \end{aligned}$$

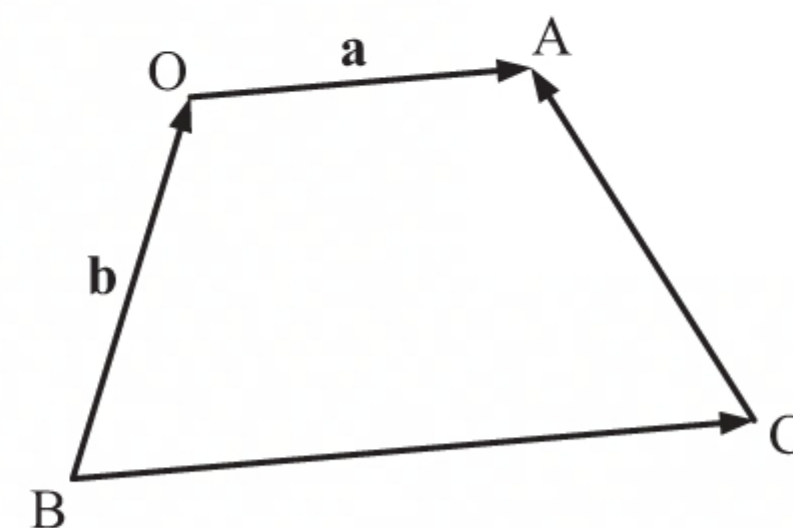
$$\begin{aligned}
 71 \quad \mathbf{a} \quad \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\
 &= \frac{2 \times 2}{1 - 2^2} \\
 &= -\frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \tan 3\theta &= \tan(2\theta + \theta) \\
 &= \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} \\
 &= \frac{-\frac{4}{3} + 2}{1 - (-\frac{4}{3}) \times 2} \quad \{\text{using } \mathbf{a}\} \\
 &= \frac{2}{11}
 \end{aligned}$$

$$\begin{aligned}
 72 \quad \sin x &= 2 \sin(x - \frac{\pi}{6}) \\
 \therefore \sin x &= 2(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}) \\
 &= 2 \sin x (\frac{\sqrt{3}}{2}) - 2 \cos x (\frac{1}{2}) \\
 \therefore \sin x(1 - \sqrt{3}) &= -\cos x \\
 \therefore \frac{\sin x}{\cos x} &= -\frac{1}{1 - \sqrt{3}} \\
 \therefore \tan x &= \left(\frac{1}{\sqrt{3} - 1} \right) \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \\
 &= \frac{\sqrt{3} + 1}{3 - 1} \\
 &= \frac{\sqrt{3} + 1}{2}
 \end{aligned}$$

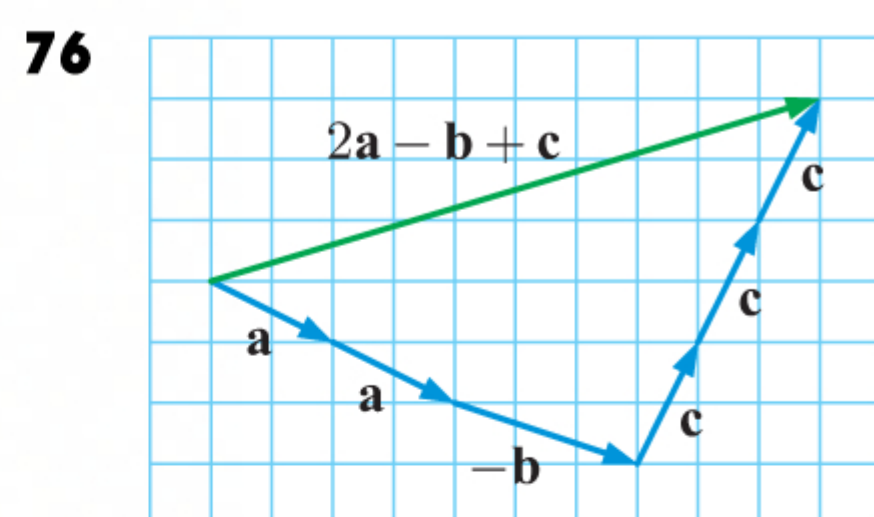
$$\begin{aligned}
 73 \quad \tan(\arctan \frac{1}{4} + \arctan \frac{3}{5}) &= \frac{\tan(\arctan \frac{1}{4}) + \tan(\arctan \frac{3}{5})}{1 - \tan(\arctan \frac{1}{4}) \tan(\arctan \frac{3}{5})} \\
 &= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \times \frac{3}{5}} \\
 &= \frac{\frac{5+12}{20}}{1 - \frac{3}{20}} \\
 &= \frac{\frac{17}{20}}{\frac{17}{20}} \\
 &= 1 \\
 \therefore \arctan \frac{1}{4} + \arctan \frac{3}{5} &= \arctan 1 \\
 &= \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 74 \quad \mathbf{a} \quad \vec{BC} &= 2\mathbf{a} \\
 \mathbf{c} \quad \vec{BA} &= \mathbf{b} + \mathbf{a} \\
 \mathbf{e} \quad \vec{AC} &= -\mathbf{a} - \mathbf{b} + 2\mathbf{a} \\
 &= \mathbf{a} - \mathbf{b} \\
 \mathbf{b} \quad \vec{CB} &= -2\mathbf{a} \\
 \mathbf{d} \quad \vec{OC} &= -\mathbf{b} + 2\mathbf{a} \\
 \mathbf{f} \quad \vec{CA} &= -(\mathbf{a} - \mathbf{b}) \\
 &= \mathbf{b} - \mathbf{a}
 \end{aligned}$$



$$\begin{aligned}
 75 \quad \mathbf{a} \quad \left| -\frac{1}{3}\mathbf{i} + k\mathbf{j} \right| &= 1 \quad \text{when} \quad \sqrt{\left(-\frac{1}{3}\right)^2 + k^2} = 1 \\
 \therefore \frac{1}{9} + k^2 &= 1 \\
 \therefore k^2 &= \frac{8}{9} \\
 \therefore k &= \pm \sqrt{\frac{8}{9}} \\
 &= \pm \frac{2\sqrt{2}}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \left| \begin{pmatrix} 3 \\ k \\ k+2 \end{pmatrix} \right| &= \sqrt{61} \quad \text{when} \\
 \sqrt{3^2 + k^2 + (k+2)^2} &= \sqrt{61} \\
 \therefore 9 + k^2 + k^2 + 4k + 4 &= 61 \\
 \therefore 2k^2 + 4k - 48 &= 0 \\
 \therefore k^2 + 2k - 24 &= 0 \\
 \therefore (k+6)(k-4) &= 0 \\
 \therefore k &= -6 \text{ or } 4
 \end{aligned}$$



$$\begin{aligned}
 2\mathbf{a} - \mathbf{b} + 3\mathbf{c} &= 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} 4 + 3 + 3 \\ -2 - 1 + 6 \end{pmatrix} \\
 &= \begin{pmatrix} 10 \\ 3 \end{pmatrix} \quad \text{which agrees with the diagram.}
 \end{aligned}$$

$$77 \quad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\begin{aligned}
 \mathbf{a} \quad \mathbf{c} - \mathbf{a} &= \begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} 4 - 1 \\ -1 - 2 \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ -3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{1}{2}\mathbf{c} + 3\mathbf{a} &= \frac{1}{2} \begin{pmatrix} 4 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2}(4) + 3(1) \\ \frac{1}{2}(-1) + 3(2) \end{pmatrix} \\
 &= \begin{pmatrix} 5 \\ \frac{11}{2} \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \mathbf{b} - 2\mathbf{c} - \mathbf{a} &= \begin{pmatrix} -2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 - 2(4) - 1 \\ 2 - 2(-1) - 2 \end{pmatrix} \\ &= \begin{pmatrix} -11 \\ 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad \mathbf{c} - 3\mathbf{a} + 2\mathbf{b} &= \begin{pmatrix} 4 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 - 3(1) + 2(-2) \\ -1 - 3(2) + 2(2) \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -3 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore |\mathbf{c} - 3\mathbf{a} + 2\mathbf{b}| &= \sqrt{(-3)^2 + (-3)^2} \\ &= \sqrt{18} = 3\sqrt{2} \text{ units}\end{aligned}$$

78 $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{k}$

$$\begin{aligned}\mathbf{a} \quad \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}) &= \frac{1}{2}(\mathbf{i} + \mathbf{j} + \mathbf{j} + \mathbf{k} + \mathbf{i} + \mathbf{k}) \\ &= \frac{1}{2}(2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k}\end{aligned}$$

$$\mathbf{b} \quad -5\mathbf{c} = -5\mathbf{i} - 5\mathbf{k} = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{aligned}\therefore |-5\mathbf{c}| &= \sqrt{(-5)^2 + 0^2 + (-5)^2} \\ &= \sqrt{50} = 5\sqrt{2} \text{ units}\end{aligned}$$

$$\mathbf{c} \quad \mathbf{b} = \mathbf{j} + \mathbf{k} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore |\mathbf{b}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \text{ units}$$

$$\begin{aligned}\text{So, } \frac{1}{|\mathbf{b}|}\mathbf{b} &= \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k}) \\ &= \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad 2\mathbf{a} - 3\mathbf{b} - \mathbf{c} &= 2(\mathbf{i} + \mathbf{j}) - 3(\mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{k}) \\ &= 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{j} - 3\mathbf{k} - \mathbf{i} - \mathbf{k} \\ &= \mathbf{i} - \mathbf{j} - 4\mathbf{k} \\ &= \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore |2\mathbf{a} - 3\mathbf{b} - \mathbf{c}| &= \sqrt{1^2 + (-1)^2 + (-4)^2} \\ &= \sqrt{18} = 3\sqrt{2} \text{ units}\end{aligned}$$

79 $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

a $2\mathbf{a} - \mathbf{x} = 4\mathbf{b}$

$$\therefore \mathbf{x} = 2\mathbf{a} - 4\mathbf{b}$$

$$\begin{aligned}&= 2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - 4\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2(3) - 4(1) \\ 2(-1) - 4(5) \\ 2(2) - 4(-1) \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -22 \\ 8 \end{pmatrix}\end{aligned}$$

b $3\mathbf{a} + 2\mathbf{x} = \mathbf{b}$

$$\therefore 2\mathbf{x} = \mathbf{b} - 3\mathbf{a}$$

$$\begin{aligned}\therefore \mathbf{x} &= \frac{1}{2}\mathbf{b} - \frac{3}{2}\mathbf{a} \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(1) - \frac{3}{2}(3) \\ \frac{1}{2}(5) - \frac{3}{2}(-1) \\ \frac{1}{2}(-1) - \frac{3}{2}(2) \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 4 \\ -\frac{7}{2} \end{pmatrix}\end{aligned}$$

80 $\overrightarrow{\text{OC}} = \begin{pmatrix} 5 \\ 22 \end{pmatrix}$

$$\text{So, } \begin{pmatrix} 5 \\ 22 \end{pmatrix} = r\begin{pmatrix} 3 \\ -6 \end{pmatrix} + s\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 3r + 7s \\ -6r + 2s \end{pmatrix}$$

$$\therefore 3r + 7s = 5 \quad \dots (1)$$

$$-6r + 2s = 22 \quad \dots (2)$$

$$\therefore 6r + 14s = 10 \quad \{(1) \times 2\}$$

$$\begin{array}{r} -6r + 2s = 22 \quad \{(2)\} \\ \hline \end{array}$$

$$\text{Adding, } 16s = 32$$

$$\therefore s = 2$$

$$\text{Substituting into (1), } 3r + 7(2) = 5$$

$$\therefore 3r = -9$$

$$\therefore r = -3$$

$$\text{So, } r = -3, s = 2.$$

$$81 \quad \mathbf{a} \quad M \text{ is } \left(\frac{1+3}{2}, \frac{-3+1}{2}, \frac{2+0}{2} \right)$$

$$\therefore M \text{ is } (2, -1, 1)$$

$$\mathbf{b} \quad \vec{AB} = \begin{pmatrix} 1 - (-2) \\ -3 - 2 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 3 \end{pmatrix}$$

$$\vec{AM} = \begin{pmatrix} 2 - (-2) \\ -1 - 2 \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

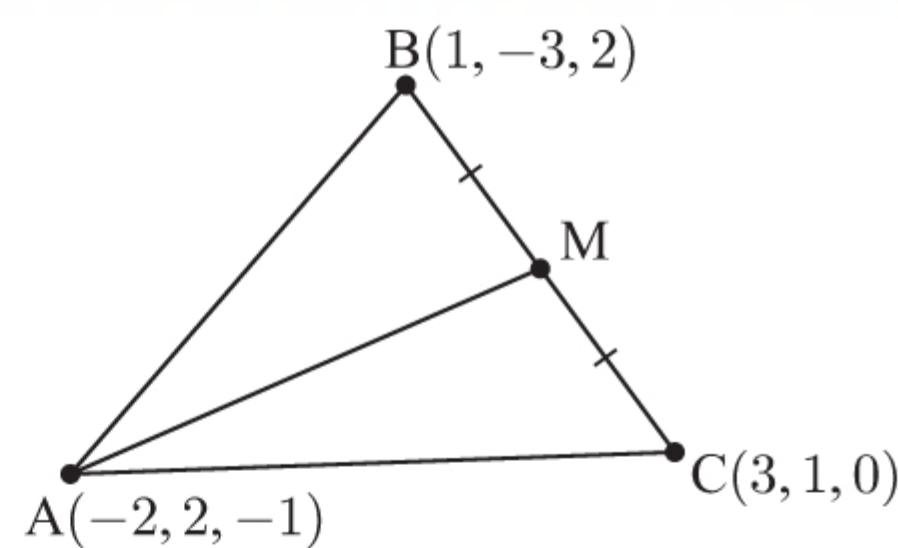
$$\vec{AC} = \begin{pmatrix} 3 - (-2) \\ 1 - 2 \\ 0 - (-1) \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{c} \quad \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AC} = \frac{1}{2} \begin{pmatrix} 3 \\ -5 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \quad \{\text{using } \mathbf{b}\}$$

$$= \begin{pmatrix} \frac{1}{2}(3) + \frac{1}{2}(5) \\ \frac{1}{2}(-5) + \frac{1}{2}(-1) \\ \frac{1}{2}(3) + \frac{1}{2}(1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

$$= \vec{AM} \quad \{\text{from } \mathbf{b}\}$$



$$82 \quad \mathbf{a} \quad \begin{pmatrix} -3 \\ 2 \end{pmatrix} \text{ has length } \sqrt{(-3)^2 + 2^2} = \sqrt{13} \text{ units}$$

$$\therefore \text{ the unit vector in the same direction is } \frac{1}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\therefore \text{ the vector of length 4 units in the same direction is } \mathbf{v} = \frac{4}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{12}{\sqrt{13}} \\ \frac{8}{\sqrt{13}} \end{pmatrix}$$

$$\mathbf{b} \quad \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix} \text{ has length } \sqrt{(-1)^2 + 4^2 + 7^2} = \sqrt{66} \text{ units}$$

$$\therefore \text{ the unit vector in the opposite direction is } -\frac{1}{\sqrt{66}} \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix} = \frac{1}{\sqrt{66}} \begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix}$$

$$\therefore \text{ the vector of length 3 units in the opposite direction is } \mathbf{v} = \frac{3}{\sqrt{66}} \begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix} = \frac{\sqrt{66}}{22} \begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{66}}{22} \\ -\frac{2\sqrt{66}}{11} \\ -\frac{7\sqrt{66}}{22} \end{pmatrix}$$

$$83 \quad \mathbf{i} + 8\mathbf{j} - 4\mathbf{k} = \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix} \text{ has length } \sqrt{1^2 + 8^2 + (-4)^2} = 9 \text{ units}$$

$$\therefore \text{ the vector of length 6 units in the same direction is } \frac{6}{9} \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} \\ \frac{16}{3} \\ -\frac{8}{3} \end{pmatrix}$$

$$\text{Now } \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{16}{3} \\ -\frac{8}{3} \end{pmatrix} = \begin{pmatrix} 2 + \frac{2}{3} \\ -1 + \frac{16}{3} \\ 3 + (-\frac{8}{3}) \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{13}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\therefore \text{ the point 6 units from } (2, -1, 3) \text{ in the direction } \mathbf{i} + 8\mathbf{j} - 4\mathbf{k} \text{ is } \left(\frac{8}{3}, \frac{13}{3}, \frac{1}{3} \right).$$

$$84 \quad \mathbf{p} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{a} \quad \mathbf{p} \bullet \mathbf{r}$$

$$\begin{aligned} &= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ &= 3 \times 2 + 0 \times 1 + 1 \times (-1) \\ &= 6 + 0 - 1 \\ &= 5 \end{aligned}$$

$$\mathbf{b} \quad \mathbf{q} \bullet (\mathbf{r} + \mathbf{p})$$

$$\begin{aligned} &= \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \bullet \left[\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 2+3 \\ 1+0 \\ -1+1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \\ &= (-1) \times 5 + 0 \times 1 + 7 \times 0 \\ &= -5 + 0 + 0 \\ &= -5 \end{aligned}$$

$$\mathbf{c} \quad (2\mathbf{p} + \mathbf{q}) \bullet \mathbf{r}$$

$$\begin{aligned} &= \left[2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \right] \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2(3) + (-1) \\ 2(0) + 0 \\ 2(1) + 7 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 0 \\ 9 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ &= 5 \times 2 + 0 \times 1 + 9 \times (-1) \\ &= 10 + 0 - 9 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad |\mathbf{q}|^2 &= (-1)^2 + 0^2 + 7^2 \\ &= 1 + 49 \\ &= 50 \end{aligned}$$

$$\mathbf{e} \quad k\mathbf{p} + \mathbf{q} \text{ is perpendicular to } \mathbf{r} \text{ if } (k\mathbf{p} + \mathbf{q}) \bullet \mathbf{r} = 0$$

$$\begin{aligned} \therefore \left[k \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \right] \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} &= 0 \\ \therefore \begin{pmatrix} 3k-1 \\ 0 \\ k+7 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} &= 0 \\ \therefore 2(3k-1) + 0 - (k+7) &= 0 \\ \therefore 6k - 2 - k - 7 &= 0 \\ \therefore 5k &= 9 \\ \therefore k &= \frac{9}{5} \end{aligned}$$

$$85 \quad \mathbf{a} \quad |\mathbf{a} + \mathbf{b}|^2$$

$$\begin{aligned} &= (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} \\ &= (\mathbf{a} \bullet \mathbf{a}) + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + (\mathbf{b} \bullet \mathbf{b}) \\ &= |\mathbf{a}|^2 + 2(\mathbf{a} \bullet \mathbf{b}) + |\mathbf{b}|^2 \end{aligned}$$

$$\mathbf{b} \quad |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a} + (-\mathbf{b})|^2$$

$$\begin{aligned} &= |\mathbf{a}|^2 + 2(\mathbf{a} \bullet (-\mathbf{b})) + |-\mathbf{b}|^2 \quad \{\text{using } \mathbf{a}\} \\ &= |\mathbf{a}|^2 - 2(\mathbf{a} \bullet \mathbf{b}) + |\mathbf{b}|^2 \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + 2(\mathbf{a} \bullet \mathbf{b}) + |\mathbf{b}|^2 + |\mathbf{a}|^2 - 2(\mathbf{a} \bullet \mathbf{b}) + |\mathbf{b}|^2 \quad \{\text{from } \mathbf{a}\} \\ &= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 \\ &= 2(3)^2 + 2(7)^2 \\ &= 18 + 98 \\ &= 116 \end{aligned}$$

$$86 \quad \text{The diagonal from O to B has direction } \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

$$\text{The diagonal from A to C has direction } \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}.$$

If the acute angle between the diagonals is θ then

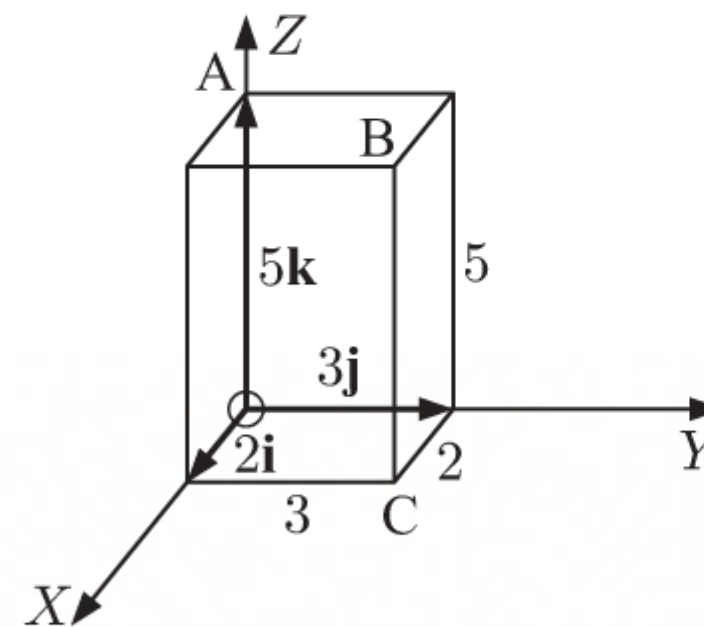
$$\left| \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \right| = \left| \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \right| \cos \theta$$

$$\therefore |4 + 9 - 25| = \sqrt{38} \sqrt{38} \cos \theta$$

$$\therefore 12 = 38 \cos \theta$$

$$\therefore \cos \theta = \frac{12}{38}$$

$$\therefore \theta \approx 71.6^\circ.$$



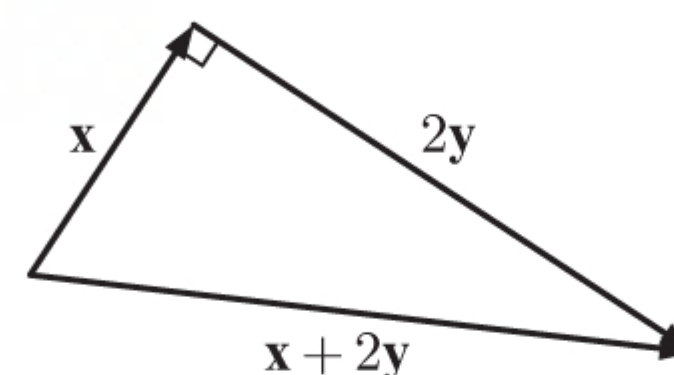
Note: This angle may depend on the diagonals you select. The two other possible angles are about 37.9° and about 58.2° .

87 a If $y = -2x$
 then $|x + 2y| = |x - 4x|$
 $= |-3x|$
 $= 3|x|$
 $= 6$ units

b $|y| = 3|x| = 6$
 $\therefore |x| = 2$

Since x and y are perpendicular, x and $2y$ are perpendicular.

Using Pythagoras' theorem, $|x + 2y| = \sqrt{|x|^2 + |2y|^2}$
 $= \sqrt{|x|^2 + 4|y|^2}$
 $= \sqrt{2^2 + 4(6)^2}$
 $= \sqrt{148}$
 $= 2\sqrt{37}$ units

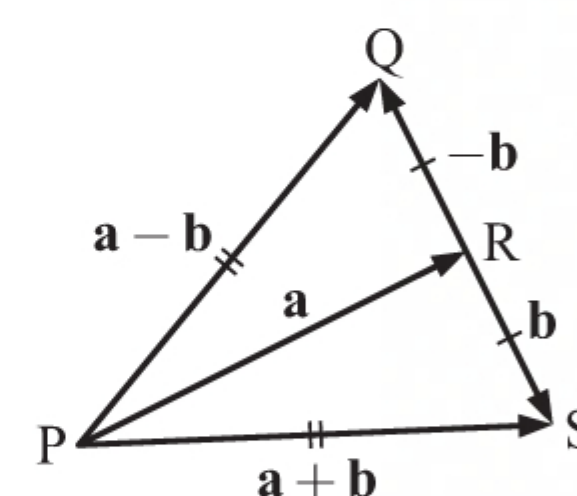


88 If $|a - b| = |a + b|$, then $PQ = PS$, and triangle PQS is isosceles.

Since $|-b| = |b|$, $QR = SR$.

Since the line joining the apex of an isosceles triangle to the midpoint of the base is perpendicular to the base, \overrightarrow{PR} is perpendicular to \overrightarrow{QS} .

So, a is perpendicular to b .



89 a $a \cdot b = |a||b|\cos\theta$ where θ is the angle between a and b . If $a \cdot b < 0$, then $\cos\theta < 0$ and so $90^\circ < \theta < 180^\circ$.

b i $a \cdot b = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = -6 - 1 + 3 = -4$

ii $|a| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}$, $|b| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$, and $\cos\theta = \frac{-4}{\sqrt{14}\sqrt{11}} \approx -0.3223$
 $\therefore \theta \approx 108.8^\circ$.

90 a If $a \cdot b = 0$, then a and b are perpendicular.

b $a \cdot b = \sqrt{15}$
 $\therefore \sqrt{15} = |a||b|\cos\theta$
 $\therefore \sqrt{15} = \sqrt{5} \times \sqrt{3} \times \cos\theta$
 $\therefore \cos\theta = 1$
 $\therefore \theta = 0^\circ$

$\therefore a$ and b are parallel and point in the same direction.

c $a \cdot b = -\sqrt{15}$
 $\therefore -\sqrt{15} = |a||b|\cos\theta$
 $\therefore -\sqrt{15} = \sqrt{5} \times \sqrt{3} \times \cos\theta$
 $\therefore \cos\theta = -1$
 $\therefore \theta = 180^\circ$

$\therefore a$ and b are parallel and point in opposite directions.

91 In rhombus $OACB$, M , N , P , and Q are midpoints of $[OA]$, $[OB]$, $[AC]$, and $[BC]$, respectively.

Let $\overrightarrow{OM} = \overrightarrow{MA} = a$ and $\overrightarrow{ON} = \overrightarrow{NB} = b$.

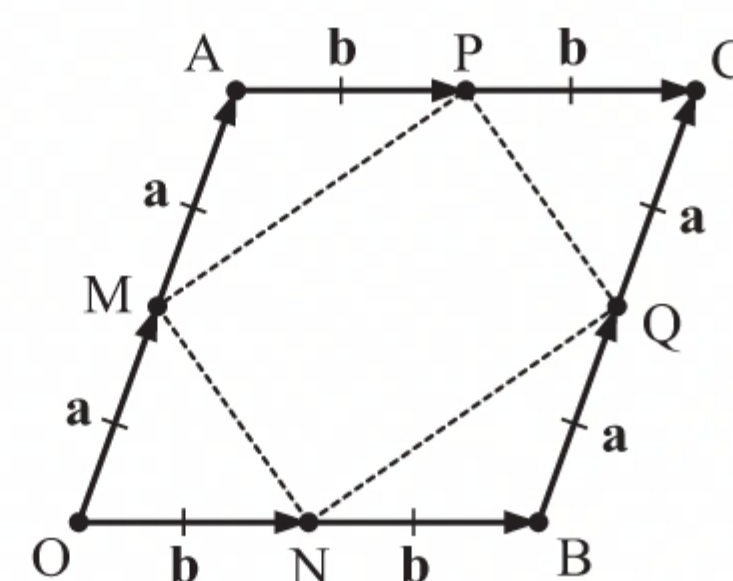
Since $OACB$ is a rhombus, $|a| = |b|$, $\overrightarrow{BQ} = \overrightarrow{QC} = a$, and $\overrightarrow{AP} = \overrightarrow{PC} = b$.

Now $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$
 $= -a + b$

Also $\overrightarrow{NQ} = \overrightarrow{NB} + \overrightarrow{BQ}$
 $= b + a$

and $\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CQ} = b - a$
 $= -a + b$
 $= \overrightarrow{MN}$

and $\overrightarrow{MP} = \overrightarrow{MA} + \overrightarrow{AP}$
 $= a + b$
 $= b + a$
 $= \overrightarrow{NQ}$



So, quadrilateral $MNQP$ has two pairs of equal and parallel sides.

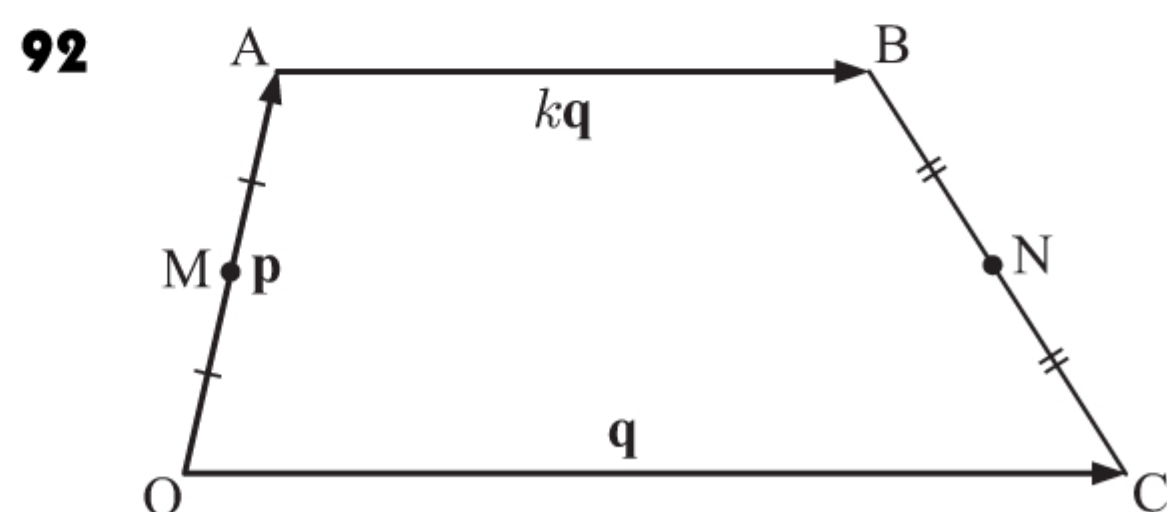
$\therefore MNQP$ is a parallelogram.

$$\begin{aligned}
 \text{But } \overrightarrow{MN} \bullet \overrightarrow{NQ} &= (-\mathbf{a} + \mathbf{b}) \bullet (\mathbf{b} + \mathbf{a}) \\
 &= -\mathbf{a} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} \\
 &= \cancel{-\mathbf{a} \bullet \mathbf{b}} - |\mathbf{a}|^2 + |\mathbf{b}|^2 + \cancel{\mathbf{a} \bullet \mathbf{b}} \\
 &= |\mathbf{b}|^2 - |\mathbf{a}|^2 \\
 &= 0 \quad \{|\mathbf{a}| = |\mathbf{b}|\}
 \end{aligned}$$

So, $\overrightarrow{MN} \perp \overrightarrow{NQ}$ and \widehat{MNQ} is a right angle.

Similarly, \widehat{NQP} , \widehat{MPQ} , and \widehat{NMP} are right angles.

\therefore MNQP is a rectangle, as required.



a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AO} + \overrightarrow{OC}$

$$\begin{aligned}
 &= -k\mathbf{q} - \mathbf{p} + \mathbf{q} \\
 &= (1-k)\mathbf{q} - \mathbf{p}
 \end{aligned}$$

b $\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{BN}$

$$\begin{aligned}
 &= \frac{1}{2}\overrightarrow{OA} + k\mathbf{q} + \frac{1}{2}\overrightarrow{BC} \\
 &= \frac{1}{2}\mathbf{p} + k\mathbf{q} + \frac{1}{2}[(1-k)\mathbf{q} - \mathbf{p}] \quad \{\text{using a}\} \\
 &= \frac{1}{2}\mathbf{p} + k\mathbf{q} + \frac{(1-k)}{2}\mathbf{q} - \frac{1}{2}\mathbf{p} \\
 &= \left(\frac{2k+1-k}{2}\right)\mathbf{q} \\
 &= \left(\frac{1+k}{2}\right)\mathbf{q} \\
 &= \left(\frac{1+k}{2}\right)\overrightarrow{OC}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \overrightarrow{MN} \parallel \overrightarrow{OC} \text{ and } |\overrightarrow{MN}| &= \left|\frac{1+k}{2}\right| |\overrightarrow{OC}| \\
 &= \left(\frac{1+k}{2}\right) |\overrightarrow{OC}| \quad \left\{\frac{1+k}{2} > 0 \text{ as } k > 0\right\}
 \end{aligned}$$

93 $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$

a $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{vmatrix}$

$$\begin{aligned}
 &= \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{k} \\
 &= \begin{pmatrix} -5 \\ -3 \\ 4 \end{pmatrix}
 \end{aligned}$$

b $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 0 & 4 & -1 \end{vmatrix}$

$$\begin{aligned}
 &= \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ 0 & 4 \end{vmatrix} \mathbf{k} \\
 &= \begin{pmatrix} -7 \\ -1 \\ -4 \end{pmatrix}
 \end{aligned}$$

So, $(\mathbf{b} \times \mathbf{c}) \bullet 2\mathbf{a} = \begin{pmatrix} -7 \\ -1 \\ 4 \end{pmatrix} \bullet \left[2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right]$

$$\begin{aligned}
 &= \begin{pmatrix} -7 \\ -1 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \\
 &= -14 - 2 - 16 \\
 &= -32
 \end{aligned}$$

94 $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = \mathbf{j} + 2\mathbf{k}$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 0 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

b Now $\mathbf{a} \times \mathbf{b}$ has length $\sqrt{5^2 + (-2)^2 + 1^2}$
 $= \sqrt{30}$ units

\therefore a vector of length 5 units perpendicular to both \mathbf{a} and

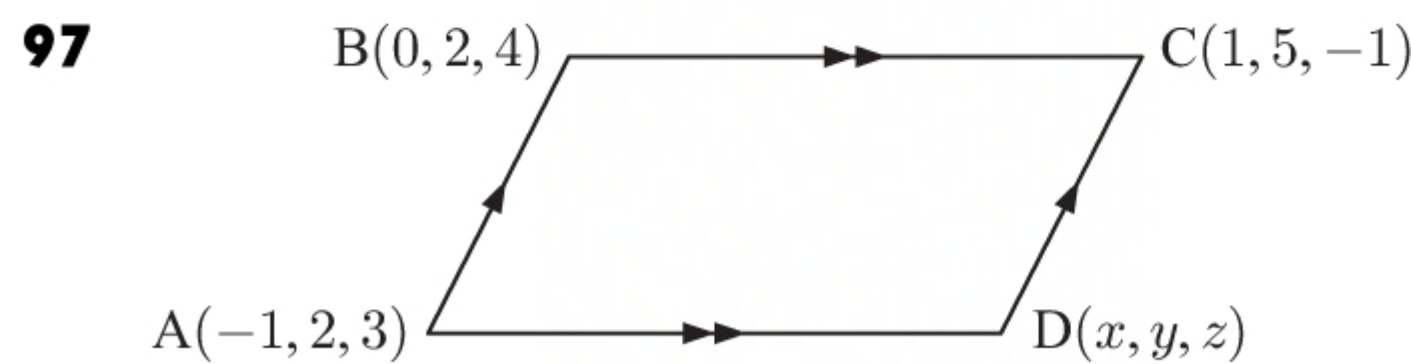
$$\begin{aligned} \mathbf{b} \text{ is } \frac{5}{\sqrt{30}} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} &= \frac{\sqrt{30}}{6} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5\sqrt{30}}{6} \\ -\frac{\sqrt{30}}{3} \\ \frac{\sqrt{30}}{6} \end{pmatrix} \end{aligned}$$

95 $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{a} \times \mathbf{b} = \mathbf{j} - 2\mathbf{k}$

Now $|\mathbf{a}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$ units
 and $|\mathbf{a} \times \mathbf{b}| = \sqrt{0^2 + 1^2 + (-2)^2} = \sqrt{5}$ units

The angle between \mathbf{a} and \mathbf{b} is $\frac{\pi}{4}$

$$\begin{aligned} \therefore |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}| |\mathbf{b}| \sin \frac{\pi}{4} \\ \therefore \sqrt{5} &= \sqrt{11} \times |\mathbf{b}| \times \frac{1}{\sqrt{2}} \\ \therefore |\mathbf{b}| &= \frac{\sqrt{5} \times \sqrt{2}}{\sqrt{11}} = \sqrt{\frac{10}{11}} \text{ units} \end{aligned}$$



a Let D have coordinates (x, y, z) .

$$\begin{aligned} \overrightarrow{AD} &= \overrightarrow{BC} \\ \therefore \begin{pmatrix} x+1 \\ y-2 \\ z-3 \end{pmatrix} &= \begin{pmatrix} 1-0 \\ 5-2 \\ -1-4 \end{pmatrix} \\ \therefore x+1 &= 1, \quad y-2 = 3, \quad z-3 = -5 \\ \therefore x &= 0, \quad y = 5, \quad z = -2 \\ \text{So, D is } &(0, 5, -2). \end{aligned}$$

b $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

$$\begin{aligned} \therefore \overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 3 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ 3 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \mathbf{k} \\ &= -3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= |-3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}| \\ &= \sqrt{(-3)^2 + 6^2 + 3^2} \\ &= 3\sqrt{6} \text{ units}^2 \end{aligned}$$

98 The defining vectors from R are $\overrightarrow{RS} = \mathbf{s} - \mathbf{r}$
 $= (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
 $= \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

and $\overrightarrow{RT} = \mathbf{t} - \mathbf{r}$
 $= (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
 $= -\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$$\begin{aligned} \text{Now } \overrightarrow{RS} \times \overrightarrow{RT} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ -1 & 4 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} \mathbf{k} \\ &= -10\mathbf{i} + \mathbf{j} + 7\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= \frac{1}{2} |\overrightarrow{RS} \times \overrightarrow{RT}| \\ &= \frac{1}{2} \sqrt{(-10)^2 + 1^2 + 7^2} \\ &= \frac{1}{2} \sqrt{150} = \frac{5}{2} \sqrt{6} \text{ units}^2 \end{aligned}$$

99 The defining vectors from P are $\vec{PQ} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$ and $\vec{PR} = \begin{pmatrix} 4 \\ k-4 \\ 1 \end{pmatrix}$.

$$\begin{aligned} \text{Now } \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -2 \\ 4 & k-4 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -2 & -2 \\ k-4 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -2 \\ 4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & -2 \\ 4 & k-4 \end{vmatrix} \mathbf{k} \\ &= \begin{pmatrix} -2 + 2(k-4) \\ -12 \\ 4(k-4) + 8 \end{pmatrix} \\ &= \begin{pmatrix} 2k-10 \\ -12 \\ 4k-8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |\vec{PQ} \times \vec{PR}| &= \sqrt{(2k-10)^2 + (-12)^2 + (4k-8)^2} \\ &= \sqrt{4k^2 - 40k + 100 + 144 + 16k^2 - 64k + 64} \\ &= \sqrt{20k^2 - 104k + 308} \end{aligned}$$

But area of triangle PQR = $\sqrt{44}$ units²

$$\therefore \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \sqrt{44}$$

$$\therefore \frac{1}{2} \sqrt{20k^2 - 104k + 308} = \sqrt{44}$$

$$\therefore \frac{1}{4} (20k^2 - 104k + 308) = 44$$

$$\therefore 20k^2 - 104k + 308 = 176$$

$$\therefore 20k^2 - 104k + 132 = 0$$

$$\therefore 5k^2 - 26k + 33 = 0$$

$$\therefore 5k^2 - 15k - 11k + 33 = 0$$

$$\therefore 5k(k-3) - 11(k-3) = 0$$

$$\therefore (5k-11)(k-3) = 0$$

$$\therefore k = \frac{11}{5} \text{ or } 3$$

100 a i $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = 2 + 3\lambda, \quad y = -1 + 2\lambda, \quad z = 3 - \lambda, \quad \lambda \in \mathbb{R}$

iii $\frac{x-2}{3} = \frac{y+1}{2} = -z+3$

b i Since the line is perpendicular to the YZ-plane, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a possible direction vector.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

ii $x = \lambda, \quad y = 1, \quad z = 2, \quad \lambda \in \mathbb{R}$

iii $y = 1, \quad z = 2$

101 a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}$

c Since the line is perpendicular to the XZ-plane, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a possible direction vector.

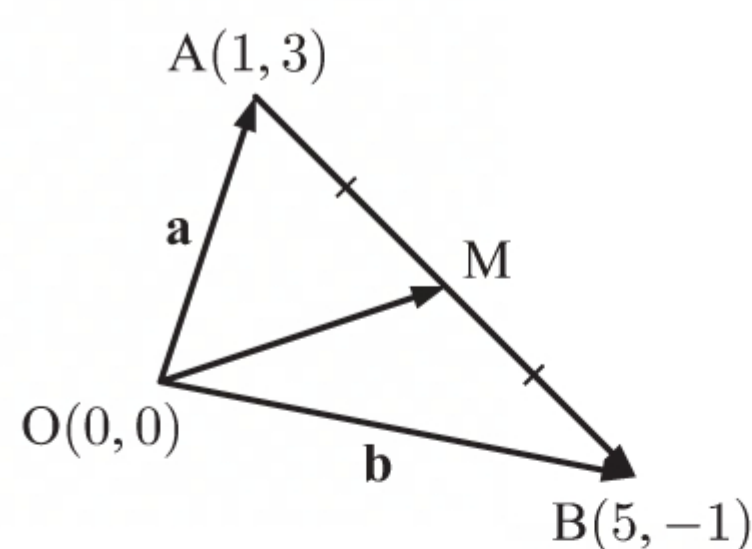
$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$102 \quad \mathbf{a} \quad \overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\overrightarrow{AB}$$

$$= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b})$$

$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

$$= \frac{1}{2}(\mathbf{a} + \mathbf{b})$$



$$\begin{aligned} \mathbf{b} \quad \overrightarrow{OM} &= \frac{1}{2}(\mathbf{a} + \mathbf{b}) \\ &= \frac{1}{2}\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} 6 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \end{aligned}$$

So, M is (3, 1).

$$\mathbf{c} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

\mathbf{d} Any point P on (OM) has coordinates $(3t, t)$.

$$\text{If } |\overrightarrow{PM}| = 2\sqrt{10}$$

$$\text{then } \sqrt{(3-3t)^2 + (1-t)^2} = 2\sqrt{10}$$

$$\therefore 9 - 18t + 9t^2 + 1 - 2t + t^2 = 40$$

$$\therefore 10t^2 - 20t + 10 = 40$$

$$\therefore t^2 - 2t - 3 = 0$$

$$\therefore (t-3)(t+1) = 0$$

$$\therefore t = -1 \text{ or } 3$$

$$\text{Letting } t = -1, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}.$$

$$\text{Letting } t = 3, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$

So, the two points are $(-3, -1)$ and $(9, 3)$.

$$103 \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{pmatrix} 0 - (-1) \\ 1 - 2 \\ 3 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\therefore \text{line (AB) has equation } \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{b} \quad \text{The line } L \text{ has direction vector } \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \text{ and (AB) has direction vector } \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

$$\begin{aligned} \text{If } \theta \text{ is the angle between the lines, then } \cos \theta &= \frac{2 + 0 - 2}{\sqrt{2^2 + 0^2 + (-1)^2} \sqrt{1^2 + (-1)^2 + 2^2}} \\ &= 0 \end{aligned}$$

$$\therefore \theta = 90^\circ$$

So, the angle between (AB) and L is 90° .

$$104 \quad \mathbf{a} \quad \text{The line has direction vector } \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\therefore \text{its vector equation is } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\mathbf{b} \quad \text{The line } x = 1 + kt, \quad y = 2 - t, \quad t \in \mathbb{R} \text{ has direction vector } \begin{pmatrix} k \\ -1 \end{pmatrix}.$$

$$\therefore \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} k \\ -1 \end{pmatrix} = \left| \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} k \\ -1 \end{pmatrix} \right| \cos \frac{\pi}{3}$$

$$\therefore 3k - 1 = \sqrt{3^2 + 1^2} \sqrt{k^2 + (-1)^2} \times \frac{1}{2}$$

$$\therefore 6k - 2 = \sqrt{10(k^2 + 1)}$$

$$\therefore (6k - 2)^2 = 10(k^2 + 1)$$

$$\therefore 36k^2 - 24k + 4 = 10k^2 + 10$$

$$\therefore 26k^2 - 24k - 6 = 0$$

$$\therefore 13k^2 - 12k - 3 = 0$$

$$\begin{aligned} \therefore k &= \frac{12 \pm \sqrt{12^2 - 4(13)(-3)}}{2(13)} \\ &= \frac{12 \pm \sqrt{300}}{26} \\ &= \frac{12 \pm 10\sqrt{3}}{26} \\ &= \frac{6 \pm 5\sqrt{3}}{13} \end{aligned}$$

- 105 a** The direction vector $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ has length $\sqrt{(-1)^2 + 2^2 + 2^2} = 3$.

Trisha moves with speed 0.5 m s^{-1} .

$$\therefore \text{Trisha's velocity vector is } \frac{0.5}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

- b** Trisha's position vector after t seconds is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \quad t \geq 0.$$

$$\begin{aligned} \text{When } t = 10, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 10 \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{pmatrix} \\ &= \begin{pmatrix} 1\frac{1}{3} \\ 4\frac{1}{3} \\ 3\frac{1}{3} \end{pmatrix} \end{aligned}$$

\therefore after 10 seconds, Trisha is at $(1\frac{1}{3}, 4\frac{1}{3}, 3\frac{1}{3})$.

- d** The escalator has direction vector $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

The horizontal corresponds to the XY -plane.

The projection of $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ onto the XY -plane is $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$.

If θ is the angle the escalator makes with the horizontal,

$$\begin{aligned} \cos \theta &= \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right| \left| \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right|} \\ &= \frac{1 + 4 + 0}{\sqrt{(-1)^2 + 2^2 + 2^2} \sqrt{(-1)^2 + 2^2 + 0^2}} \\ &= \frac{5}{\sqrt{9} \sqrt{5}} \\ &= \frac{\sqrt{5}}{3} \end{aligned}$$

$$\begin{aligned} \therefore \theta &= \cos^{-1} \left(\frac{\sqrt{5}}{3} \right) \\ &\approx 41.8^\circ \end{aligned}$$

\therefore the escalator travels at an angle of about 41.8° to the horizontal.

- 106 a** When $t = 0$, $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

\therefore ship A is initially at $(-1, 3)$ and ship B is initially at $(7, 4)$.

- c** After t seconds, Trisha's x -coordinate is $x = 3 - \frac{t}{6}$.

$$0 = 3 - \frac{t}{6}$$

$$\therefore \frac{t}{6} = 3$$

$$\therefore t = 18$$

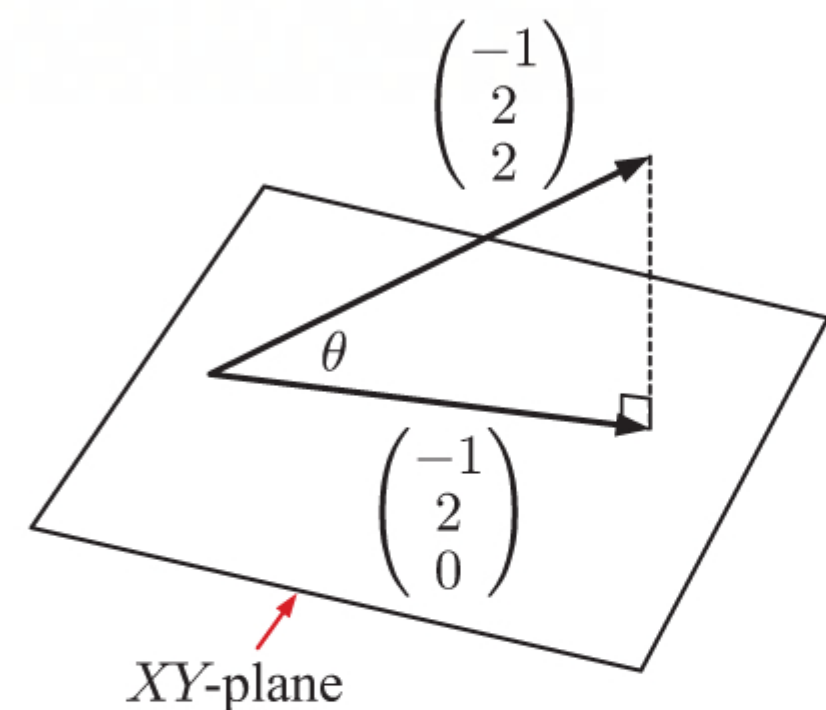
\therefore Trisha is on the escalator for 18 seconds.

$$\begin{aligned} \text{When } t = 18, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 18 \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 6 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} \end{aligned}$$

\therefore the escalator ends at $(0, 7, 6)$ and starts at $(3, 1, 0)$.

\therefore the length of the escalator

$$\begin{aligned} &= \sqrt{(0-3)^2 + (7-1)^2 + (6-0)^2} \\ &= \sqrt{(-3)^2 + 6^2 + 6^2} \\ &= 9 \text{ m} \end{aligned}$$



- b** Ship A has velocity vector $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

\therefore its speed is $\sqrt{4^2 + (-1)^2} = \sqrt{17} \text{ km h}^{-1}$.

Ship B has velocity vector $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

\therefore its speed is $\sqrt{(-2)^2 + (-1)^2} = \sqrt{5} \text{ km h}^{-1}$.

- c** Suppose the ships pass through the same point, and that ship A is there at time t_A and ship B is there at time t_B .

$$\begin{aligned} x_A = x_B &\Rightarrow -1 + 4t_A = 7 - 2t_B & y_A = y_B &\Rightarrow 3 - t_A = 4 - t_B \\ \therefore 4t_A + 2t_B &= 8 & \therefore t_A - t_B &= -1 \quad \dots (2) \\ \therefore 2t_A + t_B &= 4 \quad \dots (1) \end{aligned}$$

$$\begin{aligned} 2t_A + t_B &= 4 & \{(1)\} \\ t_A - t_B &= -1 & \{(2)\} \\ \hline \text{Adding, } 3t_A &= 3 \\ \therefore t_A &= 1 \end{aligned}$$

Substituting into (1) gives $2 + t_B = 4$
 $\therefore t_B = 2$

When $t_A = 1$, $x_A = -1 + 4 = 3$ and $y_A = 3 - 1 = 2$

So, the two ships both pass through $(3, 2)$. Ship A is there after 1 hour, and ship B is there after 2 hours.

- 107 a** The line has parametric equations $x = 1 + 2t$, $y = -3 + 3t$, $z = t$, $t \in \mathbb{R}$.

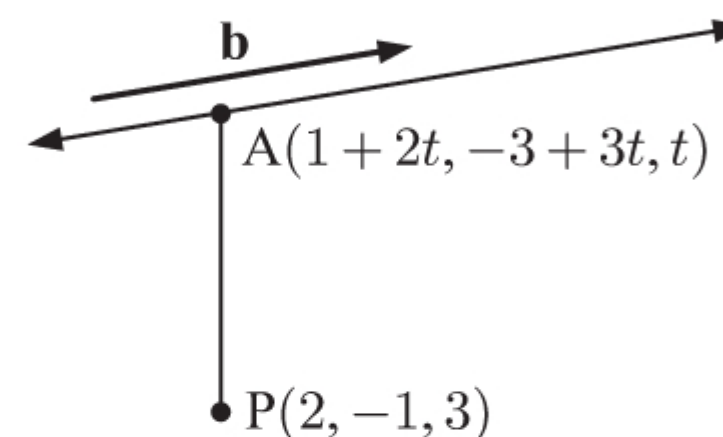
The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let $A(1 + 2t, -3 + 3t, t)$ be any point on the given line.

$$\therefore \vec{PA} = \begin{pmatrix} 1 + 2t - 2 \\ -3 + 3t - (-1) \\ t - 3 \end{pmatrix} = \begin{pmatrix} -1 + 2t \\ -2 + 3t \\ -3 + t \end{pmatrix}$$

If A is the closest point on the line to P, then $\vec{PA} \perp \mathbf{b}$.

$$\begin{aligned} \therefore \vec{PA} \cdot \mathbf{b} &= 0 \\ \therefore \begin{pmatrix} -1 + 2t \\ -2 + 3t \\ -3 + t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} &= 0 \\ \therefore 2(-1 + 2t) + 3(-2 + 3t) + 1(-3 + t) &= 0 \\ \therefore -2 + 4t - 6 + 9t - 3 + t &= 0 \\ \therefore 14t &= 11 \\ \therefore t &= \frac{11}{14} \end{aligned}$$



Substituting $t = \frac{11}{14}$ into the parametric equations gives $x = 1 + 2(\frac{11}{14}) = \frac{18}{7}$,
 $y = -3 + 3(\frac{11}{14}) = -\frac{9}{14}$,
and $z = \frac{11}{14}$

So, the foot of the perpendicular is $(\frac{18}{7}, -\frac{9}{14}, \frac{11}{14})$.

- b** Using $t = \frac{11}{14}$, $\vec{PA} = \begin{pmatrix} -1 + 2(\frac{11}{14}) \\ -2 + 3(\frac{11}{14}) \\ -3 + \frac{11}{14} \end{pmatrix} = \begin{pmatrix} \frac{4}{7} \\ \frac{5}{14} \\ -\frac{31}{14} \end{pmatrix}$
 $\therefore |\vec{PA}| = \sqrt{(\frac{4}{7})^2 + (\frac{5}{14})^2 + (-\frac{31}{14})^2}$
 $= \sqrt{\frac{75}{14}} = 5\sqrt{\frac{3}{14}} \text{ units}$

So, the shortest distance from P to the line is $5\sqrt{\frac{3}{14}}$ units.

$$108 \quad \mathbf{a} \quad \mathbf{i} \quad \overrightarrow{AB} = \begin{pmatrix} 5-1 \\ -1-(-1) \\ -1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$$\text{So, } L_1 \text{ has equation } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}.$$

ii Any point P on L_1 has coordinates $(1 + 4t, -1, 2 - 3t)$.

$$\therefore \overrightarrow{AP} = \begin{pmatrix} 4t \\ 0 \\ -3t \end{pmatrix}$$

$$\text{Now } |\overrightarrow{AP}| = 20, \text{ so } \sqrt{(4t)^2 + (-3t)^2} = 20$$

$$\therefore 16t^2 + 9t^2 = 400$$

$$\therefore t^2 = 16$$

$$\therefore t = \pm 4$$

So, letting $t = 4$, a point on L_1 which is 20 units from A is $(1 + 4(4), -1, 2 - 3(4)) = (17, -1, -10)$.

iii L_1 meets the YZ -plane when $x = 0$

$$\therefore 1 + 4t = 0$$

$$\therefore t = -\frac{1}{4}$$

When $t = -\frac{1}{4}$, P is $(0, -1, \frac{11}{4})$.

$\therefore L_1$ meets the YZ -plane at $(0, -1, \frac{11}{4})$.

$$\mathbf{b} \quad \mathbf{i} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -\frac{13}{2} \end{pmatrix} + s \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}, \quad s \in \mathbb{R}$$

ii The direction vectors of the lines are $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}$.

$$\text{Now } \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix} = -12 + 0 + 12 = 0$$

$\therefore L_1$ is perpendicular to L_2 .

c **i** The y -coordinate of P will be the same as the y -coordinate of any point on L_1 which is always -1 .

ii L_1 meets L_2 at the point on L_2 where $y = -1$.

$$\therefore 1 + 2s = -1$$

$$\therefore 2s = -2$$

$$\therefore s = -1$$

$$\begin{aligned} \therefore x &= 4 - 3s & \text{and} & & z &= -\frac{13}{2} - 4s \\ &= 4 - 3(-1) & & & &= -\frac{13}{2} - 4(-1) \\ &= 7 & & & &= -\frac{5}{2} \end{aligned}$$

$\therefore L_1$ meets L_2 at $P(7, -1, -\frac{5}{2})$.

d The shortest distance from C to the line L_1 is the distance from C to the point of intersection of L_1 and L_2 , which is $P(7, -1, -\frac{5}{2})$.

$$\begin{aligned} \therefore \text{distance} &= |\overrightarrow{CP}| \\ &= \sqrt{(7-4)^2 + (-1-1)^2 + (-\frac{5}{2} - (-\frac{13}{2}))^2} \\ &= \sqrt{9+4+16} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

109 The lines meet where $\begin{pmatrix} -3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$$\therefore -3 + 2s = -1 - 3t \quad \text{and} \quad 2 - s = 6 + 4t$$

$$\therefore 2s + 3t = 2 \quad \dots (1) \quad \text{and} \quad s + 4t = -4 \quad \dots (2)$$

$$\begin{array}{rcl} 2s + 3t & = & 2 \quad \{(1)\} \\ -2s - 8t & = & 8 \quad \{(2) \times (-2)\} \\ \hline \end{array}$$

$$\therefore -5t = 10$$

$$\therefore t = -2$$

Using (2), $s + 4(-2) = -4$

$$\therefore s = 4$$

Using line 1, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Checking in line 2, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + (-2)\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad \checkmark$

\therefore the lines meet at $(5, -2)$.

110 a $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$

b $x = -5 + 2t = -1$

$$\therefore 2t = 4$$

$$\therefore t = 2$$

Now, $y = -2 + 3t$

$$= -2 + 3(2)$$

$$= 4$$

So, the point is $(-1, 4)$.

c i L_2 is perpendicular to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, so L_2 has direction vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

$$\therefore \text{its vector equation is } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + s\begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad s \in \mathbb{R}.$$

ii L_1 meets L_2 where $\begin{pmatrix} -5 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + s\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$\begin{aligned} \therefore t\begin{pmatrix} 2 \\ 3 \end{pmatrix} - s\begin{pmatrix} -3 \\ 2 \end{pmatrix} &= \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} -5 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

$$\text{So, } 2t + 3s = 9 \quad \dots (1)$$

$$3t - 2s = 7 \quad \dots (2)$$

$$\therefore 4t + 6s = 18 \quad \{(1) \times 2\}$$

$$9t - 6s = 21 \quad \{(2) \times 3\}$$

$$\text{Adding, } 13t = 39$$

$$\therefore t = 3$$

Substituting $t = 3$ into L_1 gives $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$.

So, the lines meet at $(1, 7)$.

iii The shortest distance from $(4, 5)$ to the line L_1 is found when the line joining point $(4, 5)$ to L_1 is perpendicular to L_1 .

Since L_2 passes through $(4, 5)$ and is perpendicular to L_1 , we need the distance from $(4, 5)$ to the point of intersection of L_1 and L_2 , which is $(1, 7)$. {from **ii**}

$$\text{Distance} = \sqrt{(1-4)^2 + (7-5)^2}$$

$$= \sqrt{9+4} = \sqrt{13} \text{ units}$$

$$111 \quad \begin{cases} mx + 5y = 7 \\ 5x + my = 7 \end{cases} \quad \text{where } m \in \mathbb{R}$$

a In augmented matrix form, the system is:

$$\begin{pmatrix} m & 5 & 7 \\ 5 & m & 7 \end{pmatrix} \sim \begin{pmatrix} m & 5 & 7 \\ 0 & m^2 - 25 & 7m - 35 \end{pmatrix} \quad mR_2 - 5R_1 \rightarrow R_2 \quad \leftarrow \begin{pmatrix} 5m & m^2 & 7m \\ -5m & -25 & -35 \\ 0 & m^2 - 25 & 7m - 35 \end{pmatrix}$$

A unique solution exists provided $m^2 - 25 \neq 0$.

So, there is a unique solution when $m \neq \pm 5$.

Using row 2, $(m^2 - 25)y = 7m - 35$

$$\begin{aligned} \therefore y &= \frac{7(m-5)}{(m-5)(m+5)} \quad \{m \neq \pm 5\} \\ &= \frac{7}{m+5} \end{aligned}$$

Substituting into row 1, $mx + 5\left(\frac{7}{m+5}\right) = 7$

$$\begin{aligned} \therefore m(m+5)x + 35 &= 7(m+5) \\ \therefore m(m+5)x &= 7m + 35 - 35 \\ \therefore m(m+5)x &= 7m \\ \text{For } m \neq 0, \quad x &= \frac{7}{m+5} \end{aligned}$$

So, the unique solution is $x = \frac{7}{m+5}$, $y = \frac{7}{m+5}$ when $m \neq \pm 5$, and $m \neq 0$.

$$\begin{aligned} \text{When } m = 0, \text{ the system is } &\begin{pmatrix} 0 & 5 & 7 \\ 5 & 0 & 7 \end{pmatrix} \\ &\sim \begin{pmatrix} 5 & 0 & 7 \\ 0 & 5 & 7 \end{pmatrix} \quad R_1 \leftrightarrow R_2 \end{aligned}$$

$$\begin{aligned} \text{From } R_2, \quad 5y &= 7 \quad \text{and from } R_1, \quad 5x = 7 \\ \therefore y &= \frac{7}{5} \quad \therefore x = \frac{7}{5} \end{aligned}$$

So, for $m = 0$, the unique solution is $x = \frac{7}{5}$, $y = \frac{7}{5}$.

\therefore for $m \neq \pm 5$, the lines intersect at the point $\left(\frac{7}{m+5}, \frac{7}{m+5}\right)$.

b When $m = 5$, the equations are $5x + 5y = 7$ and $5x + 5y = 7$, so the lines are coincident.

There are infinitely many solutions of the form $x = t$, $y = \frac{7-5t}{5} = \frac{7}{5} - t$, $t \in \mathbb{R}$.

When $m = -5$, the equations are $-5x + 5y = 7$ and $5x - 5y = 7$
or $-5x + 5y = -7$

\therefore the lines are parallel and there are no solutions.

$$112 \quad \vec{AB} = \begin{pmatrix} 4-0 \\ 1-5 \\ -2-6 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix}$$

$$\therefore (AB) \text{ has equation } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} + t \begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix}, \quad t \in \mathbb{R}.$$

$$\text{Now (AB) meets the line } \mathbf{r} = \begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R} \quad \text{where} \quad \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} + t \begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix} = \begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore 4t = a + 2s \quad \text{and} \quad 5 - 4t = 3 - s \quad \text{and} \quad 6 - 8t = 2 + s$$

$$\therefore 2s - 4t = -a \quad \dots (1) \quad \therefore s - 4t = -2 \quad \dots (2) \quad \therefore s + 8t = 4 \quad \dots (3)$$

$$\begin{array}{rcl}
 2s - 8t = -4 & \{(2) \times 2\} & \\
 s + 8t = 4 & \{(3)\} & \\
 \hline
 \therefore 3s = 0 & & \\
 \therefore s = 0 & &
 \end{array}$$

$$\begin{array}{l}
 \text{Substituting into (2), } 0 - 4t = -2 \\
 \therefore t = \frac{1}{2}
 \end{array}$$

$$\begin{array}{l}
 \text{Substituting into (1), } 2(0) - 4(\frac{1}{2}) = -a \\
 \therefore -2 = -a \\
 \therefore a = 2
 \end{array}$$

$$\text{Now when } s = 0, \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

\therefore the intersection point is $(2, 3, 2)$.

113 a L_1 and L_2 have direction vectors $\begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ respectively.

$$\text{Since } \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, L_1 \text{ is parallel to } L_2.$$

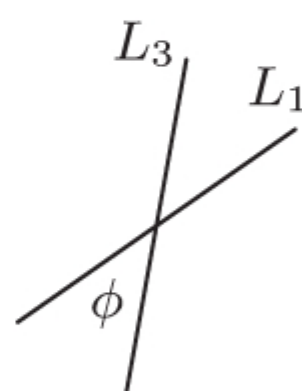
$$\text{b } L_1 \text{ and } L_3 \text{ meet if } \begin{cases} 2 + 3t = 5 - 3s \\ 1 - 6t = 5 + 4s \\ -1 - 3t = 1 + 2s \end{cases} \text{ which is } \begin{cases} 3s + 3t = 3 \\ 4s + 6t = -4 \\ 2s + 3t = -2 \end{cases}.$$

Solving this system using technology, $s = 5$ and $t = -4$.

\therefore the lines meet at $(-10, 25, 11)$.

$$L_3 \text{ has direction vector } \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}.$$

$$\begin{aligned}
 \text{Now } \cos \phi &= \frac{\left| \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|}{\sqrt{9 + 36 + 9} \sqrt{9 + 16 + 4}} \\
 &= \frac{|-9 - 24 - 6|}{\sqrt{54} \times 5} \\
 &= \frac{39}{\sqrt{1566}} \\
 \therefore \phi &= \cos^{-1} \left(\frac{39}{\sqrt{1566}} \right) \approx 9.76^\circ
 \end{aligned}$$



\therefore the angle between L_1 and L_3 is about 9.76° .

114 a Line 1: $x = 3 + 2t, y = 1 - t, z = 4 + t, t \in \mathbb{R}$

Line 2: $x = 4s, y = 3 - 2s, z = -5 + 2s, s \in \mathbb{R}$

$$\text{Line 1 has direction vector } \mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and line 2 has direction vector } \mathbf{w} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}.$$

As $\mathbf{w} = 2\mathbf{v}$ the two lines are parallel.

To see if the lines are coincident, try to find a shared point.

When $t = 0$, the point on line 1 is $(3, 1, 4)$.

The unique point on line 2 with x -coordinate 3 is the point where $4s = 3$

$$\therefore s = \frac{3}{4}$$

This point is $(3, \frac{3}{2}, -\frac{7}{2})$.

Since $(3, 1, 4) \neq (3, \frac{3}{2}, -\frac{7}{2})$, the lines are not coincident.

\therefore the lines are parallel.

The shortest distance between line 1 and line 2 is the shortest distance from *any* point on line 1 to line 2.

Let P be (3, 1, 4) which lies on line 1, and $A(4s, 3 - 2s, -5 + 2s)$ be any point on line 2.

$$\therefore \vec{PA} = \begin{pmatrix} 4s - 3 \\ 3 - 2s - 1 \\ -5 + 2s - 4 \end{pmatrix} = \begin{pmatrix} 4s - 3 \\ -2s + 2 \\ 2s - 9 \end{pmatrix}$$

If A is the closest point on line 2 to P, then $\vec{PA} \perp \mathbf{w}$.

$$\therefore \vec{PA} \bullet \mathbf{w} = 0$$

$$\therefore \begin{pmatrix} 4s - 3 \\ -2s + 2 \\ 2s - 9 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} = 0$$

$$\therefore 4(4s - 3) - 2(-2s + 2) + 2(2s - 9) = 0$$

$$\therefore 16s - 12 + 4s - 4 + 4s - 18 = 0$$

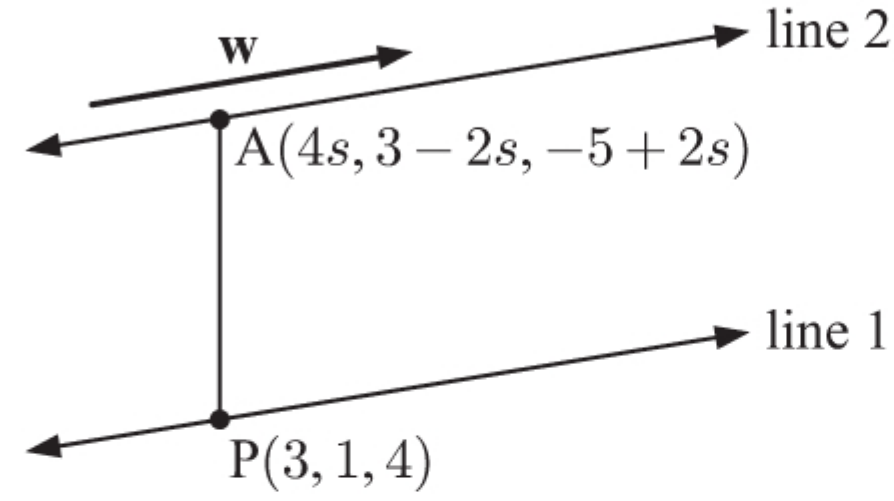
$$\therefore 24s = 34$$

$$\therefore s = \frac{17}{12}$$

$$\text{When } s = \frac{17}{12}, \quad \vec{PA} = \begin{pmatrix} 4(\frac{17}{12}) - 3 \\ -2(\frac{17}{12}) + 2 \\ 2(\frac{17}{12}) - 9 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ -\frac{5}{6} \\ -\frac{37}{6} \end{pmatrix}$$

$$\therefore |\vec{PA}| = \sqrt{\left(\frac{8}{3}\right)^2 + \left(-\frac{5}{6}\right)^2 + \left(-\frac{37}{6}\right)^2} = \frac{5\sqrt{66}}{6} \text{ units}$$

So, the shortest distance between line 1 and line 2 is $\frac{5\sqrt{66}}{6}$ units.



b Line 1: $x = 2t - 1, y = 3 - 4t, z = 4 - 3t, t \in \mathbb{R}$

Line 2: $x = -s, y = 2s - 1, z = 7 - s, s \in \mathbb{R}$

Line 1 has direction vector $\mathbf{v} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$ and line 2 has direction vector $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

As \mathbf{v} is not a scalar multiple of \mathbf{w} , the lines are not parallel.

$$\text{Now } 2t - 1 = -s \qquad 3 - 4t = 2s - 1 \qquad 4 - 3t = 7 - s$$

$$\therefore s + 2t = 1 \quad \dots (1) \qquad \therefore 2s + 4t = 4 \quad \dots (2) \qquad \therefore s - 3t = 3 \quad \dots (3)$$

$$s + 2t = 1 \quad \{(1)\}$$

$$-s + 3t = -3 \quad \{(3) \times (-1)\}$$

$$\hline \therefore 5t = -2$$

$$\therefore t = -\frac{2}{5} \text{ and } s = \frac{9}{5}$$

$$\text{Checking in (2): } 2s + 4t = 2\left(\frac{9}{5}\right) + 4\left(-\frac{2}{5}\right) = 2 \neq 4 \quad \times$$

Since the system is inconsistent, the lines do not intersect, so the lines are skew.

$$\begin{aligned} \text{Now } \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ -1 & 2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -4 & -3 \\ 2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -3 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} \mathbf{k} \\ &= 10\mathbf{i} + 5\mathbf{j} \end{aligned}$$

Let A and B be points on lines 1 and 2 such that $|\vec{AB}|$ is the shortest distance between lines 1 and 2.

\therefore A is $(2t - 1, 3 - 4t, 4 - 3t)$ and B is $(-s, 2s - 1, 7 - s)$ for some $s, t \in \mathbb{R}$.

$$\text{Now } \vec{AB} \parallel \mathbf{v} \times \mathbf{w}, \text{ so } \begin{pmatrix} -2 - 2t + 1 \\ 2s - 1 - 3 + 4t \\ 7 - s - 4 + 3t \end{pmatrix} = k \begin{pmatrix} 10 \\ 5 \\ 0 \end{pmatrix} \text{ for some } k \in \mathbb{R}.$$

$$\therefore -s - 2t + 1 = 10k \quad \dots (4)$$

$$2s + 4t - 4 = 5k \quad \dots (5)$$

$$-s + 3t + 3 = 0 \quad \dots (6)$$

$$\begin{array}{rcl} \text{Now} & -4s - 8t + 8 = -10k & \{(5) \times (-2)\} \\ & -s - 2t + 1 = 10k & \{(4)\} \\ \hline \therefore & -5s - 10t + 9 = 0 & \dots (7) \end{array}$$

$$\begin{array}{rcl} \text{and} & -5s - 10t + 9 = 0 & \{(7)\} \\ & 5s - 15t - 15 = 0 & \{(6) \times (-5)\} \\ \hline \therefore & -25t - 6 = 0 & \\ & \therefore t = -\frac{6}{25} & \end{array}$$

$$\begin{array}{rcl} \text{Using (6),} & -s + 3(-\frac{6}{25}) + 3 = 0 & \\ & \therefore s = \frac{57}{25} & \end{array}$$

$$\therefore \vec{AB} = \begin{pmatrix} -\frac{57}{25} - 2(-\frac{6}{25}) + 1 \\ 2(\frac{57}{25}) + 4(-\frac{6}{25}) - 4 \\ -\frac{57}{25} + 3(-\frac{6}{25}) + 3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{2}{5} \\ 0 \end{pmatrix}$$

$$\text{and the shortest distance between lines 1 and 2 is } |\vec{AB}| = \sqrt{(-\frac{4}{5})^2 + (-\frac{2}{5})^2 + 0^2} = \frac{2}{\sqrt{5}} \text{ units.}$$

c Line 1: $x = 1 + t, y = 2 - t, z = 3t - 1, t \in \mathbb{R}$

Line 2: $x = -2s + 4, y = 2s - 1, z = 8 - 6s, s \in \mathbb{R}$

Line 1 has direction vector $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and line 2 has direction vector $\mathbf{w} = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$.

As $\mathbf{w} = -2\mathbf{v}$ the two lines are parallel.

To see if the lines are coincident, try to find a shared point.

When $t = 0$, the point on line 1 is $(1, 2, -1)$.

$$\begin{array}{rcl} \text{The unique point on line 2 with } x\text{-coordinate 1 is the point where} & -2s + 4 = 1 & \\ & \therefore 2s = 3 & \\ & \therefore s = \frac{3}{2} & \end{array}$$

This is the point $(1, 2, -1)$.

Lines 1 and 2 are parallel and share the point $(1, 2, -1)$.

\therefore the lines are coincident, and the shortest distance between them is 0 units.

115 a $A(1, 2, 4), B(-1, 0, 3), C(2, -3, 1)$

i $\vec{AB} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$ and $\vec{CB} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$ are two non-parallel vectors in the plane.

$$\text{Using C as the known (fixed) point on the plane, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + s \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

ii The normal vector $\mathbf{n} = \vec{AB} \times \vec{CB}$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & -1 \\ -3 & 3 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -2 & -1 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -1 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -2 \\ -3 & 3 \end{vmatrix} \mathbf{k} \\ &= -\mathbf{i} + 7\mathbf{j} - 12\mathbf{k} \\ &= -(\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}) \end{aligned}$$

\therefore the vector $\begin{pmatrix} 1 \\ -7 \\ 12 \end{pmatrix}$ is also normal to the plane.

$$\begin{array}{rcl} \text{Thus the plane has equation} & x - 7y + 12z = 1(1) - 7(2) + 12(4) & \{\text{using point A}\} \\ & \therefore x - 7y + 12z = 35 & \end{array}$$

b $A(0, -1, 2)$, $B(4, 2, -1)$, $C(1, -1, 0)$

i $\vec{AB} = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}$ and $\vec{CB} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ are two non-parallel vectors in the plane.

Using C as the known (fixed) point on the plane, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$, $s, t \in \mathbb{R}$.

ii The normal vector $\mathbf{n} = \vec{AB} \times \vec{CB}$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -3 \\ 3 & 3 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -3 \\ 3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -3 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 3 \\ 3 & 3 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} - 5\mathbf{j} + 3\mathbf{k} \end{aligned}$$

Thus the plane has equation $6x - 5y + 3z = 6(0) - 5(-1) + 3(2)$ {using point A}

$$\therefore 6x - 5y + 3z = 11$$

116 Let L_1 be $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} a \\ -1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and L_2 be $\frac{x-4}{2} = 1-y = \frac{z+2}{3}$.

a L_1 meets L_2 where $1 - (-2 - \lambda) = \frac{2+2\lambda+2}{3}$

$$\therefore 3 + \lambda = \frac{4+2\lambda}{3}$$

$$\therefore 9 + 3\lambda = 4 + 2\lambda$$

$$\therefore \lambda = -5$$

$$\text{When } \lambda = -5, \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} - 5 \begin{pmatrix} a \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-5a \\ 3 \\ -8 \end{pmatrix}$$

$$\text{Now } \frac{x-4}{2} = 1-y$$

$$\therefore \frac{3-5a-4}{2} = 1-3$$

$$\therefore -1-5a = -4$$

$$\therefore 5a = 3$$

$$\therefore a = \frac{3}{5}$$

So, P has coordinates $(3 - 5(\frac{3}{5}), 3, -8)$ which is $(0, 3, -8)$.

b L_2 has parametric equations $2\mu = x - 4$, $\mu = 1 - y$, $3\mu = z + 2$, $\mu \in \mathbb{R}$
which are $x = 4 + 2\mu$, $y = 1 - \mu$, $z = -2 + 3\mu$, $\mu \in \mathbb{R}$

So, L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} \frac{3}{5} \\ -1 \\ 2 \end{pmatrix}$ and L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

$$\text{Now } \cos \theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|}$$

$$= \frac{\left| \begin{pmatrix} \frac{3}{5} \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right|}{\sqrt{\left(\frac{3}{5}\right)^2 + (-1)^2 + 2^2} \sqrt{2^2 + (-1)^2 + 3^2}}$$

$$= \frac{\left| \frac{6}{5} + 1 + 6 \right|}{\frac{\sqrt{134}}{5} \times \sqrt{14}}$$

$$= \frac{41}{\sqrt{134}\sqrt{14}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{41}{\sqrt{134}\sqrt{14}}\right) \approx 18.8^\circ$$

\therefore the acute angle between L_1 and L_2 is about 18.8° .

c The normal vector $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5} & -1 & 2 \\ 2 & -1 & 3 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{3}{5} & 2 \\ 2 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{3}{5} & -1 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\ &= -\mathbf{i} + \frac{11}{5}\mathbf{j} + \frac{7}{5}\mathbf{k} \end{aligned}$$

Since $P(0, 3, -8)$ must lie in the plane containing L_1 and L_2 , the plane has equation

$$\begin{aligned} -x + \frac{11}{5}y + \frac{7}{5}z &= -0 + \frac{11}{5}(3) + \frac{7}{5}(-8) \\ \therefore 5x - 11y - 7z &= -33 + 56 \\ \therefore 5x - 11y - 7z &= 23 \end{aligned}$$

117 a L_1 and L_2 meet where $\frac{8+3\lambda+10}{6} = \frac{-13-5\lambda-7}{-5} = \frac{-3-2\lambda-11}{-5}$

$$\begin{aligned} \therefore \frac{18+3\lambda}{6} &= \frac{-20-5\lambda}{-5} \\ \therefore 3 + \frac{1}{2}\lambda &= 4 + \lambda \\ \therefore \frac{1}{2}\lambda &= -1 \\ \therefore \lambda &= -2 \end{aligned}$$

Check: $\frac{-20-5\lambda}{-5} = \frac{-14-2\lambda}{-5}$

$$\begin{aligned} \therefore -20-5\lambda &= -14-2\lambda \\ \therefore 3\lambda &= -6 \\ \therefore \lambda &= -2 \quad \checkmark \end{aligned}$$

When $\lambda = -2$, $\mathbf{r} = \begin{pmatrix} 8 \\ -13 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$\therefore L_1$ and L_2 meet at $A(2, -3, 1)$.

b L_1 meets $3x + 2y - z = -2$ when $3(8+3\lambda) + 2(-13-5\lambda) - (-3-2\lambda) = -2$

$$\begin{aligned} \therefore 24 + 9\lambda - 26 - 10\lambda + 3 + 2\lambda &= -2 \\ \therefore \lambda &= -3 \end{aligned}$$

When $\lambda = -3$, $\mathbf{r} = \begin{pmatrix} 8 \\ -13 \\ -3 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore L_1$ meets the plane $3x + 2y - z = -2$ at $B(-1, 2, 3)$.

c $C(p, 0, q)$ lies on the plane $3x + 2y - z = -2$.

$$\begin{aligned} \therefore 3p + 2(0) - q &= -2 \\ \therefore 3p - q &= -2 \\ \therefore q &= 3p + 2 \quad \dots (*) \end{aligned}$$

$\therefore C$ is $(p, 0, 3p + 2)$.

For triangle ABC , the defining vectors from A are $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} p-2 \\ 3 \\ 3p+1 \end{pmatrix}$.

$$\begin{aligned}
\text{Now } \vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 5 & 2 \\ p-2 & 3 & 3p+1 \end{vmatrix} \\
&= \begin{vmatrix} 5 & 2 \\ 3 & 3p+1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ p-2 & 3p+1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 5 \\ p-2 & 3 \end{vmatrix} \mathbf{k} \\
&= (15p-1)\mathbf{i} - (-11p+1)\mathbf{j} + (-5p+1)\mathbf{k} \\
&= (15p-1)\mathbf{i} + (11p-1)\mathbf{j} + (1-5p)\mathbf{k} \\
\therefore |\vec{AB} \times \vec{AC}| &= \sqrt{(15p-1)^2 + (11p-1)^2 + (1-5p)^2} \\
&= \sqrt{225p^2 - 30p + 1 + 121p^2 - 22p + 1 + 1 - 10p + 25p^2} \\
&= \sqrt{371p^2 - 62p + 3}
\end{aligned}$$

But area of triangle ABC = $\frac{\sqrt{3}}{2}$ units²

$$\therefore \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{\sqrt{3}}{2}$$

$$\therefore |\vec{AB} \times \vec{AC}| = \sqrt{3}$$

$$\therefore \sqrt{371p^2 - 62p + 3} = \sqrt{3}$$

$$\therefore 371p^2 - 62p + 3 = 3$$

$$\therefore 371p^2 - 62p = 0$$

$$\therefore p(371p - 62) = 0$$

$$\therefore p = 0 \text{ or } \frac{62}{371}$$

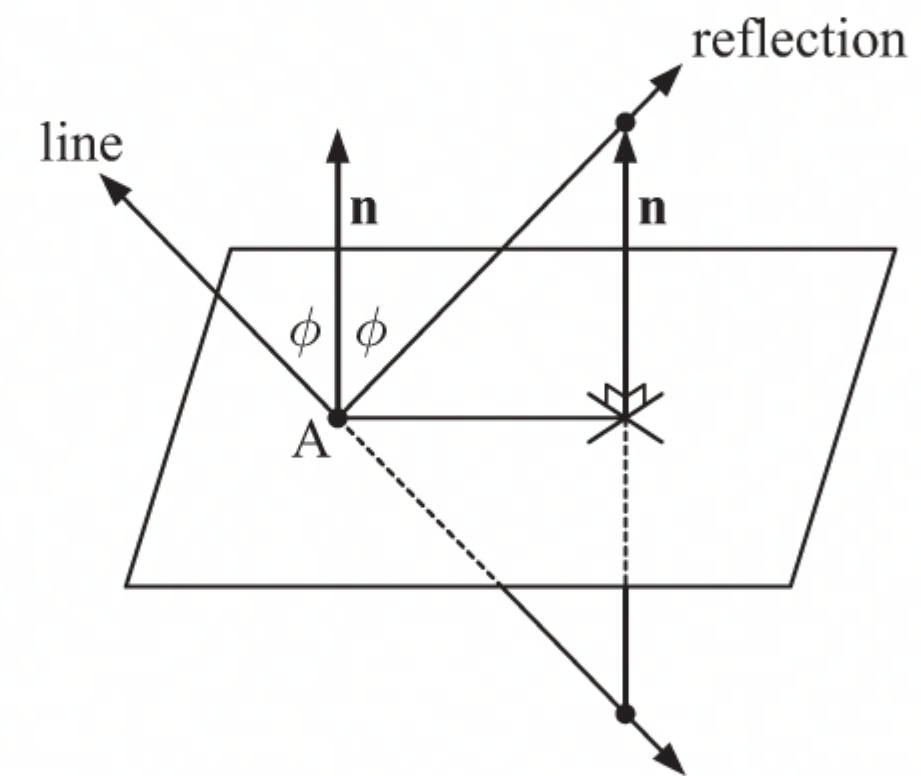
Using (*), when $p = 0$, $q = 3(0) + 2 = 2$

and when $p = \frac{62}{371}$, $q = 3(\frac{62}{371}) + 2 = \frac{928}{371}$.

118 The plane has normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

$$\begin{aligned}
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \mathbf{k} \\
&= (-2+1)\mathbf{i} - (2-0)\mathbf{j} + (-1-0)\mathbf{k} \\
&= -\mathbf{i} - 2\mathbf{j} - \mathbf{k} \\
&= \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}
\end{aligned}$$

The line has direction vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.



The acute angle ϕ between the normal and the line is given by

$$\begin{aligned}
\cos \phi &= \frac{\left| \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right|}{\sqrt{(-1)^2 + (-2)^2 + (-1)^2} \sqrt{1^2 + 2^2 + 2^2}} \\
&= \frac{|-1 - 4 - 2|}{\sqrt{6}\sqrt{9}} \\
&= \frac{7}{3\sqrt{6}} \\
\therefore \phi &\approx 0.309^c
\end{aligned}$$

Hence the acute angle between the line and its reflection is $2\phi \approx 0.618^c$.

119 a $\vec{AB} = \begin{pmatrix} 2 - (-1) \\ 1 - 2 \\ 3 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 4 - (-1) \\ -3 - 2 \\ 5 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix}$

So, an equation for the plane is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix}, \quad s, t \in \mathbb{R}$

$$\begin{aligned}
 \mathbf{b} \quad \text{A normal to the plane in } \mathbf{a} \text{ is } \mathbf{n} &= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 5 & -5 & 4 \end{vmatrix} \\
 &= \begin{vmatrix} -1 & 2 \\ -5 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ 5 & -5 \end{vmatrix} \mathbf{k} \\
 &= (-4 + 10)\mathbf{i} - (12 - 10)\mathbf{j} + (-15 + 5)\mathbf{k} \\
 &= 6\mathbf{i} - 2\mathbf{j} - 10\mathbf{k} \\
 &= 2(3\mathbf{i} - \mathbf{j} - 5\mathbf{k})
 \end{aligned}$$

So, $\begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$ is normal to the plane, and the line has direction vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

$$\begin{aligned}
 \text{If } \phi \text{ is the acute angle between the normal and the line, } \sin \phi &= \frac{\left| \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right|} \\
 \therefore \sin \phi &= \frac{|3 + 1 - 5|}{\sqrt{9 + 1 + 25} \sqrt{1 + 1 + 1}} \\
 &= \frac{1}{\sqrt{35}\sqrt{3}} \\
 \therefore \phi &= \sin^{-1}\left(\frac{1}{\sqrt{35}\sqrt{3}}\right) \\
 &\approx 5.60^\circ
 \end{aligned}$$

$$\mathbf{120} \quad \mathbf{a} \quad \mathbf{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
 \text{If } \theta \text{ is the acute angle between the planes, then } \cos \theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \\
 &= \frac{|2 - 2 - 3|}{\sqrt{14}\sqrt{6}} \\
 &= \frac{3}{\sqrt{84}} \\
 \therefore \theta &= \cos^{-1}\left(\frac{3}{\sqrt{84}}\right) \\
 &\approx 70.9^\circ
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{n}_1 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -1 & 3 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & 3 \\ -1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\
 &= 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k} \\
 \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 7 & 2 \\ 1 & 1 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} 7 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 7 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\
 &= 5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } \theta \text{ is the acute angle between the planes, then } \cos \theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \\
 &= \frac{|15 - 6 + 14|}{\sqrt{22}\sqrt{78}} \\
 &= \frac{23}{\sqrt{1716}} \\
 \therefore \theta &= \cos^{-1}\left(\frac{23}{\sqrt{1716}}\right) \\
 &\approx 56.3^\circ
 \end{aligned}$$

$$121 \quad \begin{cases} x + 2y + z = 4 \\ 2x - y + 2z = 3 \text{ where } k \text{ is a real number.} \\ 3x + 3y + kz = 1 \end{cases}$$

In augmented matrix form, the system is:

$$\begin{aligned} & \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -1 & 2 & 3 \\ 3 & 3 & k & 1 \end{array} \right) \\ & \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -5 & 0 & -5 \\ 0 & -3 & k-3 & -11 \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array} \\ & \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & k-3 & -11 \end{array} \right) \quad -\frac{1}{5}R_2 \rightarrow R_2 \\ & \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & k-3 & -8 \end{array} \right) \quad \begin{array}{l} R_3 + 3R_2 \rightarrow R_3 \end{array} \end{aligned}$$

Case 1: If $k = 3$, there are no solutions.

Since no two planes are parallel, the line of intersection of any two planes is parallel to the third plane.

Case 2: If $k \neq 3$, there is a unique solution.

Using row 3, $(k-3)z = -8$

$$\therefore z = -\frac{8}{k-3} \quad \{k \neq 3\}$$

Using row 2, $y = 1$

Substituting into row 1, $x + 2 - \left(\frac{8}{k-3}\right) = 4$

$$\begin{aligned} \therefore x &= 2 + \frac{8}{k-3} \\ &= \frac{2k-6+8}{k-3} \\ &= \frac{2k+2}{k-3} \end{aligned}$$

So, if $k \neq 3$, there is a unique solution $x = \frac{2k+2}{k-3}$, $y = 1$, $z = -\frac{8}{k-3}$.

\therefore the planes meet at the point $\left(\frac{2k+2}{k-3}, 1, -\frac{8}{k-3}\right)$.

$$122 \quad \mathbf{a} \quad \text{The planes } 2x + 4y + z = 1 \text{ and } 3x + 5y = 1 \text{ have normals } \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \text{ respectively.}$$

$$\begin{aligned} \text{Let } \theta \text{ be the acute angle between the normals. Then } & \left| \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \right| = \sqrt{4+16+1}\sqrt{9+25+0} \cos \theta \\ & \therefore |6+20+0| = \sqrt{21}\sqrt{34} \cos \theta \\ & \therefore \cos \theta = \frac{26}{\sqrt{21}\sqrt{34}} \\ & \therefore \theta \approx 13.3^\circ \end{aligned}$$

$$\mathbf{b} \quad \begin{aligned} 2x + 4y + z &= 1 \quad \dots (1) \\ 3x + 5y &= 1 \quad \dots (2) \end{aligned}$$

Let $x = t$, then in (2), $y = \frac{1-3t}{5}$

Substituting into (1), $2t + 4\left(\frac{1-3t}{5}\right) + z = 1$

$$\therefore 10t + 4 - 12t + 5z = 5$$

$$\therefore 5z = 1 + 2t$$

$$\therefore z = \frac{1+2t}{5}$$

So, the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} + t \begin{pmatrix} 1 \\ -\frac{3}{5} \\ \frac{2}{5} \end{pmatrix}, t \in \mathbb{R}$

This solution is an equation of the line of intersection of the two planes.

c If the points lie on the plane $5x + 13y + 7z = 4$ then $5t + 13\left(\frac{1-3t}{5}\right) + 7\left(\frac{1+2t}{5}\right) = 4$
 $\therefore 25t + 13 - 39t + 7 + 14t = 20$
 $\therefore 20 = 20$

Hence, the line of intersection of the first two planes lies on the third plane. This means that all points on this line lie on all three planes.

123 a P_1 has normal vector $\mathbf{n}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, and P_2 has normal vector $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

If θ is the acute angle between the plane, then $\cos \theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|}$
 $= \frac{|2 + 0 - 1|}{\sqrt{2^2 + 1^2 + 1^2} \sqrt{1^2 + (-1)^2}}$
 $= \frac{1}{\sqrt{6}\sqrt{2}}$
 $\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}\sqrt{2}}\right)$
 $\approx 73.2^\circ$

b The Cartesian equation of P_1 is $2x + y + z = 2$, and the Cartesian equation of P_2 is $x - z = 5$.

Letting $z = t$, $x = 5 + t$ and so $2(5 + t) + y + t = 2$
 $\therefore y = -8 - 3t$

So, the equation of the line of intersection is $\mathbf{r} = \begin{pmatrix} 5 \\ -8 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, t \in \mathbb{R}$.

TOPIC 4 SKILL BUILDER QUESTIONS

- 1 a** The sample size of only 10 students from a total of 500 is far too small, so this approach may produce a coverage error.
- b** The tape measure only allows Gerard to *estimate* the exact height of each student. With only 10 students being measured, any errors will significantly impact the results, so this approach may produce a measurement error.

2

	Boys	Girls
Year 8	135	140
Year 9	130	145
Year 10	125	130

- a i** A survey of 50 students can be completed in a reasonable time frame. It is also large enough that the results can be representative of the whole student body.
- ii** Surveying all students is time-consuming and often impractical. A non-response error may be produced if students are absent.

b Total number of students = $135 + 140 + 130 + 145 + 125 + 130 = 805$

i For the survey, $\frac{135}{805} \times 50 \approx 8$ Year 8 boys will be selected.

ii For the survey, $\frac{140 + 145 + 130}{805} \times 50 \approx 26$ girls will be selected.

c A stratified sample is better than a random sample in this case as a stratified sample will fairly represent each year level and gender. A random sample cannot guarantee such a fair representation.

- 3 a** This is a convenience sample because it is more convenient for Marie to sample the first 10 people to visit her office than to sample 10 random people from the whole building for example.

b The preferences of the first 10 people to visit Marie's office are likely to come from people who work with her. This may not be representative of the preferences of all people in the building, and so the sample may be biased.

c Marie could use a stratified sample where the subgroups may correspond to departments, floor number, and so on. In this way, a fair representation of preferences is more likely to be obtained.

- 4 a** The ticket inspector selects passengers at regular intervals, so the sampling method used is systematic sampling.

b The first passenger to be checked is the 8th passenger.

So, the next 6 passengers to be checked are the 28th, 48th, 68th, 88th, 108th, and 128th passengers.

c 5000 passengers left the terminal, and every 20th passenger was checked.

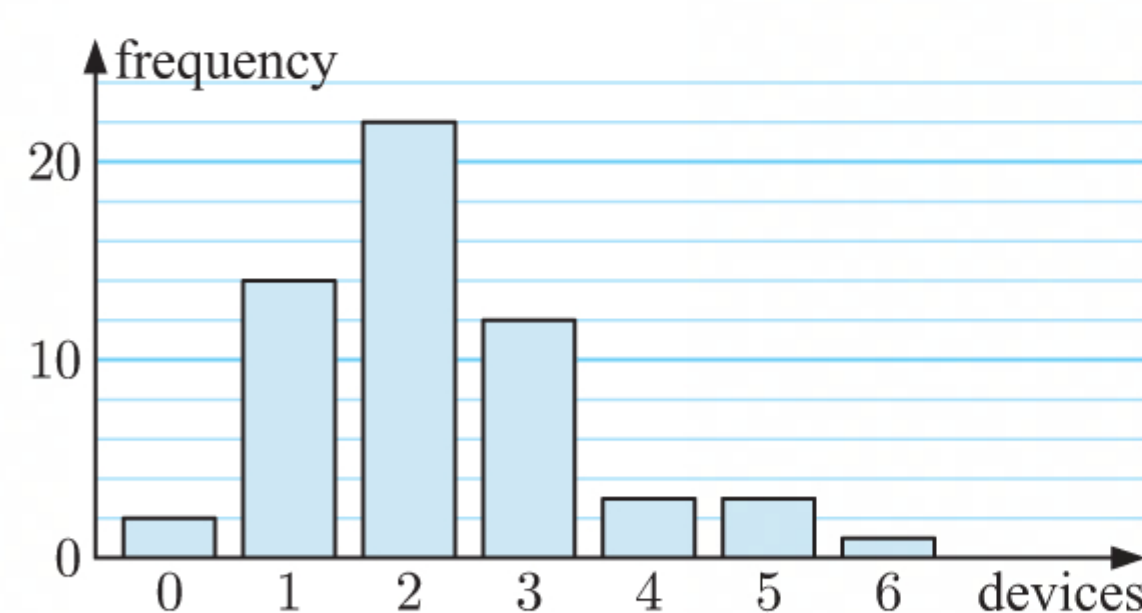
$$\therefore \frac{5000}{20} = 250 \text{ passengers were checked.}$$

- 5 a** $2 + 14 + 22 + 12 + 3 + 3 + 1 = 57$ people were surveyed.

b The mode of the data is 2 devices.

c $\frac{14 + 22}{57} \times 100\% \approx 63.2\%$ of people browsed the internet using 1 or 2 devices.

d The data is positively skewed with no outliers.



6 a

$$\frac{9 + 10 + a + 13 + b + 16 + 21}{7} = 14$$

$$\therefore \frac{69 + a + b}{7} = 14$$

$$\therefore 69 + a + b = 98$$

$$\therefore a + b = 29$$

Now, a and b are integers such that $10 \leq a \leq 13$ and $13 \leq b \leq 16$.

\therefore the only possible solution is $a = 13$ and $b = 16$.

b Since $n = 6$, $\frac{n+1}{2} = 3.5$

So the median is the average of the 3rd and 4th ordered data values.

The ordered data set is: $\cancel{1} \ \cancel{5} \ \underline{9 \ 11} \ \cancel{16} \ \cancel{p}$
two middle data values

$$\therefore \text{median} = \frac{9 + 11}{2} = 10$$

$$\text{Now, } \frac{1 + 5 + 9 + 11 + 16 + p}{6} = 10$$

$$\therefore \frac{42 + p}{6} = 10$$

$$\therefore 42 + p = 60$$

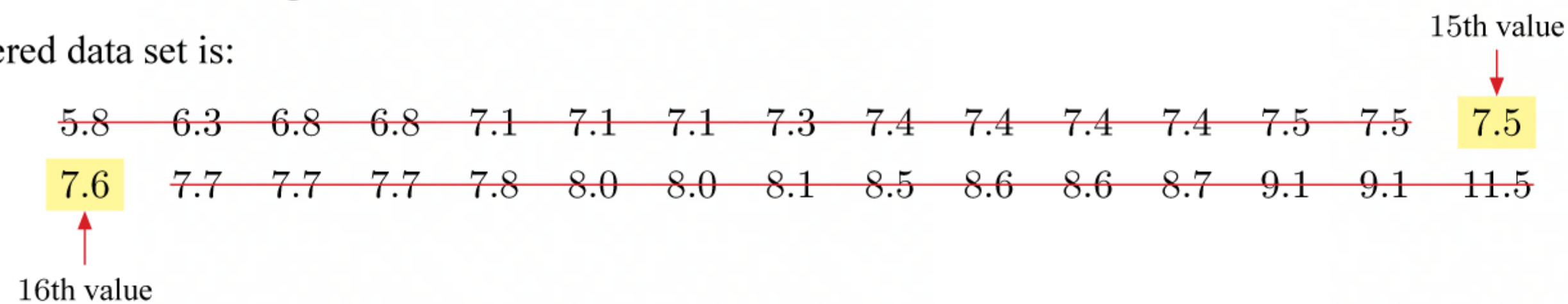
$$\therefore p = 18$$

$$\begin{aligned}
 7 \quad a \quad \text{mean} &= \frac{7.5 + 6.8 + \dots + 8.5}{30} \\
 &= \frac{233.1}{30} \\
 &= 7.77 \text{ hours}
 \end{aligned}$$

$$\text{As } n = 30, \frac{n+1}{2} = 15.5$$

So the median is the average of the 15th and 16th data values.

The ordered data set is:



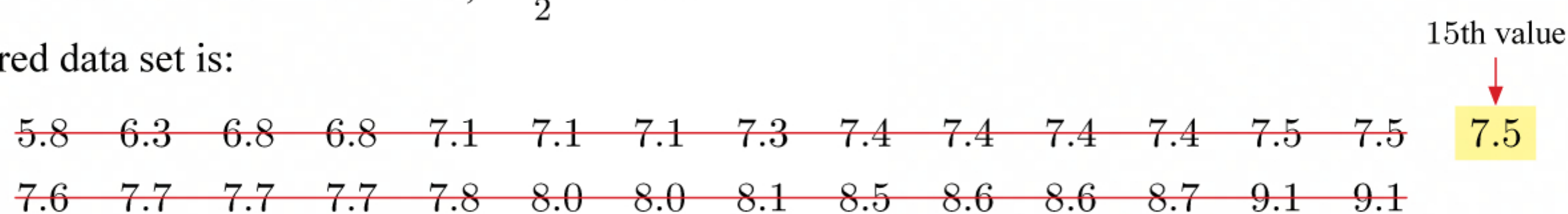
$$\therefore \text{median} = \frac{7.5 + 7.6}{2} = 7.55 \text{ hours}$$

b The outlier is 11.5 hours.

$$\begin{aligned}
 c \quad i \quad \text{mean} &= \frac{7.5 + 6.8 + \dots + 8.5}{29} \\
 &= \frac{221.6}{29} \\
 &\approx 7.64 \text{ hours}
 \end{aligned}$$

$$\text{As } n = 29 \text{ with the outlier removed, } \frac{n+1}{2} = 15.$$

The ordered data set is:



$$\therefore \text{median} = 7.5 \text{ hours}$$

ii The measure of centre which is most affected by extreme values is the mean. So, the mean is most affected if the outlier is removed.

8 a

Number of cars	Frequency	Cumulative frequency
0	78	78
1	117	195
2	69	264
3	18	282
4	2	284
Total	284	

$$\begin{aligned}
 b \quad i \quad \text{mean} &= \frac{0 \times 78 + 1 \times 117 + 2 \times 69 + 3 \times 18 + 4 \times 2}{284} \\
 &= \frac{317}{284} \\
 &\approx 1.12 \text{ cars}
 \end{aligned}$$

ii There are 284 data values, so $n = 284$. $\frac{n+1}{2} = 142.5$, so the median is the average of the 142nd and 143rd ordered data values.

From the cumulative frequency column, the 79th to 195th ordered data values are 1 car.

\therefore the 142nd and 143rd data values are 1 car.

$$\therefore \text{median} = \frac{1+1}{2} = 1 \text{ car}$$

iii Looking down the frequency column, the highest frequency is 117. This corresponds to 1 car, so the mode is 1 car.

Weekly rent (€ <i>r</i>)	Frequency (<i>f</i>)	Midpoint (<i>x</i>)	Product (<i>x f</i>)
$80 \leq r < 100$	3	90	270
$100 \leq r < 120$	15	110	1650
$120 \leq r < 140$	26	130	3380
$140 \leq r < 160$	30	150	4500
$160 \leq r < 180$	14	170	2380
$180 \leq r < 200$	1	190	190
<i>Total</i>	$\sum f = 89$		$\sum x f = 12\,370$

$$\begin{aligned} \text{a } \bar{x} &= \frac{\sum x f}{\sum f} \\ &= \frac{12\,370}{89} \\ &\approx 139 \end{aligned}$$

\therefore the mean weekly rent was about €139.

$$\begin{aligned} \text{b } P(r \geq 140) &= \frac{30 + 14 + 1}{89} \\ &\approx 0.506 \end{aligned}$$

10 a The ordered data set is:

4 9 10 12 12 14 14 15 16 16 16 17 18 18 18 20 23 26 31 {19 data values}

\downarrow \downarrow \downarrow
 $Q_1 = 12$ median = 16 $Q_3 = 18$

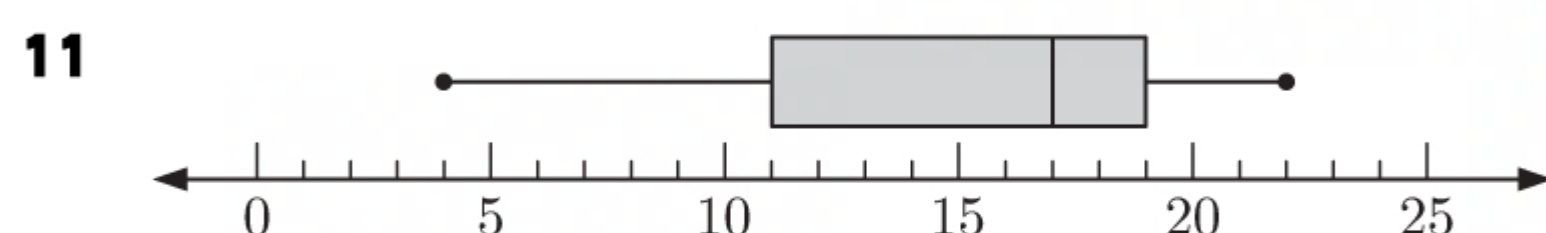
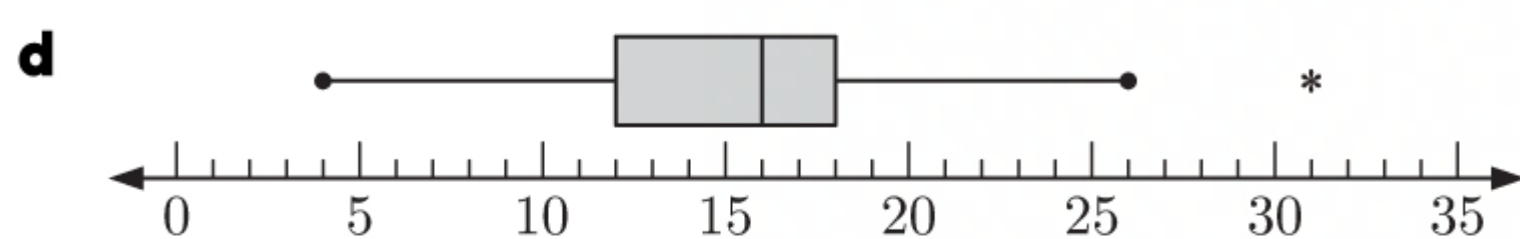
So the five-number summary is:

$$\left\{ \begin{array}{l} \text{minimum} = 4 \\ Q_1 = 12 \\ \text{median} = 16 \\ Q_3 = 18 \\ \text{maximum} = 31 \end{array} \right.$$

$$\begin{aligned} \text{b } \text{IQR} &= Q_3 - Q_1 \\ &= 18 - 12 \\ &= 6 \end{aligned}$$

$$\begin{array}{ll} \text{c } \text{upper boundary} & \text{lower boundary} \\ = \text{upper quartile} + 1.5 \times \text{IQR} & = \text{lower quartile} - 1.5 \times \text{IQR} \\ = 18 + 1.5 \times 6 & = 12 - 1.5 \times 6 \\ = 27 & = 3 \end{array}$$

31 is above the upper boundary, so it is an outlier.



a minimum value = 4 cm

c median = 17 cm

e lower quartile = 11 cm

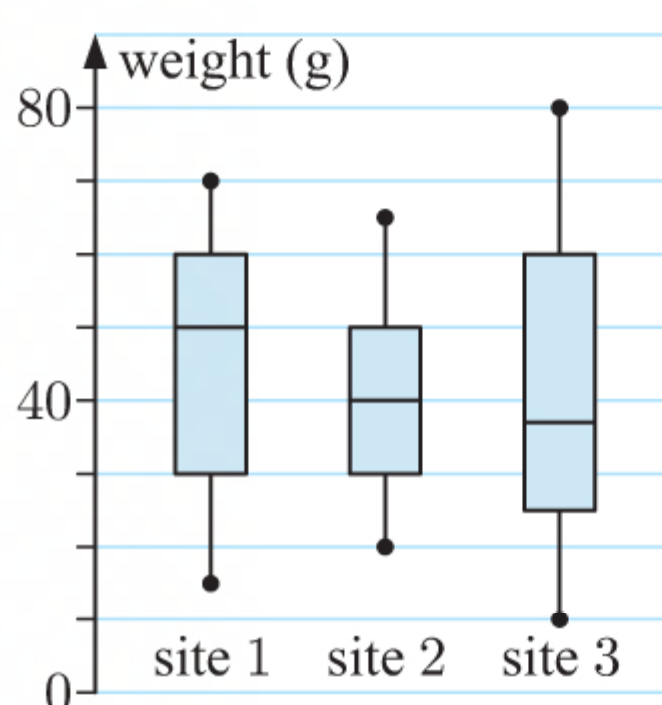
$$\begin{aligned} \text{f range} &= \text{maximum} - \text{minimum} \\ &= 22 - 4 \\ &= 18 \text{ cm} \end{aligned}$$

b maximum value = 22 cm

d upper quartile = 19 cm

$$\begin{aligned} \text{g } \text{IQR} &= Q_3 - Q_1 \\ &= 19 - 11 \\ &= 8 \text{ cm} \end{aligned}$$

12



a The five-number summary for site 1 is:

$$\begin{cases} \text{minimum} = 15 & Q_1 = 30 \\ \text{median} = 50 & Q_3 = 60 \\ \text{maximum} = 70 \end{cases}$$

b Site 3 has the greatest range of weights.

c The weights of fungi have the least variation at site 2.

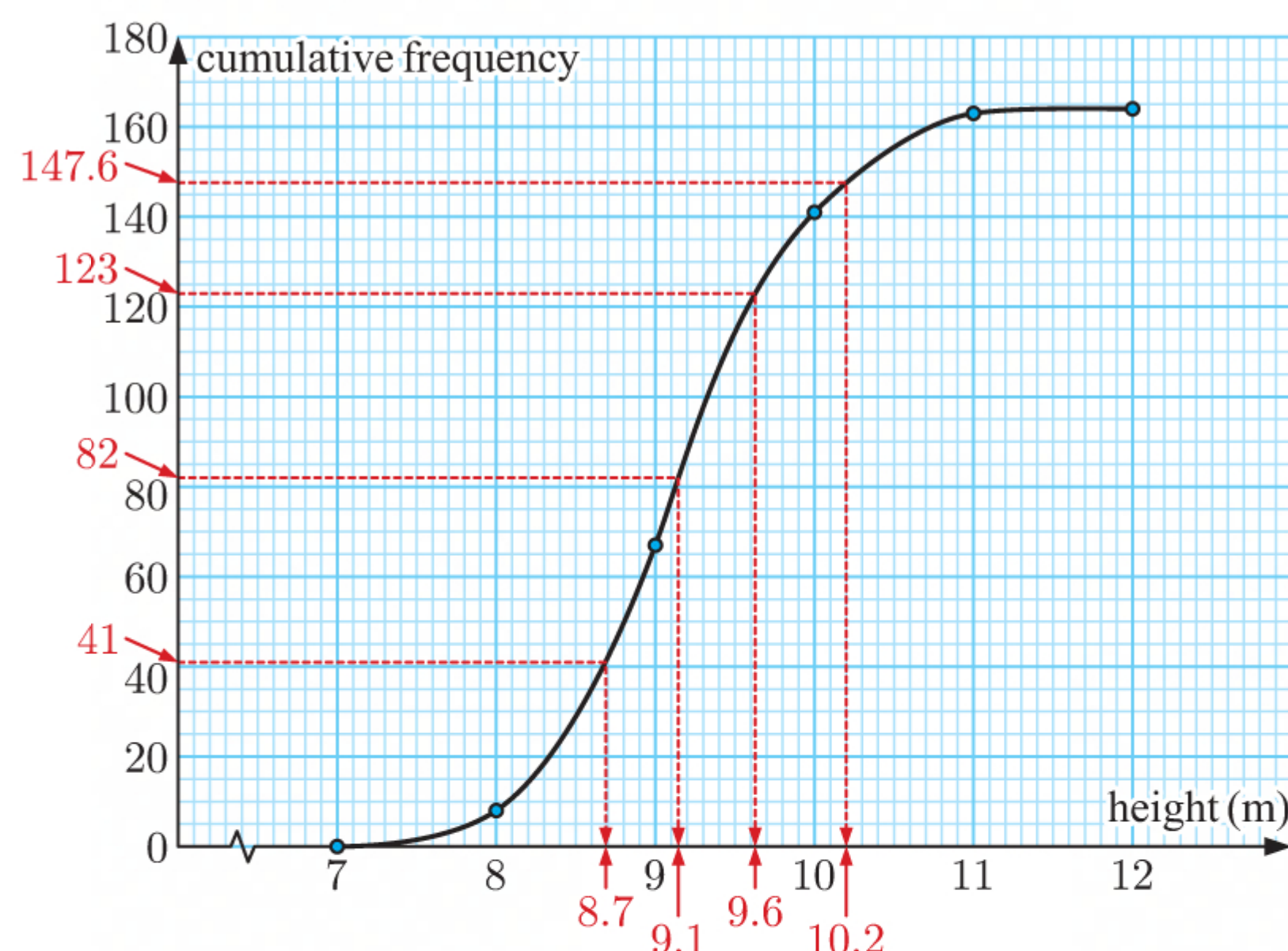
d Site 1 has the highest median weight of fungi.

e Site 1 has the highest proportion of weights above 40 grams.

13

Height (h m)	Frequency	Cumulative frequency
$7 \leq h < 8$	8	8
$8 \leq h < 9$	59	67
$9 \leq h < 10$	74	141
$10 \leq h < 11$	22	163
$11 \leq h < 12$	1	164

a



b The median is the 50th percentile. As 50% of 164 is 82, we start with the cumulative frequency 82 and find the corresponding height.

 From the graph, the median ≈ 9.1 m.

 c Q_1 is the 25th percentile. As 25% of 164 is 41, we start with the cumulative frequency 41 and find the corresponding height.

 From the graph, $Q_1 \approx 8.7$ m

 Q_3 is the 75th percentile. As 75% of 164 is 123, we start with the cumulative frequency 123 and find the corresponding height.

 From the graph, $Q_3 \approx 9.6$ m

$$\text{IQR} = Q_3 - Q_1$$

$$\approx 9.6 - 8.7$$

$$\approx 0.9 \text{ m}$$

d As 90% of 164 is 147.6, we start with the cumulative frequency 147.6 and find the corresponding height.

 The 90th percentile ≈ 10.2 m which means that 90% of trees are shorter than about 10.2 m.

 14 Anthony: $1\frac{1}{2}, 2, 2\frac{1}{2}, 4, 4\frac{1}{2}, 3, 3\frac{1}{2}, 5, 6, 6$

 Katherine: $3, 3\frac{1}{2}, 4, 3, 3, 3\frac{1}{2}, 4, 4, 4\frac{1}{2}, 4$

a Using technology:

Anthony:

1-Variable	
\bar{x}	=3.8
Σx	=38
Σx^2	=167
σx	=1.50332963
sx	=1.58464857
n	=10

Katherine:

1-Variable	
\bar{x}	=3.65
Σx	=36.5
Σx^2	=135.75
σx	=0.50249378
sx	=0.52967495
n	=10

 The mean $\mu = 3.8$ hours and the standard deviation $\sigma \approx 1.50$ hours.

 The mean $\mu = 3.65$ hours and the standard deviation $\sigma \approx 0.502$ hours.

b Anthony's mean is higher than Katherine's, so Anthony generally practised for longer.

c Katherine's standard deviation is lower than Anthony's, so there is less deviation from the mean for her data set. Katherine therefore practised more consistently than Anthony.

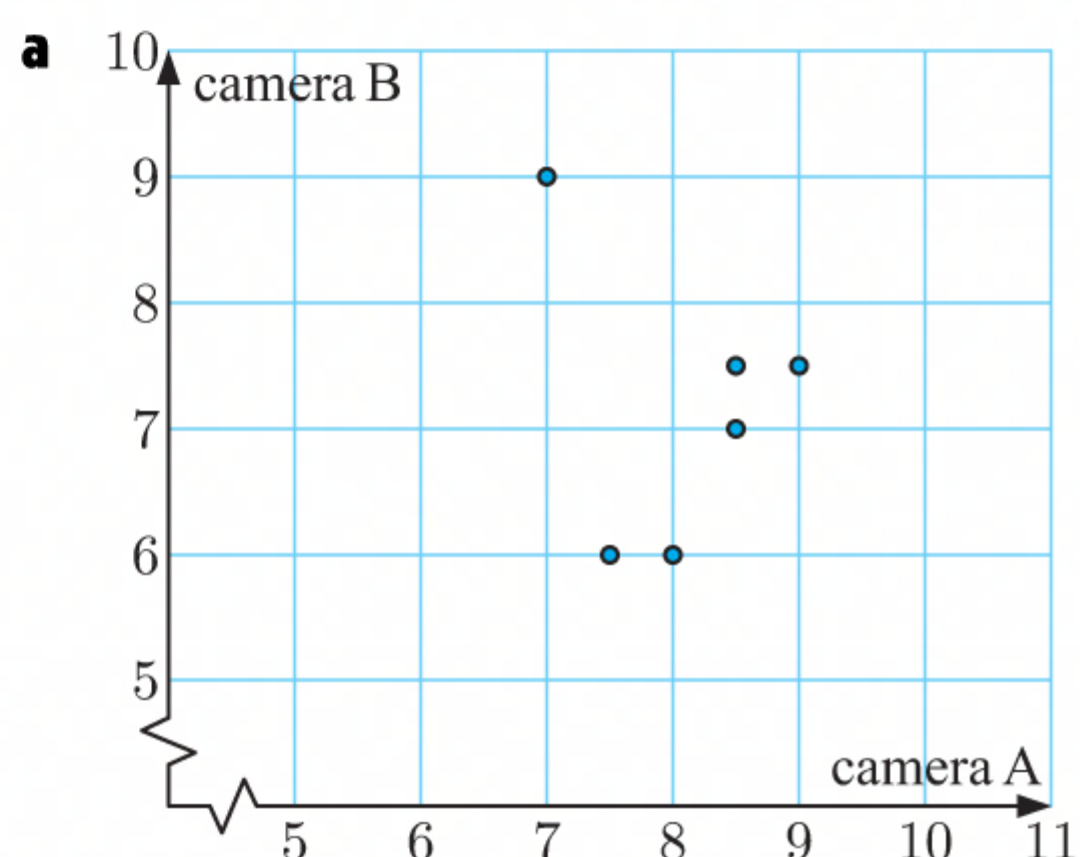
15	<i>Mark</i>	3	4	5	6	7	8	9	10
	<i>Frequency</i>	1	3	5	8	4	2	0	1

Using technology:

	1-Variable
\bar{x}	=5.91666666
Σx	=142
Σx^2	=894
σx	=1.49768339
sx	=1.52989532
n	=24

The mean test score $\mu \approx 5.92$, and the population standard deviation $\sigma \approx 1.50$.

16	<i>Camera A</i>	8.5	8	9	7	8.5	7.5
	<i>Camera B</i>	7	6	7.5	9	7.5	6

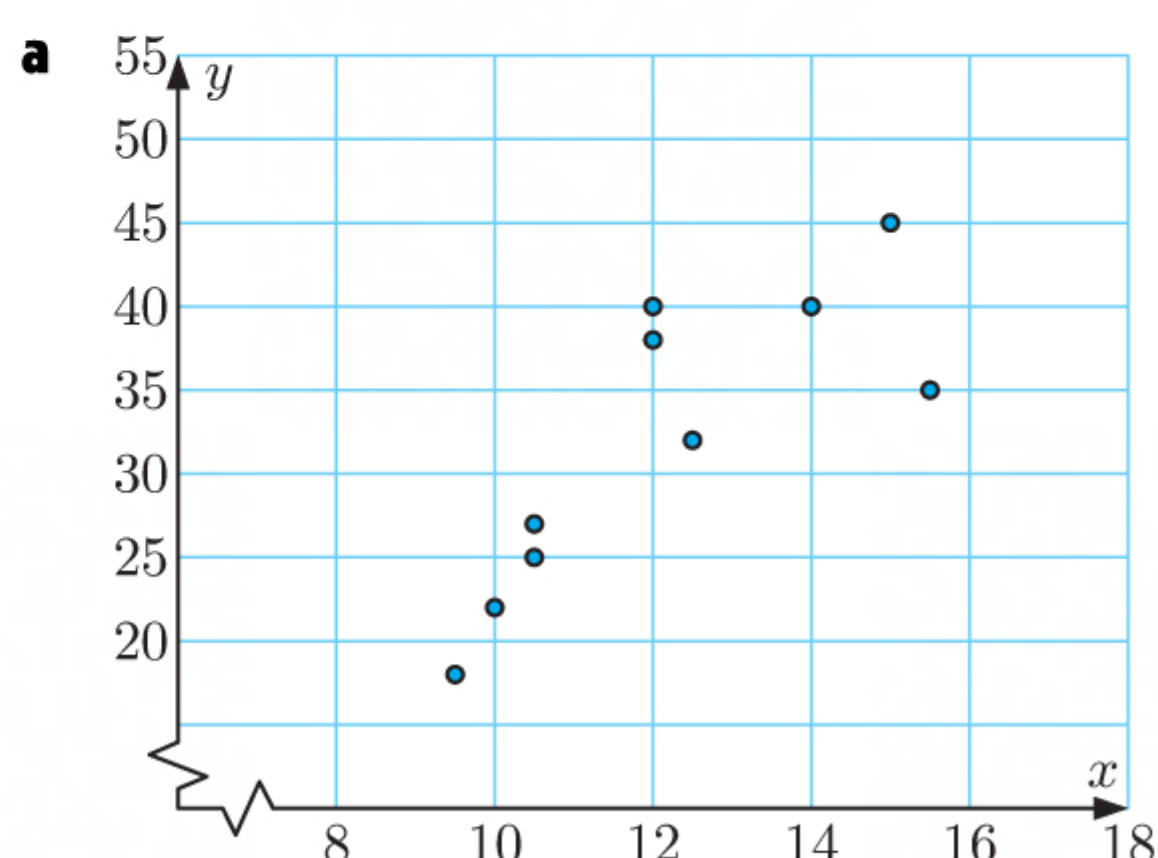
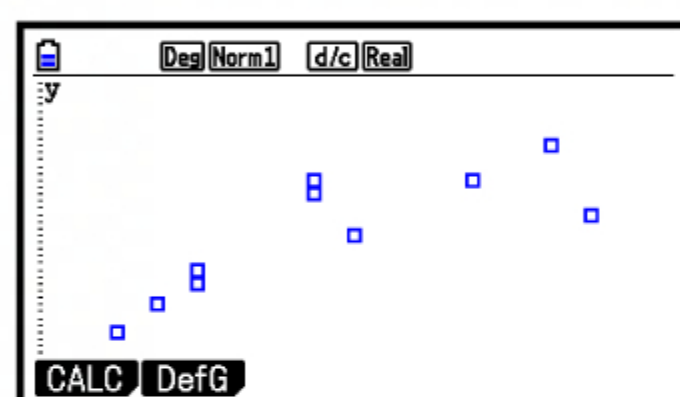


b The point (7, 9) appears to be an outlier. This corresponds to the review scoring camera A a 7, and camera B a 9.

- c**
- With the outlier removed, there appears to be a strong, positive, linear correlation between camera A's scores and camera B's scores.
 - No, an increase in camera A's scores is not likely to cause an increase in camera B's scores. It is more likely that both scores are related to the preferences of each reviewer.

17	<i>Language (x)</i>	12.5	15.0	10.5	12.0	9.5	10.5	15.5	10.0	14.0	12.0
	<i>Mathematics (y)</i>	32	45	27	38	18	25	35	22	40	40

	1-Variable
\bar{x}	=12.5
Σx	=150
Σx^2	=1825
σx	=3.45284197
sx	=3.45284197
n	=12



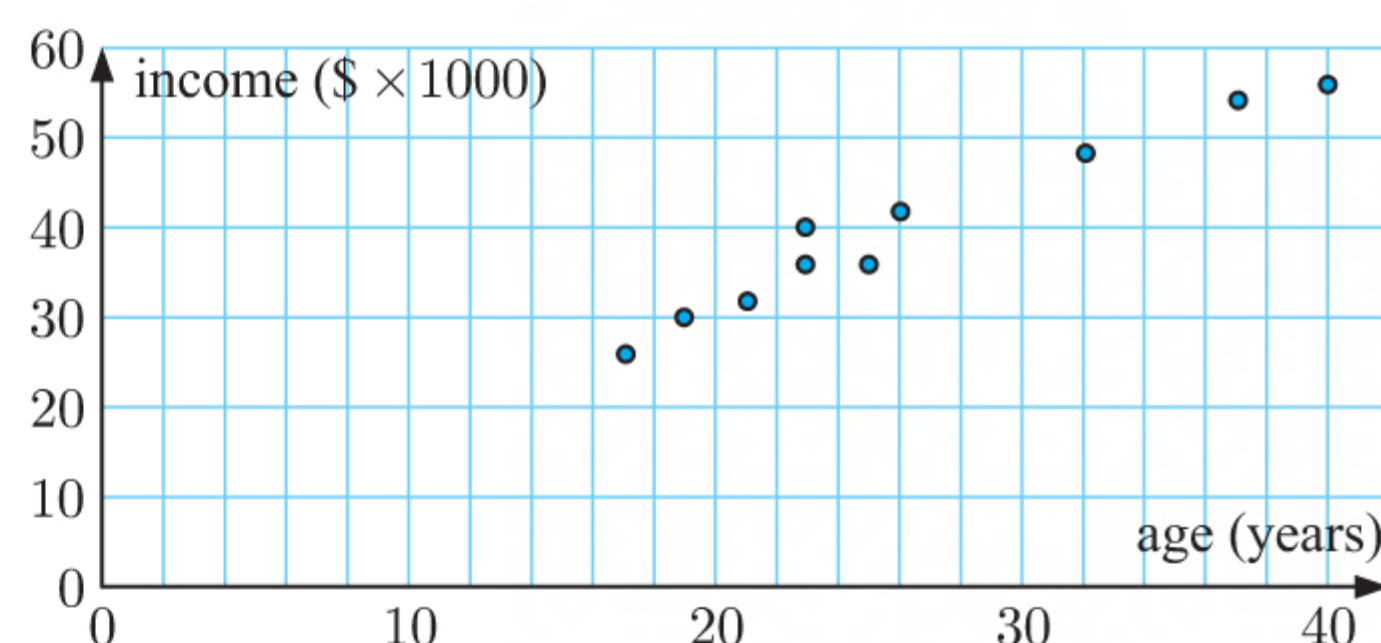
b

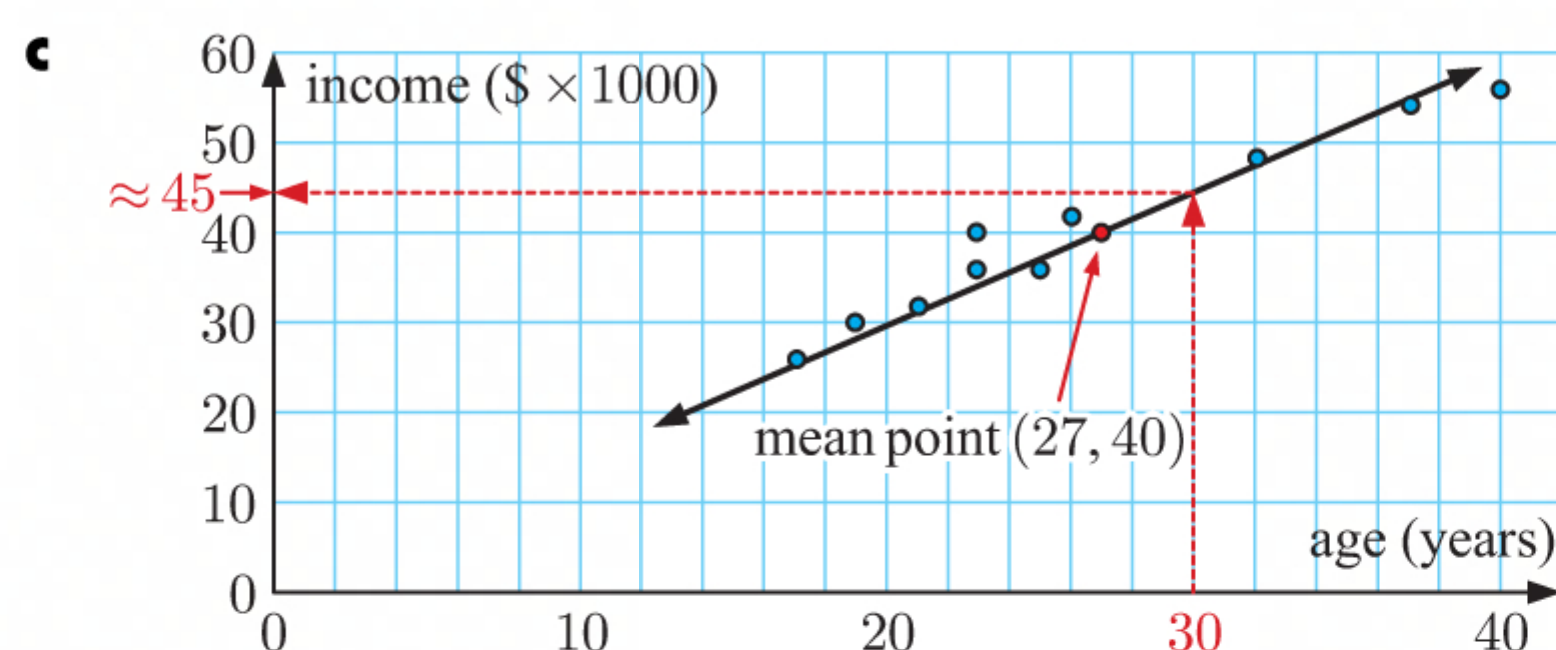
	LinearReg(ax+b)
a	=3.45284197
b	=-9.7520299
r	=0.8188878
r^2	=0.67057723
MSe	=29.3021549
$y=ax+b$	

So, $r \approx 0.819$.

- c** The data suggests that there is a moderate, positive, linear correlation between the students' language scores and their mathematics scores. So, "Those who do well in languages also do well in mathematics." is a moderately reasonable statement.

- 18**
- There is a strong, positive correlation between the age of an individual and their annual income.
 - No, the relationship is more likely dependent on the amount of professional experience or qualifications an individual has.





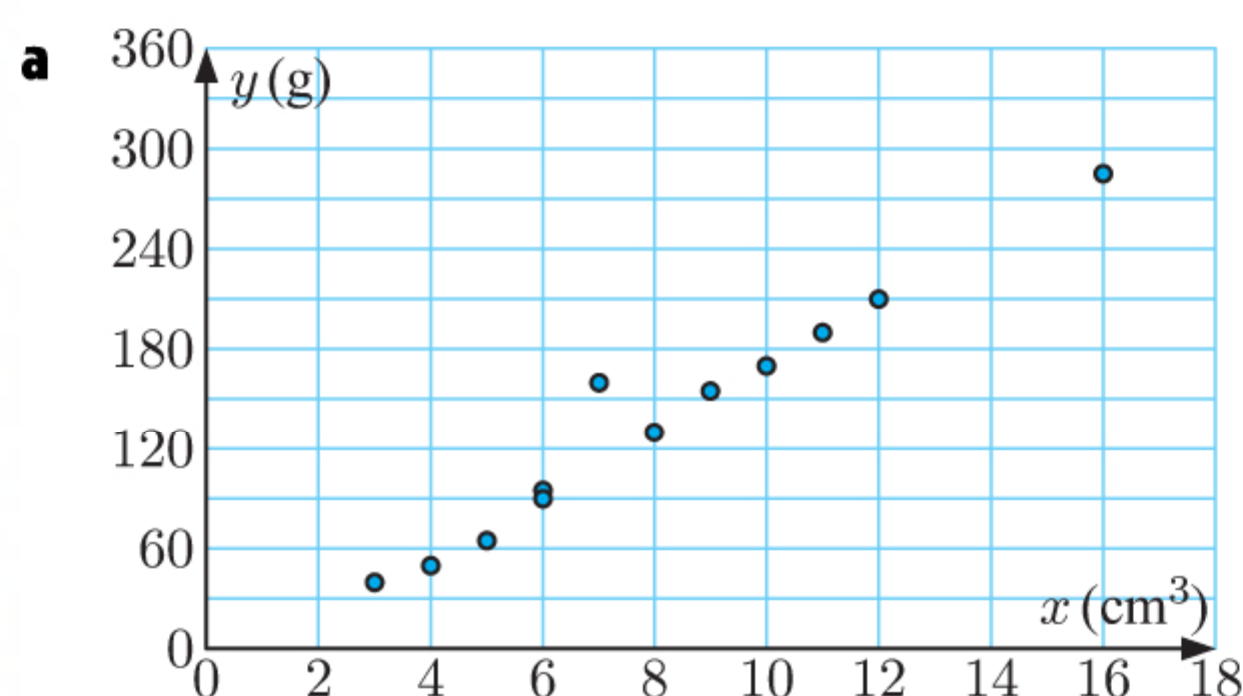
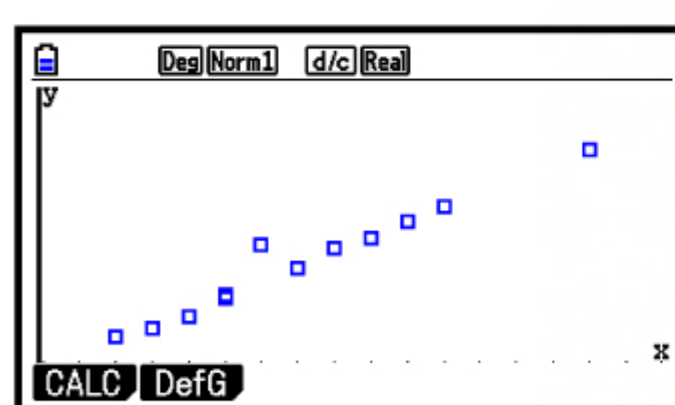
d When $x = 30$, $y \approx 45$.

The annual income of someone who is 30 years old is approximately \$45 000. This is an interpolation, so the estimate is reliable.

19

Sample	A	B	C	D	E	F	G	H	I	J	K	L
Volume ($x \text{ cm}^3$)	3	6	4	7	16	8	5	12	9	6	10	11
Mass ($y \text{ g}$)	40	95	50	160	285	130	65	210	155	90	170	190

	List 1	List 2	List 3	List 4
SUB				
1	3	40		
2	6	95		
3	4	50		
4	7	160		



b

LinearReg(ax+b)
a = 19.1171662
b = -17.86376
r = 0.98019806
r ² = 0.96078823
MSe = 228.081743
y = ax + b

c There appears to be a strong, positive correlation between the *volume* of a sample of silver and its *mass*.

d The data point (7, 160) which corresponds to sample D appears to be an outlier. We therefore agree with the jeweller that there is a fake sample.

So, $r \approx 0.980$.

e i

LinearReg(ax+b)
a = 19.4604316
b = -24.676258
r = 0.9987246
r ² = 0.99745083
MSe = 16.3069544
y = ax + b

ii When $x = 7$, $y \approx 19.5(7) - 24.7$
 ≈ 112

So, a sample of silver with volume 7 cm^3 would weigh approximately 112 g.

Using technology, the regression line is
 $y \approx 19.5x - 24.7$.

20

Study time ($x \text{ h}$)	7	6	3	16	15	11	18	32	20
Result ($y \%$)	56	42	25	80	65	60	85	96	90

a

LinearReg(ax+b)
a = 2.43264433
b = 31.9579472
r = 0.91326692
r ² = 0.83405646
MSe = 104.88158
y = ax + b

Using technology, the least squares regression line is $y \approx 2.43x + 32.0$.

b From **a**, $r \approx 0.913$.

So, there is a strong, positive correlation between the number of hours that a student studies and their examination result.

c Yes, this is a causal relationship as spending more time studying for the examination is likely to cause a better result.

- d** When $y = 70$, $70 \approx 2.43x + 32.0$
 $\therefore 38 \approx 2.43x$
 $\therefore x \approx 15.6$

So, Tony studied for approximately 15.6 hours.

- e** The y -intercept of the line of best fit ≈ 32.0 . This indicates that if a student did not spend any time studying, they would obtain a result of 32% on average.

The gradient of the line of best fit ≈ 2.43 . This indicates that for every additional hour of study, the result obtained increases by an average of 2.43%.

21

Time (t days)	0	1	2	3	4	5	6	7	8	9
Height (h mm)	5	5.7	5.7	6.2	6.8	7.1	8	8.3	9	9.3

a

So, $r \approx 0.993$.

- b** r is very close to 1 which indicates a very strong correlation between the variables.
 The sign of r is positive which indicates that the variables are positively correlated.
 An increase in one variable results in an increase in the other.

- c** $h \approx 0.4879t + 4.9145$

- i** When $t = 14$, $h \approx 0.4879(14) + 4.9145$
 ≈ 11.7

\therefore after 14 days, the grass is about 11.7 mm high.

- ii** When $h = 20$, $20 \approx 0.4879t + 4.9145$

$$\therefore 15.0855 \approx 0.4879t$$

$$\therefore t \approx 30.9$$

\therefore the grass reaches a height of 20 mm after about 30.9 days.

22

Distance from shore (x km)	3.7	1.3	4.3	2.8	0.9
Fish caught (y)	5	4	9	5	2

- a** We should use the regression line of x against y , since the number of fish caught can be more precisely measured than the distance from the shore.

b

The regression line of x against y is $x \approx 0.508y + 0.0615$.

When $x = 7$, $7 \approx 0.508y + 0.0615$

$$\therefore 6.9385 \approx 0.508y$$

$$\therefore y \approx 13.7$$

If the distance from the shore is 7 km, we expect about 14 fish to be caught.

- c** The estimate in **b** is an extrapolation, so it may not be reliable.

23

Division	2017	2018	2019
1	4	5	5
2	6	7	8
3	13	12	14
4	18	10	14
5	20	17	16
Total	61	51	57

- a** $P(\text{player in the 2017 tournament played in division 1}) \approx \frac{4}{61}$ ← number of division 1 players in 2017 tournament
← total number of players in 2017 tournament
 ≈ 0.0656

- b** There were $13 + 12 + 14 = 39$ division 3 players in total,
and $61 + 51 + 57 = 169$ players in total.

$$\therefore P(\text{player in any of the past tournaments played in division 3}) \approx \frac{39}{169} \\ \approx 0.231$$

- c** In the 2019 tournament, 8 players played in division 2 and 14 players played in division 4. So, $57 - 8 - 14 = 35$ players in the 2019 tournament did *not* play in division 2 or 4.

$$\therefore P(\text{player in the 2019 tournament did not play in division 2 or 4}) \approx \frac{35}{57} \\ \approx 0.614$$

24 a

	< 40	40 - 59	≥ 60	Total
Male	56	127	419	602
Female	75	113	230	418
Total	131	240	649	1020

- b i** 602 out of the 1020 patients were male.

$$\therefore P(\text{male}) \approx \frac{602}{1020} \approx 0.590$$

- ii** 75 out of the 1020 patients were female and younger than 40.

$$\therefore P(\text{female and younger than 40}) \approx \frac{75}{1020} \approx 0.0735$$

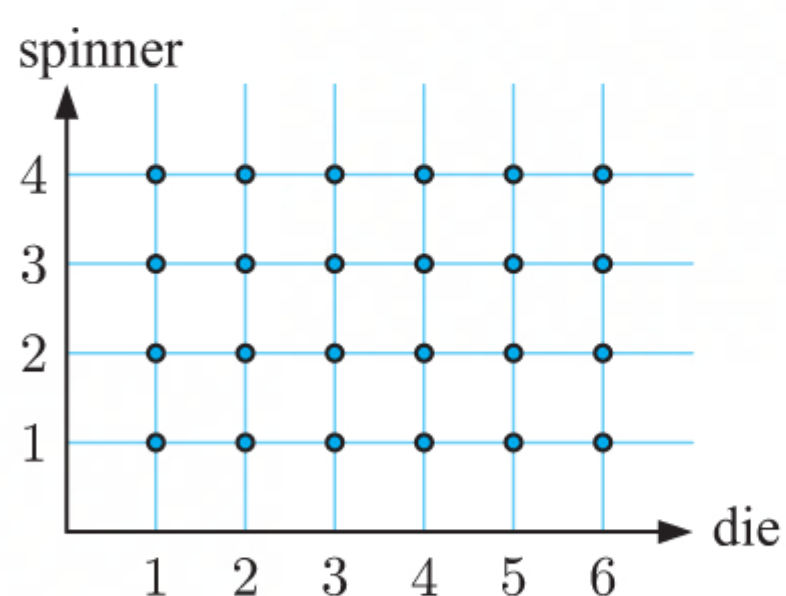
- iii** 230 out of the 418 female patients were 60 or older.

$$\therefore P(60 \text{ or older, given they were female}) \approx \frac{230}{418} \approx 0.550$$

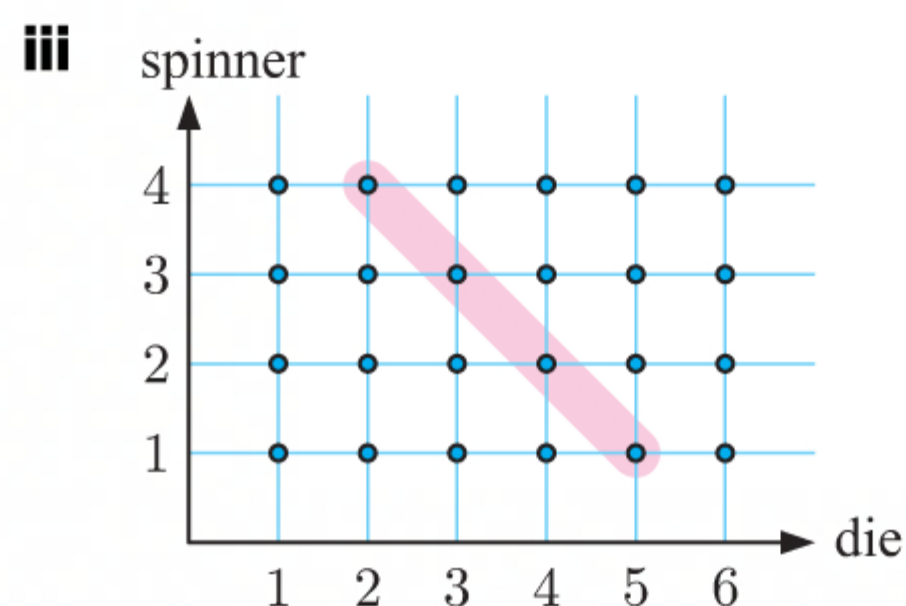
- iv** $127 + 419 = 546$ out of the $240 + 649 = 889$ patients who were 40 or older were male.

$$\therefore P(\text{male, given they were 40 or older}) \approx \frac{546}{889} \approx 0.614$$

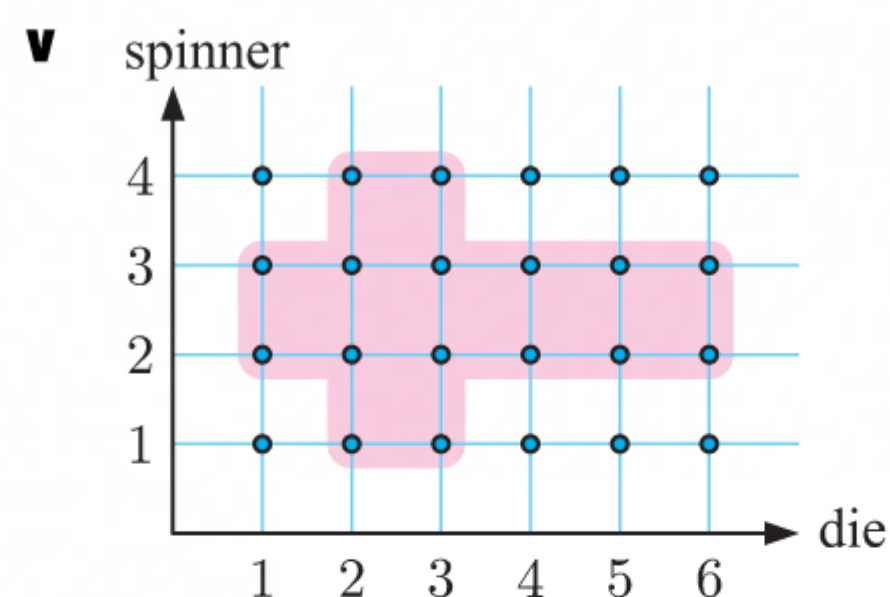
25 a



- b i** $P(\text{two 1s}) = \frac{1}{24}$

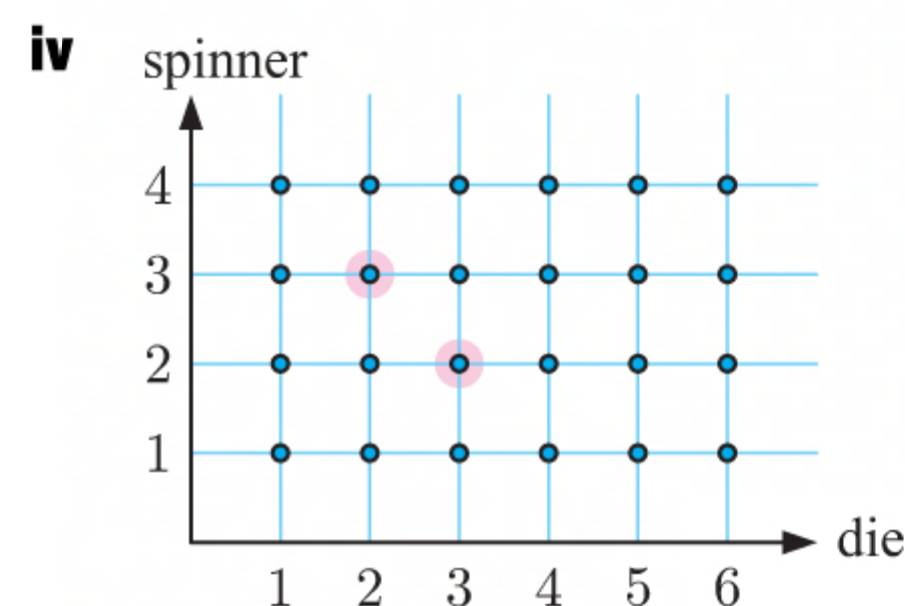


$$P(\text{a sum of 6}) = \frac{4}{24} \\ = \frac{1}{6}$$

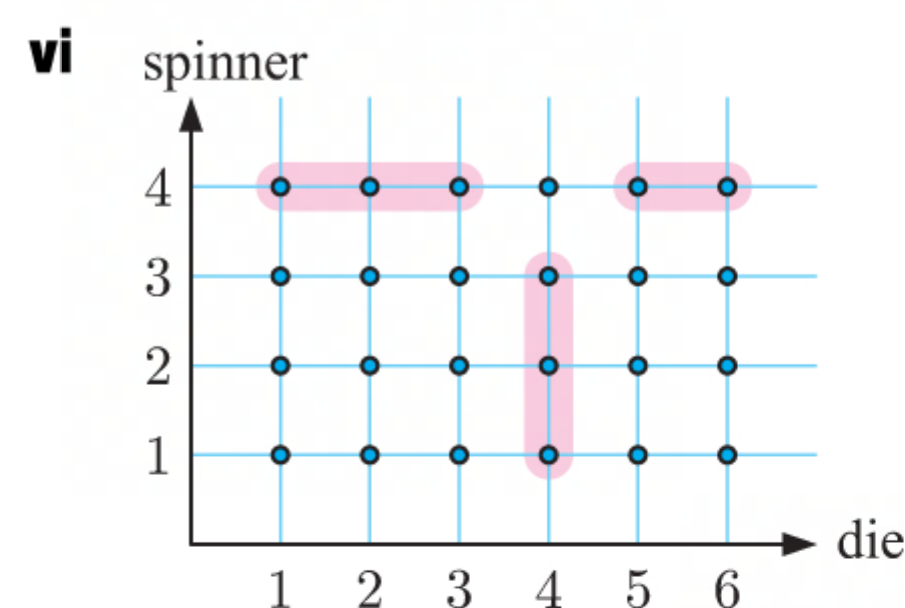


$$P(\text{a 2 or a 3 (or both)}) = \frac{16}{24} \\ = \frac{2}{3}$$

- ii** $P(\text{two 5s}) = 0$ {the spinner does not have a 5}



$$P(\text{a 2 and a 3}) = \frac{2}{24} \\ = \frac{1}{12}$$

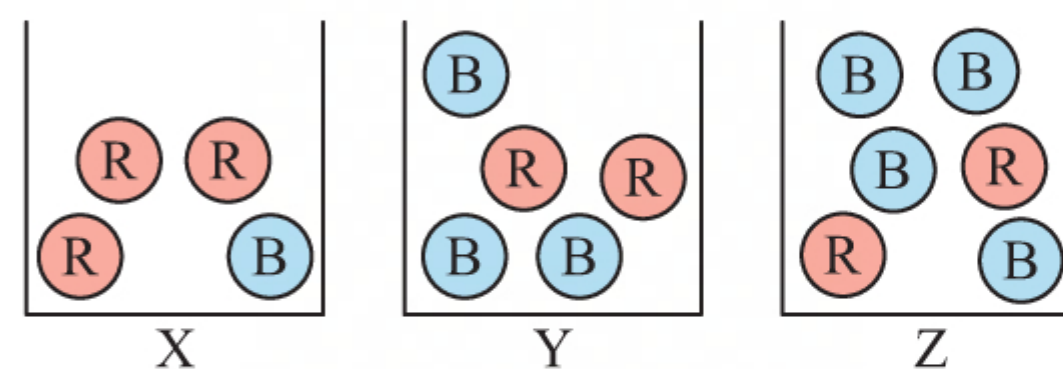
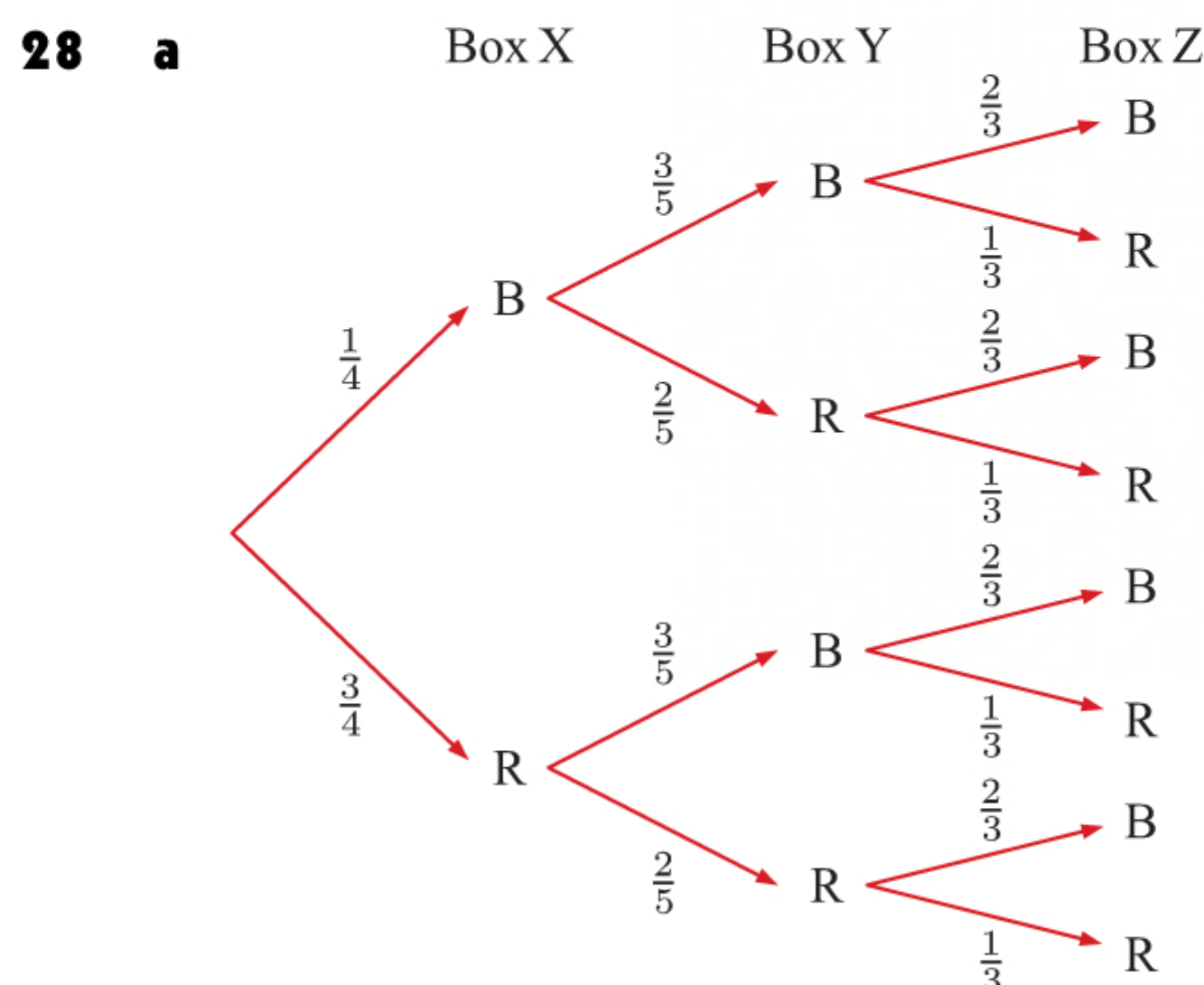


$$P(\text{exactly one 4}) = \frac{8}{24} \\ = \frac{1}{3}$$

26 a $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $\therefore 0.78 = 0.37 + 0.41 - P(A \cap B)$
 $\therefore P(A \cap B) = 0$

b Since $P(A \cap B) = 0$, A and B are mutually exclusive events.

27 $P(A \cup B) = 1 - P((A \cup B)')$ Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $= 1 - \frac{1}{12}$ $\therefore \frac{11}{12} = \frac{23}{50} + \frac{5}{7} - P(A \cap B)$
 $= \frac{11}{12}$ $\therefore P(A \cap B) = \frac{541}{2100}$

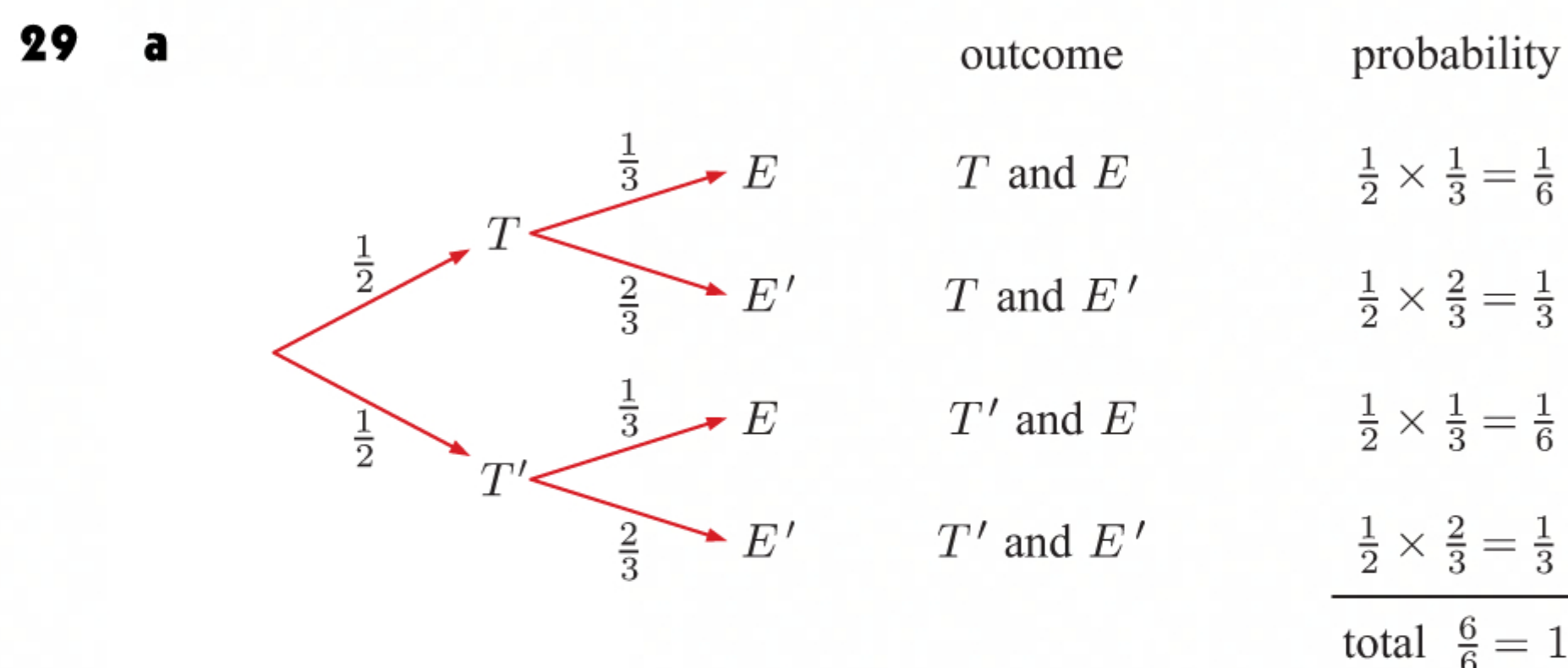


b i $P(\text{exactly 2 red balls are drawn})$
 $= P(RRB) + P(RBR) + P(BRR)$
 $= \frac{3}{4} \times \frac{2}{5} \times \frac{2}{3} + \frac{3}{4} \times \frac{3}{5} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{5} \times \frac{1}{3}$
 $= \frac{1}{5} + \frac{3}{20} + \frac{1}{30}$
 $= \frac{23}{60}$

ii $P(\text{blue balls are drawn from boxes X and Z})$
 $= P(BBB) + P(BRB)$
 $= \frac{1}{4} \times \frac{3}{5} \times \frac{2}{3} + \frac{1}{4} \times \frac{2}{5} \times \frac{2}{3}$
 $= \frac{1}{10} + \frac{1}{15}$
 $= \frac{1}{6}$

iii $P(\text{at most one blue ball is drawn})$
 $= P(\text{no blue balls are drawn}) + P(\text{exactly one blue ball is drawn})$
 $= P(RRR) + [P(BRR) + P(RBR) + P(RRB)]$
 $= \frac{3}{4} \times \frac{2}{5} \times \frac{1}{3} + [\frac{1}{4} \times \frac{2}{5} \times \frac{1}{3} + \frac{3}{4} \times \frac{3}{5} \times \frac{1}{3} + \frac{3}{4} \times \frac{2}{5} \times \frac{2}{3}]$
 $= \frac{1}{10} + \frac{1}{30} + \frac{3}{20} + \frac{1}{5}$
 $= \frac{29}{60}$

c If an extra red ball is added to box Y, the probabilities in **b i** and **b iii** will be affected.



b i $P(T \cap E') = \frac{1}{3}$

ii $P(T \cup E')$
 $= P(T) + P(E') - P(T \cap E')$
 $= \frac{1}{2} + \frac{2}{3} - \frac{1}{3} \quad \{\text{from a}\}$
 $= \frac{5}{6}$

30 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $\therefore 0.63 = P(A) + 0.36 - P(A)P(B) \quad \{A \text{ and } B \text{ are independent}\}$
 $\therefore 0.27 = P(A) - 0.36 \times P(A)$
 $\therefore 0.27 = 0.64 \times P(A)$
 $\therefore P(A) = \frac{0.27}{0.64} \approx 0.422$

$$31 \quad a \quad P(2 \text{ white truffles}) = P(\text{first is white} \cap \text{second is white})$$

$$= P(\text{first is white}) \times P(\text{second is white given first is white})$$

$$= \frac{2}{12} \times \frac{1}{11}$$

$$= \frac{2}{132}$$

$$= \frac{1}{66}$$

$$b \quad P(2 \text{ white truffles}) = \frac{1}{66} \quad \{\text{from a}\}$$

$$P(2 \text{ dark brown truffles}) = P(\text{first is dark brown} \cap \text{second is dark brown})$$

$$= P(\text{first is dark brown}) \times P(\text{second is dark brown given first is dark brown})$$

$$= \frac{6}{12} \times \frac{5}{11}$$

$$= \frac{30}{132}$$

$$= \frac{5}{22}$$

$$P(2 \text{ light brown truffles}) = P(\text{first is light brown} \cap \text{second is light brown})$$

$$= P(\text{first is light brown}) \times P(\text{second is light brown given first is light brown})$$

$$= \frac{4}{12} \times \frac{3}{11}$$

$$= \frac{12}{132}$$

$$= \frac{1}{11}$$

$$P(\text{different coloured truffles}) = 1 - P(\text{same coloured truffles})$$

$$= 1 - \left(\frac{1}{66} + \frac{5}{22} + \frac{1}{11} \right)$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

32 Let B be the event that a blue ball is drawn, and

R be the event that a red ball is drawn.

$$\text{Now } P(\text{both red}) = \frac{1}{3}$$

$$\therefore P(R \cap R) = \frac{1}{3}$$

$$\therefore \frac{n}{n+4} \times \frac{n-1}{n+3} = \frac{1}{3}$$

$$\therefore \frac{n(n-1)}{(n+4)(n+3)} = \frac{1}{3}$$

$$\therefore 3n(n-1) = (n+4)(n+3)$$

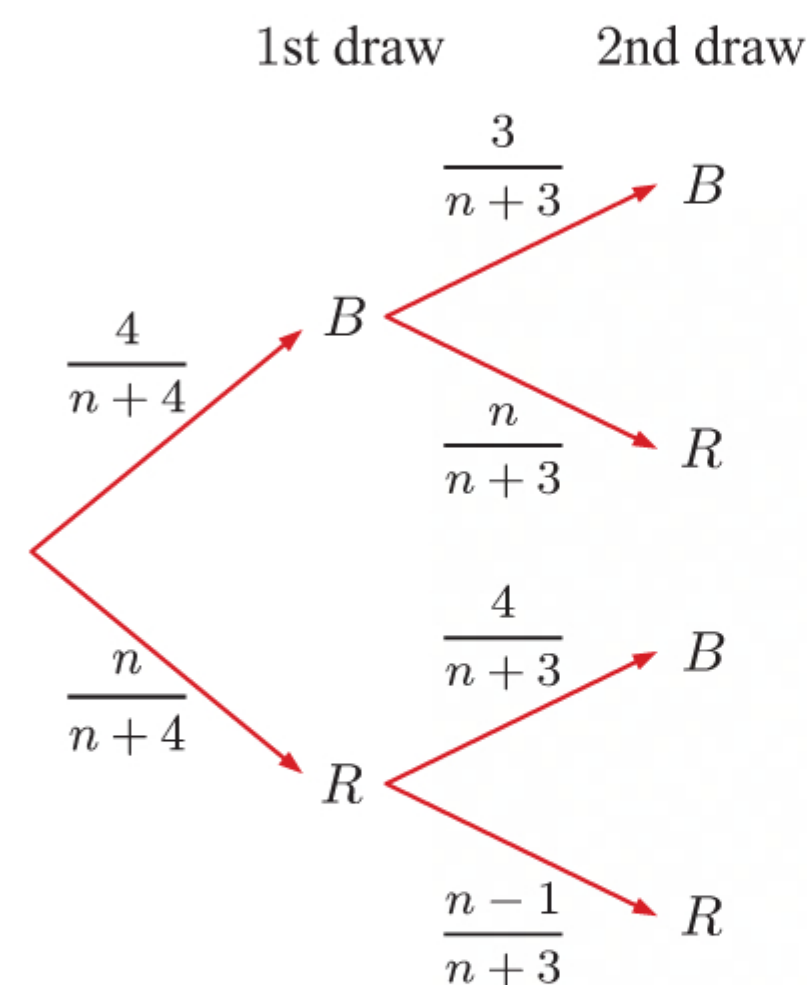
$$\therefore 3n^2 - 3n = n^2 + 7n + 12$$

$$\therefore 2n^2 - 10n - 12 = 0$$

$$\therefore n^2 - 5n - 6 = 0$$

$$\therefore (n-6)(n+1) = 0$$

$$\therefore n = 6 \quad \{n \geq 0\}$$



33 a Let O represent a student who owns an orange highlighter, and

B represent a student who owns a blue highlighter.

Let the proportion of students in $O \cap B$ be x .

\therefore the proportion in $O \cap B'$ is $0.4 - x$ and

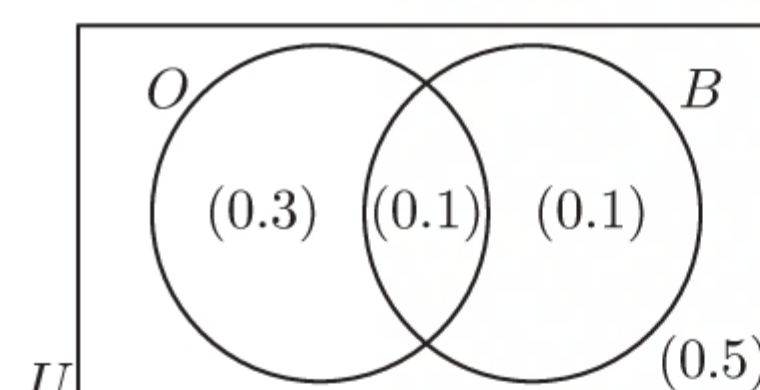
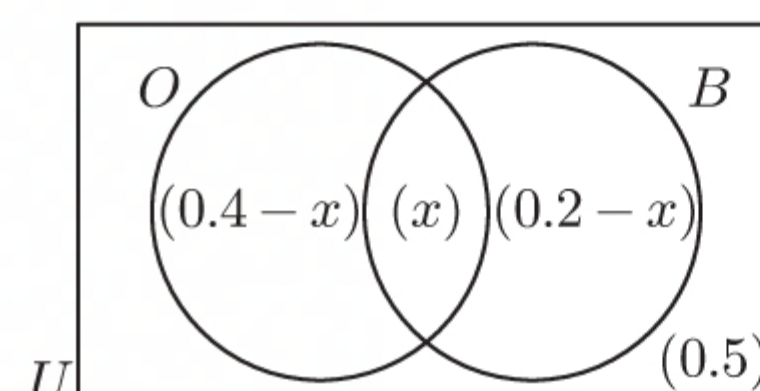
the proportion in $O' \cap B$ is $0.2 - x$.

The proportion in $O' \cap B'$ is 0.5 .

$$\therefore (0.4 - x) + x + (0.2 - x) = 1 - 0.5$$

$$\therefore 0.6 - x = 0.5$$

$$\therefore x = 0.1$$



$$\begin{aligned} \text{b i } P(B | O) &= \frac{P(B \cap O)}{P(O)} \\ &= \frac{0.1}{0.4} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{ii } P(O | B') &= \frac{P(O \cap B')}{P(B')} \\ &= \frac{0.3}{0.8} \\ &= \frac{3}{8} \end{aligned}$$

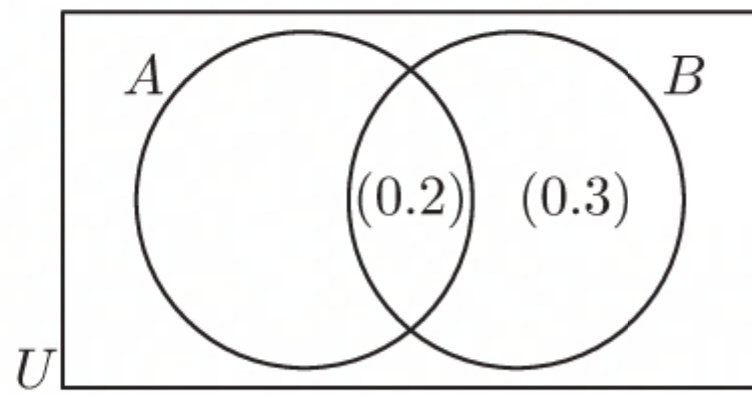
$$\begin{aligned} \text{34 a i } P(X \cap Y) &= P(X) + P(Y) - P(X \cup Y) \\ &= \frac{3}{7} + \frac{2}{9} - \frac{3}{5} \\ &= \frac{16}{315} \end{aligned}$$

$$\begin{aligned} \text{ii } P(X | Y) &= \frac{P(X \cap Y)}{P(Y)} \\ &= \frac{\frac{16}{315}}{\frac{2}{9}} \quad \{\text{using i}\} \\ &= \frac{8}{35} \end{aligned}$$

$$\begin{aligned} \text{iii } P(Y | X) &= \frac{P(Y \cap X)}{P(X)} \\ &= \frac{\frac{16}{315}}{\frac{3}{7}} \quad \{\text{using i}\} \\ &= \frac{16}{135} \end{aligned}$$

b X and Y are not independent as $P(X | Y) \neq P(X)$ and $P(Y | X) \neq P(Y)$.

35



$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A') \\ &= 0.2 + 0.3 \\ &= 0.5 \end{aligned}$$

and $P(A' \cap B) = P(A') \times P(B)$ $\{A' \text{ and } B \text{ are independent}\}$

$$\therefore 0.3 = 0.5 \times P(A')$$

$$\therefore P(A') = 0.6$$

$$\begin{aligned} \text{Now } P(A' \cup B) &= P(A') + P(B) - P(A' \cap B) \\ &= 0.6 + 0.5 - 0.3 \\ &= 0.8 \end{aligned}$$

36 Let H represent a head, and T represent a tail.

$$\begin{aligned} \text{a } P(2 \text{ heads and 1 tail}) &= P(\text{HHT, HTH, or THH}) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= \frac{3}{8} \end{aligned}$$

b $n = 400$ times

$$p = P(\text{exactly 1 tail}) = \frac{3}{8} \quad \{\text{from a}\}$$

You would expect to see exactly one tail

$$np = 400 \times \frac{3}{8} = 150 \text{ times.}$$

37 a i Events A and B are mutually exclusive if $A \cap B = \emptyset$. In this case $A \cap B$ contains the numbers 41, 42, 43, 44. So, the events are not mutually exclusive.

$$\begin{aligned} \text{ii } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{44}{100} + \frac{14}{100} - \frac{4}{100} \\ &= \frac{54}{100} \end{aligned}$$

b If the events A and B are independent, $P(A | B) = P(A)$.

If A and B are mutually exclusive, $P(A | B) = 0$ since if B occurs A cannot occur.

But $P(A) \neq 0$. Hence, the events cannot be both independent and mutually exclusive.

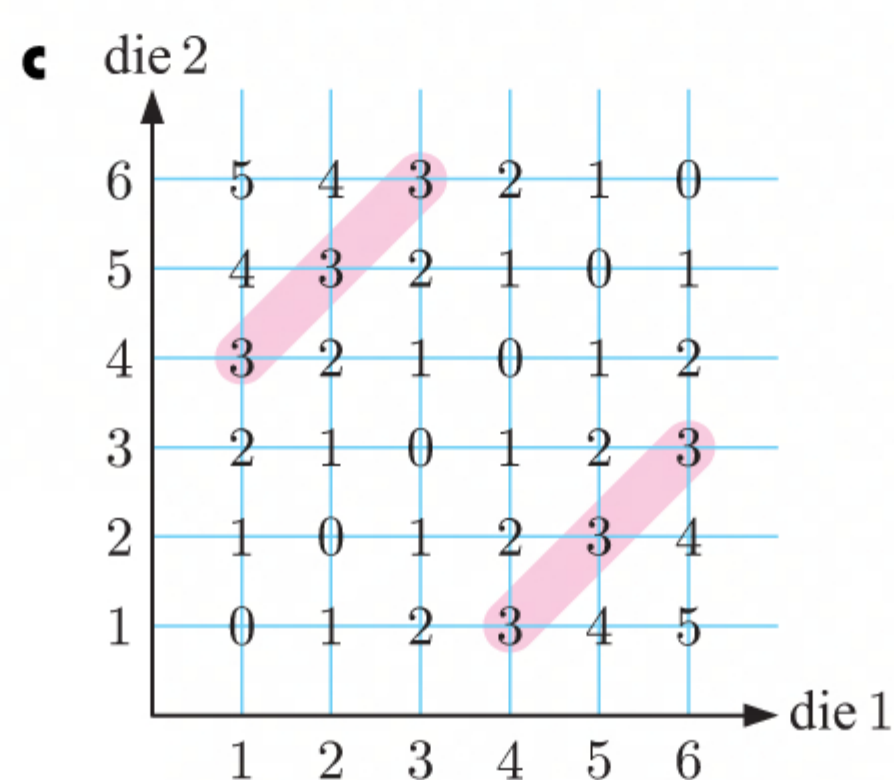
38 Let H be the event that the team played the match in their home country and W be the event that the team won the match.

$$\begin{aligned} P(H | W') &= \frac{P(W' | H) P(H)}{P(H) P(W' | H) + P(H') P(W' | H')} \quad \{\text{Bayes' theorem}\} \\ &= \frac{(0.35 + 0.25) \times 0.3}{0.3(0.35 + 0.25) + 0.7(1 - 0.15)} \\ &= \frac{0.18}{0.18 + 0.595} \\ &\approx 0.232 \end{aligned}$$

\therefore given that the team did not win their last match, the probability that the match was played in their home country is approximately 0.232.

39 a X is the difference of a number from one die and a number from the other die. So X is a discrete random variable because X has a set of distinct possible values.

b $X = 0, 1, 2, 3, 4, 5$



$$P(X = 3) = \frac{6}{36} = \frac{1}{6}$$

40 a $P(x) = k(x + 3), \quad x = 0, 1, 2, 3, 4$
 $\therefore P(0) = 3k, \quad P(1) = 4k, \quad P(2) = 5k,$
 $P(3) = 6k, \quad P(4) = 7k$

Since $P(x)$ is a probability mass function,

$$\sum_{i=1}^n P(x_i) = 1$$

$$\therefore 3k + 4k + 5k + 6k + 7k = 1$$

$$\therefore 25k = 1$$

$$\therefore k = \frac{1}{25}$$

b $P(x) = \frac{k^{x-3}}{x-1}, \quad x = 3, 4, 5$

$$\therefore P(3) = \frac{k^0}{3-1}, \quad P(4) = \frac{k^1}{4-1}, \quad P(5) = \frac{k^2}{5-1}$$

$$= \frac{1}{2} \qquad = \frac{k}{3} \qquad = \frac{k^2}{4}$$

Since $P(x)$ is a probability mass function,

$$\sum_{i=1}^n P(x_i) = 1$$

$$\therefore \frac{1}{2} + \frac{k}{3} + \frac{k^2}{4} = 1$$

$$\therefore \frac{k^2}{4} + \frac{k}{3} - \frac{1}{2} = 0$$

$$\therefore 3k^2 + 4k - 6 = 0$$

$$\therefore k = \frac{-4 \pm \sqrt{16 + 72}}{2 \times 3}$$

$$= \frac{-4 \pm \sqrt{88}}{6}$$

$$= \frac{-2 \pm \sqrt{22}}{3} \quad \{k > 0\}$$

41 a $P(x) = \frac{a}{(x-3)^2}, \quad x = 0, 1, 2$

$$\therefore P(0) = \frac{a}{9}, \quad P(1) = \frac{a}{4}, \quad P(2) = a$$

Since $P(x)$ is a probability mass function, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore \frac{a}{9} + \frac{a}{4} + a = 1$$

$$\therefore \frac{49}{36}a = 1$$

$$\therefore a = \frac{36}{49}$$

b $P(X = 2) = P(2)$
 $= a$
 $= \frac{36}{49} \quad \{\text{from a}\}$

c Since $P(X = 2) = \frac{36}{49}$ is the greatest probability, the mode of the distribution is 2.

$$p_1 = \frac{a}{9} = \frac{\frac{36}{49}}{9} = \frac{4}{49} \approx 0.0816$$

$$p_1 + p_2 = \frac{a}{9} + \frac{a}{4} = \frac{4}{49} + \frac{\frac{36}{49}}{4} = \frac{4}{49} + \frac{9}{49} = \frac{13}{49} \approx 0.265$$

Since $p_1 + p_2 + p_3 = 1 \geq 0.5$, the median is 2.

42 a $X = 2, 3, 4$

b Let B represent a blue ticket, and R represent a red ticket.

The possible selections that can be made are:

BR	BBR	RRRB
RB	RRB	↓
(X = 2)	(X = 3)	(X = 4)

So, $P(X = 2) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{3}{5}$
 $P(X = 3) = \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} + \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{3}{10}$
 $P(X = 4) = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} = \frac{1}{10}$

\therefore the probability distribution of X is

x	2	3	4
$P(X = x)$	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{1}{10}$

c It is most likely that 2 tickets are drawn, so the mode is 2.

$$\begin{aligned}
 \text{d } E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 2\left(\frac{3}{5}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{1}{10}\right) \\
 &= \frac{5}{2} = 2.5
 \end{aligned}$$

43

x	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

$$\begin{aligned}
 \text{a } \mu &= \sum x_i p_i \\
 &= 1(0.1) + 2(0.2) + 3(0.4) + 4(0.2) + 5(0.1) \\
 &= 3
 \end{aligned}$$

b Since $P(X = 3) = 0.4$ is the greatest probability, the mode of the distribution is 3.

$$\begin{aligned}
 \text{c } \sigma^2 &= \sum (x_i - \mu)^2 p_i \\
 &= (1 - 3)^2(0.1) + (2 - 3)^2(0.2) + (3 - 3)^2(0.4) + (4 - 3)^2(0.2) + (5 - 3)^2(0.1) \\
 &= 1.2
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \sigma &= \sqrt{\sigma^2} \\
 &= \sqrt{1.2} \\
 &\approx 1.10
 \end{aligned}$$

44

x	0	1	2	3
$P(X = x)$	0.3	0.2	m	n

Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.3 + 0.2 + m + n = 1$$

$$\therefore n = 0.5 - m \quad \dots (*)$$

Now if $E(X) = 1.55$, then $1.55 = \sum_{i=1}^n x_i p_i$

$$\therefore 1.55 = 0(0.3) + 1(0.2) + 2m + 3n$$

$$\therefore 1.35 = 2m + 3(0.5 - m) \quad \{\text{using } (*)\}$$

$$\therefore 1.35 = 2m + 1.5 - 3m$$

$$\therefore -0.15 = -m$$

$$\therefore m = 0.15$$

Substituting into (*), $n = 0.5 - 0.15 = 0.35$.

45

x	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	k	$\frac{1}{12}$

$$\begin{aligned}
 \text{a } \frac{1}{3} + \frac{1}{6} + k + \frac{1}{12} &= 1 \\
 \therefore k + \frac{7}{12} &= 1 \\
 \therefore k &= \frac{5}{12}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } E(X) &= -2\left(\frac{1}{3}\right) + 0\left(\frac{1}{6}\right) + 3\left(\frac{5}{12}\right) + 5\left(\frac{1}{12}\right) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= (-2)^2\left(\frac{1}{3}\right) + 0^2\left(\frac{1}{6}\right) + 3^2\left(\frac{5}{12}\right) + 5^2\left(\frac{1}{12}\right) - 1^2 \\
 &= \frac{37}{6}
 \end{aligned}$$

$$\sigma = \sqrt{\frac{37}{6}} \approx 2.48$$

c The probability distribution is

x	-2	0	3	5
$P(X = x)$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

3 is the most probable outcome, so mode = 3.

$$\text{Now } p_1 = \frac{4}{12} \approx 0.333$$

$$p_1 + p_2 = \frac{4}{12} + \frac{2}{12} = 0.5$$

Since $p_1 + p_2 \geq 0.5$, the median is 0.

$$46 \quad a \quad P(x) = \frac{x^2 + kx}{50}, \quad x = 1, 2, 3, 4$$

$$\therefore P(1) = \frac{k+1}{50}, \quad P(2) = \frac{2k+4}{50}, \quad P(3) = \frac{3k+9}{50}, \quad P(4) = \frac{4k+16}{50}$$

$$\text{Since } P(x) \text{ is a probability mass function, } \sum_{i=1}^n P(x_i) = 1$$

$$\therefore \frac{k+1}{50} + \frac{2k+4}{50} + \frac{3k+9}{50} + \frac{4k+16}{50} = 1$$

$$\therefore \frac{10k+30}{50} = 1$$

$$\therefore 10k+30 = 50$$

$$\therefore 10k = 20$$

$$\therefore k = 2$$

$$b \quad \mu = E(X)$$

$$= \sum_{i=1}^n x_i p_i$$

$$= 1\left(\frac{3}{50}\right) + 2\left(\frac{8}{50}\right) + 3\left(\frac{15}{50}\right) + 4\left(\frac{24}{50}\right)$$

$$= 3.2$$

$$c \quad P(X \geq 2) = 1 - P(X = 1)$$

$$= 1 - P(1)$$

$$= 1 - \frac{3}{50}$$

$$= \frac{47}{50}$$

$$47 \quad P(x) = \frac{x^2}{29} \text{ for } x = 2, 3, 4$$

x	2	3	4
$P(x)$	$\frac{4}{29}$	$\frac{9}{29}$	$\frac{16}{29}$

$$a \quad \text{Since } P(X = 4) = P(4) = \frac{16}{29} \text{ is the greatest probability, the mode of the distribution is 4.}$$

$$b \quad P(2) = \frac{4}{29} \approx 0.138$$

$$P(2) + P(3) = \frac{4}{29} + \frac{9}{29} = \frac{13}{29} \approx 0.448$$

Since $P(2) + P(3) + P(4) = 1 \geq 0.5$, the median is 4.

$$c \quad \text{mean} = E(X)$$

$$= \sum x_i p_i$$

$$= 2\left(\frac{4}{29}\right) + 3\left(\frac{9}{29}\right) + 4\left(\frac{16}{29}\right)$$

$$= \frac{8+27+64}{29}$$

$$= \frac{99}{29} \approx 3.41$$

$$d \quad E(X^2) = \sum x_i^2 p_i$$

$$= 2^2\left(\frac{4}{29}\right) + 3^2\left(\frac{9}{29}\right) + 4^2\left(\frac{16}{29}\right)$$

$$= \frac{16+81+256}{29}$$

$$= \frac{353}{29}$$

$$\therefore \text{variance} = \text{Var}(X)$$

$$= E(X^2) - [E(X)]^2$$

$$= \frac{353}{29} - \left(\frac{99}{29}\right)^2$$

$$= \frac{436}{841} \approx 0.518$$

$$e \quad \text{Standard deviation} = \sqrt{\text{Var}(X)}$$

$$= \sqrt{\frac{436}{841}}$$

$$= \frac{2\sqrt{109}}{29} \approx 0.720$$

$$48 \quad a \quad \text{Let } X \text{ be the return from each game.}$$

$$E(X) = 40\left(\frac{1}{12}\right) + 20\left(\frac{3}{12}\right) + 5\left(\frac{8}{12}\right)$$

$$= \frac{40+60+40}{12}$$

$$= \frac{140}{12}$$

$$\approx \$11.67$$

<i>Ticket colour</i>	Blue	Red	Yellow
<i>Winnings</i>	\$40	\$20	\$5
<i>Probability</i>	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{8}{12}$

$$b \quad \text{The expected return per game is \$11.67. It costs \$15 to play.}$$

$$\text{So, the expected gain} \approx \$11.67 - \$15 \approx -\$3.33$$

It is not advisable to play this game many times as the player can expect to lose \$3.33 on average per game.

$$c \quad \text{Let } k \text{ be the number of extra red tickets added to the bag.}$$

<i>Ticket colour</i>	Blue	Red	Yellow
<i>Winnings</i>	\$40	\$20	\$5
<i>Probability</i>	$\frac{1}{12+k}$	$\frac{k+3}{12+k}$	$\frac{8}{12+k}$

For the game to be fair, the expected return must equal the cost of each game.

$$\therefore E(X) = 40\left(\frac{1}{12+k}\right) + 20\left(\frac{k+3}{12+k}\right) + 5\left(\frac{8}{12+k}\right) = 15 \quad \{\text{the cost of the game is \$15}\}$$

$$\therefore \frac{40}{12+k} + \frac{20k+60}{12+k} + \frac{40}{12+k} = 15$$

$$\therefore \frac{20k+140}{12+k} = 15$$

$$\therefore 20k+140 = 15(12+k)$$

$$\therefore 20k+140 = 180+15k$$

$$\therefore 5k = 40$$

$$\therefore k = 8$$

So, 8 extra red tickets should be added to the bag to make the game fair.

49

x	1	2	3	4
$P(X = x)$	0.25	0.38	0.17	0.2

a $E(X) = \sum x_i p_i$
 $= 1(0.25) + 2(0.38) + 3(0.17) + 4(0.2)$
 $= 2.32$

b $E(X^2) = \sum x_i^2 p_i$
 $= 1^2(0.25) + 2^2(0.38) + 3^2(0.17) + 4^2(0.2)$
 $= 6.5$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 6.5 - (2.32)^2$$

$$= 1.1176$$

c $\sigma(X) = \sqrt{\text{Var}(X)}$
 $= \sqrt{1.1176}$
 ≈ 1.06

d $E(X+2) = E(X) + 2$
 $= 2.32 + 2$
 $= 4.32$

e $\text{Var}(2-3X) = (-3)^2 \text{Var}(X)$
 $= 9 \text{Var}(X)$
 $= 9(1.1176)$
 $= 10.0584$

f $\sigma(2X-10) = |2| \sigma(X)$
 $= 2\sqrt{1.1176}$
 ≈ 2.11

50

x	0	1	2	3	4	5	6
$P(X = x)$	0.02	0.02	0.08	0.38	0.35	0.12	0.03

a i $E(X) = \sum x_i p_i$
 $= 0(0.02) + 1(0.02) + 2(0.08) + 3(0.38) + 4(0.35) + 5(0.12) + 6(0.03)$
 $= 3.5 \text{ points}$

ii $E(X^2) = \sum x_i^2 p_i$
 $= 0^2(0.02) + 1^2(0.02) + 2^2(0.08) + 3^2(0.38) + 4^2(0.35) + 5^2(0.12) + 6^2(0.03)$
 $= 13.44$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 13.44 - (3.5)^2$$

$$= 1.19$$

iii $\sigma(X) = \sqrt{\text{Var}(X)}$
 $= \sqrt{1.19}$
 $\approx 1.09 \text{ points}$

b $Y = 3X + 20$

i $E(Y) = E(3X + 20)$
 $= 3E(X) + 20$
 $= 3(3.5) + 20$
 $= 30.5 \text{ points}$

ii $\text{Var}(Y) = \text{Var}(3X + 20)$
 $= 3^2 \text{Var}(X)$
 $= 9 \text{Var}(X)$
 $= 10.71$

iii $\sigma(Y) = \sigma(3X + 20)$
 $= |3| \sigma(X)$
 $= 3\sqrt{1.19}$
 $\approx 3.27 \text{ points}$

51 $E(X) = 7, \sigma(X) = 2 \quad \therefore \text{Var}(X) = 2^2 = 4$

a $Y = 4X + 3$

$$\begin{aligned} \therefore E(Y) &= E(4X + 3) \\ &= 4E(X) + 3 \\ &= 4(7) + 3 \\ &= 28 + 3 \\ &= 31 \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(4X + 3) \\ &= 4^2 \text{Var}(X) \\ &= 16 \text{Var}(X) \\ &= 16(4) \\ &= 64 \end{aligned}$$

b $Y = \frac{1}{2}(5 - X) = \frac{5}{2} - \frac{1}{2}X$

$$\begin{aligned} \therefore E(Y) &= E\left(\frac{5}{2} - \frac{1}{2}X\right) \\ &= \frac{5}{2} - \frac{1}{2}E(X) \\ &= \frac{5}{2} - \frac{1}{2}(7) \\ &= \frac{5}{2} - \frac{7}{2} \\ &= -\frac{2}{2} \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}\left(\frac{5}{2} - \frac{1}{2}X\right) \\ &= \left(-\frac{1}{2}\right)^2 \text{Var}(X) \\ &= \frac{1}{4} \text{Var}(X) \\ &= \frac{1}{4}(4) \\ &= 1 \end{aligned}$$

c $Y = \frac{2X - 1}{3} = \frac{2}{3}X - \frac{1}{3}$

$$\begin{aligned} \therefore E(Y) &= E\left(\frac{2}{3}X - \frac{1}{3}\right) \\ &= \frac{2}{3}E(X) - \frac{1}{3} \\ &= \frac{2}{3}(7) - \frac{1}{3} \\ &= \frac{14}{3} - \frac{1}{3} \\ &= \frac{13}{3} \end{aligned}$$

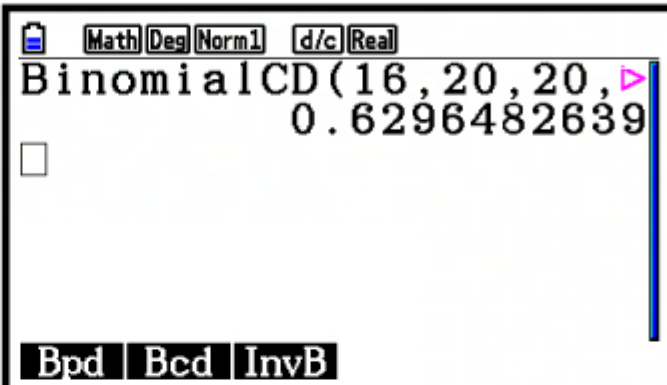
$$\begin{aligned} \text{Var}(Y) &= \text{Var}\left(\frac{2}{3}X - \frac{1}{3}\right) \\ &= \left(\frac{2}{3}\right)^2 \text{Var}(X) \\ &= \frac{4}{9} \text{Var}(X) \\ &= \frac{4}{9}(4) \\ &= \frac{16}{9} \end{aligned}$$

52 Let X be the number of residents who oppose the construction.

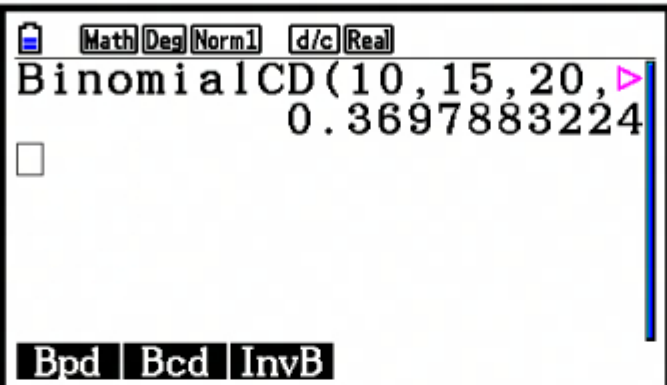
$n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = 80\% = 0.8$

$\therefore X \sim B(20, 0.8)$

a $P(X = 16) = \binom{20}{16}(0.8)^{16}(0.2)^4$
 ≈ 0.218

b 

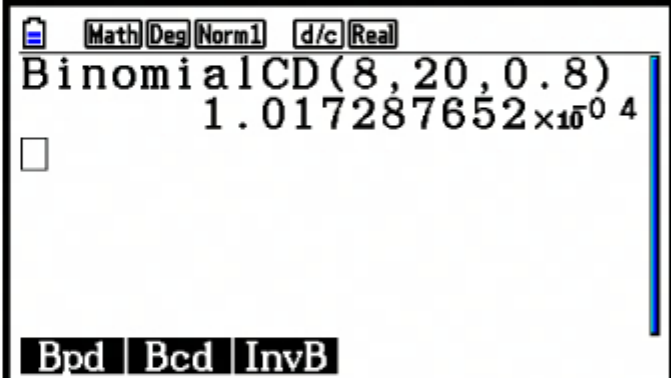
$P(X \geq 16) \approx 0.630$

c 

$P(10 \leq X \leq 15) \approx 0.370$

d If more than 8 residents support the construction, then 8 or fewer residents oppose the construction.

$P(X \leq 8) \approx 0.000102$



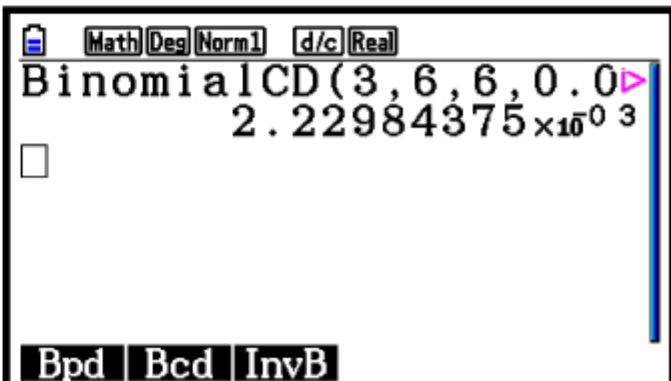
53 a Let X be the number of defective items.

$n = 6$, so $X = 0, 1, 2, 3, 4, 5$, or 6 , and $p = 5\% = 0.05$

$\therefore X \sim B(6, 0.05)$

Using technology, $P(X > 2) = P(X \geq 3)$
 ≈ 0.00223

\therefore the manufacturer will have to pay a refund on about $0.00223 \times 100\% \approx 0.223\%$ of boxes.



b Let Y be the number of boxes refunded.

$n = 10$, so $Y = 0, 1, 2, 3, \dots$, or 10 , and $p \approx 0.00223$ {from **a**}

$\therefore Y \sim B(10, 0.00223)$

$$P(Y = 1) \approx \binom{10}{1}(0.002\,23)^1(1 - 0.002\,23)^9 \\ \approx 0.0219$$

So, the probability that Patrick will get a refund for exactly 1 box is about 0.0219.

54 Let X be the number of germinations in one row.

$n = 10$, so $X = 0, 1, 2, 3, \dots$, or 10, and $p = \frac{1}{2}$

$$\therefore X \sim B(10, \frac{1}{2})$$

$$P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10) \\ = \binom{10}{8}(\frac{1}{2})^8(\frac{1}{2})^2 + \binom{10}{9}(\frac{1}{2})^9(\frac{1}{2})^1 + \binom{10}{10}(\frac{1}{2})^{10}(\frac{1}{2})^0 \\ = \frac{7}{128}$$

So, the probability that at least 8 seeds germinate in one row is $\frac{7}{128}$.

Let Y be the number of rows with at least 8 seeds germinating.

$n = 10$, so $Y = 0, 1, 2, 3, \dots$, or 10, and $p = \frac{7}{128}$

$$\therefore Y \sim B(10, \frac{7}{128})$$

$$P(Y \geq 1) = 1 - P(Y = 0) \\ = 1 - \binom{10}{0}(\frac{7}{128})^0(\frac{121}{128})^{10} \\ \approx 0.430$$

So, the probability that the row with the maximum number of germinations contains at least 8 seedlings is about 0.430.

55

Score	1	2	3	4
Probability	$\frac{1}{12}$	k	$\frac{1}{4}$	$\frac{1}{3}$

a Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore \frac{1}{12} + k + \frac{1}{4} + \frac{1}{3} = 1$$

$$\therefore k = \frac{1}{3}$$

b Let X be the number of 2s rolled.

$n = 2400$, so $X = 0, 1, 2, 3, \dots$, or 2400, and $p = \frac{1}{3}$ {from **a**}

$$\therefore X \sim B(2400, \frac{1}{3})$$

$$\text{So, } \mu = np \quad \text{and} \quad \sigma = \sqrt{np(1-p)} \\ = 2400 \times \frac{1}{3} \quad = \sqrt{2400 \times \frac{1}{3} \times \frac{2}{3}} \\ = 800 \quad = \sqrt{\frac{1600}{3}} \\ = \frac{40}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ = \frac{40\sqrt{3}}{3}$$

56 $Y \sim B(30, \frac{1}{5})$

a $\mu = np$ and $\sigma = \sqrt{np(1-p)}$

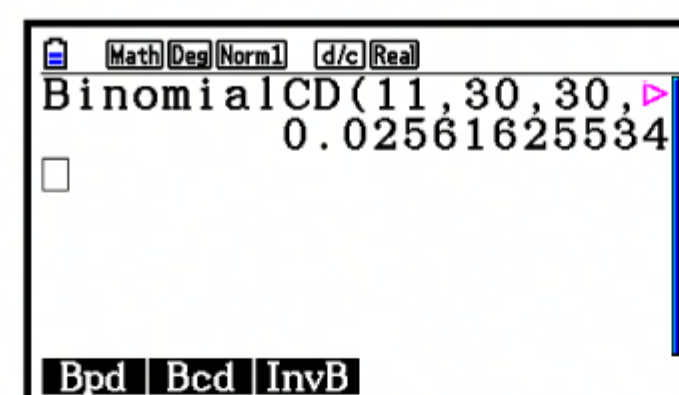
$$= 30 \times \frac{1}{5} \quad = \sqrt{30 \times \frac{1}{5} \times \frac{4}{5}} \\ = 6 \quad = \sqrt{\frac{24}{5}} \\ \approx 2.19$$

b $P(Y = 20) = \binom{30}{20}(\frac{1}{5})^{20}(\frac{4}{5})^{10}$

$$\approx 3.38 \times 10^{-8}$$

c $P(Y \geq \mu + 2\sigma) = P(Y \geq 6 + 2\sqrt{\frac{24}{5}})$ {from **a**}

$$= P(Y \geq 10.38) \\ = P(Y \geq 11) \\ \approx 0.0256 \quad \text{{using technology}}$$



57 $f(x) = ax^3 + x, \quad 0 \leq x \leq k$

Since $f(x)$ is a probability density function $\int_0^k f(x) dx = 1$

$$\therefore \int_0^k (ax^3 + x) dx = 1$$

$$\therefore \left[\frac{a}{4}x^4 + \frac{1}{2}x^2 \right]_0^k = 1$$

$$\therefore \frac{ak^4}{4} + \frac{k^2}{2} = 1$$

$$\therefore ak^4 + 2k^2 = 4 \quad \dots (*)$$

Now $P(X \leq \frac{1}{2}) = \frac{5}{32}$

$$\therefore \int_0^{\frac{1}{2}} f(x) dx = \frac{5}{32}$$

$$\therefore \int_0^{\frac{1}{2}} (ax^3 + x) dx = \frac{5}{32}$$

$$\therefore \left[\frac{a}{4}x^4 + \frac{1}{2}x^2 \right]_0^{\frac{1}{2}} = \frac{5}{32}$$

$$\therefore \frac{a}{4} \left(\frac{1}{2} \right)^4 + \frac{1}{2} \left(\frac{1}{2} \right)^2 = \frac{5}{32}$$

$$\therefore \frac{a}{64} + \frac{1}{8} = \frac{5}{32}$$

$$\therefore \frac{a}{64} = \frac{1}{32}$$

$$\therefore a = 2$$

Substituting $a = 2$ into (*) gives $2k^4 + 2k^2 = 4$

$$\therefore k^4 + k^2 = 2$$

$$\therefore k^4 + k^2 - 2 = 0$$

$$\therefore (k^2 + 2)(k^2 - 1) = 0$$

$$\therefore k = 1 \quad \{k \geq 0\}$$

58 $f(x) = \ln x, \quad 1 \leq x \leq k$

a Since $f(x)$ is a probability density function, $\int_1^k f(x) dx = 1$

$$\therefore \int_1^k \ln x dx = 1$$

$$\therefore [x \ln x]_1^k - \int_1^k \frac{1}{x} \times x dx = 1 \quad \begin{cases} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{cases}$$

$$\therefore k \ln k - 1 \ln 1 - \int_1^k 1 dx = 1$$

$$\therefore k \ln k - [x]_1^k = 1$$

$$\therefore k \ln k - (k - 1) = 1$$

$$\therefore k \ln k - k + 1 = 1$$

$$\therefore k \ln k - k = 0$$

$$\therefore k(\ln k - 1) = 0$$

$$\therefore \ln k - 1 = 0 \quad \{k \geq 1\}$$

$$\therefore \ln k = 1$$

$$\therefore k = e$$

$$\begin{aligned}
\mathbf{b} \quad \mathbf{i} \quad \mathbb{E}(X) &= \int_1^e x f(x) dx \\
&= \int_1^e x \ln x dx \\
&= \left[\frac{1}{2} x^2 \ln x \right]_1^e - \int_1^e \frac{1}{x} \times \frac{1}{2} x^2 dx & \begin{cases} u = \ln x & v' = x \\ u' = \frac{1}{x} & v = \frac{1}{2} x^2 \end{cases} \\
&= \frac{1}{2} e^2 \ln e - \frac{1}{2} \ln 1 - \int_1^e \frac{1}{2} x dx \\
&= \frac{e^2}{2} - \left[\frac{1}{4} x^2 \right]_1^e \\
&= \frac{e^2}{2} - \left(\frac{1}{4} e^2 - \frac{1}{4} \right) \\
&= \frac{e^2}{4} + \frac{1}{4} \\
&= \frac{1}{4} (e^2 + 1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{ii} \quad \mathbb{E}(X^2) &= \int_1^e x^2 \ln x dx \\
&= \left[\frac{1}{3} x^3 \ln x \right]_1^e - \int_1^e \frac{1}{x} \times \frac{1}{3} x^3 dx & \begin{cases} u = \ln x & v' = x^2 \\ u' = \frac{1}{x} & v = \frac{1}{3} x^3 \end{cases} \\
&= \frac{1}{3} e^3 \ln e - \frac{1}{3} \ln 1 - \int_1^e \frac{1}{3} x^2 dx \\
&= \frac{e^3}{3} - \left[\frac{1}{9} x^3 \right]_1^e \\
&= \frac{e^3}{3} - \left(\frac{e^3}{9} - \frac{1}{9} \right) \\
&= \frac{2e^3}{9} + \frac{1}{9} \\
&= \frac{1}{9} (2e^3 + 1)
\end{aligned}$$

$$\begin{aligned}
\therefore \text{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\
&= \frac{1}{9} (2e^3 + 1) - \left[\frac{1}{4} (e^2 + 1) \right]^2 \\
&= \frac{1}{9} (2e^3 + 1) - \frac{1}{16} (e^4 + 2e^2 + 1) \\
&= -\frac{e^4}{16} + \frac{2e^3}{9} - \frac{e^2}{8} + \frac{7}{144} \\
&\approx 0.176
\end{aligned}$$

$$\begin{aligned}
\mathbf{iii} \quad \sigma(X) &= \sqrt{\text{Var}(X)} \\
&\approx \sqrt{0.176} \\
&\approx 0.420
\end{aligned}$$

$$\mathbf{c} \quad Y = 3X - 2$$

$$\begin{aligned}
\mathbf{i} \quad \mathbb{E}(Y) &= \mathbb{E}(3X - 2) \\
&= 3\mathbb{E}(X) - 2 \\
&= 3\left(\frac{1}{4}(e^2 + 1)\right) - 2 \\
&= \frac{3}{4}(e^2 + 1) - 2 \\
&= \frac{3e^2}{4} - \frac{5}{4} \\
&= \frac{1}{4}(3e^2 - 5)
\end{aligned}$$

$$\begin{aligned}
\mathbf{ii} \quad \text{Var}(Y) &= \text{Var}(3X - 2) \\
&= 3^2 \text{Var}(X) \\
&= 9 \text{Var}(X) \\
&\approx 9(0.176) \\
&\approx 1.58
\end{aligned}$$

$$\begin{aligned}
\mathbf{iii} \quad \sigma(Y) &= \sigma(3X - 2) \\
&= |3| \sigma(X) \\
&\approx 3(0.420) \\
&\approx 1.26
\end{aligned}$$

59 $f(x) = \frac{1}{x}$, $a \leq x \leq b$ and $E(X) = 4$

a Since $f(x)$ is a probability density function, $\int_a^b f(x) dx = 1$

$$\begin{aligned} \therefore \int_a^b \frac{1}{x} dx &= 1 \\ \therefore [\ln |x|]_a^b &= 1 \\ \therefore \ln |b| - \ln |a| &= 1 \\ \therefore \ln \left| \frac{b}{a} \right| &= 1 \\ \therefore \ln \left(\frac{b}{a} \right) &= 1 \quad \dots (1) \quad \{\text{we require } b, a > 0 \text{ for } f(x) > 0\} \end{aligned}$$

Now $E(X) = 4$

$$\begin{aligned} \therefore \int_a^b x f(x) dx &= 4 \\ \therefore \int_a^b x \times \frac{1}{x} dx &= 4 \\ \therefore \int_a^b 1 dx &= 4 \\ \therefore [x]_a^b &= 4 \\ \therefore b - a &= 4 \\ \therefore b &= 4 + a \quad \dots (2) \end{aligned}$$

Substituting (2) into (1) gives $\ln \left(\frac{4+a}{a} \right) = 1$

$$\begin{aligned} \therefore \frac{4+a}{a} &= e \\ \therefore 4+a &= ae \\ \therefore a - ae &= -4 \\ \therefore a(1-e) &= -4 \\ \therefore a &= \frac{-4}{1-e} \\ \therefore a &= \frac{4}{e-1} \\ \therefore b &= 4 + \frac{4}{e-1} \quad \{\text{using (2)}\} \\ &= \frac{4e - 4 + 4}{e-1} \\ &= \frac{4e}{e-1} \end{aligned}$$

b The median is the solution of $\int_a^m f(x) dx = \frac{1}{2}$

$$\begin{aligned} \therefore \int_{\frac{4}{e-1}}^m \frac{1}{x} dx &= \frac{1}{2} \\ \therefore [\ln |x|]_{\frac{4}{e-1}}^m &= \frac{1}{2} \\ \therefore \ln |m| - \ln \left| \frac{4}{e-1} \right| &= \frac{1}{2} \\ \therefore \ln \left| \frac{m(e-1)}{4} \right| &= \frac{1}{2} \\ \therefore \frac{m(e-1)}{4} &= \sqrt{e} \quad \{m > 0\} \\ \therefore m(e-1) &= 4\sqrt{e} \\ \therefore m &= \frac{4\sqrt{e}}{e-1} \end{aligned}$$

60 a For $f(x)$ to be a probability density function, $\int_0^2 a(x^2 + 2) dx = 1$

$$\therefore a \int_0^2 (x^2 + 2) dx = 1$$

$$\therefore a \left[\frac{x^3}{3} + 2x \right]_0^2 = 1$$

$$\therefore a \left(\frac{8}{3} + 4 \right) = 1$$

$$\therefore a = \frac{3}{20}$$

b i $P(0.5 \leq X \leq 1.4) = \int_{0.5}^{1.4} \frac{3}{20}(x^2 + 2) dx$

$$= \frac{3}{20} \left[\frac{x^3}{3} + 2x \right]_{0.5}^{1.4}$$

$$= \frac{3}{20} \left(\frac{1.4^3}{3} + 2.8 - \frac{0.5^3}{3} - 1 \right)$$

$$\approx 0.401$$

ii $P(X \geq 1) = \int_1^2 \frac{3}{20}(x^2 + 2) dx$

$$= \frac{3}{20} \left[\frac{x^3}{3} + 2x \right]_1^2$$

$$= \frac{3}{20} \left(\frac{8}{3} + 4 - \frac{1}{3} - 2 \right)$$

$$= 0.65$$

c i $\int_0^m \frac{3}{20}(x^2 + 2) dx = \frac{1}{2}$

$$\therefore \frac{3}{20} \left[\frac{x^3}{3} + 2x \right]_0^m = \frac{1}{2}$$

$$\therefore \frac{3}{20} \left(\frac{m^3}{3} + 2m \right) = \frac{1}{2}$$

$$\therefore \frac{m^3}{20} + \frac{3m}{10} = \frac{1}{2}$$

$$\therefore m^3 + 6m - 10 = 0$$

ii $\mu = \int_0^2 x f(x) dx$

$$= \int_0^2 \frac{3}{20}(x^3 + 2x) dx$$

$$= \frac{3}{20} \left[\frac{x^4}{4} + x^2 \right]_0^2$$

$$= \frac{3}{20} \left(\frac{16}{4} + 4 \right)$$

$$= 1.2$$

Using technology for $m \in [0, 2]$, we find $m \approx 1.30$.

So, the median ≈ 1.30 .

iii $E(X^2) = \int_0^2 x^2 f(x) dx$

$$= \int_0^2 \frac{3}{20}(x^4 + 2x^2) dx$$

$$= \frac{3}{20} \left[\frac{x^5}{5} + \frac{2}{3}x^3 \right]_0^2$$

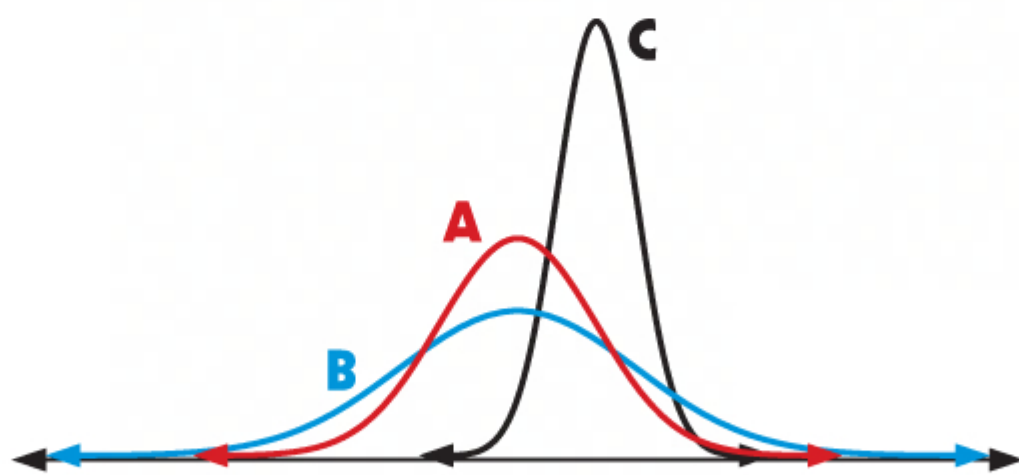
$$= \frac{3}{20} \left(\frac{32}{5} + \frac{16}{3} \right)$$

$$= 1.76$$

$$\therefore \text{Var}(X) = 1.76 - 1.2^2$$

$$= 0.32$$

61



A and **B** both have $\mu = 2$, **C** has $\mu = 4$.

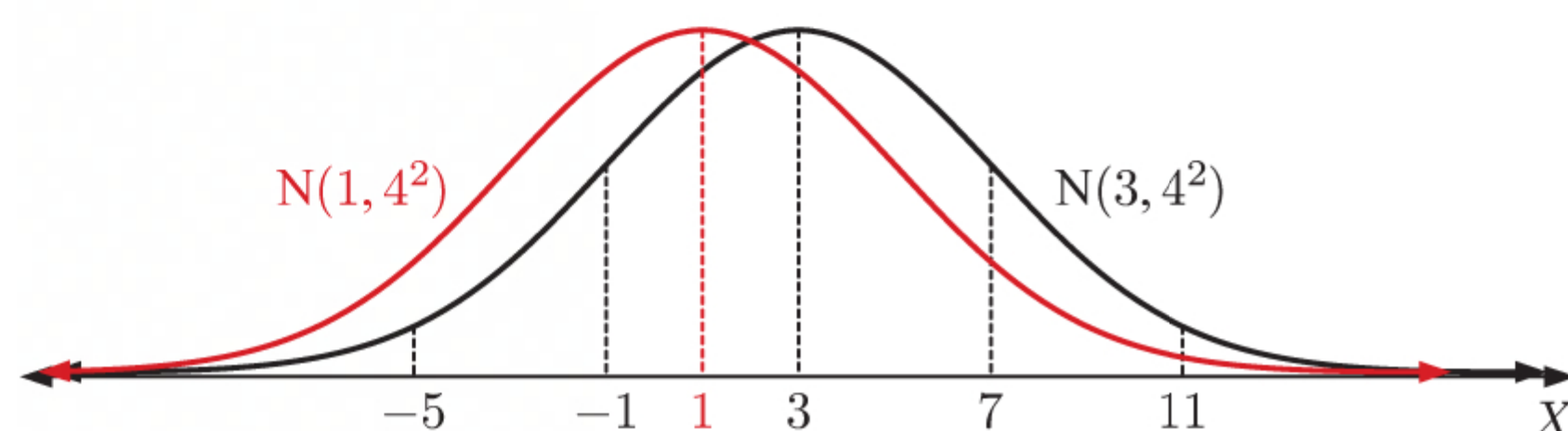
B has a greater spread, and hence a larger standard deviation than **A**.

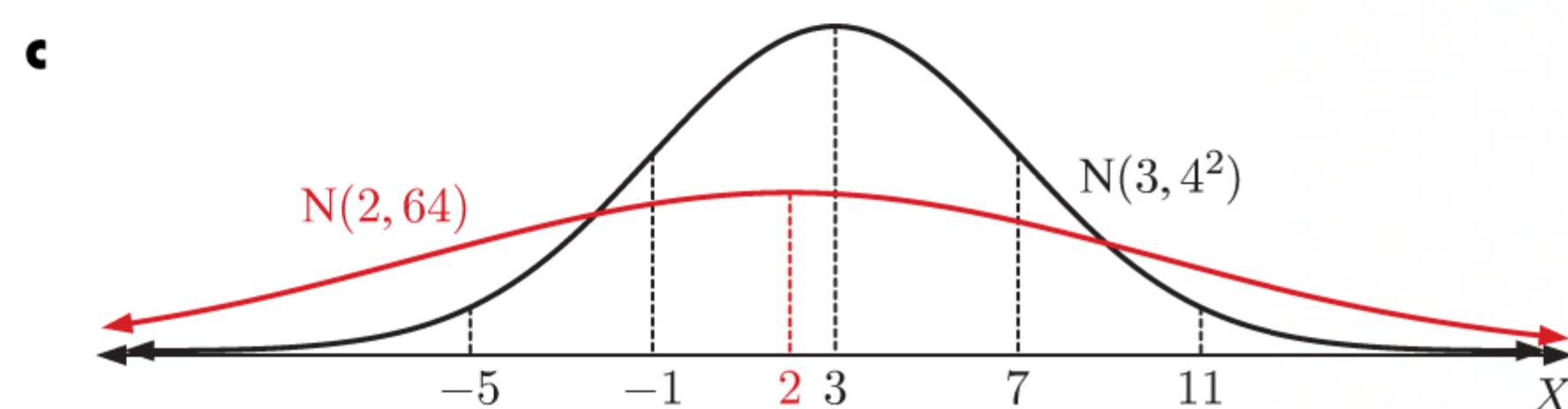
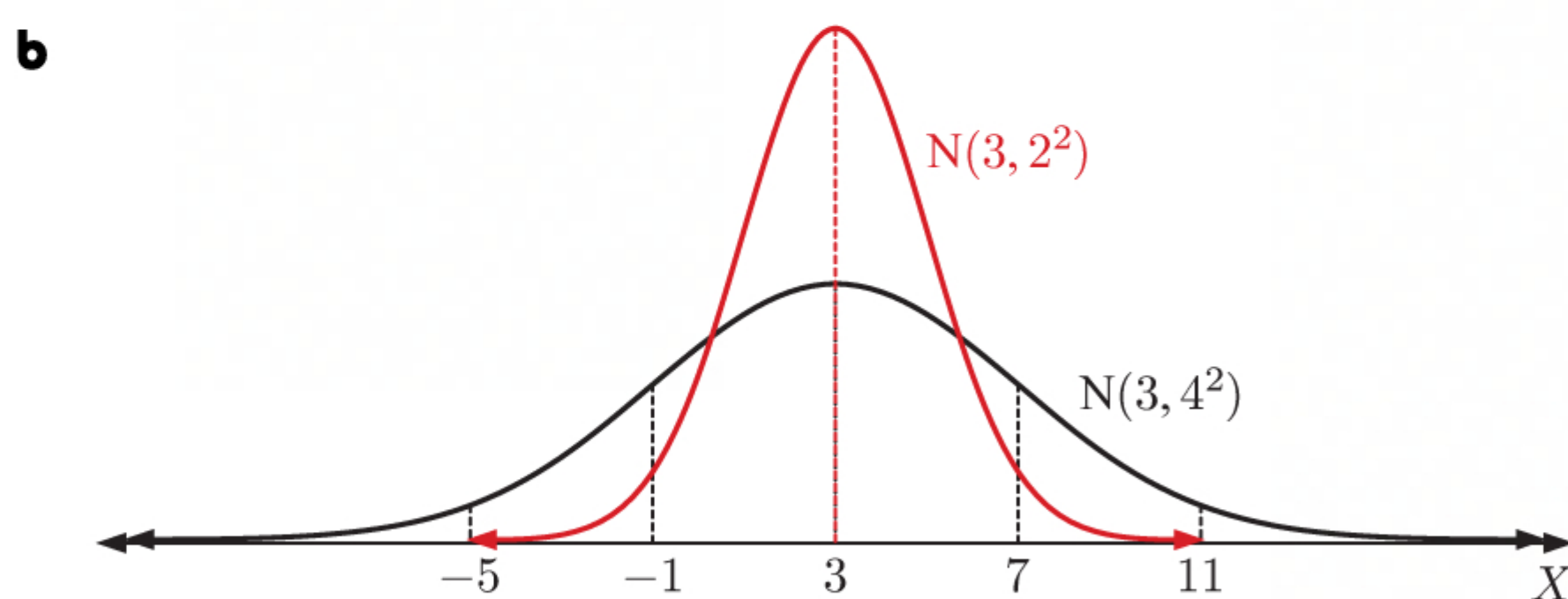
a $\mu = 4$, $\sigma = 1$ corresponds to **C**

b $\mu = 2$, $\sigma = 2$ corresponds to **A**

c $\mu = 2$, $\sigma = 3$ corresponds to **B**

62 a

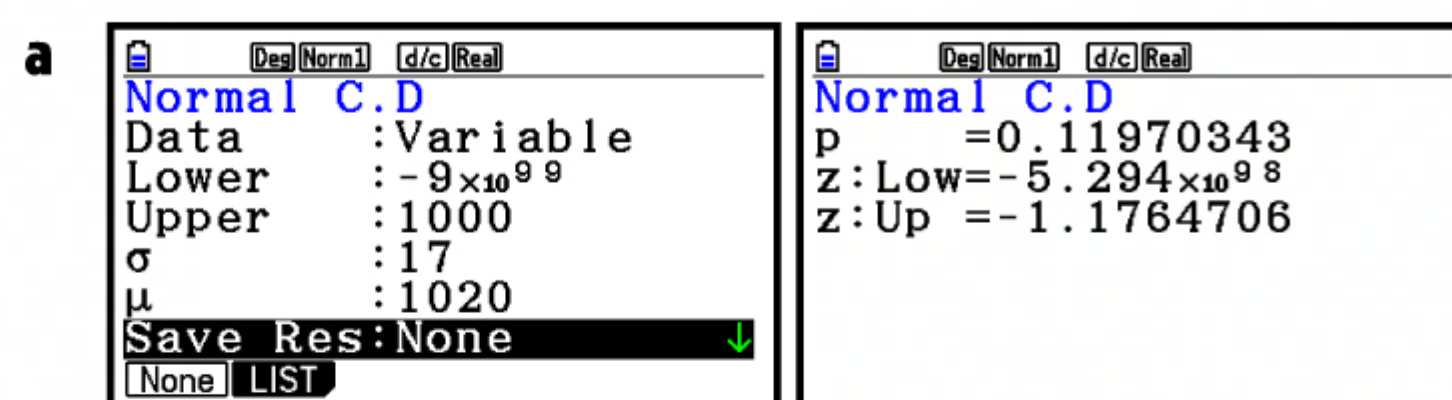




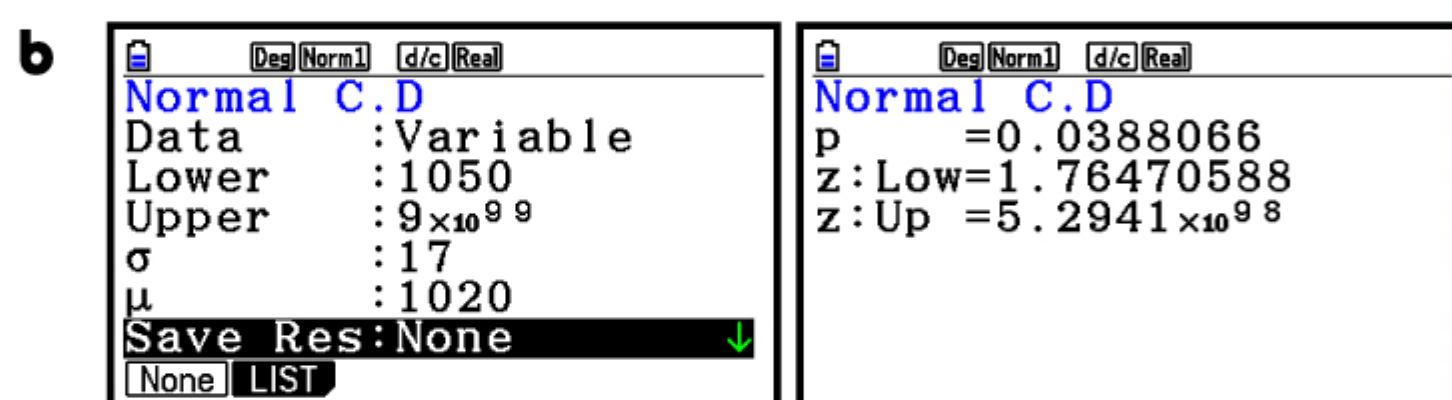
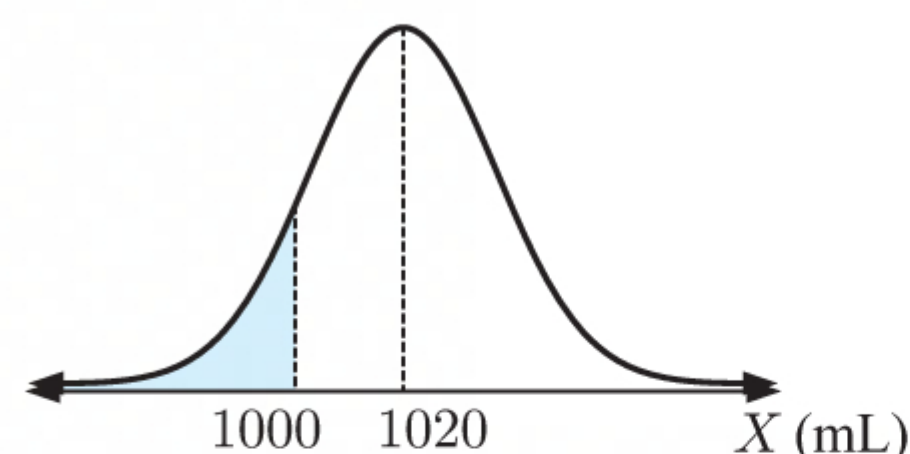
- 63**
- a** Approximately 68% of the population lies between 25 and 35.
 - b** Approximately 95% of the population lies between 20 and 40.
 - c** Approximately 99.7% of the population lies between 15 and 45.

- 64** Let the capacity of a randomly selected container be X mL.

So, $X \sim N(1020, 17^2)$.

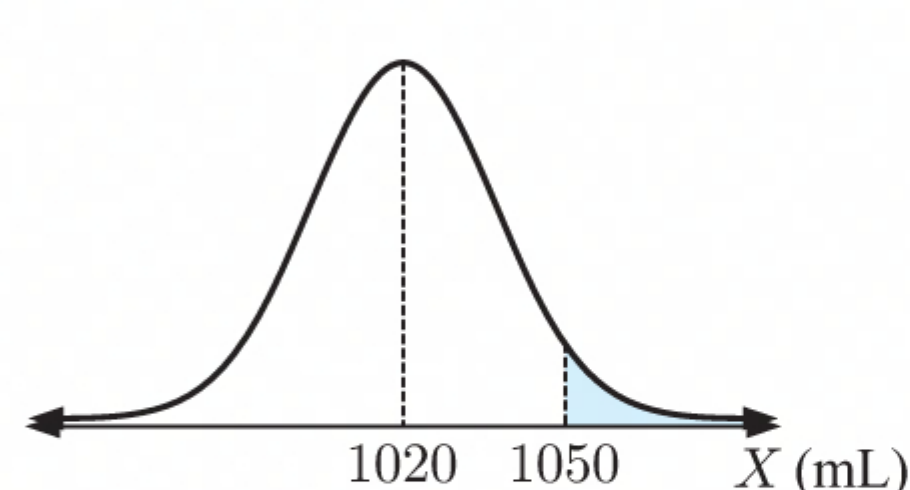


$$P(X \leq 1000) \approx 0.120$$



$$P(X \geq 1050) \approx 0.0388$$

\therefore about 3.88% of containers overflow.

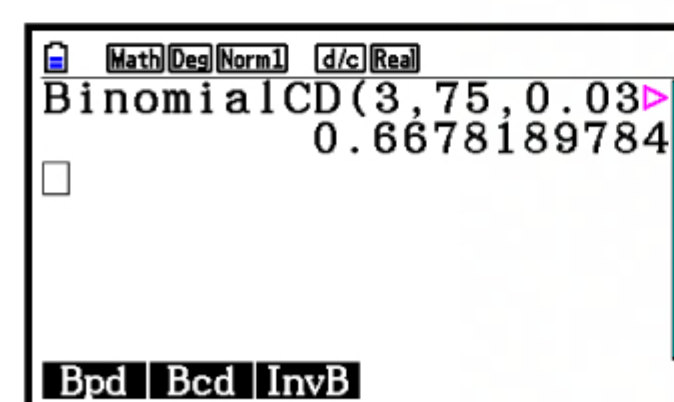


- c** Let Y be the number of containers which overflow.

$n = 75$, so $Y = 0, 1, 2, 3, \dots$, or 75 and $p \approx 0.0388$ {from **b**}

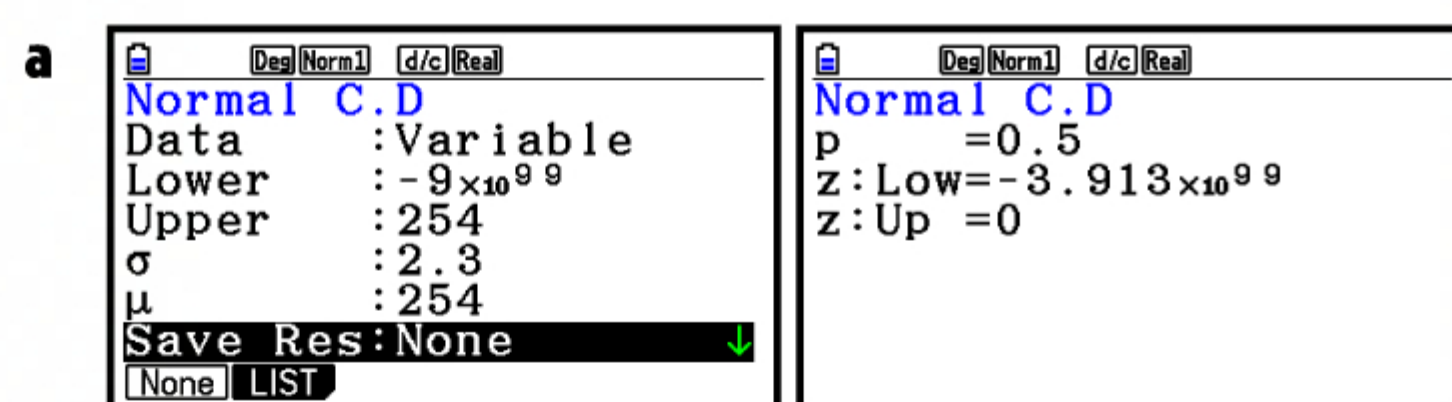
$$\therefore Y \sim B(75, 0.0388)$$

Using technology, $P(Y \leq 3) \approx 0.668$

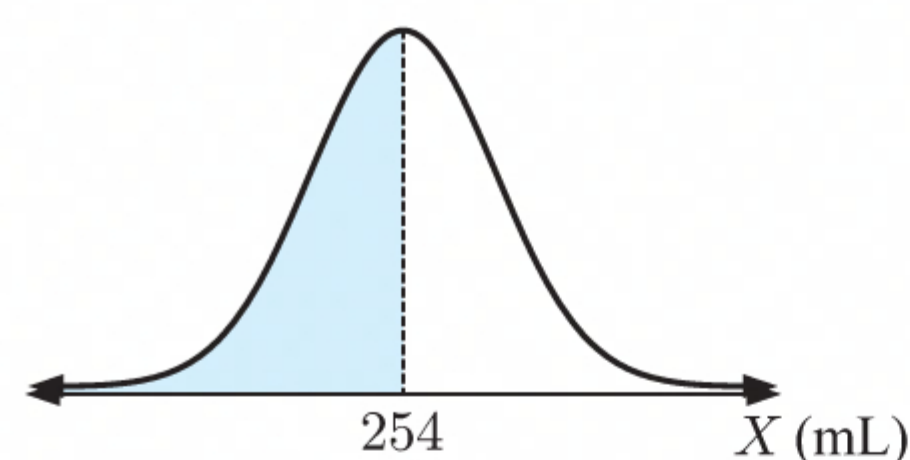


- 65** Let the volume of a randomly selected drink be X mL.

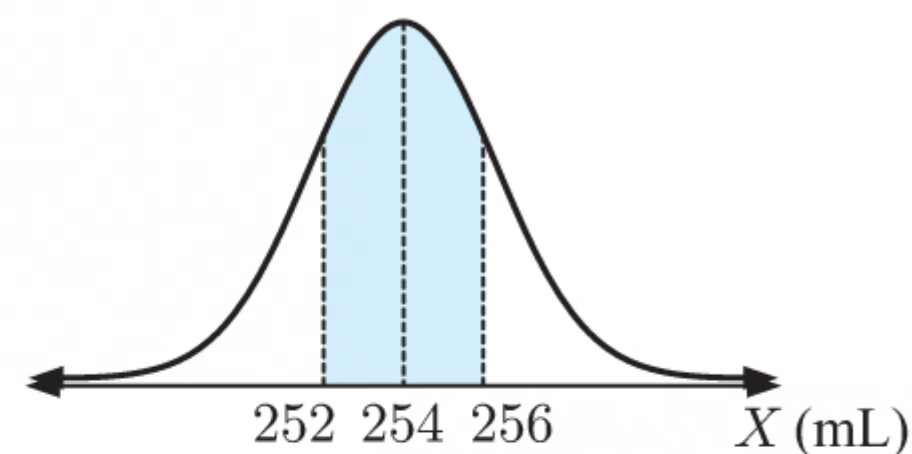
So, $X \sim N(254, (2.3)^2)$.



$$P(X < 254) = 0.5$$



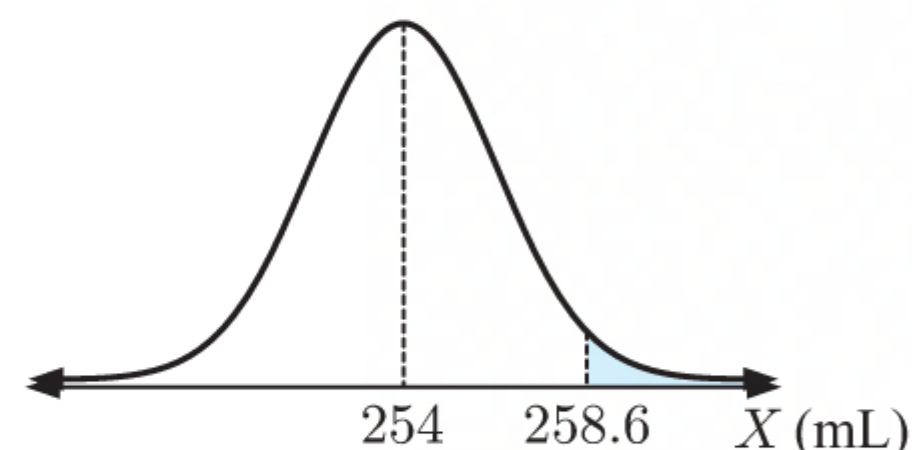
b	<pre> Normal C.D Data :Variable Lower :252 Upper :256 σ :2.3 μ :254 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.61546194 z:Low=-0.8695652 z:Up =0.86956521 </pre>
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$$P(252 \leq X \leq 256) \approx 0.615$$

\therefore about 61.5% of drinks dispensed by the machine have volume between 252 mL and 256 mL.

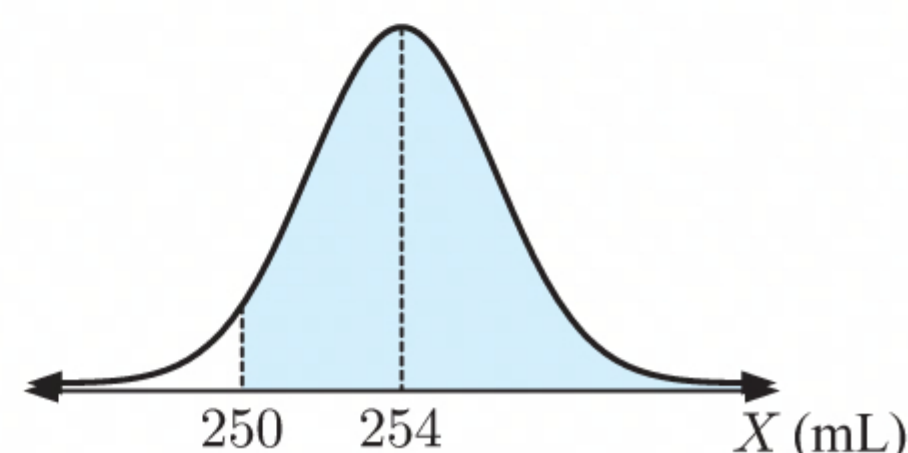
c	<pre> Normal C.D Data :Variable Lower :258.6 Upper :9×10⁹⁹ σ :2.3 μ :254 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.02275013 z:Low=2 z:Up =3.913×10⁹⁹ </pre>
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$$P(X \geq 254 + 2 \times 2.3) = P(X \geq 258.6) \\ \approx 0.0228$$

\therefore we expect about $0.0228 \times 80 \approx 2$ drinks to have volume at least two standard deviations above the mean.

d i	<pre> Normal C.D Data :Variable Lower :250 Upper :9×10⁹⁹ σ :2.3 μ :254 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.95899408 z:Low=-1.7391304 z:Up =3.913×10⁹⁹ </pre>
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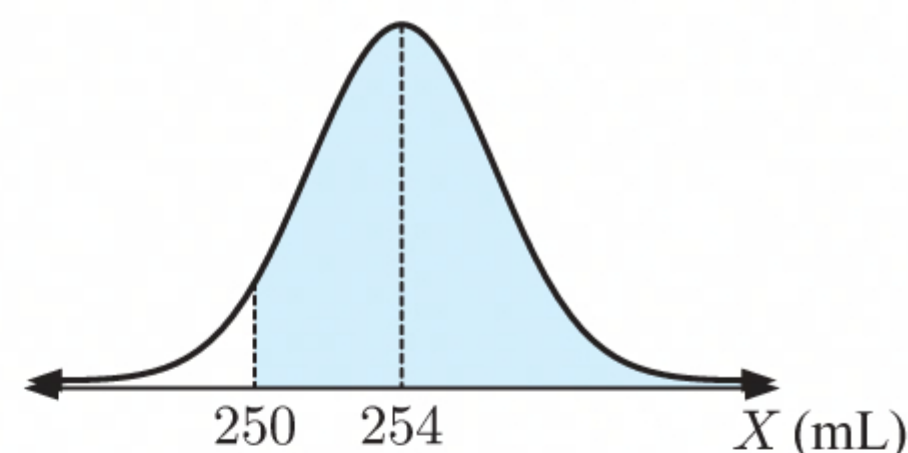


$$P(X \geq 250) \approx 0.959$$

\therefore the operator's guarantee that at least 95% of drinks will have volume at least 250 mL is valid.

ii Suppose $X \sim N(254, (2.5)^2)$.

<pre> Normal C.D Data :Variable Lower :250 Upper :9×10⁹⁹ σ :2.5 μ :254 Save Res:None None LIST </pre>	<pre> Normal C.D p =0.9452007 z:Low=-1.6 z:Up =3.6×10⁹⁹ </pre>
--	---



$$P(X \geq 250) \approx 0.945$$

\therefore about 94.5% of drinks will have volume at least 250 mL.

So, the operator's guarantee is no longer valid.

66 a Let the volume of sauce in a randomly selected bottle be X mL.

$$X \sim N(500, (2.5)^2)$$

$$\therefore P(X < 495) \approx 0.022750 \approx 0.0228$$

<pre> Math Des Norm1 d/c Real NormCD(-9×10⁹⁹, 495, 2.5, 2.5) 0.02275013195 </pre>
Npd Ncd InvN

b Let Y be the number of bottles which require extra sauce.

$$Y \sim B(200, 0.022750) \quad \{\text{from a}\}$$

$$\therefore P(Y \geq 8) \approx 0.0884$$

<pre> Math Des Norm1 d/c Real NormCD(-9×10⁹⁹, 495, 2.5, 2.5) 0.02275013195 BinomialCD(8, 200, 0.022750) 0.08837407712 </pre>
Bpd Bcd InvB

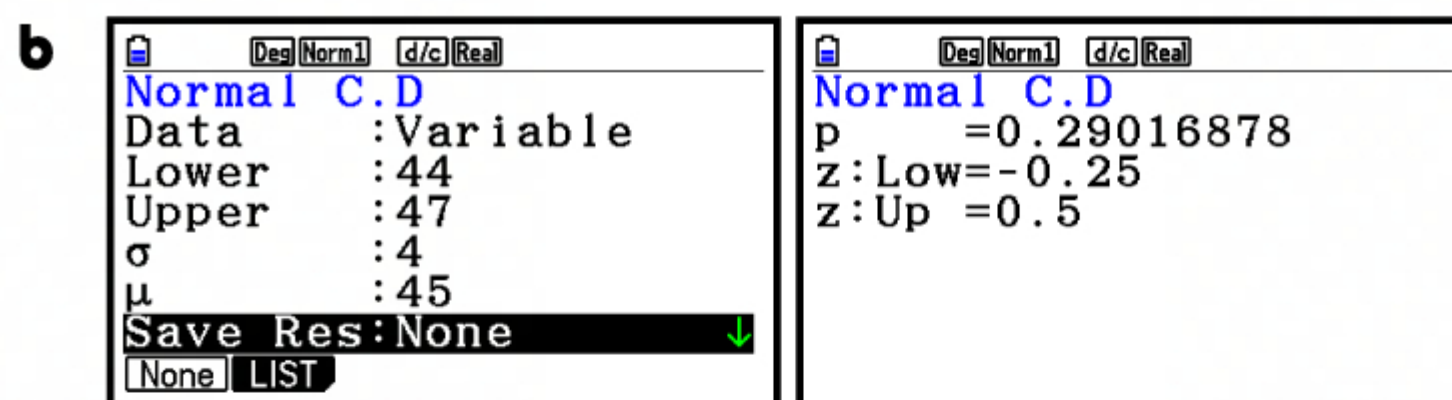
67 a Let the completion time of a randomly selected run be X seconds.

$$X \sim N(45, 4^2)$$

i $P(X < 40) \approx 0.106$

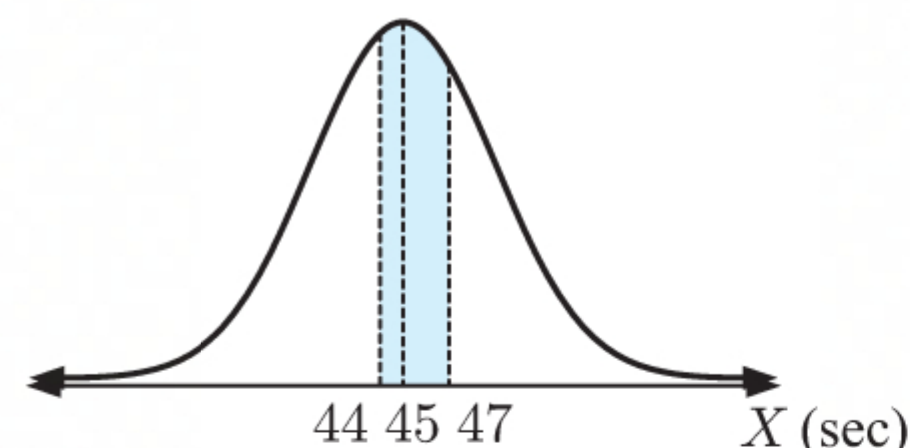
ii $P(\text{two consecutive runs under 40 seconds})$
 $= [P(X < 40)]^2$
 ≈ 0.0112

<pre> Math Des Norm1 d/c Real NormCD(-9×10⁹⁹, 40, 4, 4) 0.1056497737 Ans^2 0.01116187468 </pre>
Npd Ncd InvN



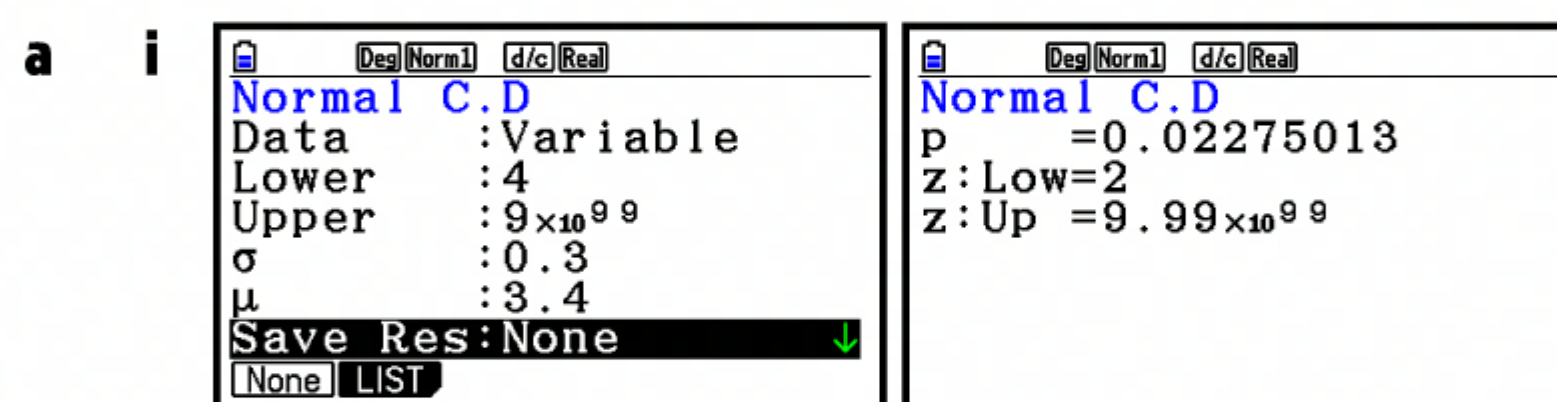
$$P(44 \leq X \leq 47) \approx 0.290$$

\therefore we expect about $0.290 \times 60 \approx 17$ runs to take between 44 seconds and 47 seconds.



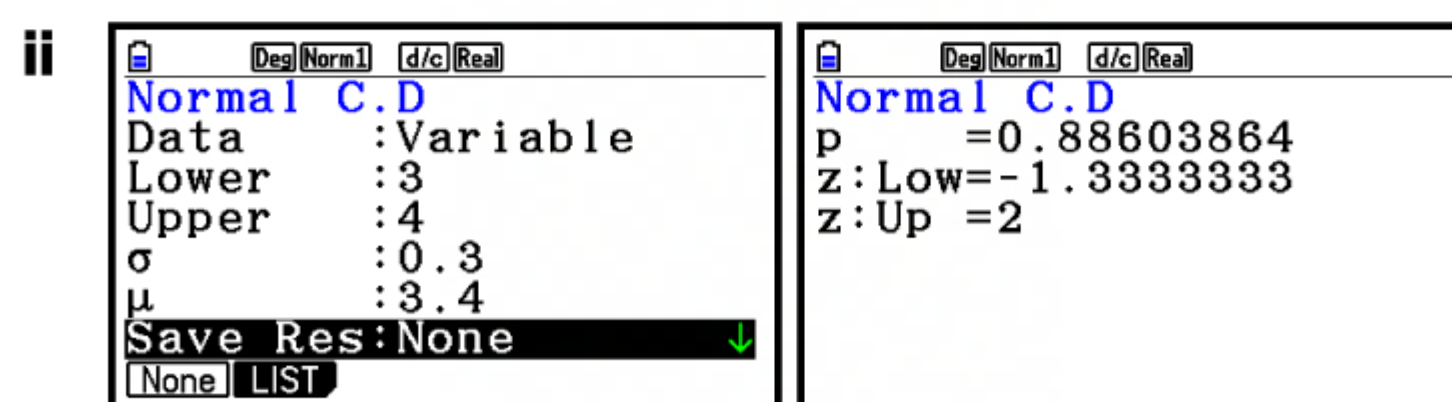
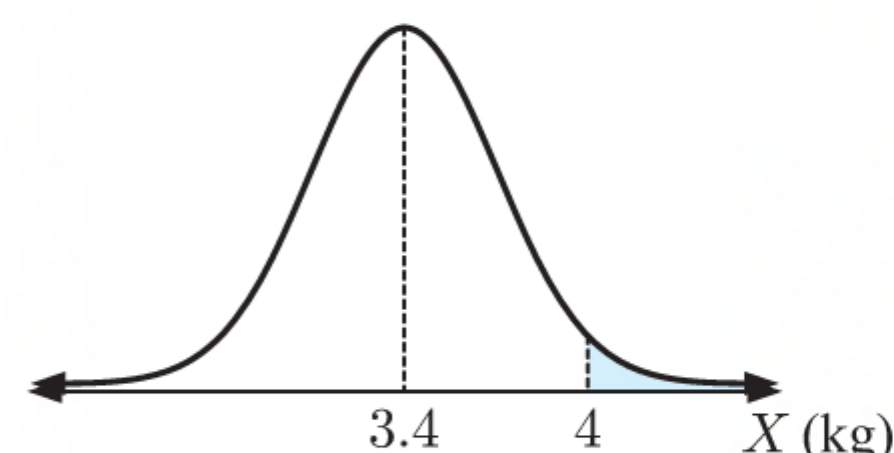
68 Let the birth weight of a randomly selected baby be X kg.

So, $X \sim N(3.4, (0.3)^2)$.



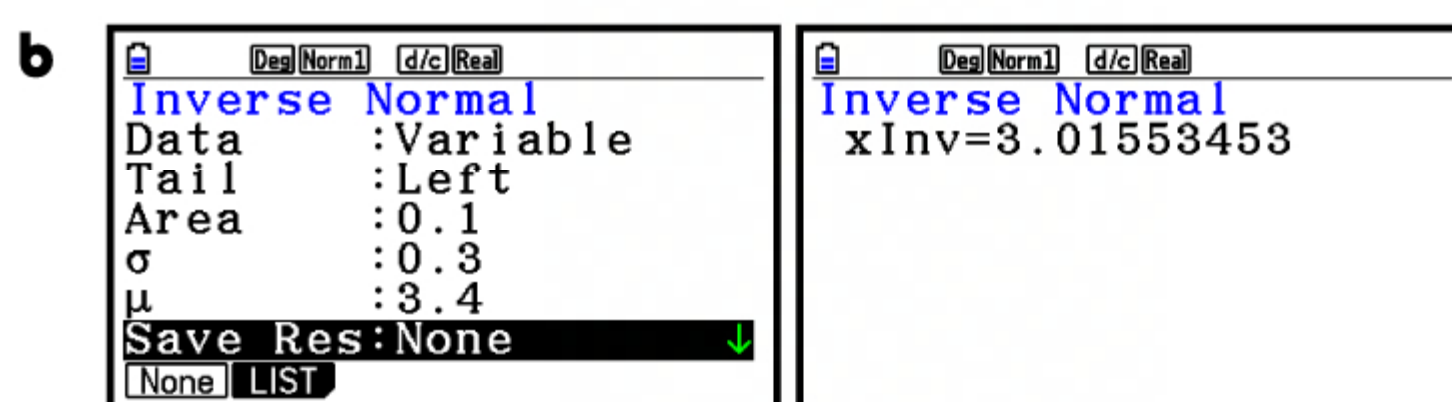
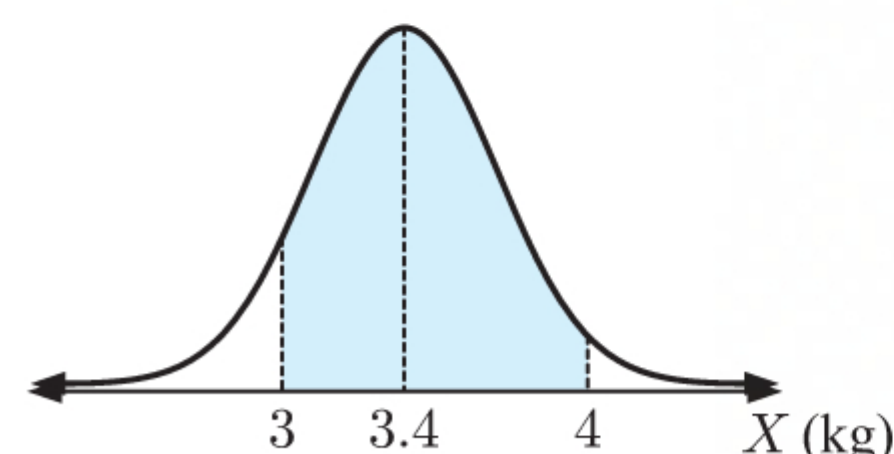
$$P(X > 4) \approx 0.0228$$

\therefore about 2.28% of babies have birth weights in excess of 4 kg.



$$P(3 \leq X \leq 4) \approx 0.886$$

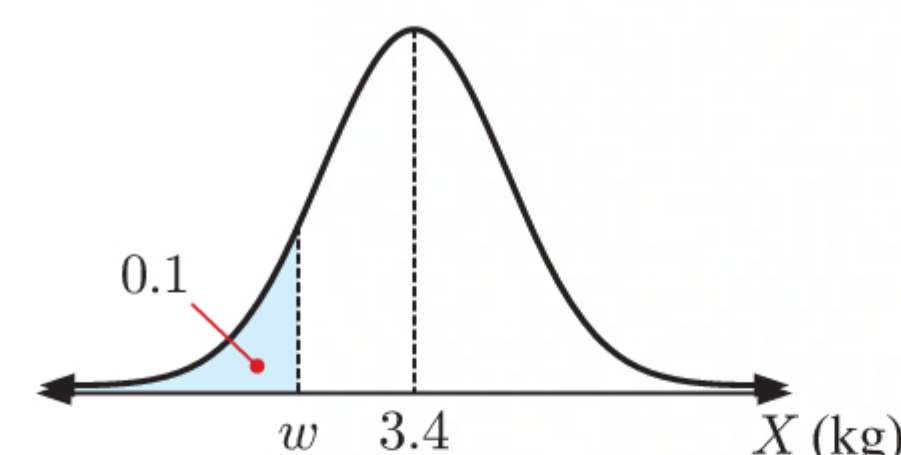
\therefore about 88.6% of babies have birth weights between 3 kg and 4 kg.



$$P(X \leq w) = 0.1$$

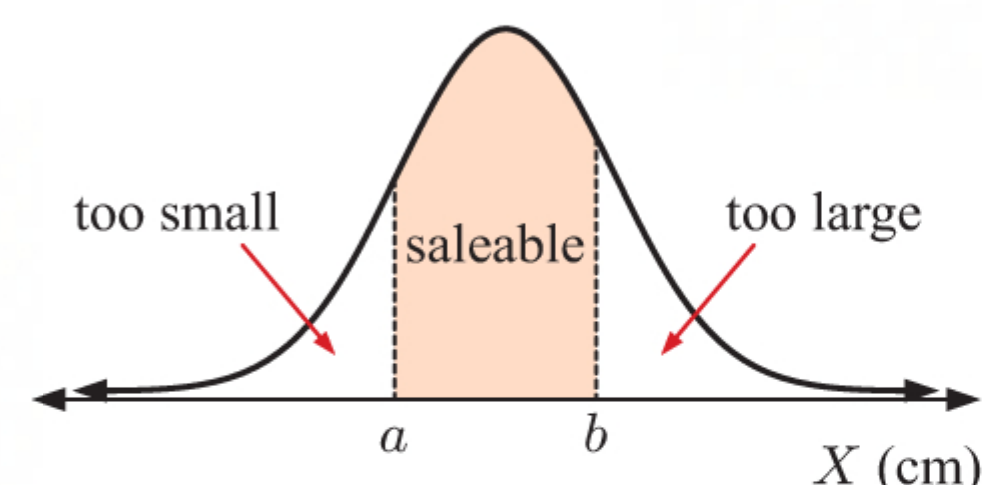
$$\therefore w \approx 3.02$$

\therefore the weight below which a baby is classified as having *low birth weight* is about 3.02 kg.



69 Let the length of a randomly selected zucchini be X cm.

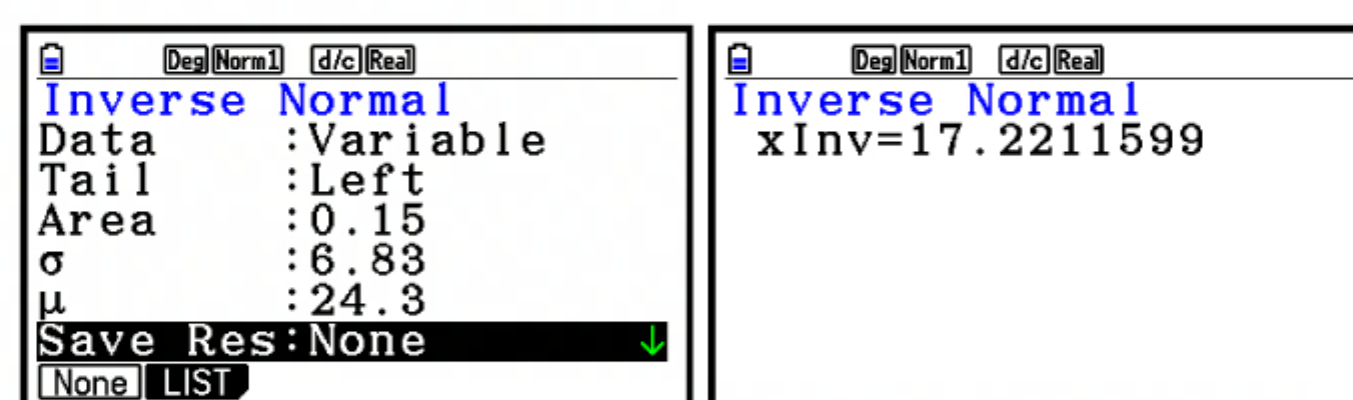
So, $X \sim N(24.3, (6.83)^2)$.



a 15% of zucchinis are too small.

$$P(X < a) = 0.15$$

$$\therefore a \approx 17.2$$

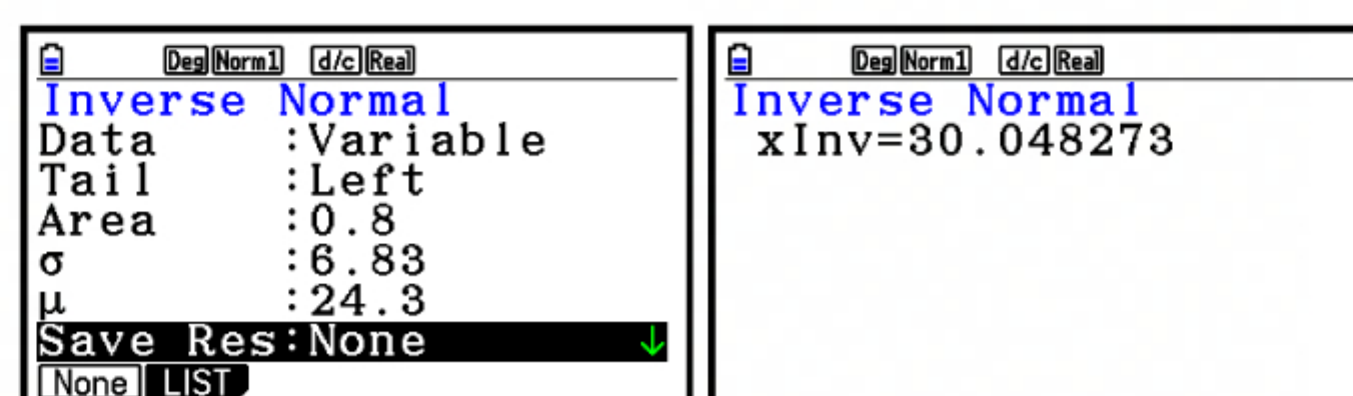


20% of zucchinis are too large.

$$P(X > b) = 0.2$$

$$\therefore P(X \leq b) = 0.8$$

$$\therefore b \approx 30.0$$

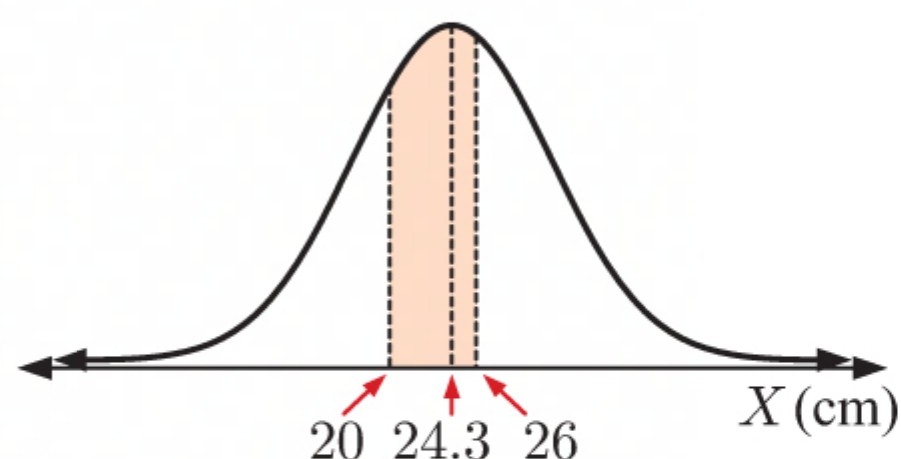


b i $P(\text{saleable length}) = 1 - 0.2 - 0.15$
 $= 0.65$

ii

Normal C.D	Normal C.D
Data : Variable	p = 0.33379545
Lower : 20	z: Low = -0.6295754
Upper : 26	z: Up = 0.2489019
σ : 6.83	
μ : 24.3	
Save Res: None	
None LIST	

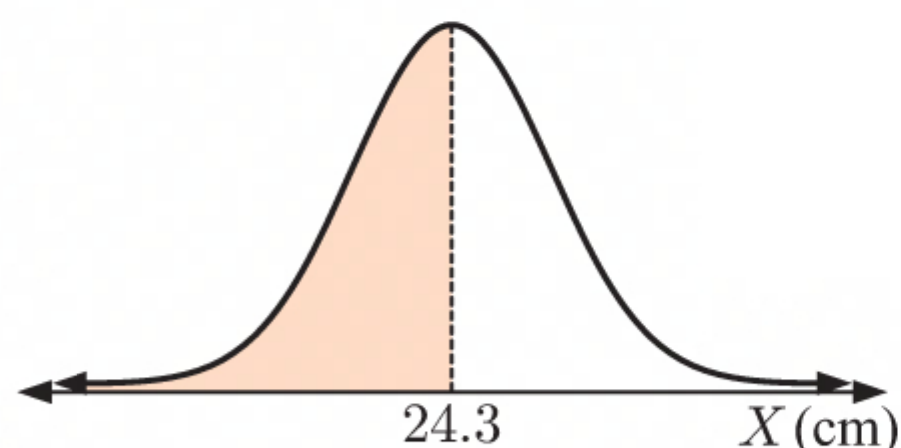
$$P(20 \leq X \leq 26) \approx 0.334$$



iii

Normal C.D	Normal C.D
Data : Variable	p = 0.5
Lower : -9×10^9	z: Low = -1.318×10^9
Upper : 24.3	z: Up = 0
σ : 6.83	
μ : 24.3	
Save Res: None	
None LIST	

$$P(X < 24.3) = 0.5$$



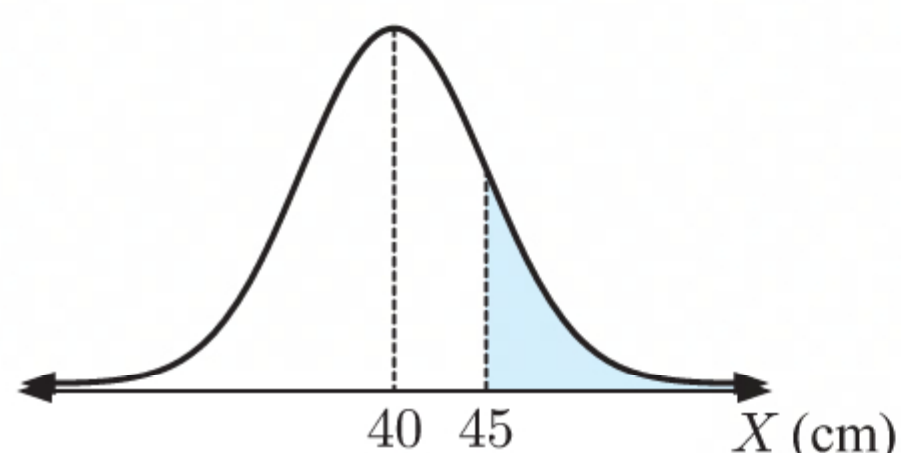
70 Let the length of a randomly selected adult fish be X cm.

So, $X \sim N(40, 5^2)$.

a i

Normal C.D	Normal C.D
Data : Variable	p = 0.15865525
Lower : 45	z: Low = 1
Upper : 9×10^9	z: Up = 1.8×10^9
σ : 5	
μ : 40	
Save Res: None	
None LIST	

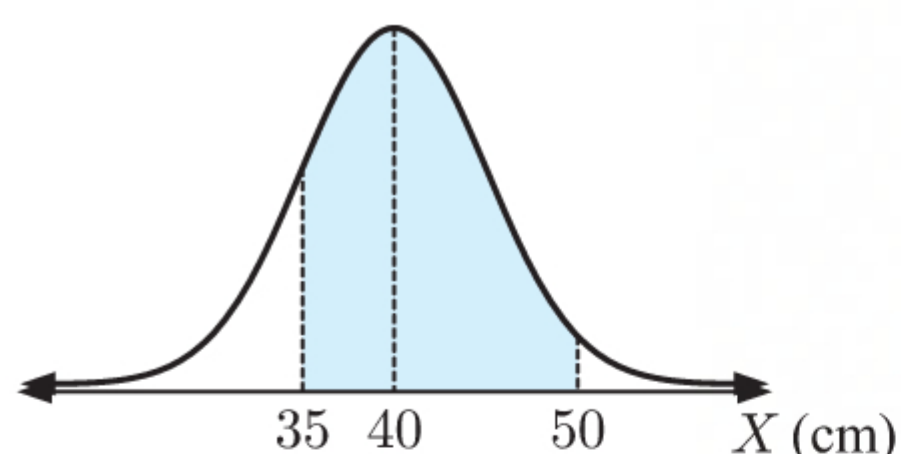
$$P(X > 45) \approx 0.159$$



ii

Normal C.D	Normal C.D
Data : Variable	p = 0.81859461
Lower : 35	z: Low = -1
Upper : 50	z: Up = 2
σ : 5	
μ : 40	
Save Res: None	
None LIST	

$$P(35 \leq X \leq 50) \approx 0.819$$



b

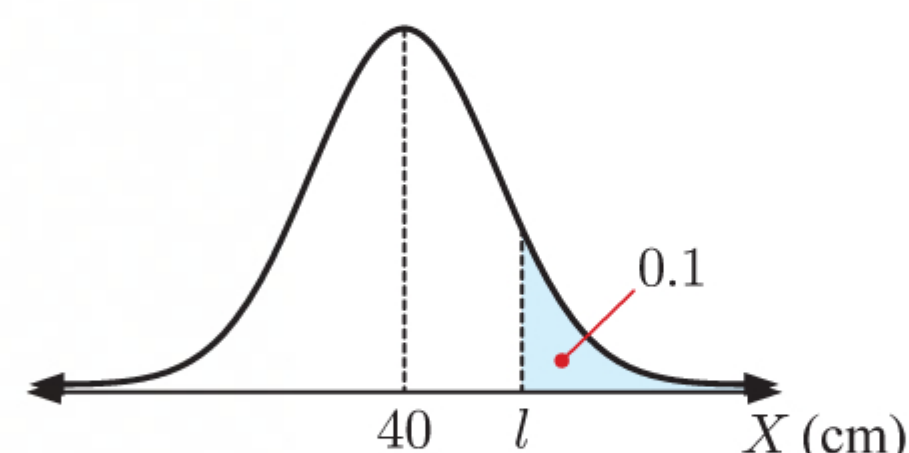
Inverse Normal	Inverse Normal
Data : Variable	xInv = 46.4077578
Tail : Left	
Area : 0.9	
σ : 5	
μ : 40	
Save Res: None	
None LIST	

$$P(X > l) = 0.1$$

$$\therefore P(X \leq l) = 0.9$$

$$\therefore l = 46.4$$

\therefore the minimum length of the longest 10% of fish is about 46.4 cm.



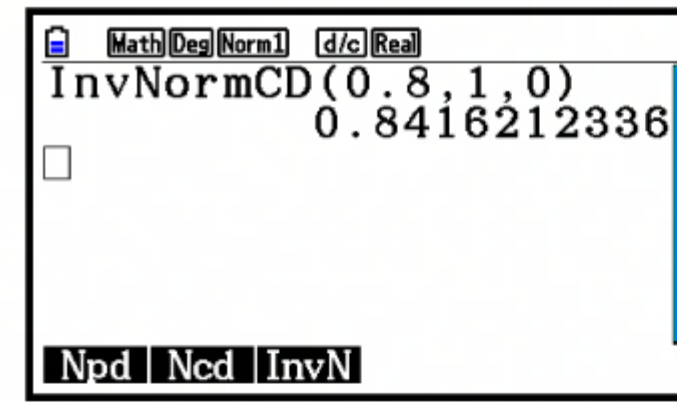
c

Normal C.D	Normal C.D	Normal C.D	Normal C.D
Data : Variable	p = 0.2881446	Data : Variable	p = 0.9452007
Lower : 40	z: Low = 0	Lower : -9×10^9	z: Low = -1.8×10^9
Upper : 44	z: Up = 0.8	Upper : 48	z: Up = 1.6
σ : 5		σ : 5	
μ : 40		μ : 40	
Save Res: None		Save Res: None	
None LIST		None LIST	

$$\begin{aligned}
 P(40 \leq X \leq 44 \mid X < 48) &= \frac{P((40 \leq X \leq 44) \cap (X < 48))}{P(X < 48)} \\
 &= \frac{P(40 \leq X \leq 44)}{P(X < 48)} \\
 &\approx \frac{0.288}{0.945} \\
 &\approx 0.305
 \end{aligned}$$

71 $X \sim N(\mu, \sigma^2)$

$$\begin{aligned} \mathbf{a} \quad & P(X < 56) = 0.8 \\ \therefore & P\left(\frac{X - \mu}{\sigma} < \frac{56 - \mu}{\sigma}\right) = 0.8 \\ \therefore & P\left(Z < \frac{56 - \mu}{\sigma}\right) = 0.8 \quad \left\{ Z = \frac{X - \mu}{\sigma} \right\} \\ & \therefore \frac{56 - \mu}{\sigma} \approx 0.842 \quad \{Z \sim N(0, 1^2)\} \\ & \therefore 56 - \mu \approx 0.842\sigma \quad \dots (*) \end{aligned}$$



So, a score of 56 is about 0.842 standard deviations from the mean.

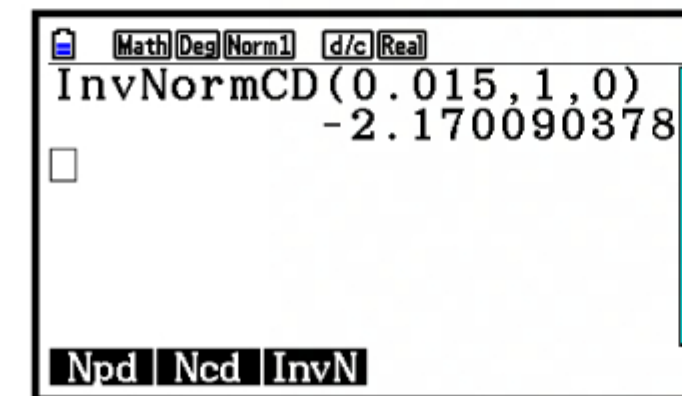
$$\begin{aligned} \mathbf{b} \quad & \text{If } \sigma = 4, \text{ then } (*) \text{ gives } 56 - \mu \approx 0.842 \times 4 \\ & \therefore \mu \approx 56 - 0.842 \times 4 \\ & \therefore \mu \approx 52.6 \end{aligned}$$

72 Let the length of a randomly selected steel rod be X cm.

So, $X \sim N(13.8, \sigma^2)$.

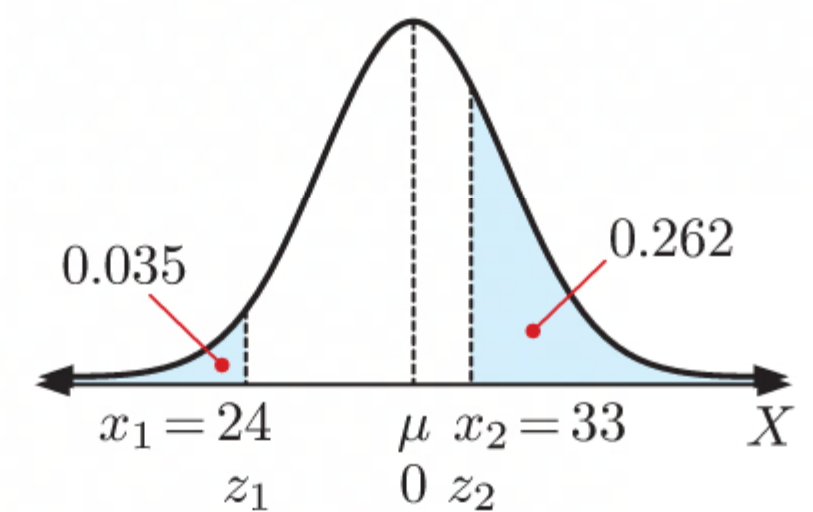
$$\begin{aligned} \mathbf{a} \quad & P(13.2 < X < 13.8) = P(X < 13.8) - P(X < 13.2) \\ & = 0.5 - 0.015 \\ & = 0.485 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(X < 13.2) = 0.015 \\ \therefore & P\left(\frac{X - 13.8}{\sigma} < \frac{13.2 - 13.8}{\sigma}\right) = 0.015 \\ \therefore & P\left(Z < \frac{-0.6}{\sigma}\right) = 0.015 \quad \left\{ Z = \frac{X - 13.8}{\sigma} \right\} \\ & \therefore \frac{-0.6}{\sigma} \approx -2.17 \quad \{Z \sim N(0, 1^2)\} \\ & \therefore \sigma \approx \frac{-0.6}{-2.17} \\ & \therefore \sigma \approx 0.276 \end{aligned}$$

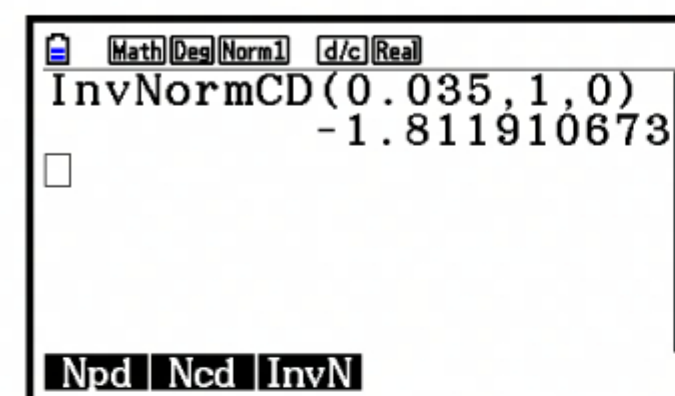


73 $X \sim N(\mu, \sigma^2)$ where we have to find μ and σ .

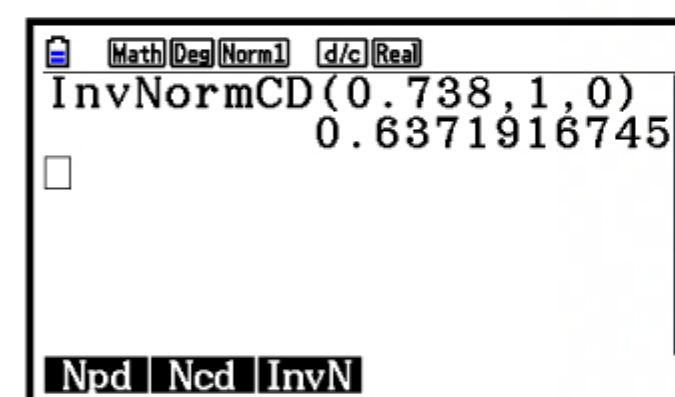
We start by finding z_1 and z_2 which correspond to $x_1 = 24$ and $x_2 = 33$.



$$\begin{aligned} \text{Now } & P(X \leq x_1) = 0.035 \\ \therefore & P\left(\frac{X - \mu}{\sigma} \leq \frac{24 - \mu}{\sigma}\right) = 0.035 \\ \therefore & P\left(Z \leq \frac{24 - \mu}{\sigma}\right) = 0.035 \quad \left\{ Z = \frac{X - \mu}{\sigma} \right\} \\ & \therefore z_1 = \frac{24 - \mu}{\sigma} \approx -1.8119 \quad \{Z \sim N(0, 1^2)\} \\ & \therefore 24 - \mu \approx -1.8119\sigma \quad \dots (1) \end{aligned}$$



$$\begin{aligned} \text{and } & P(X \geq x_2) = 0.262 \\ \therefore & P(X < x_2) = 0.738 \\ \therefore & P\left(\frac{X - \mu}{\sigma} < \frac{33 - \mu}{\sigma}\right) = 0.738 \\ \therefore & P\left(Z < \frac{33 - \mu}{\sigma}\right) = 0.738 \quad \left\{ Z = \frac{X - \mu}{\sigma} \right\} \\ & \therefore z_2 = \frac{33 - \mu}{\sigma} \approx 0.6372 \quad \{Z \sim N(0, 1^2)\} \\ & \therefore 33 - \mu \approx 0.6372\sigma \quad \dots (2) \end{aligned}$$



Solving (1) and (2) simultaneously, $\mu \approx 30.7$ and $\sigma \approx 3.67$.

TOPIC 5 SKILL BUILDER QUESTIONS

- 1 a** When $x = 1$, the denominator of $\frac{x^3 - x^2 + 2x - 2}{x - 1}$ is zero. So, the function is undefined at $x = 1$.

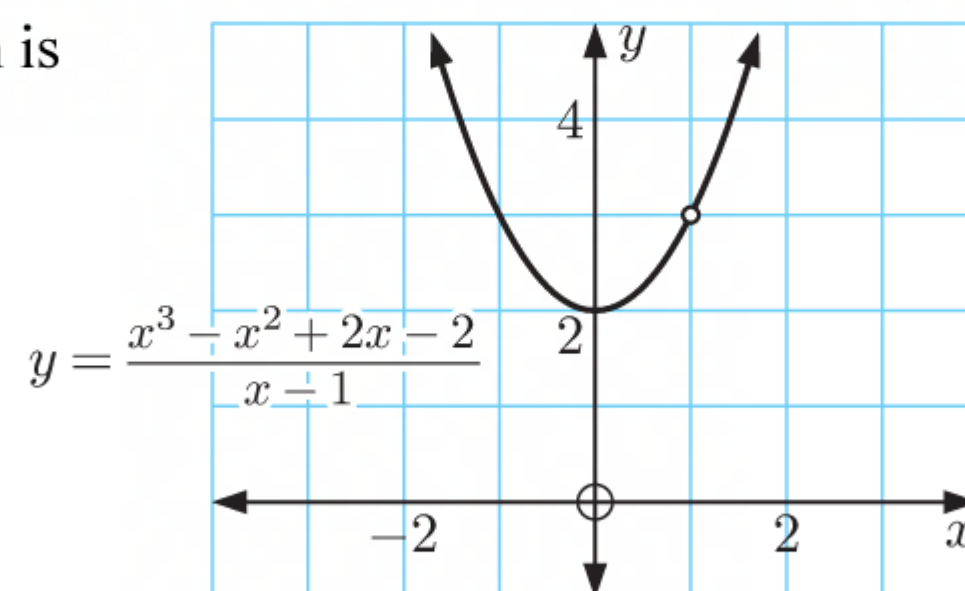
- b** From **a**, there is a discontinuity at $x = 1$.

From the graph, we can see that $y \rightarrow 3$ as $x \rightarrow 1$ from either direction.

So, $\lim_{x \rightarrow 1} f(x) = 3$.

\therefore there is a removable discontinuity when $x = 1$.

f is continuous for all $x \in \mathbb{R}$, $x \neq 1$.



- 2 a** $f(x) = \frac{x+3}{x^2+3x}$ is undefined where $x^2 + 3x = 0$

$$\therefore x(x+3) = 0$$

$$\therefore x = 0 \text{ or } -3$$

- b i** As $x \rightarrow 0^+$, $f(x) \rightarrow \infty$

As $x \rightarrow 0^-$, $f(x) \rightarrow -\infty$

$\therefore \lim_{x \rightarrow 0} f(x)$ does not exist.

$$\text{iii} \quad \lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} \frac{x+3}{x^2+3x}$$

$$= \lim_{x \rightarrow -3} \frac{\cancel{x+3}}{x(\cancel{x+3})}$$

$$= \lim_{x \rightarrow -3} \frac{1}{x} \quad \{\text{since } x \neq -3\}$$

$$= \frac{1}{-3}$$

$$\therefore \lim_{x \rightarrow -3} f(x) = -\frac{1}{3}$$

- ii** As $x \rightarrow 1$, $f(x) \rightarrow \frac{1+3}{1^2+3(1)} = \frac{4}{4} = 1$

$$\therefore \lim_{x \rightarrow 1} f(x) = 1$$

- 3 a** As $x \rightarrow -1$,

$$x^2 - 3x + 1 \rightarrow (-1)^2 - 3(-1) + 1 = 5$$

$$\therefore \lim_{x \rightarrow -1} (x^2 - 3x + 1) = 5$$

- b** As $x \rightarrow 2$,

$$\frac{x-2}{x^2+3x-15} \rightarrow \frac{2-2}{2^2+3(2)-15} = 0$$

$$\therefore \lim_{x \rightarrow 2} (x^2 - 3x + 1) = 0$$

- 4 a** $\lim_{x \rightarrow \infty} \frac{4x-1}{x+3}$

$$= \lim_{x \rightarrow \infty} \frac{4 - \frac{1}{x}}{1 + \frac{3}{x}} \quad \{\text{dividing each term by } x, \text{ since } x \neq 0\}$$

$$= \frac{4}{1} \quad \left\{ \text{as } x \rightarrow \infty, \frac{1}{x} \rightarrow 0 \text{ and } \frac{3}{x} \rightarrow 0 \right\}$$

$$= 4$$

- b** $\lim_{x \rightarrow \infty} \frac{2x+3}{x^2-x-1}$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + \frac{3}{x^2}}{1 - \frac{1}{x} - \frac{1}{x^2}} \quad \{\text{dividing each term by } x^2, \text{ since } x^2 \neq 0\}$$

$$= \frac{0}{1} \quad \left\{ \text{as } x \rightarrow \infty, \frac{2}{x} \rightarrow 0, \frac{3}{x^2} \rightarrow 0, \frac{1}{x} \rightarrow 0, \text{ and } \frac{1}{x^2} \rightarrow 0 \right\}$$

$$= 0$$

- 5 a** $\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta}$

$$= \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta} \times 4$$

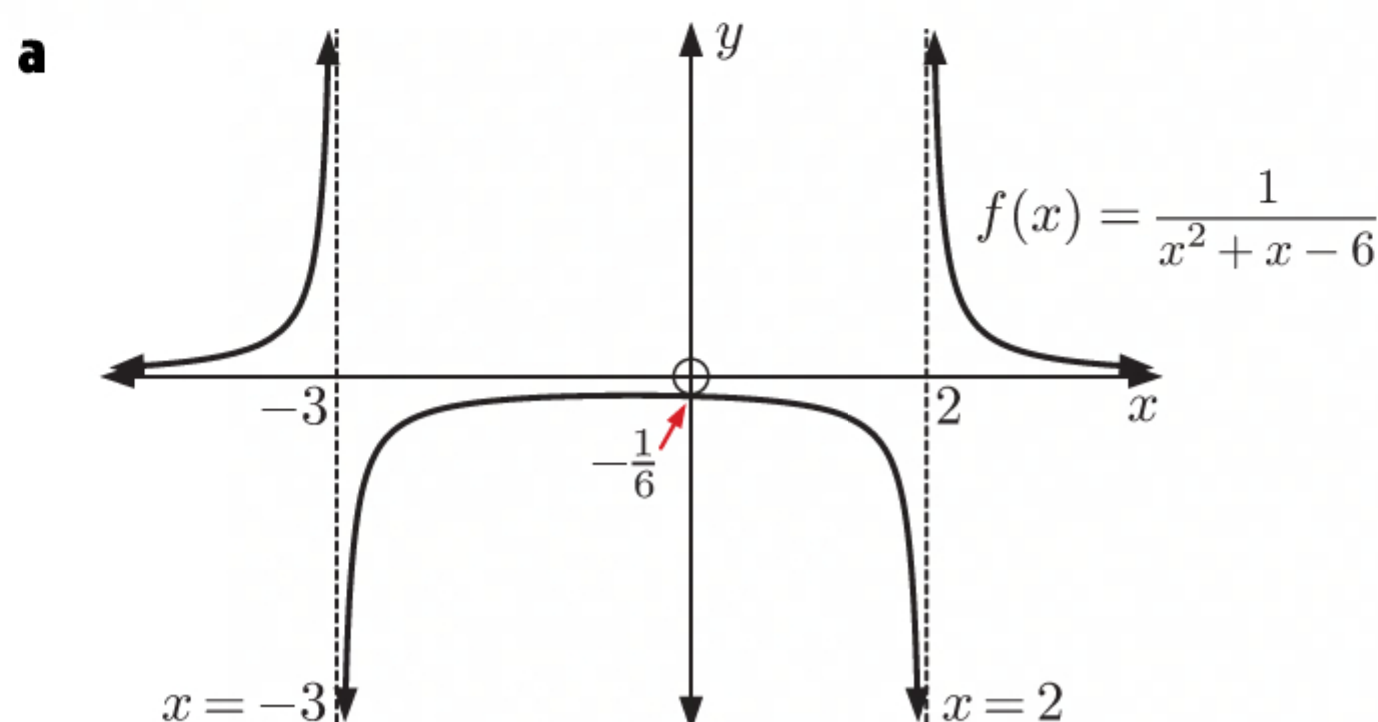
$$= 4 \times \lim_{4\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \quad \{\text{as } \theta \rightarrow 0, 4\theta \rightarrow 0 \text{ also}\}$$

$$= 4 \times 1$$

$$= 4$$

$$\begin{aligned}
 \text{b} \quad & \lim_{\theta \rightarrow 0} \frac{\theta^2 - 3\theta}{\sin \theta} \\
 &= \lim_{\theta \rightarrow 0} \left(\frac{\theta^2}{\sin \theta} - \frac{3\theta}{\sin \theta} \right) \\
 &= \lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin \theta} - \lim_{\theta \rightarrow 0} \frac{3\theta}{\sin \theta} \\
 &= \lim_{\theta \rightarrow 0} \left(\theta \times \frac{\theta}{\sin \theta} \right) - 3 \times \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \\
 &= \lim_{\theta \rightarrow 0} \theta \times \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} - 3 \times \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \\
 &= \lim_{\theta \rightarrow 0} \theta \times \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} - 3 \times \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} \\
 &= \lim_{\theta \rightarrow 0} \theta \times \frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} - 3 \times \frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} \\
 &= 0 \times \frac{1}{1} - 3 \times \frac{1}{1} \\
 &= -3
 \end{aligned}$$

$$6 \quad f(x) = \frac{1}{x^2 + x - 6} = \frac{1}{(x+3)(x-2)}$$



b f is not defined at $x = 3$ or $x = 2$, and $\lim_{x \rightarrow -3} f(x)$ and $\lim_{x \rightarrow 2} f(x)$ do not exist.

f has essential discontinuities at $x = -3$ and $x = 2$.

f is continuous for all $x \in \mathbb{R}$, $x \neq -3$, $x \neq 2$.

$$7 \quad f(x) = \begin{cases} kx^2, & x \leq 1 \\ 2x + 3, & x > 1 \end{cases}$$

$$\begin{aligned}
 \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} kx^2 & \text{and} & \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x + 3) \\
 &= k(1)^2 & & \quad = 2(1) + 3 \\
 &= k & & \quad = 5
 \end{aligned}$$

f is continuous on \mathbb{R} if $\lim_{x \rightarrow 1} f(x)$ exists.

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^+} f(x) \\
 \therefore k &= 5
 \end{aligned}$$

$$8 \quad \text{a} \quad y = x^2$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh - h^2 - \cancel{x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh - h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x - h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x - h) \quad \{\text{since } h \neq 0\} \\
 &= 2x
 \end{aligned}$$

$$\text{b} \quad y = 3x + 5$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{3(x+h) + 5 - (3x + 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{3x} + 3h + \cancel{5} - \cancel{3x} - \cancel{5}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3\cancel{h}}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} 3 \quad \{\text{since } h \neq 0\} \\
 &= 3
 \end{aligned}$$

$$\mathbf{c} \quad y = -3x^2 + x - 1$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{-3(x+h)^2 + (x+h) - 1 - (-3x^2 + x - 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-3(x^2 + 2xh + h^2) + \cancel{x} + h - \cancel{1} + 3x^2 - \cancel{x} + \cancel{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-\cancel{3x^2} - 6xh - 3h^2 + h + \cancel{3x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-6x - 3h + 1)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (-6x - 3h + 1) \quad \{\text{since } h \neq 0\} \\ &= -6x + 1 \end{aligned}$$

$$\mathbf{9} \quad f(x) = x^3 + 8x^2 + 5x + 3$$

$$\begin{aligned} \mathbf{a} \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^3 + 8(x+h)^2 + 5(x+h) + 3 - (x^3 + 8x^2 + 5x + 3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{8x^2} + 16xh + 8h^2 + \cancel{5x} + 5h + \cancel{3} - \cancel{x^3} - \cancel{8x^2} - \cancel{5x} - \cancel{3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 + 16xh + 8h^2 + 5h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 16x + 8h + 5)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 + 16x + 8h + 5) \quad \{\text{since } h \neq 0\} \\ &= 3x^2 + 16x + 5 \end{aligned}$$

$$\mathbf{b} \quad \mathbf{i} \quad \text{Gradient of tangent} = 0$$

$$\therefore f'(x) = 0$$

$$\therefore 3x^2 + 16x + 5 = 0 \quad \{\text{using } \mathbf{a}\}$$

$$\therefore 3x^2 + x + 15x + 5 = 0$$

$$\therefore x(3x + 1) + 5(3x + 1) = 0$$

$$\therefore (3x + 1)(x + 5) = 0$$

$$\therefore x = -\frac{1}{3} \text{ or } -5$$

$$\begin{aligned} f(-\frac{1}{3}) &= (-\frac{1}{3})^3 + 8(-\frac{1}{3})^2 + 5(-\frac{1}{3}) + 3 & \text{and} & \quad f(-5) = (-5)^3 + 8(-5)^2 + 5(-5) + 3 \\ &= -\frac{1}{27} + \frac{8}{9} - \frac{5}{3} + 3 & & \quad = -125 + 200 - 25 + 3 \\ &= \frac{59}{27} & & \quad = 53 \end{aligned}$$

\therefore the tangent has gradient 0 at $(-\frac{1}{3}, \frac{59}{27})$ and $(-5, 53)$.

$$\mathbf{ii} \quad \text{Gradient of tangent} = 17$$

$$\therefore f'(x) = 17$$

$$\therefore 3x^2 + 16x + 5 = 17 \quad \{\text{using } \mathbf{a}\}$$

$$\therefore 3x^2 + 16x - 12 = 0$$

$$\therefore 3x^2 + 18x - 2x - 12 = 0$$

$$\therefore 3x(x + 6) - 2(x + 6) = 0$$

$$\therefore (x + 6)(3x - 2) = 0$$

$$\therefore x = -6 \text{ or } \frac{2}{3}$$

$$\begin{aligned} f(-6) &= (-6)^3 + 8(-6)^2 + 5(-6) + 3 & \text{and} & \quad f(\frac{2}{3}) = (\frac{2}{3})^3 + 8(\frac{2}{3})^2 + 5(\frac{2}{3}) + 3 \\ &= -216 + 288 - 30 + 3 & & \quad = \frac{8}{27} + \frac{32}{9} + \frac{10}{3} + 3 \\ &= 45 & & \quad = \frac{275}{27} \end{aligned}$$

\therefore the tangent has gradient 17 at $(-6, 45)$ and $(\frac{2}{3}, \frac{275}{27})$.

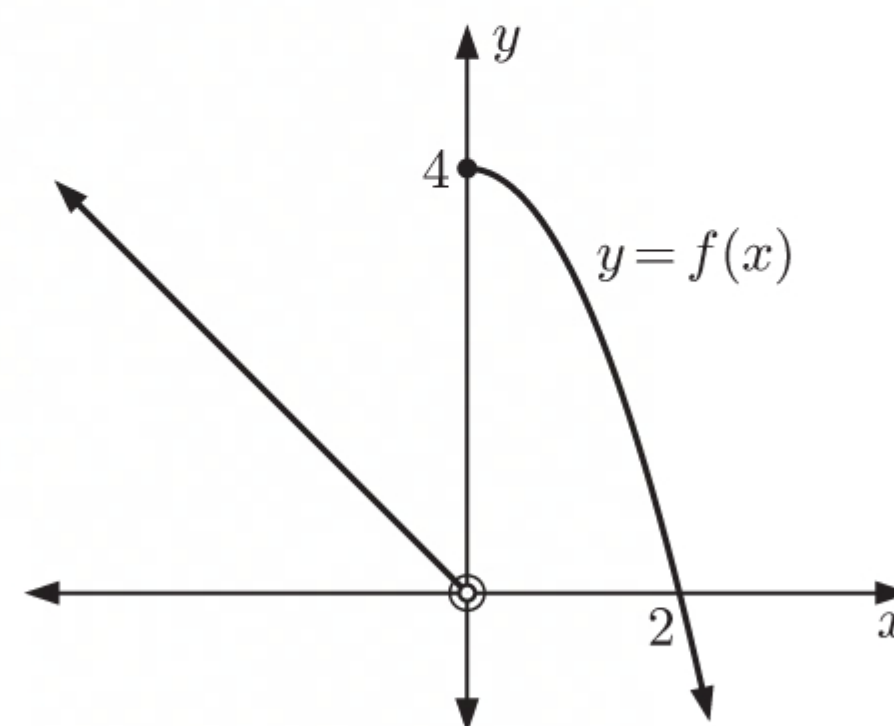
10 a From the graph, there is a “jump” at $x = 0$.

$\therefore f(x)$ is not continuous at $x = 0$.

So, $k = 0$.

b From **a**, $f(x)$ is not continuous at $x = 0$.

$\therefore f(x)$ is not differentiable at $x = 0$.



11 a $f(x) = ax + \frac{b}{x^2}$, $f(1) = 8$, and $f'(1) = -7$

$$= ax + bx^{-2}$$

$$\therefore f'(x) = a + b(-2x^{-3})$$

$$= a - \frac{2b}{x^3}$$

But $f'(1) = -7$, so $a - \frac{2b}{(1)^3} = -7$

$$\therefore a - 2b = -7$$

$$\therefore a = 2b - 7 \quad \dots (*)$$

and $f(1) = 8$, so $a(1) + \frac{b}{(1)^2} = 8$

$$\therefore a + b = 8$$

$$\therefore 2b - 7 + b = 8 \quad \{\text{using } (*)\}$$

$$\therefore 3b - 7 = 8$$

$$\therefore 3b = 15$$

$$\therefore b = 5$$

and so $a = 2(5) - 7 \quad \{\text{using } (*)\}$
 $= 3$

12 a $y = (x^2 - 3x)^5$
 $\therefore \frac{dy}{dx} = 5(x^2 - 3x)^4(2x - 3) \quad \{\text{chain rule}\}$

c $y = \sqrt{x^2 - 3x} = (x^2 - 3x)^{\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = \frac{1}{2}(x^2 - 3x)^{-\frac{1}{2}}(2x - 3) \quad \{\text{chain rule}\}$
 $= \frac{2x - 3}{2\sqrt{x^2 - 3x}}$

13 $f(x) = (ax + b)^c$
 $\therefore f'(x) = c(ax + b)^{c-1}(a) \quad \{\text{chain rule}\}$
 $= ac(ax + b)^{c-1}$

Now $f'(x) = 81x^2 + 108x + 36$ is a polynomial of degree 2, so we must have $c - 1 = 2 \quad \therefore c = 3$

$$\therefore 3a(ax + b)^2 = 81x^2 + 108x + 36$$

$$\therefore 3a(a^2x^2 + 2abx + b^2) = 81x^2 + 108x + 36$$

$$\therefore 3a^3x^2 + 6a^2bx + 3ab^2 = 81x^2 + 108x + 36$$

Equating coefficients gives: $3a^3 = 81 \quad \dots (1) \quad 6a^2b = 108 \quad \dots (2) \quad 3ab^2 = 36 \quad \dots (3)$

From (1), $3a^3 = 81$

$$\therefore a^3 = 27$$

$$\therefore a = 3$$

Substituting $a = 3$ into (2) gives $6(3)^2b = 108$

$$\therefore 54b = 108$$

$$\therefore b = 2$$

Check: $3ab^2 = 3(3)(2)^2 = 36 \quad \checkmark$

b $f(x) = ax^b$, $f(2) = \frac{32}{b}$, and $f'(1) = 8$
 $\therefore f'(x) = abx^{b-1}$
 But $f'(1) = 8$, so $ab(1)^{b-1} = 8$
 $\therefore ab = 8 \quad \dots (*)$
 and $f(2) = \frac{32}{b}$, so $a(2)^b = \frac{32}{b}$
 $\therefore 2^b = \frac{32}{ab}$
 $\therefore 2^b = \frac{32}{8} \quad \{\text{using } (*)\}$
 $\therefore 2^b = 4$
 $\therefore 2^b = 2^2$
 $\therefore b = 2 \quad \{\text{equating indices}\}$
 and so $a(2) = 8 \quad \{\text{using } (*)\}$
 $\therefore a = 4$

b $y = \frac{3}{(x^2 + 3)^3} = 3(x^2 + 3)^{-3}$
 $\therefore \frac{dy}{dx} = 3 \times (-3)(x^2 + 3)^{-4} \times (2x) \quad \{\text{chain rule}\}$
 $= -18x(x^2 + 3)^{-4}$
 $= \frac{-18x}{(x^2 + 3)^4}$

$$\begin{aligned}
 \mathbf{14} \quad \mathbf{a} \quad y &= x^2 \sqrt{x^2 + 2x} = x^2(x^2 + 2x)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= 2x(x^2 + 2x)^{\frac{1}{2}} + x^2 \times \frac{1}{2}(x^2 + 2x)^{-\frac{1}{2}}(2x + 2) \quad \{\text{product rule}\} \\
 &= 2x\sqrt{x^2 + 2x} + \frac{x^2(x+1)}{\sqrt{x^2 + 2x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \sqrt{x}(2x+3)^4 = x^{\frac{1}{2}} \times (2x+3)^4 \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}}(2x+3)^4 + x^{\frac{1}{2}} \times 4(2x+3)^3(2) \quad \{\text{product rule}\} \\
 &= \frac{(2x+3)^4}{2\sqrt{x}} + 8\sqrt{x}(2x+3)^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= (2x+1)^3(x-5)^2 \\
 \therefore \frac{dy}{dx} &= 3(2x+1)^2(2)(x-5)^2 + (2x+1)^3 \times 2(x-5)(1) \quad \{\text{product rule}\} \\
 &= 6(2x+1)^2(x-5)^2 + 2(2x+1)^3(x-5)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{15} \quad y &= (x-2)^2(2x-1) \\
 \therefore \frac{dy}{dx} &= 2(x-2)(2x-1) + (x-2)^2(2) \quad \{\text{product rule}\} \\
 &= 2(x-2)[(2x-1) + (x-2)] \\
 &= 2(x-2)(3x-3) \\
 &= 6(x-2)(x-1)
 \end{aligned}$$

Now $\frac{dy}{dx} = 36$ where $6(x-2)(x-1) = 36$

$$\begin{aligned}
 \therefore (x-2)(x-1) &= 6 \\
 \therefore x^2 - 3x + 2 &= 6 \\
 \therefore x^2 - 3x - 4 &= 0 \\
 \therefore (x+1)(x-4) &= 0 \\
 \therefore x &= -1 \text{ or } 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16} \quad \mathbf{a} \quad y &= \frac{x^3}{x^2-1} \\
 \therefore \frac{dy}{dx} &= \frac{3x^2(x^2-1) - (2x)x^3}{(x^2-1)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2} \\
 &= \frac{x^4 - 3x^2}{(x^2-1)^2}
 \end{aligned}$$

When $x = 2$, $\frac{dy}{dx} = \frac{(2)^4 - 3(2)^2}{(2^2-1)^2} = \frac{16-12}{9} = \frac{4}{9}$.

So, the tangent has gradient $\frac{4}{9}$.

$$\begin{aligned}
 \mathbf{b} \quad y &= \frac{\sqrt{x}}{2x+5} = \frac{x^{\frac{1}{2}}}{2x+5} \\
 \therefore \frac{dy}{dx} &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(2x+5) - (2) \times x^{\frac{1}{2}}}{(2x+5)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{2x+5-4x}{2\sqrt{x}(2x+5)^2} \\
 &= \frac{5-2x}{2\sqrt{x}(2x+5)^2}
 \end{aligned}$$

When $x = 4$,

$$\frac{dy}{dx} = \frac{5-2(4)}{2\sqrt{4}(2(4)+5)^2} = \frac{-3}{2 \times 2 \times 169} = \frac{-3}{676}$$

So, the tangent has gradient $-\frac{3}{676}$.

$$\begin{aligned}
 \mathbf{17} \quad \mathbf{a} \quad f(t) &= 20te^{-0.1t} \\
 \therefore f'(t) &= 20(1)e^{-0.1t} + 20t(-0.1e^{-0.1t}) \quad \{\text{product rule}\} \\
 &= 20e^{-0.1t} - 2te^{-0.1t}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(t) &= \frac{100}{1+7e^{-\frac{t}{4}}} = 100\left(1+7e^{-\frac{t}{4}}\right)^{-1} \\
 \therefore f'(t) &= -100\left(1+7e^{-\frac{t}{4}}\right)^{-2} \times \left(-\frac{7}{4}e^{-\frac{t}{4}}\right) \quad \{\text{chain rule}\} \\
 &= \frac{175e^{-\frac{t}{4}}}{\left(1+7e^{-\frac{t}{4}}\right)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad f(t) &= \frac{t+9}{e^t} \\
 \therefore f'(t) &= \frac{(1)e^t - e^t(t+9)}{(e^t)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{1-(t+9)}{e^t} \\
 &= \frac{-t-8}{e^t}
 \end{aligned}$$

18 $f(x) = e^{ax+2} + x^2$ and $f(2) = f'(2)$

$$\therefore f'(x) = ae^{ax+2} + 2x$$

Now $f(2) = e^{2a+2} + 4$ and $f'(2) = ae^{2a+2} + 4$

$$\therefore e^{2a+2} + 4 = ae^{2a+2} + 4$$

$$\therefore e^{2a+2} = ae^{2a+2}$$

$$\therefore (a-1)e^{2a+2} = 0$$

$$\therefore a-1 = 0 \quad \{e^x > 0 \text{ for all } x\}$$

$$\therefore a = 1$$

19 a $\frac{d}{dx} \left(\ln \left(\frac{x-4}{x^2+4} \right) \right) = \frac{d}{dx} (\ln(x-4) - \ln(x^2+4))$ **b** $\frac{d}{dx} \left(\ln \left(x\sqrt{x^2+4} \right) \right) = \frac{d}{dx} \left(\ln x + \ln \left((x^2+4)^{\frac{1}{2}} \right) \right)$

$$= \frac{1}{x-4} - \frac{2x}{x^2+4} \qquad \qquad \qquad = \frac{d}{dx} \left(\ln x + \frac{1}{2} \ln(x^2+4) \right)$$

$$= \frac{1}{x} + \frac{1}{2} \left(\frac{2x}{x^2+4} \right)$$

$$= \frac{1}{x} + \frac{x}{x^2+4}$$

c $\frac{d}{dx} \left(\ln \left(\frac{\sqrt{x^2+1}}{(x+3)(x-2)} \right) \right) = \frac{d}{dx} \left(\ln \left((x^2+1)^{\frac{1}{2}} \right) - \ln((x+3)(x-2)) \right)$

$$= \frac{d}{dx} \left(\frac{1}{2} \ln(x^2+1) - \ln(x+3) - \ln(x-2) \right)$$

$$= \frac{1}{2} \left(\frac{2x}{x^2+1} \right) - \frac{1}{x+3} - \frac{1}{x-2}$$

$$= \frac{x}{x^2+1} - \frac{1}{x+3} - \frac{1}{x-2}$$

20 a $\frac{d}{dx} (3 \sin(x-4)) = 3 \cos(x-4)$

b $\frac{d}{dx} (12x - 2 \cos \frac{x}{3}) = 12 - 2(-\sin \frac{x}{3} \times \frac{1}{3})$

$$= 12 + \frac{2}{3} \sin \frac{x}{3}$$

c $\frac{d}{dx} (x^2 \sin 3x) = (2x) \sin 3x + x^2 (3 \cos 3x) \quad \{\text{product rule}\}$

$$= 2x \sin 3x + 3x^2 \cos 3x$$

d $\frac{d}{dx} ((\sin x)e^{\cos x}) = (\cos x)e^{\cos x} + (\sin x)((-\sin x)e^{\cos x}) \quad \{\text{product rule}\}$

$$= (\cos x)e^{\cos x} - (\sin^2 x)e^{\cos x}$$

$$= e^{\cos x}(\cos x - \sin^2 x)$$

21 a $f(x) = \sqrt{\sin(2x+1)} = (\sin(2x+1))^{\frac{1}{2}}$

$$\therefore f'(x) = \frac{1}{2}(\sin(2x+1))^{-\frac{1}{2}}(2 \cos(2x+1)) \quad \{\text{chain rule}\}$$

$$= \frac{\cos(2x+1)}{\sqrt{\sin(2x+1)}}$$

b $f(x) = \cos \frac{x}{2} \sin \frac{x}{3}$

$$\therefore f'(x) = \left(-\frac{1}{2} \sin \frac{x}{2}\right) \sin \frac{x}{3} + \cos \frac{x}{2} \left(\frac{1}{3} \cos \frac{x}{3}\right) \quad \{\text{product rule}\}$$

$$= -\frac{1}{2} \sin \frac{x}{2} \sin \frac{x}{3} + \frac{1}{3} \cos \frac{x}{2} \cos \frac{x}{3}$$

c $f(x) = \ln \left(\frac{\sin x}{x} \right) = \ln(\sin x) - \ln x$

$$\therefore f'(x) = \frac{\cos x}{\sin x} - \frac{1}{x}$$

$$= \frac{1}{\tan x} - \frac{1}{x}$$

$$\begin{aligned}
 \mathbf{22} \quad \mathbf{a} \quad f(x) &= \cos^4 x \\
 \therefore f'(x) &= 4 \cos^3 x (-\sin x) \quad \{\text{chain rule}\} \\
 &= -4 \cos^3 x \sin x \\
 f'\left(\frac{3\pi}{4}\right) &= -4 \cos^3\left(\frac{3\pi}{4}\right) \sin \frac{3\pi}{4} \\
 &= -4 \left(-\frac{1}{\sqrt{2}}\right)^3 \left(\frac{1}{\sqrt{2}}\right) \\
 &= 1
 \end{aligned}$$

So, the gradient of the tangent is 1.

$$\begin{aligned}
 \mathbf{b} \quad f(x) &= \frac{3 \sin^2 x}{\cos 2x} \\
 \therefore f'(x) &= \frac{3(2 \sin x \cos x) \cos 2x - (-2 \sin 2x)(3 \sin^2 x)}{\cos^2 2x} \\
 &= \frac{3 \sin 2x \cos 2x + 6 \sin 2x \sin^2 x}{\cos^2 2x} \\
 &= \frac{3 \sin 2x (\cos 2x + 2 \sin^2 x)}{\cos^2 2x} \\
 &= \frac{3 \sin 2x (1 - \cancel{2 \sin^2 x} + \cancel{2 \sin^2 x})}{\cos^2 2x} \\
 &\quad \{\text{double angle formula}\} \\
 &= \frac{3 \sin 2x}{\cos^2 2x} \\
 \therefore f'\left(-\frac{\pi}{3}\right) &= \frac{3 \sin\left(-\frac{2\pi}{3}\right)}{\cos^2\left(-\frac{2\pi}{3}\right)} = \frac{3\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)^2} = -6\sqrt{3}
 \end{aligned}$$

So, the gradient of the tangent is $-6\sqrt{3}$.

$$\begin{aligned}
 \mathbf{23} \quad \mathbf{a} \quad f(x) &= 4^x \\
 \therefore f'(x) &= 4^x \ln 4 \\
 \mathbf{c} \quad f(x) &= 3^{x^2-x-2} \\
 \therefore f'(x) &= 3^{x^2-x-2} \ln 3 (2x-1) \quad \{\text{chain rule}\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad f(x) &= \frac{4^{\sqrt{x}-x}}{5^{3x}} \\
 \therefore f'(x) &= \frac{4^{\sqrt{x}-x} \ln 4 \times \left(\frac{1}{2\sqrt{x}} - 1\right) \times 5^{3x} - 4^{\sqrt{x}-x} \times 5^{3x} \ln 5 \times 3}{(5^{3x})^2} \quad \{\text{quotient rule}\} \\
 &= \frac{4^{\sqrt{x}-x} \times \cancel{5^{3x}} \left(\left(\frac{1}{2\sqrt{x}} - 1\right) \ln 4 - 3 \ln 5\right)}{(5^{3x})^{\cancel{2}}} \\
 &= \frac{4^{\sqrt{x}-x} \left(\left(\frac{1}{2\sqrt{x}} - 1\right) \ln 4 - 3 \ln 5\right)}{5^{3x}}
 \end{aligned}$$

$$\mathbf{24} \quad y = k^{k^x}, \quad k > 0$$

$$\begin{aligned}
 \mathbf{a} \quad y &= k^{k^x} \\
 \therefore \ln y &= k^x \ln k \\
 \therefore \frac{1}{y} \frac{dy}{dx} &= k^x (\ln k)^2 \\
 \therefore \frac{dy}{dx} &= k^x (\ln k)^2 \times y \quad \dots (*) \\
 &= k^x (\ln k)^2 \times k^{k^x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{If } \frac{dy}{dx} &= 2k^x \times y, \text{ then} \\
 k^x (\ln k)^2 \times y &= 2k^x \times y \quad \{\text{using } (*)\} \\
 \therefore (\ln k)^2 &= 2 \quad \{y \neq 0 \text{ and } k^x \neq 0 \text{ as } k > 0\} \\
 \therefore \ln k &= \sqrt{2} \text{ or } -\sqrt{2} \\
 \therefore k &= e^{\sqrt{2}} \text{ or } \frac{1}{e^{\sqrt{2}}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{25} \quad \mathbf{a} \quad y &= \log_2 x + \log_3 (x^3) \\
 &= \log_2 x + 3 \log_3 x \\
 \therefore \frac{dy}{dx} &= \frac{1}{x \ln 2} + \frac{3}{x \ln 3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \log_5 ((x+1)^{\log_2 x}) \\
 &= \log_2 x \log_5 (x+1) \\
 \therefore \frac{dy}{dx} &= \frac{1}{x \ln 2} \times \log_5 (x+1) + \log_2 x \times \frac{1}{(x+1) \ln 5} \\
 &= \frac{\log_5 (x+1)}{x \ln 2} + \frac{\log_2 x}{(x+1) \ln 5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= x \log_5 \left(\frac{1}{x}\right) \\
 &= x \log_5 (x^{-1}) \\
 &= -x \log_5 x \\
 \therefore \frac{dy}{dx} &= -\log_5 x - x \times \frac{1}{x \ln 5} \\
 &= -\log_5 x - \frac{1}{\ln 5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= \frac{x}{\log_2 (\sqrt{x})} \\
 &= \frac{x}{\log_2 (x^{\frac{1}{2}})} \\
 &= \frac{x}{\frac{1}{2} \log_2 x} \\
 &= \frac{2x}{\log_2 x} \\
 \therefore \frac{dy}{dx} &= \frac{2 \times \log_2 x - 2x \times \frac{1}{x \ln 2}}{(\log_2 x)^2} \\
 &= \frac{2 \log_2 x - \frac{2}{\ln 2}}{(\log_2 x)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{26 \quad a} \quad y &= \left[\log_3 \left(\frac{x-3}{2+x^2} \right) \right]^5 \\
 &= [\log_3(x-3) - \log_3(2+x^2)]^5 \\
 \therefore \frac{dy}{dx} &= 5 [\log_3(x-3) - \log_3(2+x^2)]^4 \times \left(\frac{1}{(x-3)\ln 3} - \frac{2x}{(2+x^2)\ln 3} \right) \\
 &= 5 \left[\log_3 \left(\frac{x-3}{2+x^2} \right) \right]^4 \times \left(\frac{2+x^2 - 2x(x-3)}{(x-3)(2+x^2)\ln 3} \right) \\
 &= \frac{5 \left[\log_3 \left(\frac{x-3}{2+x^2} \right) \right]^4 \times (2+6x-x^2)}{(x-3)(2+x^2)\ln 3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \log_2(e^{\log_2 x} \log_3(x+1)) \\
 &= \log_2(e^{\log_2 x}) + \log_2(\log_3(x+1)) \\
 \therefore \frac{dy}{dx} &= \log_2 e \times \frac{1}{x \ln 2} + \frac{1}{\log_3(x+1) \ln 2} \\
 &= \frac{\log_2 e}{x \ln 2} + \frac{1}{(x+1) \log_3(x+1) \ln 3 \ln 2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{27 \quad a} \quad y &= \tan 2x & \mathbf{b} \quad y &= \tan(3x-4) & \mathbf{c} \quad y &= \tan(2^x + x^2) \\
 \therefore \frac{dy}{dx} &= \sec^2(2x) \times 2 & \therefore \frac{dy}{dx} &= \sec^2(3x-4) \times 3 & \therefore \frac{dy}{dx} &= \sec^2(2^x + x^2) \times (2^x \ln 2 + 2x) \\
 &= 2 \sec^2(2x) & &= 3 \sec^2(3x-4) & &= (2^x \ln 2 + 2x) \sec^2(2^x + x^2)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{28 \quad a} \quad \frac{d}{dx}(\sec 5x) &= \sec 5x \tan 5x \times 5 \\
 &= 5 \sec 5x \tan 5x \\
 \mathbf{b} \quad \frac{d}{dx}(\sqrt{\cot x}) &= \frac{d}{dx}((\cot x)^{\frac{1}{2}}) \\
 &= \frac{1}{2}(\cot x)^{-\frac{1}{2}} \times (-\operatorname{cosec}^2 x) \\
 &= \frac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}} \\
 &= -\frac{1}{2} \operatorname{cosec}^2 x \sqrt{\tan x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \frac{d}{dx}(e^{3x} \operatorname{cosec}(x^2)) &= 3e^{3x} \operatorname{cosec}(x^2) + e^{3x}(-\operatorname{cosec}(x^2) \cot(x^2) \times 2x) \\
 &= e^{3x} \operatorname{cosec}(x^2)(3 - 2x \cot(x^2))
 \end{aligned}$$

$$\mathbf{29} \quad f(x) = \cot(2x+c), \quad f'\left(\frac{\pi}{4}\right) = -\frac{8}{3}, \quad 0 < c < 2\pi$$

$$\begin{aligned}
 \therefore f'(x) &= -\operatorname{cosec}^2(2x+c) \times 2 \\
 &= -2 \operatorname{cosec}^2(2x+c)
 \end{aligned}$$

$$\text{Now } f'\left(\frac{\pi}{4}\right) = -\frac{8}{3}$$

$$\therefore -\frac{8}{3} = -2 \operatorname{cosec}^2\left(2\left(\frac{\pi}{4}\right) + c\right)$$

$$\therefore -\frac{8}{3} = \frac{-2}{\sin^2\left(\frac{\pi}{2} + c\right)}$$

$$\therefore \sin^2\left(\frac{\pi}{2} + c\right) = \frac{3}{4}$$

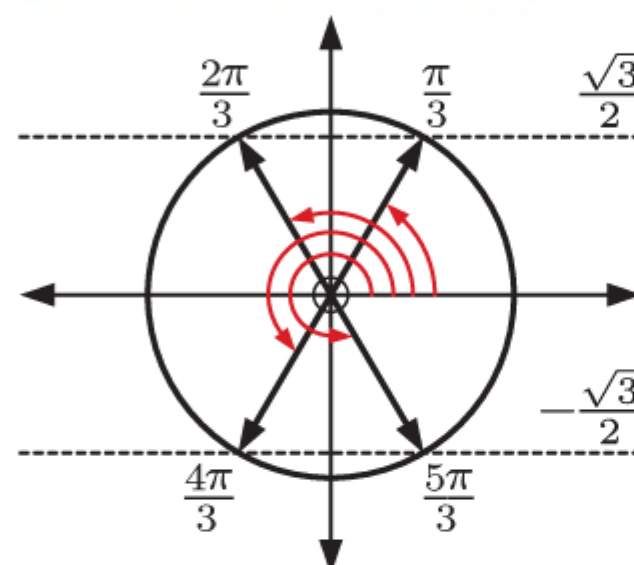
$$\therefore \sin\left(\frac{\pi}{2} + c\right) = \pm \frac{\sqrt{3}}{2}$$

$$\text{Now } 0 < c < 2\pi$$

$$\therefore \frac{\pi}{2} < \frac{\pi}{2} + c < \frac{5\pi}{2}$$

$$\text{So, } \frac{\pi}{2} + c = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$\therefore c = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$



$$\begin{aligned}
 \mathbf{30 \quad a \quad i} \quad f(x) &= \frac{1}{2} \arcsin x \\
 \therefore f'(x) &= \frac{1}{2} \frac{1}{\sqrt{1-x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad g(x) &= \sin 2x \\
 \therefore g'(x) &= 2 \cos 2x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f'(g(x))g'(x) &= \frac{1}{2} \frac{1}{\sqrt{1-[g(x)]^2}} \times 2 \cos 2x \\
 &= \frac{\cos 2x}{\sqrt{1-\sin^2 2x}} \\
 &= \frac{\cos 2x}{\sqrt{\cos^2 2x}} \\
 &= \frac{\cos 2x}{\cos 2x} \quad \left\{ -\frac{\pi}{4} < x < \frac{\pi}{4} \right\} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{31} \quad \mathbf{a} \quad y &= \arccos\left(\frac{1}{2} \sin x\right) \\
 \therefore \frac{dy}{dx} &= \frac{-1}{\sqrt{1-\left(\frac{1}{2} \sin x\right)^2}} \times \frac{1}{2} \cos x \\
 &= \frac{-\cos x}{\sqrt{4-\sin^2 x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= (2-x^2) \arcsin(5^x) \\
 \therefore \frac{dy}{dx} &= -2x \arcsin(5^x) + (2-x^2) \times \frac{1}{\sqrt{1-(5^x)^2}} \times 5^x \ln 5 \\
 &= -2x \arcsin(5^x) + \frac{(2-x^2)5^x \ln 5}{\sqrt{1-5^{2x}}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{32} \quad \mathbf{a} \quad y &= \frac{3}{x^2} = 3x^{-2} \\
 \therefore \frac{dy}{dx} &= -6x^{-3} \\
 \therefore \frac{d^2y}{dx^2} &= 18x^{-4} \\
 &= \frac{18}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \frac{x+3}{6-x} \\
 \therefore \frac{dy}{dx} &= \frac{(1)(6-x) - (-1)(x+3)}{(6-x)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{6-x+x+3}{(6-x)^2} \\
 &= \frac{9}{(6-x)^2} \\
 &= 9(6-x)^{-2} \\
 \therefore \frac{d^2y}{dx^2} &= -18(6-x)^{-3} \times (-1) \quad \{\text{chain rule}\} \\
 &= \frac{18}{(6-x)^3}
 \end{aligned}$$

$$\mathbf{33} \quad f(x) = \ln(\cos x)$$

$$\begin{aligned}
 \mathbf{a} \quad f\left(\frac{\pi}{4}\right) &= \ln\left(\cos \frac{\pi}{4}\right) \\
 &= \ln\left(\frac{1}{\sqrt{2}}\right) \\
 &= \ln\left(2^{-\frac{1}{2}}\right) \\
 &= -\frac{1}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad f'(x) &= \frac{-\sin x}{\cos x} \quad \{\text{from } \mathbf{b}\} \\
 \therefore f''(x) &= \frac{-\cos x(\cos x) - (-\sin x)(-\sin x)}{\cos^2 x} \quad \{\text{quotient rule}\} \\
 &= \frac{-\cos^2 x - \sin^2 x}{\cos^2 x} \\
 &= \frac{-1}{\cos^2 x} \\
 \therefore f''\left(\frac{\pi}{4}\right) &= \frac{-1}{\cos^2\left(\frac{\pi}{4}\right)} \\
 &= \frac{-1}{\left(\frac{1}{\sqrt{2}}\right)^2} \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= e^{-2x} \arctan 2x \\
 \therefore \frac{dy}{dx} &= -2e^{-2x} \arctan 2x + e^{-2x} \times \frac{1}{1+(2x)^2} \times 2 \\
 &= -e^{-2x} \left(2 \arctan 2x - \frac{2}{1+4x^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= 2x^3 + 3x^2 + 2 \\
 \therefore \frac{dy}{dx} &= 6x^2 + 6x \\
 \therefore \frac{d^2y}{dx^2} &= 12x + 6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f'(x) &= \frac{-\sin x}{\cos x} \quad \{\text{chain rule}\} \\
 &= -\tan x \\
 \therefore f'\left(\frac{\pi}{4}\right) &= -\tan \frac{\pi}{4} \\
 &= -1
 \end{aligned}$$

$$34 \quad \mathbf{a} \quad y = x2^x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 2^x + x2^x \ln 2 \\ &= 2^x(1 + x \ln 2) \\ \therefore \frac{d^2y}{dx^2} &= 2^x \ln 2(1 + x \ln 2) + 2^x \ln 2 \\ &= 2^x \ln 2(2 + x \ln 2) \end{aligned}$$

$$\mathbf{c} \quad y = \tan(x+1)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \sec^2(x+1) \\ \therefore \frac{d^2y}{dx^2} &= 2 \sec(x+1)[\sec(x+1) \tan(x+1)] \\ &= 2 \sec^2(x+1) \tan(x+1) \end{aligned}$$

$$35 \quad f(x) = \arcsin(\sqrt{1-x})$$

$$\begin{aligned} \mathbf{a} \quad f(x) &= \arcsin((1-x)^{\frac{1}{2}}) \\ \therefore f'(x) &= \frac{1}{\sqrt{1 - [(1-x)^{\frac{1}{2}}]^2}} \times \frac{1}{2}(1-x)^{-\frac{1}{2}} \times (-1) \\ &= -\frac{1}{2} \times \frac{1}{\sqrt{1-(1-x)}} \times \frac{1}{\sqrt{1-x}} \\ &= \frac{-1}{2\sqrt{x}\sqrt{1-x}} \\ &= \frac{-1}{2\sqrt{x-x^2}} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \log_3(2x^2) \\ &= \log_3 2 + \log_3(x^2) \\ &= \log_3 2 + 2 \log_3 x \\ \therefore \frac{dy}{dx} &= \frac{2}{x \ln 3} = \frac{2}{\ln 3} x^{-1} \\ \therefore \frac{d^2y}{dx^2} &= -\frac{2}{\ln 3} x^{-2} \\ &= -\frac{2}{x^2 \ln 3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f'(x) &= -\frac{1}{2}(x-x^2)^{-\frac{1}{2}} \\ \therefore f''(x) &= \frac{1}{4}(x-x^2)^{\frac{3}{2}}(1-2x) \\ &= \frac{1-2x}{4(x-x^2)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} 36 \quad \mathbf{a} \quad x^2 - xy^2 + y &= 21 \\ \therefore \frac{d}{dx}(x^2) - \frac{d}{dx}(xy^2) + \frac{d}{dx}(y) &= \frac{d}{dx}(21) \\ \therefore 2x - \left[y^2 + x \left(2y \frac{dy}{dx} \right) \right] + \frac{dy}{dx} &= 0 \\ \therefore \frac{dy}{dx}(1 - 2xy) &= y^2 - 2x \\ \therefore \frac{dy}{dx} &= \frac{y^2 - 2x}{1 - 2xy} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad e^y \sin 2x &= 1 \\ \therefore \frac{d}{dx}(e^y \sin 2x) &= \frac{d}{dx}(1) \\ \therefore e^y \frac{dy}{dx} \sin 2x + e^y(2 \cos 2x) &= 0 \\ \therefore \frac{dy}{dx} &= \frac{-2e^y \cos 2x}{e^y \sin 2x} \\ &= -2 \cot 2x \end{aligned}$$

$$\begin{aligned} 37 \quad \mathbf{a} \quad x^3 + y^2 &= 7 \\ \therefore 3x^2 + 2y \frac{dy}{dx} &= 0 \\ \therefore \frac{dy}{dx} &= -\frac{3x^2}{2y} \\ \therefore \frac{d^2y}{dx^2} &= \frac{-6x(2y) - 2 \frac{dy}{dx}(-3x^2)}{4y^2} \quad \{\text{quotient rule}\} \\ &= \frac{-12xy + 6x^2 \frac{dy}{dx}}{4y^2} \\ &= \frac{-12xy + 6x^2 \left(-\frac{3x^2}{2y} \right)}{4y^2} \\ &= \frac{-12xy^2 - 9x^2}{4y^3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2^x + 2^y &= 1 \\ \therefore 2^x \ln 2 + 2^y \ln 2 \times \frac{dy}{dx} &= 0 \\ \therefore 2^x + 2^y \frac{dy}{dx} &= 0 \\ \therefore \frac{dy}{dx} &= -\frac{2^x}{2^y} = -2^{x-y} \\ \therefore \frac{d^2y}{dx^2} &= -2^{x-y} \ln 2 \left(1 - \frac{dy}{dx} \right) \\ &= -2^{x-y} \ln 2 (1 + 2^{x-y}) \end{aligned}$$

38 \mathbf{a} As $x \rightarrow \infty$, $\ln x \rightarrow \infty$ and $x^2 + x \rightarrow \infty$, so we can use l'Hôpital's rule.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln x}{x^2 + x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^2 + x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{2x + 1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{2x^2 + x} \\ &= 0 \end{aligned}$$

b As $x \rightarrow 0$, $e^{x^2} - \ln(x + e) \rightarrow 0$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{x^2} - \ln(x + e)}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^{x^2} - \ln(x + e))}{\frac{d}{dx}(x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \left(2xe^{x^2} - \frac{1}{x + e} \right) \\ &= 2(0)e^0 - \frac{1}{0 + e} \\ &= -\frac{1}{e}\end{aligned}$$

c As $x \rightarrow -\infty$, $x - 2 \rightarrow -\infty$ and $3^x \rightarrow 0$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow -\infty} (x - 2)3^x &= \lim_{x \rightarrow -\infty} \frac{x - 2}{3^{-x}} \quad \left\{ \text{converting to the form } -\frac{\infty}{\infty} \right\} \\ &= \lim_{x \rightarrow -\infty} \frac{\frac{d}{dx}(x - 2)}{\frac{d}{dx}(3^{-x})} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow -\infty} \frac{1}{3^{-x} \ln 3 \times (-1)} \\ &= -\frac{1}{\ln 3} \lim_{x \rightarrow -\infty} 3^x \\ &= -\frac{1}{\ln 3} \times 0 \\ &= 0\end{aligned}$$

39 a As $x \rightarrow \infty$, $x^2 \rightarrow \infty$ and $2^x \rightarrow \infty$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x^2}{2^x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(2^x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \frac{2x}{2^x \ln 2} \\ &= \frac{2}{\ln 2} \lim_{x \rightarrow \infty} \frac{x}{2^x} \\ &= \frac{2}{\ln 2} \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(2^x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{2}{\ln 2} \lim_{x \rightarrow \infty} \frac{1}{2^x \ln 2} \\ &= \frac{2}{(\ln 2)^2} \times \lim_{x \rightarrow \infty} \frac{1}{2^x} \\ &= \frac{2}{(\ln 2)^2} \times 0 \\ &= 0\end{aligned}$$

b As $x \rightarrow 0$, $x^2 - \sin x \rightarrow 0$ and $x - 3 \sin x \rightarrow 0$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x^2 - \sin x}{x - 3 \sin x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x^2 - \sin x)}{\frac{d}{dx}(x - 3 \sin x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{2x - \cos x}{1 - 3 \cos x} \\ &= \frac{2(0) - 1}{1 - 3(1)} \\ &= \frac{1}{2}\end{aligned}$$

c As $x \rightarrow 0$, $5^x - 2^x \rightarrow 0$ and $x^2 - 2x \rightarrow 0$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{5^x - 2^x}{x^2 - 2x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(5^x - 2^x)}{\frac{d}{dx}(x^2 - 2x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{5^x \ln 5 - 2^x \ln 2}{2x - 2} \\ &= \frac{5^0 \ln 5 - 2^0 \ln 2}{2(0) - 2} \\ &= \frac{\ln 5 - \ln 2}{-2} \\ &= -\frac{1}{2} \ln\left(\frac{5}{2}\right)\end{aligned}$$

40 a As $x \rightarrow 0^+$, $\arctan(\sqrt{x}) \rightarrow 0$ so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\arctan(\sqrt{x})}{x} &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\arctan(\sqrt{x}))}{\frac{d}{dx}(x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{1 + (\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2} \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x} + x\sqrt{x}}\end{aligned}$$

As $x \rightarrow 0^+$, $\sqrt{x} + x\sqrt{x} \rightarrow 0$

$\therefore \lim_{x \rightarrow 0^+} \frac{\arctan(\sqrt{x})}{x}$ does not exist.

b As $x \rightarrow \frac{\pi}{2}$, $\ln(\sin^2 x) \rightarrow 0$ and $\cos^2 x \rightarrow 0$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\sin^2 x)}{\cos^2 x} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{d}{dx}(\ln(\sin^2 x))}{\frac{d}{dx}(\cos^2 x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{2 \sin x \cos x}{\sin^2 x}}{2 \cos x (-\sin x)} \\ &= - \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\sin^2 x} \quad \{x \neq \frac{\pi}{2}\} \\ &= -\frac{1}{1^2} \\ &= -1\end{aligned}$$

41 a As $x \rightarrow \infty$, $\ln\left(1 - \frac{2}{x}\right) \rightarrow 0$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow \infty} x \ln\left(1 - \frac{2}{x}\right) &= \lim_{x \rightarrow \infty} \frac{\ln\left(1 - \frac{2}{x}\right)}{\frac{1}{x}} \quad \left\{\text{converting to the form } \frac{0}{0}\right\} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}\left(\ln\left(1 - \frac{2}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{\frac{2}{x^2}}{1 - \frac{2}{x}}}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{-2}{1 - \frac{2}{x}} \quad \{x \neq 0\} \\ &= \frac{-2}{1 - 0} \\ &= -2\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{x \ln\left(1 - \frac{2}{x}\right)} \\ &= e^{\left[\lim_{x \rightarrow \infty} x \ln\left(1 - \frac{2}{x}\right)\right]} \quad \{e^x \text{ is continuous for all } x \in \mathbb{R}\} \\ &= e^{-2} \quad \{\text{from a}\} \\ &= \frac{1}{e^2}\end{aligned}$$

42 As $x \rightarrow 0$, $\tan x - x \rightarrow 0$ and $x^3 \rightarrow 0$, so we can use l'Hôpital's rule.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(x^3)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}\end{aligned}$$

Now as $x \rightarrow 0$, $\sec^2 x - 1 \rightarrow 0$ and $3x^2 \rightarrow 0$, so we can use l'Hôpital's rule again.

$$\begin{aligned}\therefore \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}(3x^2)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec x (\sec x \tan x)}{6x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x}\end{aligned}$$

Now as $x \rightarrow 0$, $2 \sec^2 x \tan x \rightarrow 0$ and $6x \rightarrow 0$, so we can use l'Hôpital's rule again.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(2 \sec^2 x \tan x)}{\frac{d}{dx}(6x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{4 \sec^2 x (\sec x \tan x) \tan x + 2 \sec^2 x \sec^2 x}{6} \\ &= \frac{1}{3} \lim_{x \rightarrow 0} (2 \sec^3 x \tan^2 x + \sec^4 x) \\ &= \frac{1}{3} (2(1)^3(0)^2 + (1)^4) \\ &= \frac{1}{3}\end{aligned}$$

43 a $g(x) = -x \cos x$

$$\begin{aligned}\therefore g'(x) &= (-1) \cos x - x(-\sin x) \quad \{\text{product rule}\} \\ &= -\cos x + x \sin x\end{aligned}$$

b Since $g\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} \cos \frac{\pi}{3} = -\frac{\pi}{3} \times \frac{1}{2} = -\frac{\pi}{6}$, the point of contact is $\left(\frac{\pi}{3}, -\frac{\pi}{6}\right)$.

$$\begin{aligned}\text{Now } g'\left(\frac{\pi}{3}\right) &= -\cos \frac{\pi}{3} + \frac{\pi}{3} \sin \frac{\pi}{3} \\ &= -\frac{1}{2} + \frac{\pi}{3} \times \frac{\sqrt{3}}{2} \\ &= -\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\end{aligned}$$

$$\begin{aligned}\text{The tangent has equation } y &= g'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + g\left(\frac{\pi}{3}\right) \\ \therefore y &= \left(-\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)\left(x - \frac{\pi}{3}\right) - \frac{\pi}{6} \\ \therefore y &= \left(-\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)x + \frac{\pi}{6} - \frac{\pi^2\sqrt{3}}{6} - \frac{\pi}{6} \\ \therefore y &= \left(-\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)x - \frac{\pi^2\sqrt{3}}{6}\end{aligned}$$

44 a $f(x) = -x^2 + 4x$

$$\therefore f'(x) = -2x + 4$$

b Since $f(k) = -k^2 + 4k$, the point of contact is $(k, -k^2 + 4k)$.

$$\text{Now } f'(k) = -2k + 4.$$

$$\begin{aligned}\text{The tangent has equation } y &= f'(k)(x - k) + f(k) \\ \therefore y &= (-2k + 4)(x - k) - k^2 + 4k \\ \therefore y &= (-2k + 4)x + 2k^2 - 4k - k^2 + 4k \\ \therefore y &= (-2k + 4)x + k^2\end{aligned}$$

c The tangent passes through $(4, 9)$, so $(-2k + 4)(4) + k^2 = 9$

$$\begin{aligned}\therefore -8k + 16 + k^2 &= 9 \\ \therefore k^2 - 8k + 7 &= 0 \\ \therefore (k - 7)(k - 1) &= 0 \\ \therefore k &= 1 \text{ or } 7\end{aligned}$$

The gradient of the tangent is positive, so $-2k + 4 > 0$.

$$\text{Now if } k = 1, \quad -2k + 4 = -2 + 4 = 2 \quad \checkmark$$

$$\text{if } k = 7, \quad -2k + 4 = -14 + 4 = -10 \quad \times$$

So $k = 1$.

45 $y = \sqrt{3x + 1}$ and $y = \sqrt{5x - x^2}$

a The curves meet where $\sqrt{3x + 1} = \sqrt{5x - x^2}$

$$\begin{aligned}\therefore 3x + 1 &= 5x - x^2 \quad \{\text{squaring both sides}\} \\ \therefore x^2 - 2x + 1 &= 0 \\ \therefore (x - 1)^2 &= 0 \\ \therefore x &= 1\end{aligned}$$

$$\text{Now when } x = 1, \quad y = \sqrt{3(1) + 1} = \sqrt{4} = 2$$

So the curves meet at $(1, 2)$.

b For $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(3x+1)^{-\frac{1}{2}} \times 3 \quad \{\text{chain rule}\} \\ &= \frac{3}{2\sqrt{3x+1}}\end{aligned}$$

When $x = 1$, $\frac{dy}{dx} = \frac{3}{2\sqrt{3(1)+1}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$.

For $y = \sqrt{5x-x^2} = (5x-x^2)^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(5x-x^2)^{-\frac{1}{2}}(5-2x) \quad \{\text{chain rule}\} \\ &= \frac{5-2x}{2\sqrt{5x-x^2}}\end{aligned}$$

When $x = 1$, $\frac{dy}{dx} = \frac{5-2(1)}{2\sqrt{5(1)-(1)^2}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$.

So, the tangents to the curves have the same gradient at the intersection point.

c The equation of the common tangent is $y = \frac{3}{4}(x-1) + 2$

$$\therefore y = \frac{3}{4}x - \frac{3}{4} + 2$$

$$\therefore y = \frac{3}{4}x + \frac{5}{4}$$

46 a $y = \frac{a}{x} - x^2 + 1 = ax^{-1} - x^2 + 1$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -ax^{-2} - 2x \\ &= \frac{-a}{x^2} - 2x\end{aligned}$$

Now the gradient of the tangent at $x = 2$ is -5 .

$$\therefore \frac{-a}{2^2} - 2(2) = -5$$

$$\therefore \frac{-a}{4} - 4 = -5$$

$$\therefore \frac{-a}{4} = -1$$

$$\therefore a = 4$$

47 a $f(x) = \frac{x+2}{\sqrt{x-1}}$

$f(x)$ is undefined when $x-1 < 0$ and $x = 1$

$$\therefore x < 1$$

So, the domain of $f(x)$ is $\{x \mid x > 1\}$.

b $f(10) = \frac{10+2}{\sqrt{10-1}} = \frac{12}{\sqrt{9}} = \frac{12}{3} = 4$, so the point of contact is $(10, 4)$.

Now $f(x) = \frac{x+2}{\sqrt{x-1}} = \frac{x+2}{(x-1)^{\frac{1}{2}}}$ has derivative

$$f'(x) = \frac{(1)(x-1)^{\frac{1}{2}} - \frac{1}{2}(x-1)^{-\frac{1}{2}}(x+2)}{x-1} \quad \{\text{quotient rule}\}$$

$$= \frac{\sqrt{x-1} - \frac{x+2}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{2x-2-x-2}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{x-4}{2(x-1)^{\frac{3}{2}}}$$

$$\therefore f'(10) = \frac{10-4}{2(10-1)^{\frac{3}{2}}} = \frac{6}{2 \times 9^{\frac{3}{2}}} = \frac{6}{2 \times 27} = \frac{1}{9}$$

So, the normal has gradient -9 .

The normal has equation $y = -9(x-10) + 4$

$$\therefore y = -9x + 90 + 4$$

$$\therefore y = -9x + 94$$

b $y = \frac{a}{x} - x^2 + 1 = \frac{4}{x} - x^2 + 1$ {from **a**}

$$\begin{aligned}\text{When } x = 2, \quad y &= \frac{4}{2} - 2^2 + 1 \\ &= 2 - 4 + 1 \\ &= -1\end{aligned}$$

So, the point of contact is $(2, -1)$.

The equation of tangent is $y = -5(x-2) + (-1)$

$$\therefore y = -5x + 10 - 1$$

$$\therefore y = -5x + 9$$

48 a $y = x^3 + ax^2 + bx + 3$

The function passes through $(1, 8)$, so $(1)^3 + a(1)^2 + b(1) + 3 = 8$

$$\therefore 1 + a + b + 3 = 8$$

$$\therefore a + b + 4 = 8$$

$$\therefore a + b = 4$$

$$\therefore a = 4 - b \quad \dots (*)$$

Now $\frac{dy}{dx} = 3x^2 + 2ax + b$, and at $(1, 8)$ the tangent has equation $y = 2x + 6$ which has gradient 2.

$$\therefore 3(1)^2 + 2a(1) + b = 2$$

$$\therefore 3 + 2a + b = 2$$

$$\therefore 2a + b = -1$$

$$\therefore 2(4 - b) + b = -1 \quad \{\text{using } (*)\}$$

$$\therefore 8 - 2b + b = -1$$

$$\therefore -b = -9$$

$$\therefore b = 9$$

$$\therefore a = 4 - 9 = -5$$

b From **a**, $y = x^3 - 5x^2 + 9x + 3$ and $\frac{dy}{dx} = 3x^2 - 10x + 9$.

Now when $x = -1$, $y = (-1)^3 - 5(-1)^2 + 9(-1) + 3 = -1 - 5 - 9 + 3 = -12$

$$\text{and } \frac{dy}{dx} = 3(-1)^2 - 10(-1) + 9 = 3 + 10 + 9 = 22$$

So the point of contact is $(-1, -12)$ and the gradient of the normal is $-\frac{1}{22}$.

The equation of the normal is $y = -\frac{1}{22}(x - (-1)) - 12$

$$\therefore y = -\frac{1}{22}x - \frac{265}{22}$$

49 $f(x) = \arcsin(-2\sqrt{x})$

a $\arcsin(-2\sqrt{x})$ is defined when $x \geq 0$ and when $-1 \leq -2\sqrt{x} \leq 1$

$$\therefore -1 \leq 2\sqrt{x} \leq 1$$

$$\therefore -\frac{1}{2} \leq \sqrt{x} \leq \frac{1}{2}$$

$$\therefore x \leq \frac{1}{4} \quad \{x \geq 0\}$$

\therefore the domain of $f(x)$ is $\{x \mid 0 \leq x \leq \frac{1}{4}\}$.

Now if $0 \leq x \leq \frac{1}{4}$, then

$$0 \leq \sqrt{x} \leq \frac{1}{2}$$

$$\therefore -1 \leq -2\sqrt{x} \leq 0$$

$$\therefore -\frac{\pi}{2} \leq \arcsin(-2\sqrt{x}) \leq 0$$

\therefore the range of $f(x)$ is $\{y \mid -\frac{\pi}{2} \leq y \leq 0\}$.

b i $f(\frac{1}{16}) = \arcsin\left(-2\sqrt{\frac{1}{16}}\right)$
 $= \arcsin\left(-2\left(\frac{1}{4}\right)\right)$
 $= \arcsin\left(-\frac{1}{2}\right)$
 $= -\frac{\pi}{6}$

$$\begin{aligned}
 \text{ii} \quad f(x) &= \arcsin(-2\sqrt{x}) & \therefore f'(\tfrac{1}{16}) &= \frac{-2}{\sqrt{\frac{1}{16} - 4(\frac{1}{16})^2}} \\
 \therefore f'(x) &= \frac{1}{\sqrt{1 - (-2\sqrt{x})^2}} \times \left(-2 \times \frac{1}{2\sqrt{x}}\right) & &= \frac{-1}{\sqrt{\frac{1}{16} - \frac{4}{16^2}}} \\
 &= \frac{-1}{\sqrt{1 - 4x\sqrt{x}}} & &= \frac{-1}{\sqrt{\frac{16-4}{16^2}}} \\
 &= \frac{-1}{\sqrt{x - 4x^2}} & &= \frac{-1}{\frac{1}{16}\sqrt{12}} \\
 & & &= \frac{-16}{2\sqrt{3}} \\
 & & &= -\frac{8}{\sqrt{3}}
 \end{aligned}$$

$$\text{c} \quad f'(x) = \frac{-1}{\sqrt{x - 4x^2}}$$

The tangent to $y = f(x)$ has gradient -4 where $f'(x) = -4$

$$\begin{aligned}
 \therefore \frac{-1}{\sqrt{x - 4x^2}} &= -4 \\
 \therefore \sqrt{x - 4x^2} &= \frac{1}{4} \\
 \therefore x - 4x^2 &= \frac{1}{16} \\
 \therefore 16x - 64x^2 &= 1 \\
 \therefore 64x^2 - 16x + 1 &= 0 \\
 \therefore (8x - 1)^2 &= 0 \\
 \therefore x &= \frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f(\tfrac{1}{8}) &= \arcsin\left(-2\sqrt{\tfrac{1}{8}}\right) \\
 &= \arcsin\left(-2\left(\tfrac{1}{2\sqrt{2}}\right)\right) \\
 &= \arcsin\left(-\tfrac{1}{\sqrt{2}}\right) \\
 &= -\tfrac{\pi}{4}
 \end{aligned}$$

\therefore P has coordinates $(\frac{1}{8}, -\frac{\pi}{4})$.

$$\text{50 a} \quad y = \operatorname{cosec} \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2} \operatorname{cosec} \frac{x}{2} \cot \frac{x}{2} \quad \{\text{chain rule}\}$$

$$\begin{aligned}
 \text{When } x = \frac{\pi}{3}, \quad y &= \operatorname{cosec} \frac{\pi}{6} \quad \text{and} \quad \frac{dy}{dx} = -\frac{1}{2} \operatorname{cosec} \frac{\pi}{6} \cot \frac{\pi}{6} \\
 &= \frac{1}{\sin \frac{\pi}{6}} & &= -\frac{1}{2} \times \frac{1}{\sin \frac{\pi}{6}} \times \frac{1}{\tan \frac{\pi}{6}} \\
 &= \frac{1}{\frac{1}{2}} & &= -\frac{1}{2} \times \frac{1}{\frac{1}{2}} \times \frac{1}{\frac{1}{\sqrt{3}}} \\
 &= 2 & &= -\sqrt{3}
 \end{aligned}$$

\therefore the gradient of the tangent at $(\frac{\pi}{3}, 2)$ is $-\sqrt{3}$, and the equation of the tangent is $y = -\sqrt{3}(x - \frac{\pi}{3}) + 2$
 $= -\sqrt{3}x + \frac{\pi}{\sqrt{3}} + 2$

$$\text{b} \quad y = \arcsin 2x$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \frac{2}{\sqrt{1 - (2x)^2}} \quad \{\text{chain rule}\} \\
 &= \frac{2}{\sqrt{1 - 4x^2}}
 \end{aligned}$$

$$\text{When } y = \frac{\pi}{4}, \quad \arcsin 2x = \frac{\pi}{4}$$

$$\begin{aligned}
 \therefore 2x &= \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \\
 \therefore x &= \frac{1}{2\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \frac{1}{2\sqrt{2}}, \quad \frac{dy}{dx} &= \frac{2}{\sqrt{1 - 4\left(\frac{1}{2\sqrt{2}}\right)^2}} \\
 &= \frac{2}{\sqrt{1 - \frac{4}{8}}} \\
 &= \frac{2}{\sqrt{\frac{1}{2}}} \\
 &= 2\sqrt{2}
 \end{aligned}$$

\therefore the gradient of the tangent at $\left(\frac{1}{2\sqrt{2}}, \frac{\pi}{4}\right)$ is $2\sqrt{2}$, and the equation of the tangent is $y = 2\sqrt{2}\left(x - \frac{1}{2\sqrt{2}}\right) + \frac{\pi}{4}$
 $= 2\sqrt{2}x - 1 + \frac{\pi}{4}$

51 a $y = 5^{2x} + x^2$

$$\therefore \frac{dy}{dx} = 2 \times 5^{2x} \ln 5 + 2x \quad \{\text{chain rule}\}$$

$$\begin{aligned}
 \text{When } x = 0, \quad y &= 5^0 + 0^2 \quad \text{and} \quad \frac{dy}{dx} = 2 \times 5^0 \ln 5 + 2(0) \\
 &= 1 \qquad \qquad \qquad = 2 \ln 5
 \end{aligned}$$

\therefore the gradient of the normal at $(0, 1)$ is $-\frac{1}{2 \ln 5}$, and the equation of the normal is $y = -\frac{1}{2 \ln 5}(x - 0) + 1$
 $= -\frac{1}{2 \ln 5}x + 1$

b $y = x \log_3(x + 1)$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \log_3(x + 1) + x \times \frac{1}{(x + 1) \ln 3} \quad \{\text{product rule}\} \\
 &= \log_3(x + 1) + \frac{x}{(x + 1) \ln 3}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = 2, \quad y &= 2 \log_3(2 + 1) \quad \text{and} \quad \frac{dy}{dx} = \log_3(2 + 1) + \frac{2}{(2 + 1) \ln 3} \\
 &= 2 \log_3 3 \qquad \qquad \qquad = \log_3 3 + \frac{2}{3 \ln 3} \\
 &= 2 \qquad \qquad \qquad = 1 + \frac{2}{3 \ln 3} \\
 & \qquad \qquad \qquad = \frac{3 \ln 3 + 2}{3 \ln 3}
 \end{aligned}$$

\therefore the gradient of the normal at $(2, 2)$ is $-\frac{3 \ln 3}{3 \ln 3 + 2}$, and the equation of the normal is $y = -\frac{3 \ln 3}{3 \ln 3 + 2}(x - 2) + 2$.

52 $\frac{x^2}{4} + \frac{y^2}{9} = 1$

a $\frac{2x}{4} + \frac{2y}{9} \frac{dy}{dx} = 0 \quad \{\text{implicit differentiation}\}$

$$\therefore \frac{2y}{9} \frac{dy}{dx} = -\frac{x}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{9x}{4y}$$

b When $x = -1$, $\frac{(-1)^2}{4} + \frac{y^2}{9} = 1$

$$\therefore \frac{1}{4} + \frac{y^2}{9} = 1$$

$$\therefore \frac{y^2}{9} = \frac{3}{4}$$

$$\therefore y^2 = \frac{27}{4}$$

$$\therefore y = \pm \frac{3\sqrt{3}}{2}$$

\therefore the points of contact are $\left(-1, \frac{3\sqrt{3}}{2}\right)$ and $\left(-1, -\frac{3\sqrt{3}}{2}\right)$.

$$\begin{aligned}\text{At } \left(-1, \frac{3\sqrt{3}}{2}\right), \text{ the gradient of the tangent is } \frac{dy}{dx} &= -\frac{9(-1)}{4\left(\frac{3\sqrt{3}}{2}\right)} \\ &= \frac{9}{6\sqrt{3}} \\ &= \frac{3}{2\sqrt{3}} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\therefore \text{ the equation of the tangent at } \left(-1, \frac{3\sqrt{3}}{2}\right) \text{ is } y &= \frac{\sqrt{3}}{2}(x+1) + \frac{3\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{2}x + 2\sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{At } \left(-1, -\frac{3\sqrt{3}}{2}\right), \text{ the gradient of the tangent is } \frac{dy}{dx} &= -\frac{9(-1)}{4\left(-\frac{3\sqrt{3}}{2}\right)} \\ &= -\frac{9}{6\sqrt{3}} \\ &= -\frac{3}{2\sqrt{3}} \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\therefore \text{ the equation of the tangent at } \left(-1, -\frac{3\sqrt{3}}{2}\right) \text{ is } y &= -\frac{\sqrt{3}}{2}(x+1) - \frac{3\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2}x - 2\sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{c The tangents intersect where } \frac{\sqrt{3}}{2}x + 2\sqrt{3} &= -\frac{\sqrt{3}}{2}x - 2\sqrt{3} \\ \therefore \sqrt{3}x &= -4\sqrt{3} \\ \therefore x &= -4\end{aligned}$$

$$\begin{aligned}\text{When } x = -4, y &= \frac{\sqrt{3}}{2}(-4) + 2\sqrt{3} \\ &= -2\sqrt{3} + 2\sqrt{3} \\ &= 0\end{aligned}$$

\therefore the tangents intersect at $(-4, 0)$.

$$\mathbf{53 \quad a} \quad ax \sec y = 4, \quad 0 \leq y \leq 2\pi$$

$$\begin{aligned}\therefore a \sec y + ax \sec y \tan y \frac{dy}{dx} &= 0 \\ \therefore ax \sec y \tan y \frac{dy}{dx} &= -a \sec y \\ \therefore \frac{dy}{dx} &= -\frac{1}{x \tan y}\end{aligned}$$

$$\begin{aligned}\text{When } x = 4, \frac{dy}{dx} &= \frac{\sqrt{3}}{12} \\ \therefore \frac{\sqrt{3}}{12} &= -\frac{1}{4 \tan y} \\ \therefore \tan y &= -\frac{3}{\sqrt{3}} = -\sqrt{3} \\ \therefore y &= \frac{2\pi}{3} \quad \{0 \leq y \leq \pi\}\end{aligned}$$

So, the curve passes through $(4, \frac{2\pi}{3})$.

$$\begin{aligned}\therefore a(4) \sec \frac{2\pi}{3} &= 4 \\ \therefore \frac{a}{\cos \frac{2\pi}{3}} &= 1 \\ \therefore a &= \cos \frac{2\pi}{3} = \frac{1}{2}\end{aligned}$$

$$\mathbf{b} \quad \frac{1}{2}x \sec y = 4, \quad 0 \leq y \leq 2\pi \quad \{\text{from a}\}$$

$$\begin{aligned}\therefore x \sec y &= 8 \\ \therefore x &= 8 \cos y \quad \dots (*)\end{aligned}$$

Now, the tangent to the curve has gradient $-\frac{1}{4}$ where

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{4} \\ \therefore -\frac{1}{4} &= -\frac{1}{x \tan y} \\ \therefore x \tan y &= 4 \\ \therefore 8 \cos y \times \frac{\sin y}{\cos y} &= 4 \quad \{\text{using } (*)\} \\ \therefore \sin y &= \frac{1}{2} \quad \{\cos y \neq 0\} \\ \therefore y &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \quad \{0 \leq y \leq 2\pi\}\end{aligned}$$

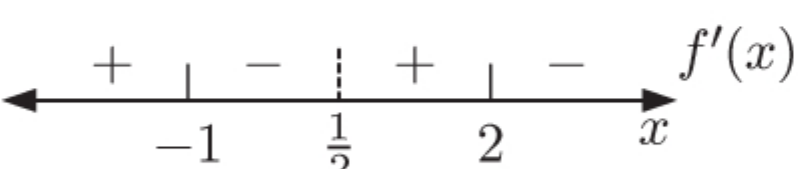
$$\text{When } y = \frac{\pi}{6}, \quad x = 8 \cos \frac{\pi}{6} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}.$$

$$\text{When } y = \frac{5\pi}{6}, \quad x = 8 \cos \frac{5\pi}{6} = 8\left(-\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}.$$

\therefore the points on the curve at which the tangent has gradient $-\frac{1}{4}$ are $(4\sqrt{3}, \frac{\pi}{6})$ and $(-4\sqrt{3}, \frac{5\pi}{6})$.

54 a $f(x) = \ln\left(\frac{1-2x}{x^2+2}\right) = \ln(1-2x) - \ln(x^2+2)$

$$\begin{aligned}\therefore f'(x) &= \frac{-2}{1-2x} - \frac{2x}{x^2+2} \\ &= \frac{-2(x^2+2) - 2x(1-2x)}{(1-2x)(x^2+2)} \\ &= \frac{-2x^2 - 4 - 2x + 4x^2}{(x^2+2)(1-2x)} \\ &= \frac{2x^2 - 2x - 4}{(x^2+2)(1-2x)} \\ &= \frac{2(x-2)(x+1)}{(x^2+2)(1-2x)}\end{aligned}$$

b The sign diagram of $f'(x)$ is 

$f(x)$ is decreasing whenever $f'(x) \leq 0$.

But $f(x)$ is undefined when $1-2x \leq 0$ which is when $x \geq \frac{1}{2}$.

$\therefore f(x)$ is decreasing for $-1 \leq x < \frac{1}{2}$.

55 a $y = xe^{-x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (1)e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\} \\ &= e^{-x} - xe^{-x}\end{aligned}$$

Stationary points occur where $\frac{dy}{dx} = 0$


$$\therefore e^{-x} - xe^{-x} = 0$$

$$\therefore e^{-x}(1-x) = 0$$

$$\therefore 1-x = 0 \quad \{e^{-x} > 0\}$$

$$\therefore x = 1$$

When $x = 1$, $y = (1)e^{-1} = \frac{1}{e}$

Now $f'(x)$ has sign diagram 

So, $\left(1, \frac{1}{e}\right)$ is a local maximum.

b $y = \frac{x-3}{x^2-5}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{(1)(x^2-5) - (2x)(x-3)}{(x^2-5)^2} \quad \{\text{quotient rule}\} \\ &= \frac{x^2 - 5 - 2x^2 + 6x}{(x^2-5)^2} \\ &= \frac{-x^2 + 6x - 5}{(x^2-5)^2}\end{aligned}$$

Stationary points occur where $\frac{dy}{dx} = 0$

$$\therefore -x^2 + 6x - 5 = 0$$

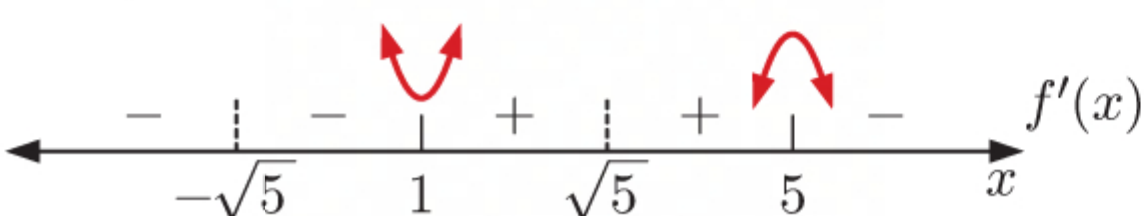
$$\therefore x^2 - 6x + 5 = 0$$

$$\therefore (x-5)(x-1) = 0$$

$$\therefore x = 1 \text{ or } 5$$

When $x = 1$, $y = \frac{1-3}{1^2-5} = \frac{-2}{-4} = \frac{1}{2}$.

When $x = 5$, $y = \frac{5-3}{5^2-5} = \frac{2}{20} = \frac{1}{10}$.

Now $f'(x)$ has sign diagram 

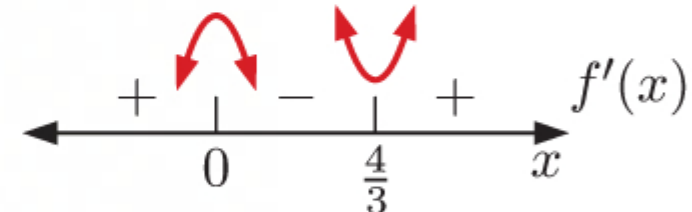
So, $\left(1, \frac{1}{2}\right)$ is a local minimum and $\left(5, \frac{1}{10}\right)$ is a local maximum.

56 a $f(x) = x^3 - 2x^2, \quad -1 \leq x \leq 1$

$$\begin{aligned}\therefore f'(x) &= 3x^2 - 4x \\ &= x(3x - 4)\end{aligned}$$

which is 0 when $x = 0$ or $\frac{4}{3}$.

The sign diagram of $f'(x)$ is



\therefore there is a local maximum at $x = 0$, and a local minimum at $x = \frac{4}{3}$.

Critical value (x)	$f(x)$
-1 (end point)	-3
0 (local maximum)	0
1 (end point)	-1

The greatest of these values is 0 when $x = 0$.

The least of these values is -3 when $x = -1$.

b $f(x) = x^2 - \frac{27}{x} = x^2 - 27x^{-1}, \quad -6 \leq x \leq -1$

$$\therefore f'(x) = 2x + 27x^{-2}$$

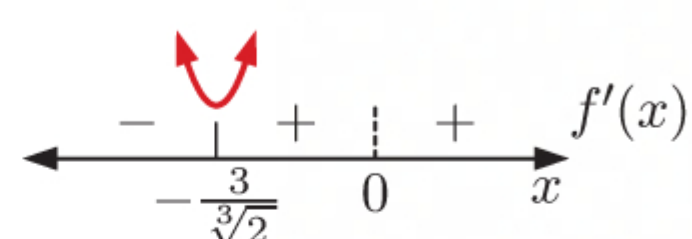
which is 0 when $2x + 27x^{-2} = 0$

$$\therefore 2x = -\frac{27}{x^2}$$

$$\therefore x^3 = -\frac{27}{2}$$

$$\therefore x = -\frac{3}{\sqrt[3]{2}}$$

The sign diagram of $f'(x)$ is



\therefore there is a local minimum at $x = -\frac{3}{\sqrt[3]{2}}$.

Critical value (x)	$f(x)$
-6 (end point)	40.5
$-\frac{3}{\sqrt[3]{2}}$ (local minimum)	≈ 17.0
-1 (end point)	28

The greatest of these values is 40.5 when $x = -6$.

The least of these values is ≈ 17.0 when $x = -\frac{3}{\sqrt[3]{2}}$.

c $f(x) = x^3 - 6x^2 + 12x - 10, \quad 0 \leq x \leq 5$

$$\therefore f'(x) = 3x^2 - 12x + 12$$

which is 0 when $3x^2 - 12x + 12 = 0$


$$\therefore x^2 - 4x + 4 = 0$$

$$\therefore (x - 2)^2 = 0$$

$$\therefore x - 2 = 0$$

$$\therefore x = 2$$

The sign diagram of $f'(x)$ is



\therefore there is a stationary inflection at $x = 2$.

Critical value (x)	$f(x)$
0 (end point)	-10
5 (end point)	25

The greatest of these values is 25 when $x = 5$.

The least of these values is -10 when $x = 0$.

57 a $f(x) = \frac{e^{3x}}{kx}, \quad k \neq 0$

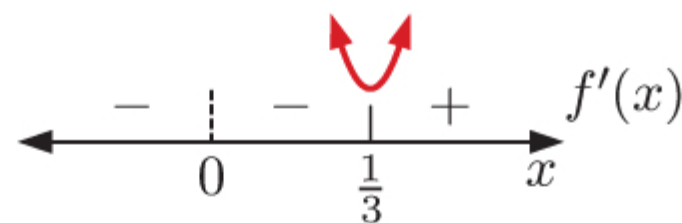
$$\begin{aligned} \therefore f'(x) &= \frac{3e^{3x}(kx) - ke^{3x}}{(kx)^2} \quad \{\text{quotient rule}\} \\ &= \frac{ke^{3x}(3x-1)}{k^2x^2} \\ &= \frac{e^{3x}(3x-1)}{kx^2} \quad \{k \neq 0\} \end{aligned}$$

Stationary points occur where $f'(x) = 0$

$$\begin{aligned} \therefore e^{3x}(3x-1) &= 0 \\ \therefore 3x-1 &= 0 \quad \{e^{3x} > 0 \text{ for all } x\} \\ \therefore x &= \frac{1}{3} \end{aligned}$$

So, the stationary point has x -coordinate $\frac{1}{3}$.

b i If the stationary point is a local minimum, then the sign diagram of $f'(x)$ should be



This occurs when $k > 0$.

ii Using **b i**, if the stationary point is a local maximum, then $k < 0$.

Note: It is not possible for the stationary point to be an inflection point because the factor $(3x-1)$ in $f'(x)$ is raised to an odd power. So, the sign of $f'(x)$ will always be different on either side of $x = \frac{1}{3}$.

c The stationary point has y -coordinate $-\frac{e}{2}$.

$$\therefore f\left(\frac{1}{3}\right) = -\frac{e}{2} \quad \{\text{using a}\}$$

$$\therefore \frac{e^{3(\frac{1}{3})}}{k(\frac{1}{3})} = -\frac{e}{2}$$

$$\therefore \frac{3e}{k} = -\frac{e}{2}$$

$$\therefore \frac{k}{2} = -3$$

$$\therefore k = -6$$

Since $k < 0$, the stationary point is a local maximum.

d $g(x) = -f(2x)$

$$f(x) \xrightarrow[\text{reflection in } x\text{-axis}]{\text{horizontal stretch scale factor } \frac{1}{2}} -f(x) \xrightarrow[\text{horizontal stretch scale factor } \frac{1}{2}]{\text{horizontal stretch scale factor } \frac{1}{2}} -f(2x)$$

So, a reflection in the x -axis, followed by a horizontal stretch with scale factor $\frac{1}{2}$ maps $y = f(x)$ onto $y = g(x)$.

The stationary point of $f(x)$ is $\left(\frac{1}{3}, -\frac{e}{2}\right)$, so for the stationary point of $g(x)$:

$$\left(\frac{1}{3}, -\frac{e}{2}\right) \xrightarrow[\text{reflection in } x\text{-axis}]{\text{horizontal stretch scale factor } \frac{1}{2}} \left(\frac{1}{3}, \frac{e}{2}\right) \xrightarrow[\text{horizontal stretch scale factor } \frac{1}{2}]{\text{horizontal stretch scale factor } \frac{1}{2}} \left(\frac{1}{6}, \frac{e}{2}\right)$$

So, the stationary point of $g(x)$ is $\left(\frac{1}{6}, \frac{e}{2}\right)$ which is a local minimum due to the reflection in the x -axis.

58 a Let $y = x^{\frac{1}{x}}$, so $\ln y = \frac{1}{x} \ln x = \frac{\ln x}{x}$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\therefore \frac{dy}{dx} = \left(\frac{1 - \ln x}{x^2}\right)x^{\frac{1}{x}}$$

b A stationary point occurs when $\frac{dy}{dx} = 0$

$$\therefore 1 - \ln x = 0$$

$$\therefore \ln x = 1$$

$$\therefore x = e$$

When $x = e$, $y = e^{\frac{1}{e}}$

$\therefore (e, e^{\frac{1}{e}})$ is a stationary point.

$$\begin{aligned}
 59 \quad a \quad y &= \frac{\sin x}{\tan x + 1}, \quad -\pi \leq x \leq \frac{\pi}{2} \\
 \therefore \frac{dy}{dx} &= \frac{\cos x(\tan x + 1) - \sin x \sec^2 x}{(\tan x + 1)^2} \\
 \therefore \frac{dy}{dx} &= \frac{\sin x + \cos x - \frac{\sin x}{\cos^2 x}}{(\tan x + 1)^2}
 \end{aligned}$$

which is 0 when $\sin x + \cos x = \frac{\sin x}{\cos^2 x}$

$$\begin{aligned}
 \therefore \sin x \cos^2 x + \cos^3 x &= \sin x \\
 \therefore \cos^3 x &= \sin x(1 - \cos^2 x) \\
 \therefore \cos^3 x &= \sin^3 x \\
 \therefore \tan^3 x &= 1 \\
 \therefore \tan x &= 1 \\
 \therefore x &= \frac{\pi}{4}, -\frac{3\pi}{4} \quad \{-\pi \leq x \leq \frac{\pi}{2}\}
 \end{aligned}$$

$$\text{When, } x = \frac{\pi}{4}, \quad y = \frac{\sin \frac{\pi}{4}}{\tan \frac{\pi}{4} + 1} = \frac{\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}}{4}.$$

$$\text{When, } x = -\frac{3\pi}{4}, \quad y = \frac{\sin(-\frac{3\pi}{4})}{\tan(-\frac{3\pi}{4}) + 1} = \frac{-\frac{1}{\sqrt{2}}}{2} = -\frac{\sqrt{2}}{4}.$$

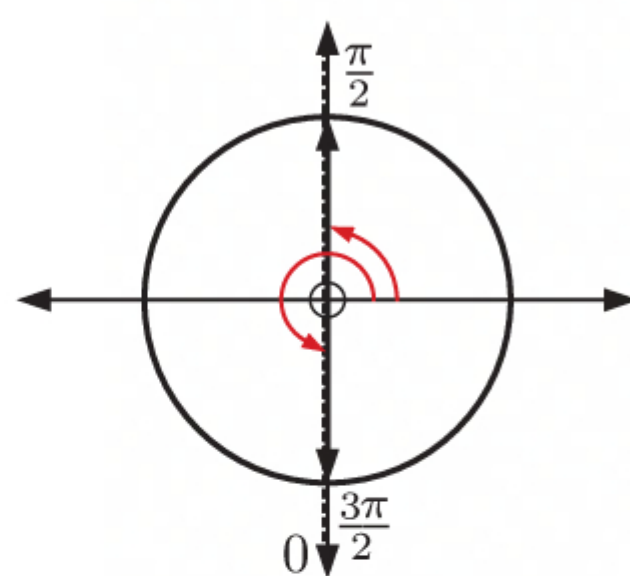
\therefore the stationary points are at $(\frac{\pi}{4}, \frac{\sqrt{2}}{4})$ and $(-\frac{3\pi}{4}, -\frac{\sqrt{2}}{4})$.

$$b \quad y = \operatorname{cosec} 2x, \quad 0 \leq x \leq \pi$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= -2 \operatorname{cosec} 2x \cot 2x \quad \text{which is 0 when } \cot 2x = 0 \quad \{\operatorname{cosec} 2x \neq 0 \text{ for all } x\} \\
 \therefore \frac{\cos 2x}{\sin 2x} &= 0 \\
 \therefore \cos 2x &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{Since } 0 &\leq x \leq \pi \\
 \therefore 0 &\leq 2x \leq 2\pi
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } 2x &= \frac{\pi}{2}, \frac{3\pi}{2} \\
 \therefore x &= \frac{\pi}{4}, \frac{3\pi}{4}
 \end{aligned}$$



$$\text{When, } x = \frac{\pi}{4}, \quad y = \operatorname{cosec} \frac{\pi}{2} = \frac{1}{\sin \frac{\pi}{2}} = 1.$$

$$\text{When, } x = \frac{3\pi}{4}, \quad y = \operatorname{cosec} \frac{3\pi}{2} = \frac{1}{\sin \frac{3\pi}{2}} = -1.$$

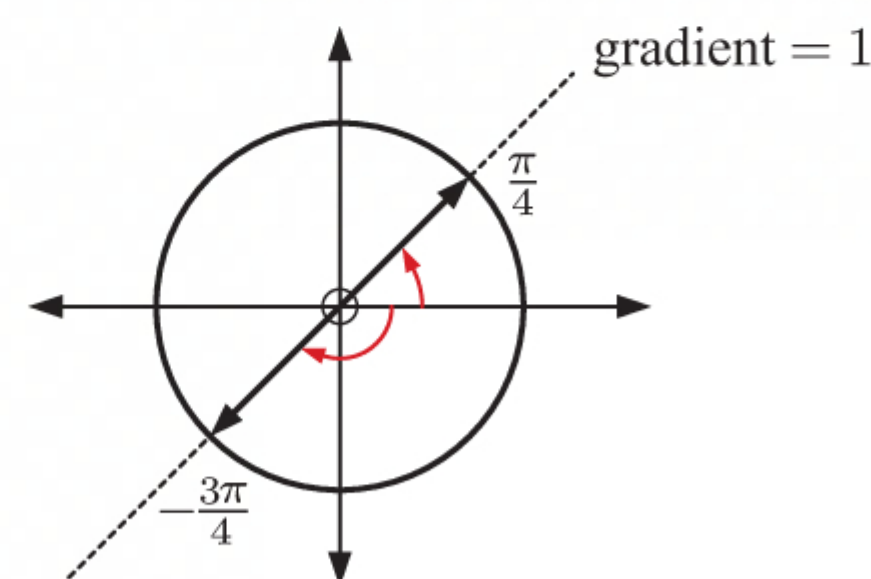
\therefore the stationary points are at $(\frac{\pi}{4}, 1)$ and $(\frac{3\pi}{4}, -1)$.

$$c \quad y = 4^x - 2^x$$

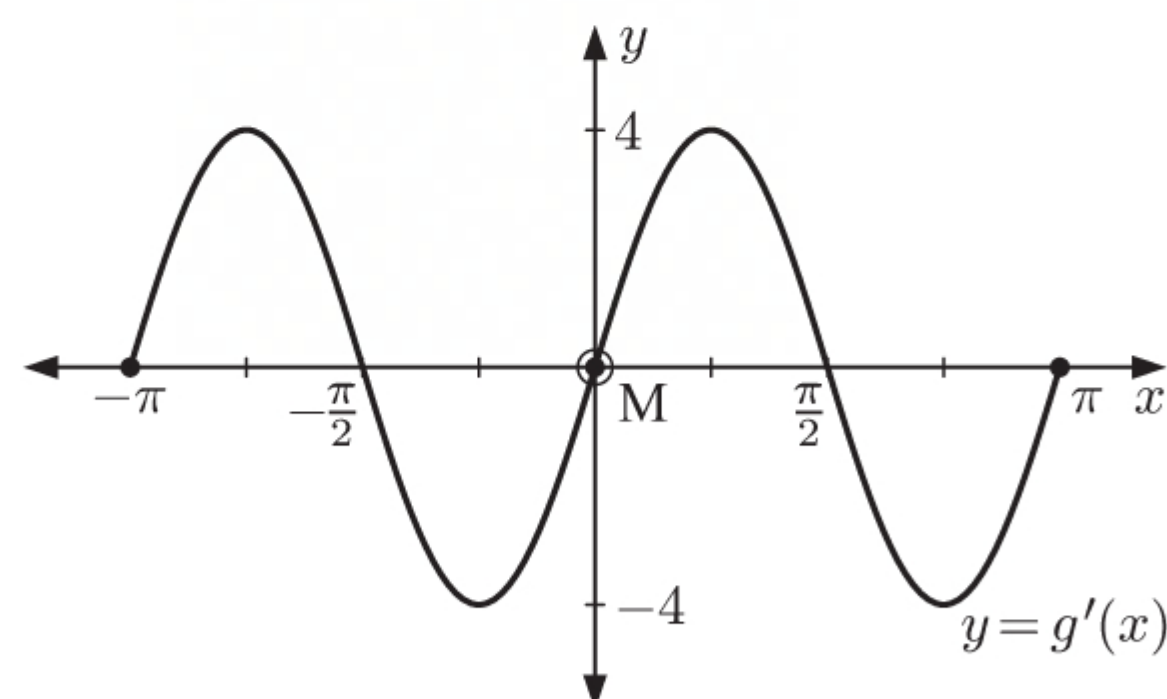
$$\begin{aligned}
 \therefore \frac{dy}{dx} &= 4^x \ln 4 - 2^x \ln 2 \quad \text{which is 0 when } 4^x \ln 4 = 2^x \ln 2 \\
 \therefore 2^{2x} \ln(2^2) &= 2^x \ln 2 \\
 \therefore 2^{2x+1} \ln 2 &= 2^x \ln 2 \\
 \therefore 2^{2x+1} &= 2^x \\
 \therefore 2x + 1 &= x \\
 \therefore x &= -1
 \end{aligned}$$

$$\text{When } x = -1, \quad y = 4^{-1} - 2^{-1} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$$

\therefore the stationary point is at $(-1, -\frac{1}{4})$.



60 a $g(x) = 3 - 2 \cos 2x$
 $\therefore g'(x) = -2(-2 \sin 2x)$
 $= 4 \sin 2x$

b, d

- c** From **b**, the graph of $y = g'(x)$ cuts the x -axis 5 times.
 \therefore there are 5 solutions to $g'(x) = 0$ for $-\pi \leq x \leq \pi$.

61 a $y = \frac{x^2 - 5}{x + 2}$
 $\therefore \frac{dy}{dx} = \frac{2x(x+2) - (x^2 - 5)(1)}{(x+2)^2}$
 $= \frac{2x^2 + 4x - x^2 + 5}{(x+2)^2}$
 $= \frac{x^2 + 4x + 5}{(x+2)^2}$
 $\therefore \frac{d^2y}{dx^2} = \frac{(2x+4)(x+2)^2 - (x^2 + 4x + 5) \times 2(x+2)}{(x+2)^4}$
 $= \frac{2(x+2)^2 - 2(x^2 + 4x + 5)}{(x+2)^3}$
 $= \frac{2(\cancel{x^2} + 4\cancel{x} + 4 - \cancel{x^2} - 4\cancel{x} - 5)}{(x+2)^3}$
 $= \frac{-2}{(x+2)^3}$

which has sign diagram $\begin{array}{c} + \quad | \quad - \\ -2 \quad x \end{array} \frac{d^2y}{dx^2}$

i $y = \frac{x^2 - 5}{x + 2}$ is concave up for $x < -2$.

ii $y = \frac{x^2 - 5}{x + 2}$ is concave down for $x > -2$.

b $y = \log_2 x + \frac{x^2}{2}$
 $\therefore \frac{dy}{dx} = \frac{1}{x \ln 2} + x$
 $\therefore \frac{d^2y}{dx^2} = -\frac{1}{x^2 \ln 2} + 1$
 $= \frac{x^2 \ln 2 - 1}{x^2 \ln 2}$

Now $\frac{d^2y}{dx^2} = 0$ when $x^2 \ln 2 = 1$

$$\therefore x^2 = \frac{1}{\ln 2}$$

$$\therefore x = \frac{1}{\sqrt{\ln 2}} \quad \{x > 0\}$$

So, $\frac{d^2y}{dx^2}$ has sign diagram: $\begin{array}{c} - \quad | \quad + \\ 0 \quad \frac{1}{\sqrt{\ln 2}} \quad x \end{array} \frac{d^2y}{dx^2}$

i $y = \log_2 x + \frac{x^2}{2}$ is concave up for $x \geq \frac{1}{\sqrt{\ln 2}}$.

ii $y = \log_2 x + \frac{x^2}{2}$ is concave down for $0 < x \leq \frac{1}{\sqrt{\ln 2}}$.

62 a $f(x) = xe^{1-2x^2}$

$$\begin{aligned}\therefore f'(x) &= (1)e^{1-2x^2} + x(-4xe^{1-2x^2}) && \{\text{product rule}\} \\ &= e^{1-2x^2} - 4x^2e^{1-2x^2} \\ &= e^{1-2x^2}(1 - 4x^2) \\ \therefore f''(x) &= -4xe^{1-2x^2}(1 - 4x^2) + e^{1-2x^2}(-8x) && \{\text{product rule}\} \\ &= -4xe^{1-2x^2}(1 - 4x^2 + 2) \\ &= -4xe^{1-2x^2}(3 - 4x^2)\end{aligned}$$

b Stationary points occur where $f'(x) = 0$

$$\begin{aligned}\therefore e^{1-2x^2}(1 - 4x^2) &= 0 && \{\text{using a}\} \\ \therefore 1 - 4x^2 &= 0 && \{e^{1-2x^2} > 0 \text{ for all } x\} \\ \therefore x^2 &= \frac{1}{4} \\ \therefore x &= \pm \frac{1}{2}\end{aligned}$$

The sign diagram of $f'(x)$ is

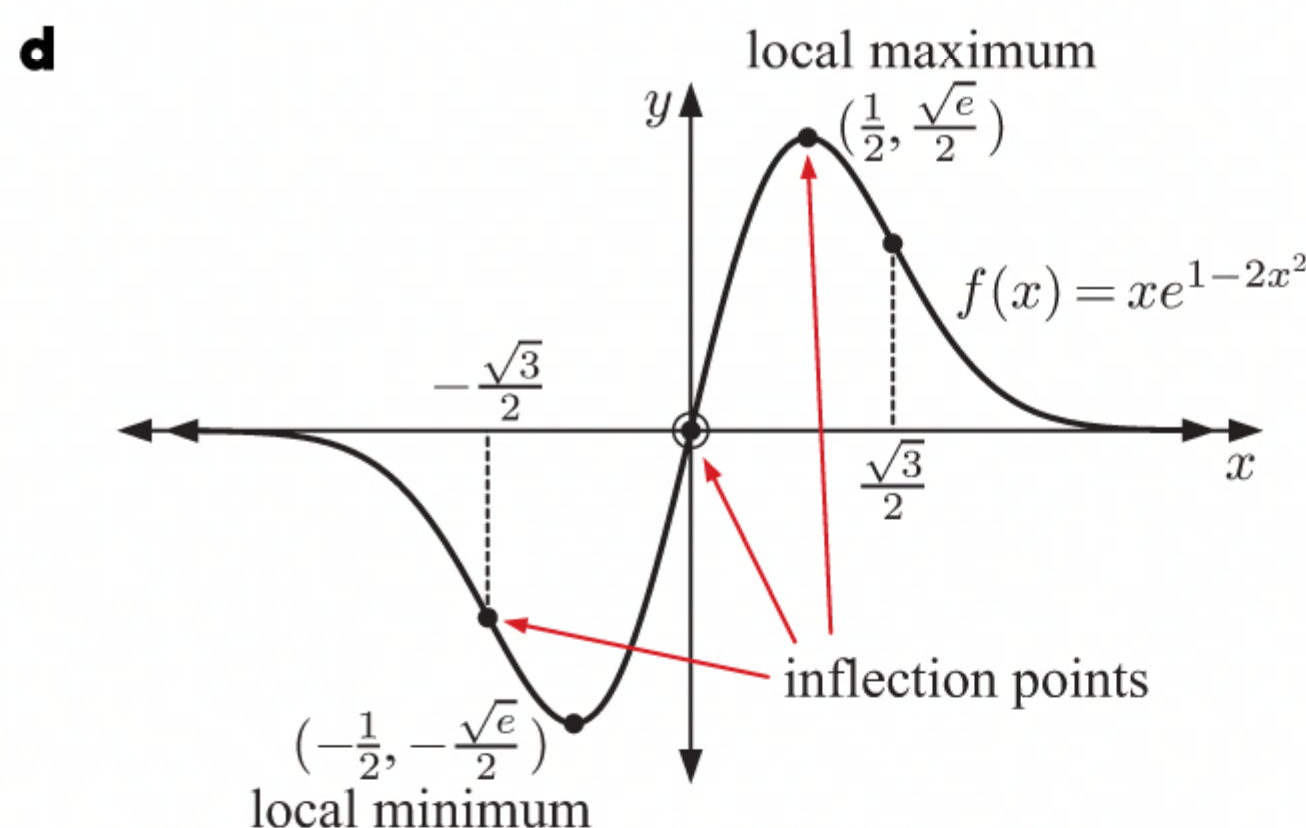
Now $f(-\frac{1}{2}) = (-\frac{1}{2})e^{1-2(\frac{1}{4})} = -\frac{1}{2}e^{\frac{1}{2}} = -\frac{\sqrt{e}}{2}$ and $f(\frac{1}{2}) = (\frac{1}{2})e^{1-2(\frac{1}{4})} = \frac{1}{2}e^{\frac{1}{2}} = \frac{\sqrt{e}}{2}$

So $(-\frac{1}{2}, -\frac{\sqrt{e}}{2})$ is a local minimum and $(\frac{1}{2}, \frac{\sqrt{e}}{2})$ is a local maximum.

c Inflection points occur where $f''(x) = 0$

$$\begin{aligned}\therefore -4xe^{1-2x^2}(3 - 4x^2) &= 0 && \{\text{using a}\} \\ \therefore x(3 - 4x^2) &= 0 && \{e^{1-2x^2} > 0 \text{ for all } x\} \\ \therefore x &= 0 \text{ or } x^2 = \frac{3}{4} \\ \therefore x &= \pm \frac{\sqrt{3}}{2}\end{aligned}$$

So, the x -coordinates of the inflection points are 0 , $-\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{3}}{2}$.



63 $f(x) = \frac{a \ln bx}{x}$

$$\begin{aligned}\therefore f'(x) &= \frac{\frac{ab}{bx}(x) - (1)a \ln bx}{x^2} && \{\text{quotient rule}\} \\ &= \frac{a - a \ln bx}{x^2} \\ \therefore f''(x) &= \frac{-\frac{ab}{bx}(x^2) - 2x(a - a \ln bx)}{x^4} && \{\text{quotient rule}\} \\ &= \frac{-ax - 2x(a - a \ln bx)}{x^4} \\ &= \frac{-a - 2a + 2a \ln bx}{x^3} \\ &= \frac{-3a + 2a \ln bx}{x^3}\end{aligned}$$

If $f(x)$ has an inflection point at $\left(\frac{e\sqrt{e}}{2}, \frac{9}{e\sqrt{e}}\right)$, then

$$f\left(\frac{e\sqrt{e}}{2}\right) = \frac{9}{e\sqrt{e}} \quad \text{and} \quad f''\left(\frac{e\sqrt{e}}{2}\right) = 0$$

$$\therefore \frac{a \ln\left[b\left(\frac{e\sqrt{e}}{2}\right)\right]}{\frac{e\sqrt{e}}{2}} = \frac{9}{e\sqrt{e}} \quad \therefore \frac{-3a + 2a \ln\left[b\left(\frac{e\sqrt{e}}{2}\right)\right]}{\left(\frac{e\sqrt{e}}{2}\right)^3} = 0$$

$$\therefore a \ln\left(\frac{b}{2}e^{\frac{3}{2}}\right) = \frac{9}{2} \quad \dots (1) \quad \therefore -3a + 2a \ln\left(\frac{b}{2}e^{\frac{3}{2}}\right) = 0 \quad \dots (2)$$

Substituting (1) into (2) gives: $-3a + 2\left(\frac{9}{2}\right) = 0$

$$\therefore -3a + 9 = 0$$

$$\therefore -3a = -9$$

$$\therefore a = 3$$

Substituting $a = 3$ into (1) gives: $3 \ln\left(\frac{b}{2}e^{\frac{3}{2}}\right) = \frac{9}{2}$

$$\therefore \ln\left(\frac{b}{2}e^{\frac{3}{2}}\right) = \frac{3}{2}$$

$$\therefore \ln\left(\frac{b}{2}\right) + \ln e^{\frac{3}{2}} = \frac{3}{2}$$

$$\therefore \ln\left(\frac{b}{2}\right) + \frac{3}{2} = \frac{3}{2}$$

$$\therefore \ln\left(\frac{b}{2}\right) = 0$$

$$\therefore \frac{b}{2} = e^0 = 1$$

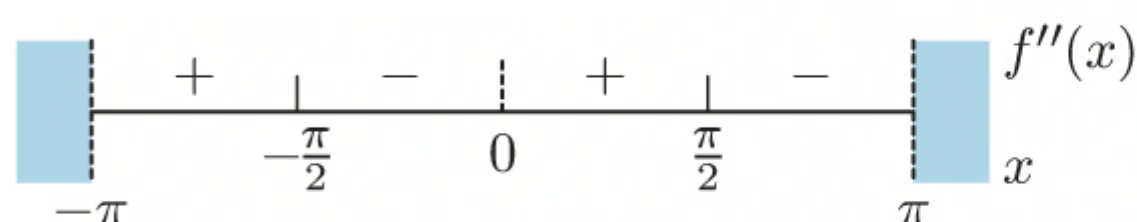
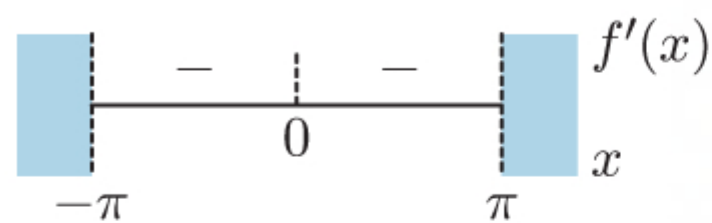
$$\therefore b = 2$$

So, $a = 3$ and $b = 2$.

64 a $f(x) = \cot x, \quad -\pi \leq x \leq \pi$

$$\therefore f'(x) = -\operatorname{cosec}^2 x$$

$$\begin{aligned} \therefore f''(x) &= -2 \operatorname{cosec} x (-\operatorname{cosec} x \cot x) \\ &= 2 \operatorname{cosec}^2 x \cot x \end{aligned}$$



Since the sign of $f''(x)$ changes about $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, both of these points are points of inflection.

Now $f(-\frac{\pi}{2}) = \cot(-\frac{\pi}{2}) = 0$ and $f(\frac{\pi}{2}) = \cot \frac{\pi}{2} = 0$.

Also $f'(x) \neq 0$ for all $-\pi \leq x \leq \pi$.

$\therefore (-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, 0)$ are non-stationary points of inflection.

b $f(x) = \arctan(x^2)$

$$\therefore f'(x) = \frac{2x}{1+(x^2)^2}$$

$$= \frac{2x}{1+x^4} \quad \begin{array}{c} - \quad | \quad + \\ \leftarrow \quad 0 \quad \rightarrow \end{array} \quad \begin{array}{c} f'(x) \\ x \end{array}$$

$$\therefore f''(x) = \frac{2(1+x^4) - 2x \times 4x^3}{(1+x^4)^2}$$

$$= \frac{2 + 2x^4 - 8x^4}{(1+x^4)^2}$$

$$= \frac{2 - 6x^4}{(1+x^4)^2}$$

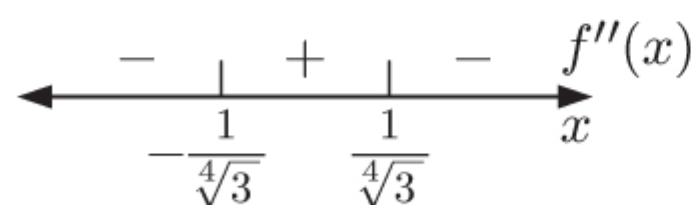
Now $f''(x) = 0$ when $2 - 6x^4 = 0$

$$\therefore 6x^4 = 2$$

$$\therefore x^4 = \frac{1}{3}$$

$$\therefore x = \pm \frac{1}{\sqrt[4]{3}}$$

So, $f''(x)$ has sign diagram:



Since the sign of $f''(x)$ changes about $x = -\frac{1}{\sqrt[4]{3}}$ and $x = \frac{1}{\sqrt[4]{3}}$, both of these points are points of inflection.

Now $f\left(-\frac{1}{\sqrt[4]{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ and $f\left(\frac{1}{\sqrt[4]{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

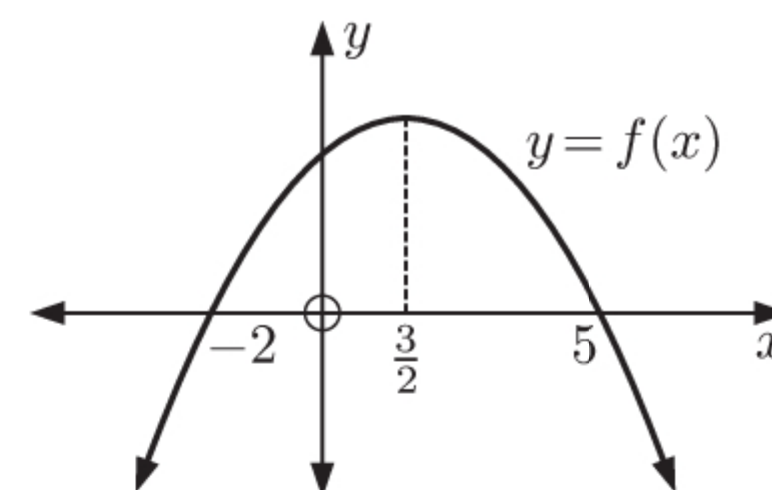
Also $f'(x) \neq 0$ whenever $x \neq 0$.

$\therefore \left(-\frac{1}{\sqrt[4]{3}}, \frac{\pi}{6}\right)$ and $\left(\frac{1}{\sqrt[4]{3}}, \frac{\pi}{6}\right)$ are non-stationary points of inflection.

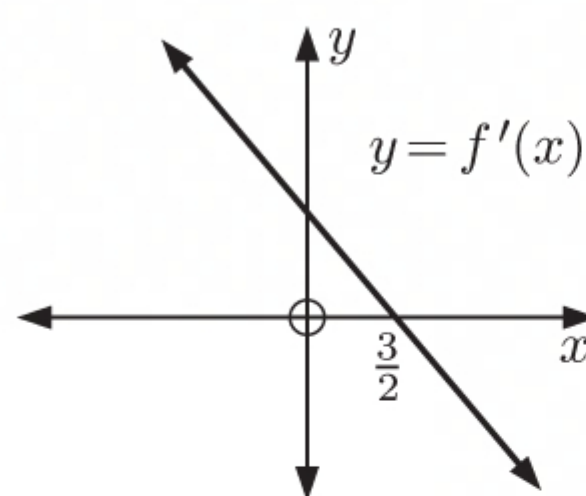
- 65 a** The turning point occurs at the average of the x -intercepts which is

$$x = \frac{5 + (-2)}{2} = \frac{3}{2}.$$

$f(x)$ is increasing for $x \leq \frac{3}{2}$, and decreasing for $x \geq \frac{3}{2}$.

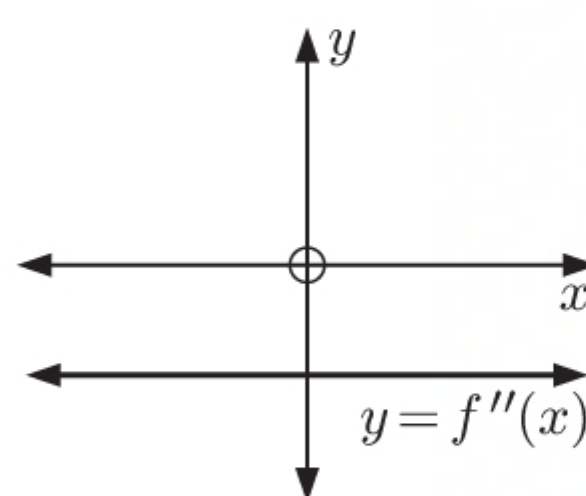


So, the graph of $y = f'(x)$ is:



- b** $f'(x)$ has a constant negative gradient.

So, the graph of $y = f''(x)$ is:

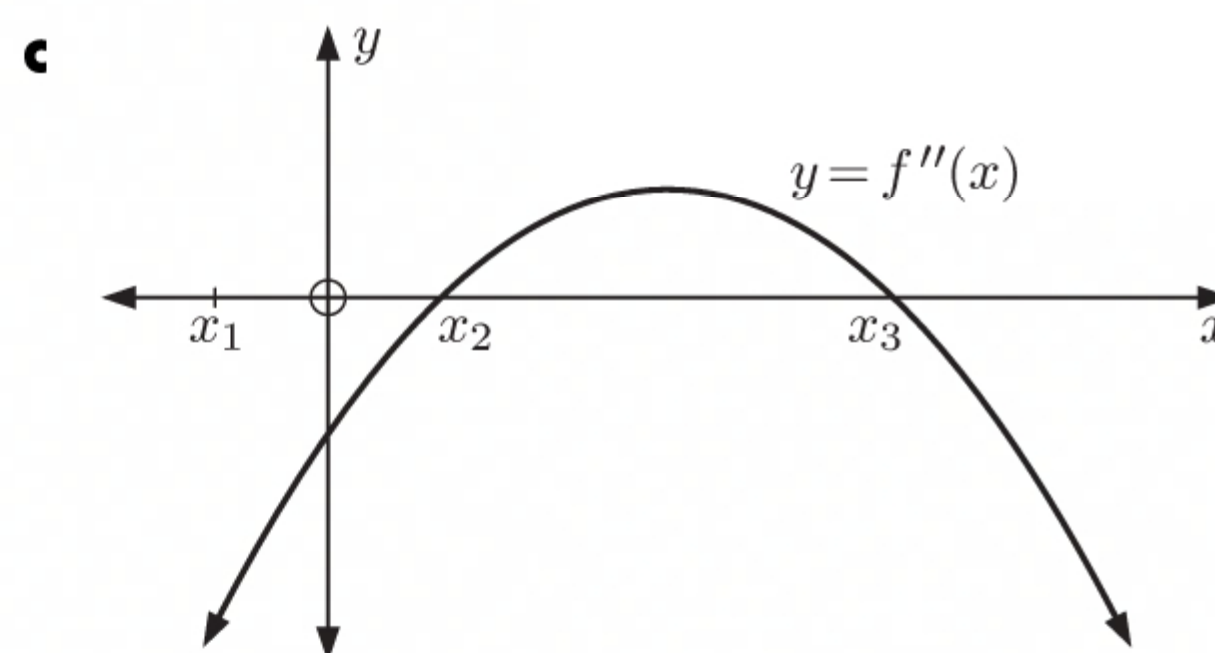
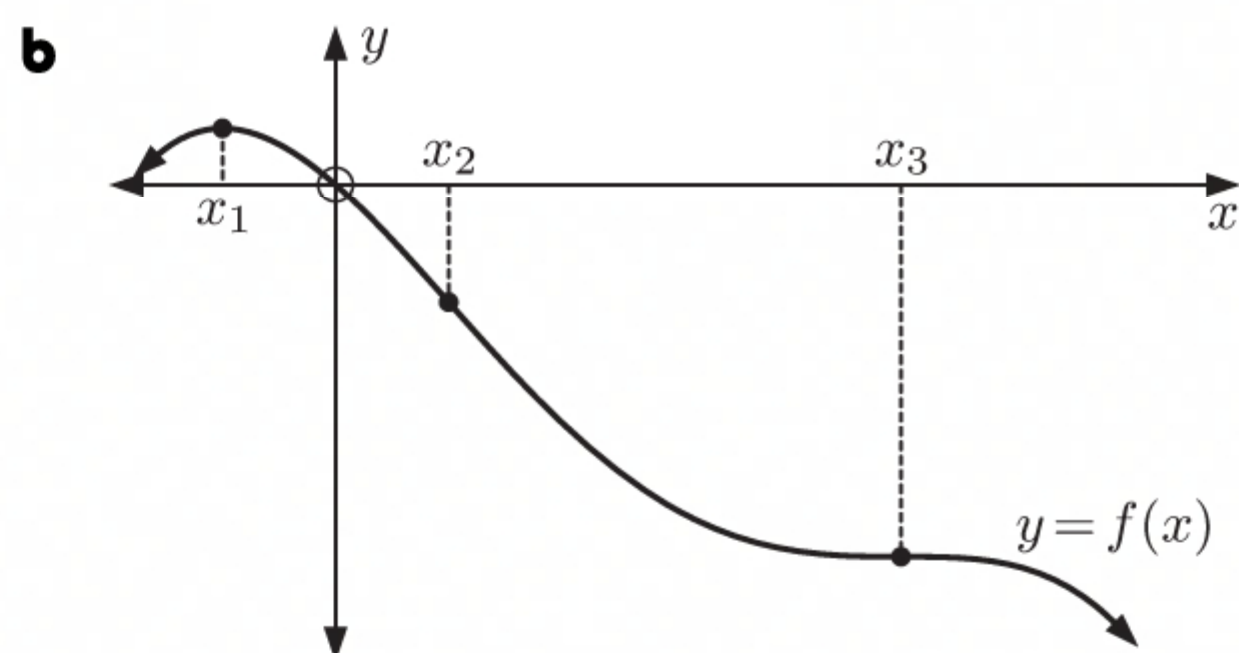
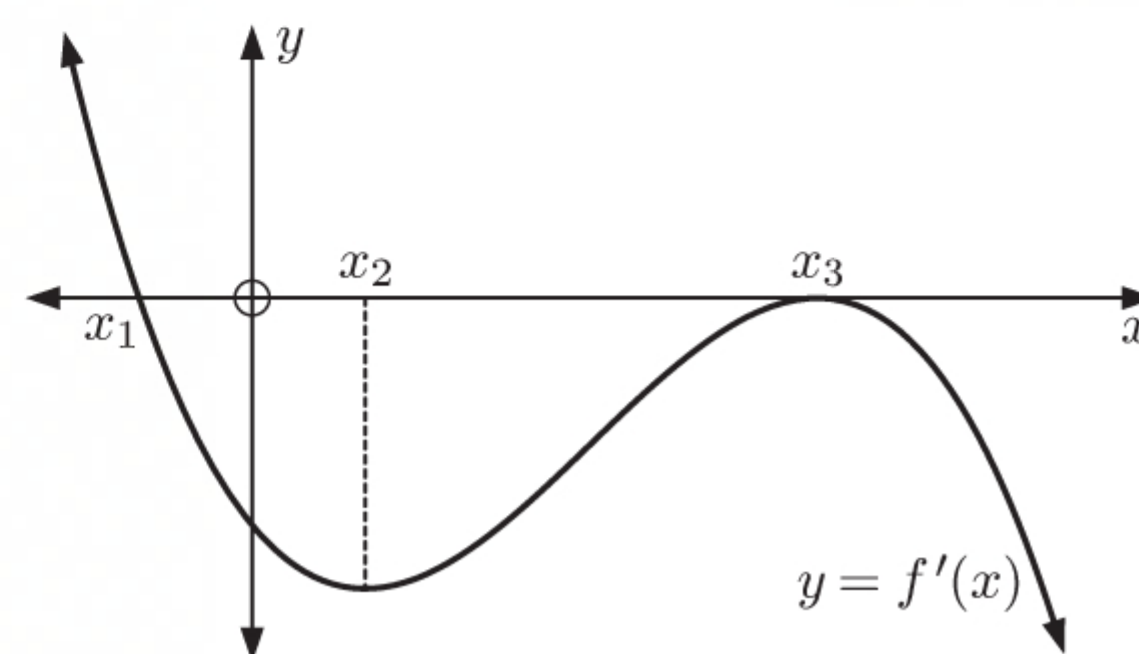


- 66 a i** $f(x)$ is increasing when $f'(x) \geq 0$.

$\therefore f(x)$ is increasing for $x \leq x_1$.

- ii** $f(x)$ is concave down when $f'(x)$ is decreasing.

$\therefore f(x)$ is concave down for $x \leq x_2$ and $x \geq x_3$.



67 $V(t) = 10t^2 - \frac{1}{3}t^3, \quad 0 \leq t \leq 30$

a $V(5) = 10(5)^2 - \frac{1}{3}(5)^3$
 $= 250 - \frac{1}{3}(125)$
 $= 208\frac{1}{3} \text{ litres}$

This represents the volume of water in the tank after 5 minutes.

c $V'(t) = 0$
 $\therefore t(20 - t) = 0$
 $\therefore t = 0 \text{ or } 20$

b $V'(t) = 20t - t^2$
 $= t(20 - t) \text{ litres per minute}$

d $V'(5) = 5(20 - 5)$
 $= 5(15)$
 $= 75 \text{ litres per minute}$

$V'(25) = 25(20 - 25)$
 $= 25(-5)$
 $= -125 \text{ litres per minute}$

e $V'(t) = 75$
 $\therefore t(20 - t) = 75$
 $\therefore 20t - t^2 = 75$
 $\therefore t^2 - 20t + 75 = 0$
 $\therefore (t - 15)(t - 5) = 0$
 $\therefore t = 5 \text{ or } 15$

So, the volume is increasing by 75 litres per minute after 5 minutes and 15 minutes.

68 C has coordinates $(x, \cos x)$.

Let A be the area of rectangle ABCD.

$\therefore A = 2x \cos x \text{ units}^2, \quad 0 \leq x \leq \frac{\pi}{2}$

Now $\frac{dA}{dx} = 2 \cos x + 2x(-\sin x) \quad \{\text{product rule}\}$
 $= 2 \cos x - 2x \sin x$
 $= 2(\cos x - x \sin x)$

A is maximised when $\frac{dA}{dx} = 0$

$\therefore 2(\cos x - x \sin x) = 0$

$\therefore \cos x - x \sin x = 0$

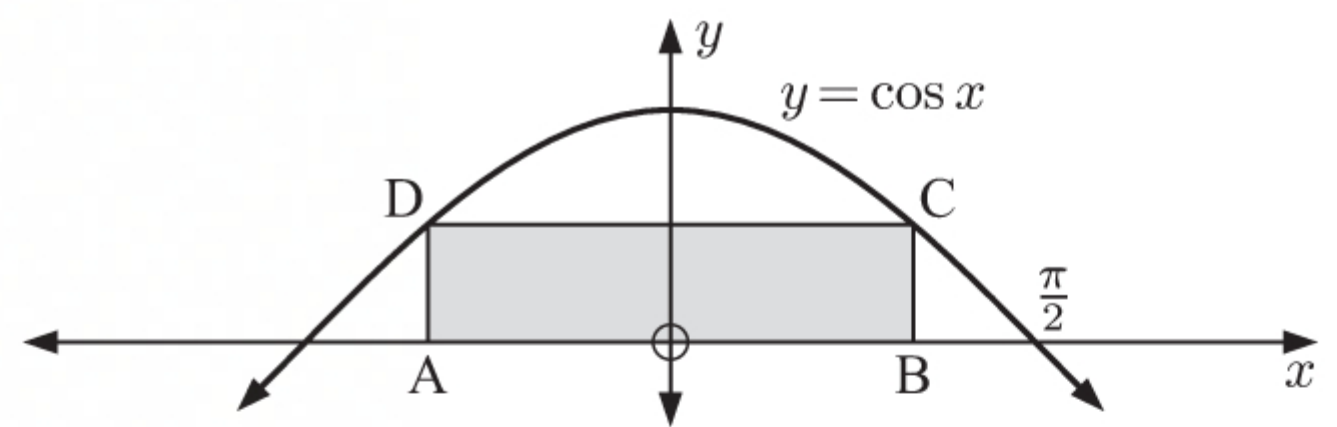
$\therefore \cos x = x \sin x$

$\therefore x \approx 0.860 \quad \{\text{using technology}\}$

The sign diagram of $\frac{dA}{dx}$ is

$\therefore A$ is maximised when $x \approx 0.860$.

\therefore when ABCD has maximum area, C has coordinates $(\approx 0.860, \approx \cos 0.860)$ which is $(\approx 0.860, \approx 0.652)$.



69 a $N = (8 - t)e^{t-6}, \quad 0 \leq t \leq 8$

$\frac{dN}{dt} = (-1)e^{t-6} + (8 - t)e^{t-6} \quad \{\text{product rule}\}$
 $= e^{t-6}(-1 + 8 - t)$
 $= (7 - t)e^{t-6}$

- b i** The turning point occurs when

$$\begin{aligned}\frac{dN}{dt} &= 0 \\ \therefore (7-t)e^{t-6} &= 0 \\ \therefore 7-t &= 0 \quad \{e^{t-6} > 0 \text{ for all } t\} \\ \therefore t &= 7\end{aligned}$$

$$\begin{aligned}\text{When } t = 7, \quad N &= (8-7)e^{7-6} \\ &= (1)e^1 \\ &= e\end{aligned}$$

\therefore the turning point has coordinates $(7, e)$.

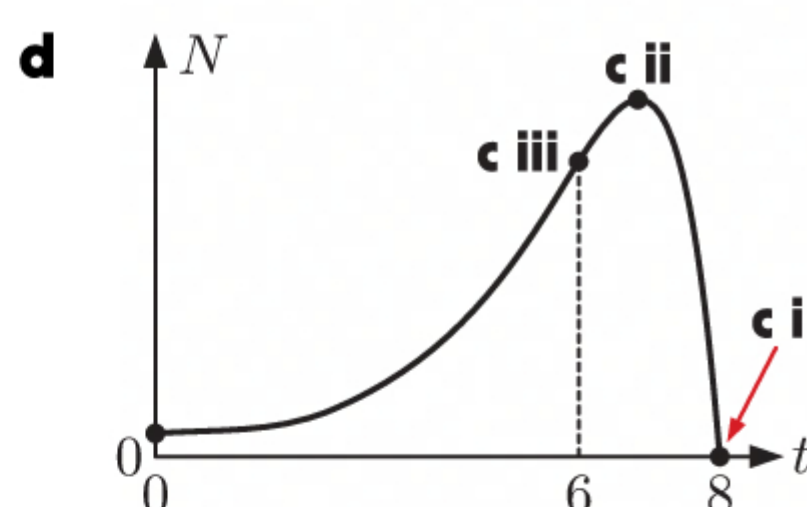
$$\begin{aligned}\text{iii} \quad N &= 0 \\ \therefore (8-t)e^{t-6} &= 0 \\ \therefore 8-t &= 0 \quad \{e^{t-6} > 0 \text{ for all } t\} \\ \therefore t &= 8\end{aligned}$$

\therefore the t -intercept has coordinates $(8, 0)$.

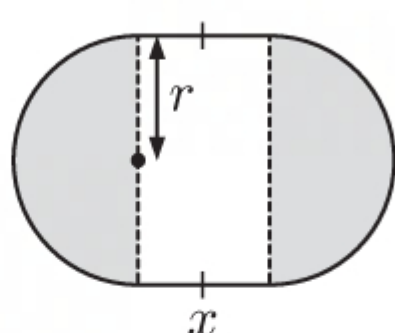
- c i** Using the t -intercept, the time when all the bacteria are dead is $t = 8$ hours.

- ii** Using the turning point, the maximum number of bacteria reached in the sample was $N = e \approx 2.71$ million bacteria.

- iii** Using the point of inflection, the time at which the rate of increase of the bacteria is a maximum is $t = 6$ hours.



70



Let the length of the straight sides be x units and the radius of the semi-circular ends be r units.

$$\begin{aligned}\therefore A &= 2xr + \pi r^2 \\ \therefore 2xr &= A - \pi r^2 \\ \therefore x &= \frac{A}{2r} - \frac{\pi r}{2} \quad \dots (*)\end{aligned}$$

Let c be the cost of tiling the straight sides per unit length.

$$\begin{aligned}\therefore \text{the total tiling cost } C &= 2cx + 2\pi r \times 1.25c \\ &= 2cx + \frac{5c}{2}\pi r \\ &= 2c\left(\frac{A}{2r} - \frac{\pi r}{2}\right) + \frac{5c}{2}\pi r \quad \{\text{using } (*)\} \\ &= \frac{Ac}{r} - c\pi r + \frac{5c}{2}\pi r \\ &= \frac{Ac}{r} + \frac{3c}{2}\pi r\end{aligned}$$

$$\begin{aligned}\text{ii} \quad \frac{d^2N}{dt^2} &= (-1)e^{t-6} + (7-t)e^{t-6} \quad \{\text{product rule}\} \\ &= e^{t-6}(-1+7-t) \\ &= (6-t)e^{t-6}\end{aligned}$$

The point of inflection occurs when

$$\begin{aligned}\frac{d^2N}{dt^2} &= 0 \\ \therefore (6-t)e^{t-6} &= 0 \\ \therefore 6-t &= 0 \quad \{e^{t-6} > 0 \text{ for all } t\} \\ \therefore t &= 6\end{aligned}$$

$$\begin{aligned}\text{When } t = 6, \quad N &= (8-6)e^{6-6} \\ &= 2e^0 \\ &= 2\end{aligned}$$

\therefore the point of inflection has coordinates $(6, 2)$.

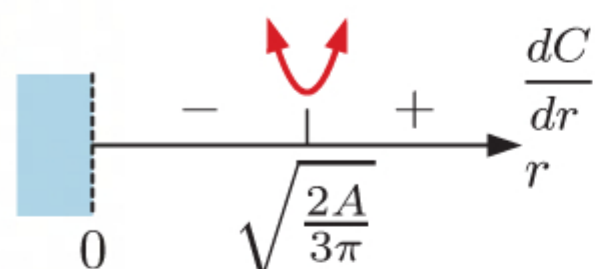
Now $\frac{dC}{dr} = -\frac{Ac}{r^2} + \frac{3c}{2}\pi$ which is 0 when $-\frac{Ac}{r^2} + \frac{3c}{2}\pi = 0$

$$\therefore -\frac{Ac}{r^2} = -\frac{3c}{2}\pi$$

$$\therefore r^2 = \frac{2A}{3\pi}$$

$$\therefore r = \sqrt{\frac{2A}{3\pi}} \quad \{r > 0\}$$

The sign diagram of $\frac{dC}{dr}$ is



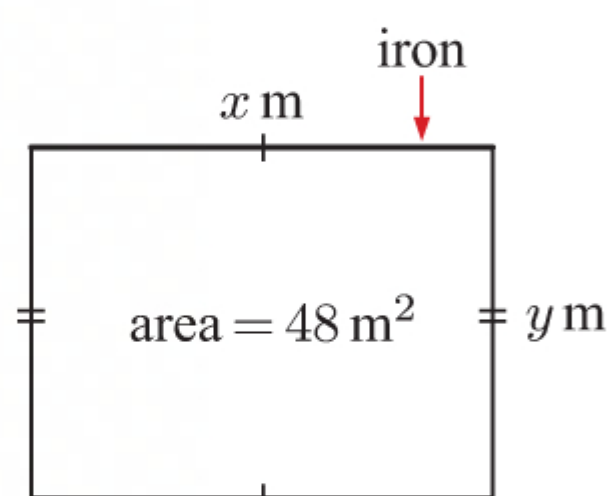
So, C is minimised when $r = \sqrt{\frac{2A}{3\pi}}$.

Now when $r = \sqrt{\frac{2A}{3\pi}}$, the shaded area $= \pi r^2$

$$= \pi \left(\sqrt{\frac{2A}{3\pi}} \right)^2$$

$$= \frac{2}{3}A$$

71 a



Let the length of the side adjacent to the corrugated iron fence be y m.

Now area $= xy$

$$\therefore xy = 48$$

$$\therefore y = \frac{48}{x}$$

Cost of fencing $= 18(2y + x) + 30x$

$$\therefore C = 18\left(\frac{2 \times 48}{x} + x\right) + 30x$$

$$\therefore C = \frac{2 \times 48 \times 18}{x} + 18x + 30x$$

$$\therefore C = \frac{2 \times 48 \times 18}{x} + 48x$$

$$\therefore C = 48\left(\frac{36}{x} + x\right) \text{ dollars}$$

b $\frac{dC}{dx} = 48\left(\frac{-36}{x^2} + 1\right)$

Now C is minimised when $\frac{dC}{dx} = 0$

$$\therefore 48\left(\frac{-36}{x^2} + 1\right) = 0$$

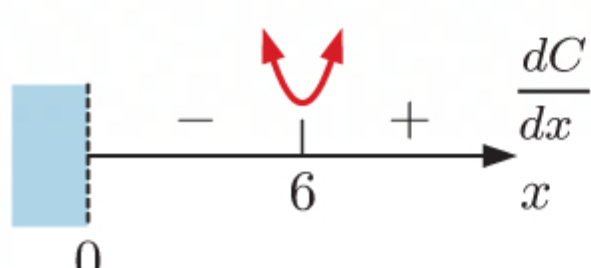
$$\therefore \frac{-36}{x^2} + 1 = 0$$

$$\therefore \frac{36}{x^2} = 1$$

$$\therefore x^2 = 36$$

$$\therefore x = 6 \quad \{x > 0\}$$

The sign diagram of $\frac{dC}{dx}$ is



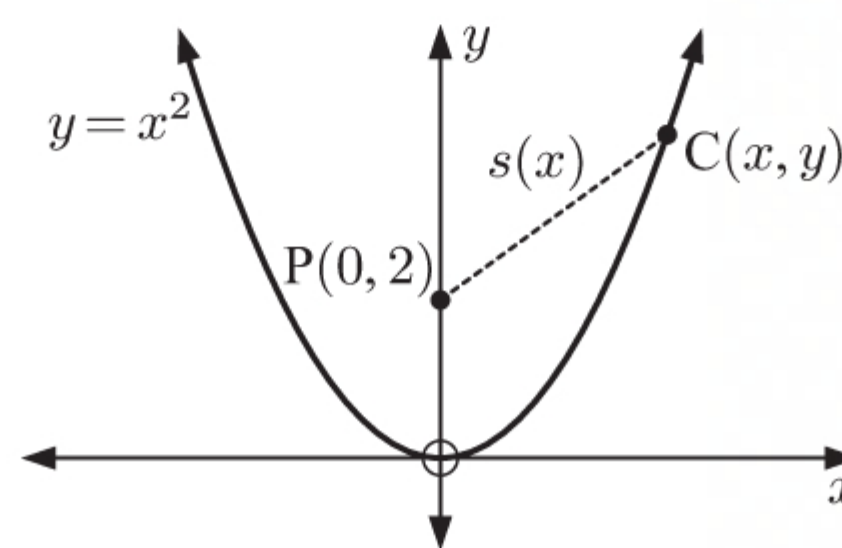
$\therefore C$ is minimised when $x = 6$.

When $x = 6$, $y = \frac{48}{6} = 8$.

The dimensions that minimise the cost of fencing are $6 \text{ m} \times 8 \text{ m}$, where one of the 6 m sides is fenced with corrugated iron.

- 72 a** C has coordinates (x, x^2) .

$$\begin{aligned}\text{Now } s(x) &= \text{CP} \\ &= \sqrt{(x-0)^2 + (x^2-2)^2} \\ &= \sqrt{x^2 + x^4 - 4x^2 + 4} \\ &= \sqrt{x^4 - 3x^2 + 4}\end{aligned}$$



$$\begin{aligned}\text{b } s'(x) &= \frac{1}{2}(x^4 - 3x^2 + 4)^{-\frac{1}{2}}(4x^3 - 6x) \quad \{\text{chain rule}\} \\ &= \frac{2x^3 - 3x}{\sqrt{x^4 - 3x^2 + 4}}\end{aligned}$$

which is 0 when $2x^3 - 3x = 0$

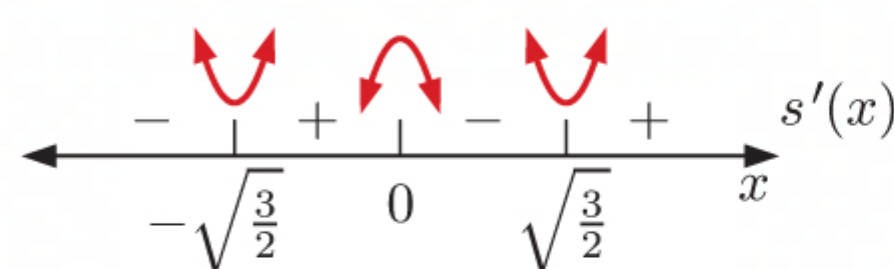
$$\therefore x(2x^2 - 3) = 0$$

$$\therefore x = 0 \quad \text{or} \quad 2x^2 - 3 = 0$$

$$\therefore x^2 = \frac{3}{2}$$

$$\therefore x = \pm\sqrt{\frac{3}{2}}$$

The sign diagram of $s'(x)$ is



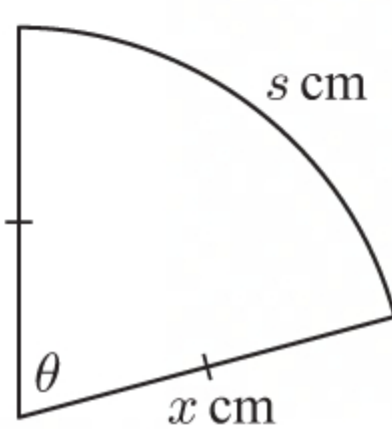
\therefore there is a local maximum at $x = 0$, and local minima at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$.

Critical point (x)	$s(x)$
-2 (end point)	≈ 2.83
$-\sqrt{\frac{3}{2}}$ (local minimum)	≈ 1.32
0 (local maximum)	2
$\sqrt{\frac{3}{2}}$ (local minimum)	≈ 1.32
2 (end point)	≈ 2.83

The greatest distance between the comet and the observer is ≈ 2.83 units when $x = \pm 2$.

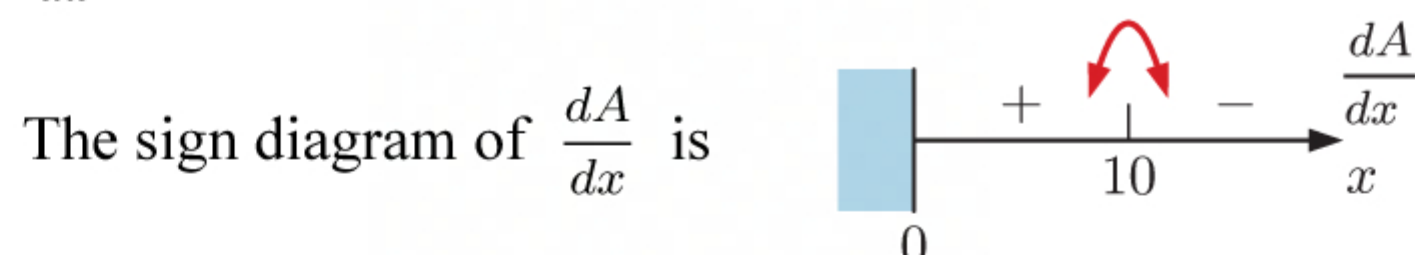
The shortest distance between the comet and the observer is ≈ 1.32 units when $x = \pm\sqrt{\frac{3}{2}}$.

73 a Perimeter $= 2x + s$
 $= 2x + \theta x$
 $\therefore 2x + \theta x = 40$
 $\therefore \theta x = 40 - 2x$
 $\therefore \theta = \frac{40}{x} - 2$



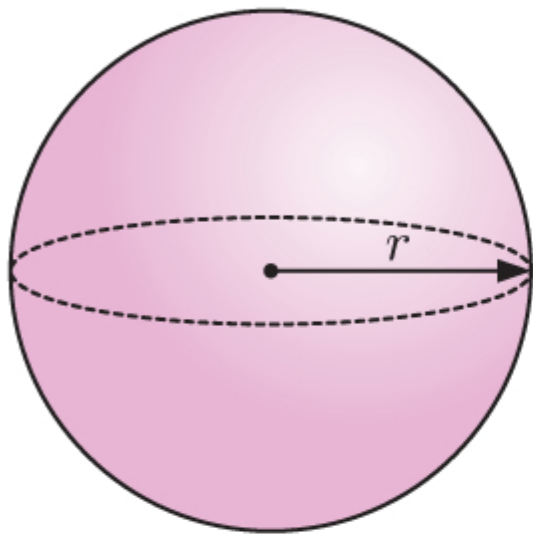
b Area $A = \frac{1}{2}\theta x^2$
 $= \frac{1}{2}\left(\frac{40}{x} - 2\right)x^2 \quad \{\text{using a}\}$
 $= 20x - x^2 \text{ cm}^2$

c $\frac{dA}{dx} = 20 - 2x$ which is 0 when $x = 10$.



When $x = 10$, $\theta = \frac{40}{10} - 2 = 4 - 2 = 2$.

$\therefore A$ is a maximum when $x = 10$ and $\theta = 2$.

74 a

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\text{But } \frac{dr}{dt} = -\frac{8}{5} \text{ cm min}^{-1}$$

$$\therefore \frac{dV}{dt} = 4\pi r^2 \left(-\frac{8}{5}\right) = -6.4\pi r^2$$

$$\text{At time } t = 2.5, \quad r = 8 - \frac{8}{5} \times 2.5 = 4 \text{ cm}$$

$$\therefore \frac{dV}{dt} = -6.4\pi \times 4^2 \\ \approx -322$$

\therefore the volume is decreasing at about $322 \text{ cm}^3 \text{ min}^{-1}$.

$$\begin{aligned} \text{b The average change in volume} &= \frac{V(5) - V(1)}{5 - 1} \\ &= \frac{0 - \frac{4}{3}\pi(6.4)^3}{4} \quad \{\text{when } t = 1, r = 6.4\} \\ &= -\frac{\pi}{3}(6.4)^3 \\ &\approx -275 \end{aligned}$$

\therefore on average, the volume is decreasing at about $275 \text{ cm}^3 \text{ min}^{-1}$.

$$\mathbf{75} \quad K = \frac{1}{2}mv^2$$

$$\text{Differentiating with respect to } t \text{ gives } \frac{dK}{dt} = \frac{1}{2} \left[\frac{dm}{dt} v^2 + m \left(2v \frac{dv}{dt} \right) \right]$$

$$\therefore 50\,000 = \frac{1}{2} \left(-10v^2 + 2mv \frac{dv}{dt} \right)$$

Particular case:

$$\text{When } m = 4000, v = 8, \quad 50\,000 = \frac{1}{2} \left(-10(64) + 2(4000)(8) \frac{dv}{dt} \right)$$

$$\therefore 100\,000 = -640 + 64\,000 \frac{dv}{dt}$$

$$\therefore 100\,640 = 64\,000 \frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} \approx 1.57$$

So, the velocity is increasing at about 1.57 km s^{-2} .

76 Let x m be the distance between the car and A, and y m be the height of the skydiver above A.

The car drives toward A at 12 m s^{-1} , so $\frac{dx}{dt} = -12$.

The skydiver descends towards A at 50 m s^{-1} , so $\frac{dy}{dt} = -50$.

$$\text{Now } \tan \theta = \frac{y}{x}$$

$$\therefore \theta = \arctan \frac{y}{x}$$

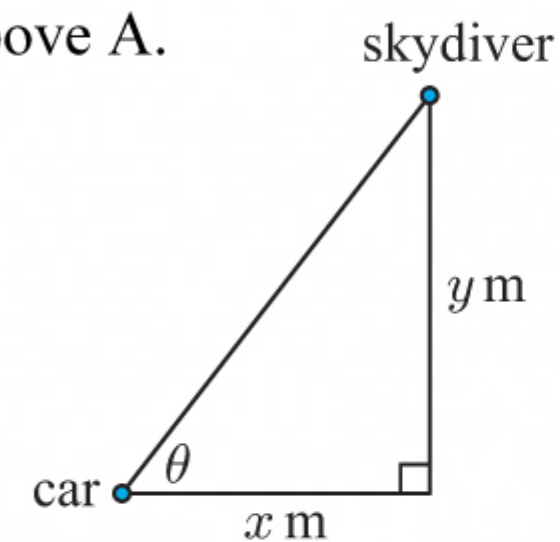
$$\begin{aligned} \text{Differentiating with respect to } t \text{ gives } \frac{d\theta}{dt} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \times \left(\frac{\frac{dy}{dt}x - y\frac{dx}{dt}}{x^2} \right) \\ &= \frac{-50x + 12y}{x^2 + y^2} \end{aligned}$$

Particular case:

After 1 minute, $y = 4000 - 50 \times 60 = 1000$ and $x = 1500 - 12 \times 60 = 780$.

$$\begin{aligned} \therefore \frac{d\theta}{dt} &= \frac{-50(780) + 12(1000)}{780^2 + 1000^2} \\ &= -\frac{135}{8042} \approx -0.0168 \end{aligned}$$

So, after 1 minute, the angle of elevation from the car to the skydiver is decreasing at about 0.0168 radians per second.



$$\begin{aligned} \mathbf{77} \quad \text{a} \quad \int (3x^2 + 2x + 1) dx &= \frac{3x^3}{3} + \frac{2x^2}{2} + x + c \\ &= x^3 + x^2 + x + c \end{aligned} \quad \begin{aligned} \text{b} \quad \int e^{4x} dx &= \frac{1}{4}e^{4x} + c \end{aligned} \quad \begin{aligned} \text{c} \quad \int \cos(2x + 1) dx &= \frac{1}{2} \sin(2x + 1) + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{78} \quad \mathbf{a} \quad f(x) &= \sqrt{xe^x} = (xe^x)^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}(xe^x)^{-\frac{1}{2}} \times \frac{d}{dx}(xe^x) \quad \{\text{chain rule}\} \\
 &= \frac{1}{2\sqrt{xe^x}} \times ((1)e^x + xe^x) \quad \{\text{product rule}\} \\
 &= \frac{e^x + xe^x}{2\sqrt{xe^x}} \\
 &= \frac{e^x(1+x)}{2e^{\frac{x}{2}}\sqrt{x}} \\
 &= \frac{e^{\frac{x}{2}}(1+x)}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{79} \quad \frac{d}{dx}(x^2 \ln x) &= 2x \ln x + x^2 \left(\frac{1}{x}\right) \quad \{\text{product rule}\} \\
 &= 2x \ln x + x
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \int x \ln x \, dx &= \int \frac{\frac{d}{dx}(x^2 \ln x) - x}{2} \, dx \\
 &= \frac{1}{2} \int \left(\frac{d}{dx}(x^2 \ln x) - x \right) \, dx \\
 &= \frac{1}{2} \left(x^2 \ln x - \frac{1}{2}x^2 \right) + c \\
 &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{80} \quad f'(x) &= (x^2 + 2)^2 = x^4 + 4x^2 + 4, \quad f(1) = \frac{8}{15} \\
 \therefore f(x) &= \int (x^4 + 4x^2 + 4) \, dx \\
 &= \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f(1) &= \frac{8}{15}, \quad \therefore \frac{1}{5}(1)^5 + \frac{4}{3}(1)^3 + 4(1) + c = \frac{8}{15} \\
 &\therefore \frac{1}{5} + \frac{4}{3} + 4 + c = \frac{8}{15} \\
 &\therefore \frac{83}{15} + c = \frac{8}{15} \\
 &\therefore c = -5
 \end{aligned}$$

$$\therefore f(x) = \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x - 5$$

$$\mathbf{81} \quad f'(x) = \sqrt{4x+5} = (4x+5)^{\frac{1}{2}}, \quad f(0) = -\frac{\sqrt{5}}{6}$$

$$\begin{aligned}
 \mathbf{a} \quad f'(x) \text{ is defined when } 4x+5 &\geq 0 \\
 \therefore 4x &\geq -5 \\
 \therefore x &\geq -\frac{5}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \frac{2e^{\frac{x}{2}}(1+x)}{\sqrt{x}} \, dx &= \int 4 \left(\frac{e^{\frac{x}{2}}(1+x)}{2\sqrt{x}} \right) \, dx \\
 &= \int 4f'(x) \, dx \quad \{\text{using a}\} \\
 &= 4 \int f'(x) \, dx \\
 &= 4f(x) + c \\
 &= 4\sqrt{xe^x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) &= \int (4x+5)^{\frac{1}{2}} \, dx \\
 &= \frac{2}{3} \times \frac{1}{4}(4x+5)^{\frac{3}{2}} + c \\
 &= \frac{1}{6}(4x+5)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\text{Now } f(0) = -\frac{\sqrt{5}}{6},$$

$$\begin{aligned}
 \therefore \frac{1}{6}(5)^{\frac{3}{2}} + c &= -\frac{\sqrt{5}}{6} \\
 \therefore \frac{5\sqrt{5}}{6} + c &= -\frac{\sqrt{5}}{6} \\
 \therefore c &= -\frac{6\sqrt{5}}{6} = -\sqrt{5}
 \end{aligned}$$

$$\therefore f(x) = \frac{1}{6}(4x+5)^{\frac{3}{2}} - \sqrt{5}$$

$$\mathbf{82} \quad \mathbf{a} \quad f''(x) = e^x + 2x - 1, \quad f'(0) = 4, \quad f(0) = 1$$

$$\begin{aligned} \therefore f'(x) &= \int (e^x + 2x - 1) dx \\ &= e^x + x^2 - x + c \end{aligned}$$

$$\text{Now } f'(0) = 4, \quad \therefore e^0 + 0^2 - 0 + c = 4$$

$$\therefore 1 + c = 4$$

$$\therefore c = 3$$

$$\therefore f'(x) = e^x + x^2 - x + 3$$

$$\begin{aligned} \therefore f(x) &= \int (e^x + x^2 - x + 3) dx \\ &= e^x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x + d \end{aligned}$$

$$\text{Now } f(0) = 1,$$

$$\therefore e^0 + \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 + 3(0) + d = 1$$

$$\therefore 1 + d = 1$$

$$\therefore d = 0$$

$$\therefore f(x) = e^x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x$$

$$\mathbf{b} \quad f''(x) = 2 + \sin x, \quad f'(\pi) = 1, \quad f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$\begin{aligned} \therefore f'(x) &= \int (2 + \sin x) dx \\ &= 2x - \cos x + c \end{aligned}$$

$$\text{Now } f'(\pi) = 1, \quad \therefore 2\pi - \cos \pi + c = 1$$

$$\therefore 2\pi - (-1) + c = 1$$

$$\therefore 2\pi + 1 + c = 1$$

$$\therefore 2\pi + c = 0$$

$$\therefore c = -2\pi$$

$$\therefore f'(x) = 2x - \cos x - 2\pi$$

$$\begin{aligned} \therefore f(x) &= \int (2x - \cos x - 2\pi) dx \\ &= x^2 - \sin x - 2\pi x + d \end{aligned}$$

$$\text{Now } f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4},$$

$$\therefore \frac{\pi^2}{4} - \sin \frac{\pi}{2} - 2\pi\left(\frac{\pi}{2}\right) + d = \frac{\pi^2}{4}$$

$$\therefore -1 - \pi^2 + d = 0$$

$$\therefore d = \pi^2 + 1$$

$$\therefore f(x) = x^2 - \sin x - 2\pi x + \pi^2 + 1$$

$$\mathbf{c} \quad f''(x) = \frac{2}{\sqrt{x}} + 3x = 2x^{-\frac{1}{2}} + 3x, \quad f(1) = -\frac{19}{3}, \quad f(4) = \frac{64}{3}$$

$$\begin{aligned} \therefore f'(x) &= \int (2x^{-\frac{1}{2}} + 3x) dx \\ &= 4x^{\frac{1}{2}} + \frac{3}{2}x^2 + c \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= \int (4x^{\frac{1}{2}} + \frac{3}{2}x^2 + c) dx \\ &= \frac{8}{3}x^{\frac{3}{2}} + \frac{x^3}{2} + cx + d \end{aligned}$$

$$\text{Now } f(1) = -\frac{19}{3}$$

$$\therefore \frac{8}{3}(1)^{\frac{3}{2}} + \frac{1^3}{2} + c(1) + d = -\frac{19}{3}$$

$$\therefore \frac{8}{3} + \frac{1}{2} + c + d = -\frac{19}{3}$$

$$\therefore \frac{19}{6} + c + d = -\frac{19}{3}$$

$$\therefore c + d = -\frac{19}{2} \quad \dots (1)$$

and

$$f(4) = \frac{64}{3}$$

$$\therefore \frac{8}{3}(4)^{\frac{3}{2}} + \frac{4^3}{2} + c(4) + d = \frac{64}{3}$$

$$\therefore \frac{64}{3} + \frac{64}{2} + 4c + d = \frac{64}{3}$$

$$\therefore 32 + 4c + d = 0$$

$$\therefore 4c + d = -32 \quad \dots (2)$$

$$\begin{array}{rcl} 4c + 4d & = & -38 \quad \{(1) \times 4\} \\ -4c - d & = & 32 \quad \{(2) \times (-1)\} \\ \hline 3d & = & -6 \end{array}$$

Adding,

$$\therefore d = -2$$

$$\text{Substituting } d = -2 \text{ into (1) gives } c - 2 = -\frac{19}{2}$$

$$\therefore c = -\frac{15}{2}$$

$$\begin{aligned} \therefore f(x) &= \frac{8}{3}x^{\frac{3}{2}} + \frac{1}{2}x^3 - \frac{15}{2}x - 2 \\ &= \frac{8}{3}x\sqrt{x} + \frac{1}{2}x^3 - \frac{15}{2}x - 2 \end{aligned}$$

$$\mathbf{83} \quad \mathbf{a} \quad \int (2 \sin(x-3) + e^{3x}) dx$$

$$= 2(-\cos(x-3)) + \frac{1}{3}e^{3x} + c$$

$$= -2 \cos(x-3) + \frac{1}{3}e^{3x} + c$$

$$\mathbf{b} \quad \int \frac{2}{5x-1} dx$$

$$= \frac{2}{5} \ln|5x-1| + c$$

$$\mathbf{c} \quad \int \cos(5-7x) dx$$

$$= -\frac{1}{7} \sin(5-7x) + c$$

$$\begin{aligned}
 \mathbf{84} \quad \mathbf{a} \quad & \int (2 \sin^2 x - 1) dx \\
 &= \int (-(1 - 2 \sin^2 x)) dx \\
 &= \int (-\cos 2x) dx \quad \{\text{double angle formula}\} \\
 &= -\frac{1}{2} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (\sin 2x - \cos 2x)^2 dx \\
 &= \int (\sin^2 2x - 2 \sin 2x \cos 2x + \cos^2 2x) dx \\
 &= \int (1 - 2 \sin 2x \cos 2x) dx \\
 &= \int (1 - \sin 4x) dx \quad \{\text{double angle formula}\} \\
 &= x + \frac{1}{4} \cos 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (\cos x + 2)^2 dx \\
 &= \int (\cos^2 x + 4 \cos x + 4) dx \\
 &= \int \left(\frac{1}{2} \cos 2x + \frac{1}{2} + 4 \cos x + 4 \right) dx \quad \{\text{double angle formula}\} \\
 &= \int \left(\frac{1}{2} \cos 2x + 4 \cos x + \frac{9}{2} \right) dx \\
 &= \frac{1}{4} \sin 2x + 4 \sin x + \frac{9}{2} x + c
 \end{aligned}$$

$$\mathbf{85} \quad \mathbf{a} \quad \int 6^x dx = \frac{6^x}{\ln 6} + c$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \left(\frac{2}{x} - \frac{5}{x \ln 3} \right) dx \\
 &= 2 \ln |x| - \frac{5 \ln |x|}{\ln 3} + c \\
 &= 2 \ln |x| - 5 \log_3 |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (\cos x - \sec x \tan x) dx \\
 &= \sin x - \sec x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{86} \quad & f'(x) = \frac{2}{1+x^2} \\
 \therefore f(x) &= \int \frac{2}{1+x^2} dx \\
 &= 2 \arctan x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } f(-\sqrt{3}) &= \pi, \text{ so } 2 \arctan(-\sqrt{3}) + c = \pi \\
 \therefore 2\left(-\frac{\pi}{3}\right) + c &= \pi \\
 \therefore c &= \frac{5\pi}{3}
 \end{aligned}$$

$$\therefore f(x) = 2 \arctan x + \frac{5\pi}{3}$$

$$\begin{aligned}
 \mathbf{87} \quad & f'(x) = a \times 2^x \\
 \therefore f(x) &= \int (a \times 2^x) dx \\
 &= \frac{a \times 2^x}{\ln 2} + c
 \end{aligned}$$

$$\text{Now } f(0) = -1, \text{ so } \frac{a}{\ln 2} + c = -1$$

$$\therefore c = -1 - \frac{a}{\ln 2} \quad \dots (*)$$

$$\text{and } f(1) = 2, \text{ so } \frac{2a}{\ln 2} + c = 2$$

$$\therefore \frac{2a}{\ln 2} - 1 - \frac{a}{\ln 2} = 2 \quad \{\text{using } (*)\}$$

$$\therefore \frac{a}{\ln 2} = 3$$

$$\therefore a = 3 \ln 2$$

$$\text{Substituting } a = 3 \ln 2 \text{ into } (*) \text{ gives } \frac{3 \ln 2}{\ln 2} + c = -1$$

$$\therefore 3 + c = -1$$

$$\therefore c = -4$$

$$\begin{aligned}
 \therefore f(x) &= \frac{3 \ln 2 \times 2^x}{\ln 2} + (-4) \\
 &= 3 \times 2^x - 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{88} \quad \mathbf{a} \quad \int \frac{-10}{\sqrt{36-4x^2}} dx &= -10 \int \frac{1}{\sqrt{4}\sqrt{9-x^2}} dx \\
 &= -5 \int \frac{1}{\sqrt{9-x^2}} dx
 \end{aligned}$$

$$\text{Let } x = 3 \sin \theta \quad \therefore \frac{dx}{d\theta} = 3 \cos \theta \text{ and } \theta = \arcsin \frac{x}{3}$$

$$\begin{aligned}
 \therefore \int \frac{-10}{\sqrt{36-4x^2}} dx &= -5 \int \frac{1}{\sqrt{9-9\sin^2 \theta}} 3 \cos \theta d\theta \\
 &= -5 \int \frac{3 \cos \theta}{\sqrt{9}\sqrt{1-\sin^2 \theta}} d\theta \\
 &= -5 \int \frac{\cos \theta}{\sqrt{\cos^2 \theta}} d\theta \\
 &= -5 \int \frac{\cos \theta}{\cos \theta} d\theta \\
 &= -5 \int 1 d\theta \\
 &= -5\theta + c \\
 &= -5 \arcsin \frac{x}{3} + c
 \end{aligned}$$

$$\mathbf{b} \quad \int \frac{2}{9+4x^2} dx = 2 \int \frac{1}{9+(2x)^2} dx$$

$$\text{Let } 2x = 3 \tan \theta$$

$$\therefore x = \frac{3}{2} \tan \theta, \quad \frac{dx}{d\theta} = \frac{3}{2} \sec^2 \theta, \quad \text{and } \theta = \arctan \frac{2x}{3}$$

$$\begin{aligned}
 \therefore \int \frac{2}{9+4x^2} dx &= 2 \int \frac{1}{9+9\tan^2 \theta} \times \frac{3}{2} \sec^2 \theta d\theta \\
 &= \int \frac{3 \sec^2 \theta}{9(1+\tan^2 \theta)} d\theta \\
 &= \frac{1}{3} \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta \\
 &= \frac{1}{3} \int 1 d\theta \\
 &= \frac{1}{3} \theta + c \\
 &= \frac{1}{3} \arctan \frac{2x}{3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int \frac{3}{(x+4)^2 + (x-4)^2} dx &= 3 \int \frac{1}{x^2 + 8x + 16 + x^2 - 8x + 16} dx \\
 &= 3 \int \frac{1}{2x^2 + 32} dx \\
 &= \frac{3}{2} \int \frac{1}{x^2 + 16} dx
 \end{aligned}$$

$$\text{Let } x = 4 \tan \theta \quad \therefore \frac{dx}{d\theta} = 4 \sec^2 \theta \text{ and } \theta = \arctan \frac{x}{4}$$

$$\begin{aligned}
 \therefore \int \frac{3}{(x+4)^2 + (x-4)^2} dx &= \frac{3}{2} \int \frac{1}{16 \tan^2 \theta + 16} 4 \sec^2 \theta d\theta \\
 &= \frac{12}{32} \int \frac{\sec^2 \theta}{\tan^2 \theta + 1} d\theta \\
 &= \frac{3}{8} \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta \\
 &= \frac{3}{8} \int 1 d\theta \\
 &= \frac{3}{8} \theta + c \\
 &= \frac{3}{8} \arctan \frac{x}{4} + c
 \end{aligned}$$

89

$$\begin{array}{r}
 x^2 + 8x + 21 \\
 x - 3 \overline{) x^3 + 5x^2 - 3x + 2} \\
 \underline{-(x^2 - 3x^2)} \\
 8x^2 - 3x \\
 \underline{-(8x^2 - 24x)} \\
 21x + 2 \\
 \underline{-(21x - 63)} \\
 65
 \end{array}$$

$$\begin{aligned}
 \therefore \frac{x^3 + 5x^2 - 3x + 2}{x - 3} &= x^2 + 8x + 21 + \frac{65}{x - 3} \\
 \therefore \int \frac{x^3 + 5x^2 - 3x + 2}{x - 3} dx &= \int \left(x^2 + 8x + 21 + \frac{65}{x - 3} \right) dx \\
 &= \frac{1}{3}x^3 + 4x^2 + 21x + 65 \ln |x - 3| + c
 \end{aligned}$$

90 a

$$\begin{aligned}
 &\int 3x^2(5 + x^3)^4 dx \\
 &= \int u^4 \frac{du}{dx} dx \quad \left\{ u = 5 + x^3 \quad \therefore \frac{du}{dx} = 3x^2 \right\} \\
 &= \int u^4 du \\
 &= \frac{1}{5}u^5 + c \\
 &= \frac{1}{5}(5 + x^3)^5 + c
 \end{aligned}$$

b

$$\begin{aligned}
 &\int x e^{x^2+2} dx \\
 &= \int e^u \left(\frac{1}{2} \frac{du}{dx} \right) dx \quad \left\{ u = x^2 + 2 \quad \therefore \frac{du}{dx} = 2x \right\} \\
 &= \int \frac{1}{2} e^u du \\
 &= \frac{1}{2} e^u + c \\
 &= \frac{1}{2} e^{x^2+2} + c
 \end{aligned}$$

c

$$\begin{aligned}
 &\int \frac{2(\ln x)^2}{x} dx \\
 &= \int 2u^2 \frac{du}{dx} dx \quad \left\{ u = \ln x \quad \therefore \frac{du}{dx} = \frac{1}{x} \right\} \\
 &= \int 2u^2 du \\
 &= \frac{2}{3}u^3 + c \\
 &= \frac{2}{3}(\ln x)^3 + c
 \end{aligned}$$

91 a

$$\begin{aligned}
 &\int \sqrt{x^2 + 3x - 1} (2x + 3) dx \\
 &= \int \sqrt{u} \frac{du}{dx} dx \quad \left\{ u = x^2 + 3x - 1 \quad \therefore \frac{du}{dx} = 2x + 3 \right\} \\
 &= \int u^{\frac{1}{2}} du \\
 &= \frac{2}{3}u^{\frac{3}{2}} + c \\
 &= \frac{2}{3}(x^2 + 3x - 1)^{\frac{3}{2}} + c
 \end{aligned}$$

b

$$\begin{aligned}
 &\int \frac{e^x + 2}{e^x + 2x} dx \\
 &= \int \frac{1}{u} \frac{du}{dx} dx \quad \left\{ u = e^x + 2x \quad \therefore \frac{du}{dx} = e^x + 2 \right\} \\
 &= \int \frac{1}{u} du \\
 &= \ln |u| + c \\
 &= \ln |e^x + 2x| + c
 \end{aligned}$$

c

$$\begin{aligned}
 &\int \frac{6 - 8x}{2x^2 - 3x + 2} dx \\
 &= \int \frac{1}{u} \left(-2 \frac{du}{dx} \right) dx \quad \left\{ u = 2x^2 - 3x + 2 \quad \therefore \frac{du}{dx} = 4x - 3 \right\} \\
 &= \int -\frac{2}{u} du \\
 &= -2 \ln |u| + c \\
 &= -2 \ln |2x^2 - 3x + 2| + c
 \end{aligned}$$

92 a

$$\begin{aligned}
 &\int \sin^4 x \cos x dx \\
 &= \int u^4 \frac{du}{dx} dx \quad \left\{ u = \sin x \quad \therefore \frac{du}{dx} = \cos x \right\} \\
 &= \int u^4 du \\
 &= \frac{1}{5}u^5 + c \\
 &= \frac{1}{5} \sin^5 x + c
 \end{aligned}$$

b

$$\begin{aligned}
 &\int 3x^3 \sin(x^4) dx \\
 &= \int \sin u \left(\frac{3}{4} \frac{du}{dx} \right) dx \quad \left\{ u = x^4 \quad \therefore \frac{du}{dx} = 4x^3 \right\} \\
 &= \int \frac{3}{4} \sin u du \\
 &= -\frac{3}{4} \cos u + c \\
 &= -\frac{3}{4} \cos(x^4) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int \frac{\sin 2x}{(3 - \cos 2x)^3} dx &= \int \frac{1}{u^3} \left(\frac{1}{2} \frac{du}{dx} \right) dx \quad \left\{ u = 3 - \cos 2x \quad \therefore \frac{du}{dx} = 2 \sin 2x \right\} \\
 &= \int \frac{1}{2} u^{-3} du \\
 &= \frac{1}{2} \times \frac{1}{-2} u^{-2} + c \\
 &= -\frac{1}{4u^2} + c \\
 &= -\frac{1}{4(3 - \cos 2x)^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{93 a} \quad \cos^4 x &= (\cos^2 x)^2 \\
 &= \left(\frac{1}{2} \cos 2x + \frac{1}{2} \right)^2 && \{\text{double angle formula}\} \\
 &= \frac{1}{4} \cos^2 2x + \frac{1}{2} \cos 2x + \frac{1}{4} \\
 &= \frac{1}{4} \left(\frac{1}{2} \cos 4x + \frac{1}{2} \right) + \frac{1}{2} \cos 2x + \frac{1}{4} && \{\text{double angle formula}\} \\
 &= \frac{1}{8} \cos 4x + \frac{1}{8} + \frac{1}{2} \cos 2x + \frac{1}{4} \\
 &= \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int \cos^4 x dx &= \int \left(\frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) dx \quad \{\text{using a}\} \\
 &= \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{94 a} \quad \int (e^{-3x} + 2 \sec^2(\frac{\pi}{3} - x)) dx \\
 &= -\frac{1}{3}e^{-3x} + \frac{2}{-1} \tan(\frac{\pi}{3} - x) + c \\
 &= -\frac{1}{3}e^{-3x} - 2 \tan(\frac{\pi}{3} - x) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int (7^{3x+2} - \sec^2(\frac{x}{2} + \frac{\pi}{6})) dx \\
 &= \frac{1}{3} \times \frac{7^{3x+2}}{\ln 7} - \frac{1}{\frac{1}{2}} \tan(\frac{x}{2} + \frac{\pi}{6}) + c \\
 &= \frac{7^{3x+2}}{3 \ln 7} - 2 \tan(\frac{x}{2} + \frac{\pi}{6}) + c
 \end{aligned}$$

$$\text{95 Let } x = 2 \sin \theta \quad \therefore \frac{dx}{d\theta} = 2 \cos \theta \text{ and } \theta = \arcsin \frac{x}{2}$$

$$\begin{aligned}
 \therefore \int \sqrt{4 - x^2} dx &= \int \sqrt{4 - 4 \sin^2 \theta} \times 2 \cos \theta d\theta \\
 &= \int \sqrt{4} \sqrt{1 - \sin^2 \theta} \times 2 \cos \theta d\theta \\
 &= 4 \int \sqrt{\cos^2 \theta} \cos \theta d\theta \\
 &= 4 \int \cos^2 \theta d\theta \\
 &= 4 \int \frac{1}{2} (\cos 2\theta + 1) d\theta \\
 &= 2 \int (\cos 2\theta + 1) d\theta \\
 &= 2 \left(\frac{1}{2} \sin 2\theta + \theta \right) + c \\
 &= \sin 2\theta + 2\theta + c \\
 &= \sin(2 \arcsin \frac{x}{2}) + 2 \arcsin \frac{x}{2} + c
 \end{aligned}$$

$$\text{96 Let } u = x + 4 \quad \therefore \frac{du}{dx} = 1$$

$$\begin{aligned}
 \int 3x^2 \sqrt{x+4} dx &= \int 3(u-4)^2 \sqrt{u} \frac{du}{dx} dx \\
 &= 3 \int (u^2 - 8u + 16) u^{\frac{1}{2}} du \\
 &= 3 \int (u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}) du \\
 &= 3 \left(\frac{2}{7} u^{\frac{7}{2}} - \frac{16}{5} u^{\frac{5}{2}} + \frac{32}{3} u^{\frac{3}{2}} \right) + c \\
 &= \frac{6}{7} u^{\frac{7}{2}} - \frac{48}{5} u^{\frac{5}{2}} + 32 u^{\frac{3}{2}} + c \\
 &= \frac{6}{7} (x+4)^{\frac{7}{2}} - \frac{48}{5} (x+4)^{\frac{5}{2}} + 32 (x+4)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\mathbf{97} \quad \mathbf{a} \quad \text{Let } u = 3 - x \quad \therefore \frac{du}{dx} = -1$$

$$\begin{aligned} \therefore \int x\sqrt{3-x} \, dx &= \int (3-u)\sqrt{u} \left(-\frac{du}{dx}\right) dx \\ &= \int (u-3)u^{\frac{1}{2}} \, du \\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right) \, du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{1}{2}u^{\frac{3}{2}} + c \\ &= \frac{2}{5}(3-x)^{\frac{5}{2}} - \frac{1}{2}(3-x)^{\frac{3}{2}} + c \end{aligned}$$

$$\mathbf{c} \quad \text{Let } u = 4 - x^2 \quad \therefore \frac{du}{dx} = -2x$$

$$\begin{aligned} \therefore \int x^3\sqrt{4-x^2} \, dx &= \int x \times x^2\sqrt{4-x^2} \, dx \\ &= \int \left(-\frac{1}{2} \frac{du}{dx}\right) \times (4-u) \times \sqrt{u} \, dx \\ &= \frac{1}{2} \int (u-4)u^{\frac{1}{2}} \, du \\ &= \frac{1}{2} \int \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right) \, du \\ &= \frac{1}{2} \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right) + c \\ &= \frac{1}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + c \\ &= \frac{1}{5}(4-x^2)^{\frac{5}{2}} - \frac{4}{3}(4-x^2)^{\frac{3}{2}} + c \end{aligned}$$

$$\mathbf{98} \quad \text{Let } x = \cos^2 \theta \quad \therefore \frac{dx}{d\theta} = -2 \cos \theta \sin \theta$$

$$\begin{aligned} \therefore \int x^{-\frac{3}{2}}\sqrt{1-x} \, dx &= \int (\cos^2 \theta)^{-\frac{3}{2}} \sqrt{1-\cos^2 \theta} (-2 \cos \theta \sin \theta) \, d\theta \\ &= \int (\cos \theta)^{-3} \sqrt{\sin^2 \theta} (-2 \cos \theta \sin \theta) \, d\theta \\ &= -2 \int \frac{\sin^2 \theta}{\cos^2 \theta} \, d\theta \\ &= -2 \int \frac{1-\cos^2 \theta}{\cos^2 \theta} \, d\theta \\ &= -2 \int \left(\frac{1}{\cos^2 \theta} - 1\right) \, d\theta \\ &= -2 \int (\sec^2 \theta - 1) \, d\theta \\ &= -2(\tan \theta - \theta) + c \\ &= 2\theta - 2 \tan \theta + c \end{aligned}$$

$$\text{Now } x = \cos^2 \theta, \quad 1-x = 1-\cos^2 \theta = \sin^2 \theta$$

$$\therefore \sqrt{\frac{1-x}{x}} = \sqrt{\frac{\sin^2 \theta}{\cos^2 \theta}} = \sqrt{\tan^2 \theta} = \tan \theta$$

$$\text{Also } \sqrt{x} = \cos \theta, \text{ so } \theta = \arccos(\sqrt{x})$$

$$\begin{aligned} \therefore \int x^{-\frac{3}{2}}\sqrt{1-x} \, dx &= 2 \arccos(\sqrt{x}) - 2\sqrt{\frac{1-x}{x}} + c \\ &= 2 \arccos(\sqrt{x}) - 2\sqrt{\frac{1}{x} - 1} + c \end{aligned}$$

$$\mathbf{b} \quad \text{Let } u = x - 4 \quad \therefore \frac{du}{dx} = 1$$

$$\begin{aligned} \therefore \int \frac{2x}{\sqrt{x-4}} \, dx &= \int \frac{2(u+4)}{\sqrt{u}} \frac{du}{dx} \, dx \\ &= 2 \int (u+4)u^{-\frac{1}{2}} \, du \\ &= 2 \int \left(u^{\frac{1}{2}} + 4u^{-\frac{1}{2}}\right) \, du \\ &= 2 \left(\frac{2}{3}u^{\frac{3}{2}} + 8u^{\frac{1}{2}}\right) + c \\ &= \frac{4}{3}u^{\frac{3}{2}} + 16u^{\frac{1}{2}} + c \\ &= \frac{4}{3}(x-4)^{\frac{3}{2}} + 16(x-4)^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \text{99 a Let } \frac{4-x}{(x-3)(x-5)} &= \frac{A}{x-3} + \frac{B}{x-5} \\ \therefore 4-x &= A(x-5) + B(x-3) \end{aligned}$$

$$\text{Substituting } x=5, \quad 4-5=2B$$

$$\therefore B = -\frac{1}{2}$$

$$\text{Substituting } x=3, \quad 4-3=-2A$$

$$\therefore A = -\frac{1}{2}$$

$$\therefore \frac{4-x}{(x-3)(x-5)} = -\frac{1}{2(x-3)} - \frac{1}{2(x-5)}$$

$$\begin{aligned} \text{b } \int \frac{4-x}{(x-3)(x-5)} dx &= \int \left(-\frac{1}{2(x-3)} - \frac{1}{2(x-5)} \right) dx \quad \{\text{using a}\} \\ &= -\frac{1}{2} \int \left(\frac{1}{x-3} + \frac{1}{x-5} \right) dx \\ &= -\frac{1}{2} (\ln|x-3| + \ln|x-5|) + c \\ &= -\frac{1}{2} \ln|x-3| - \frac{1}{2} \ln|x-5| + c \end{aligned}$$

$$\begin{aligned} \text{100 a Let } \frac{x^2+x-5}{(x+2)(x+1)^2} &= \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\ \therefore x^2+x-5 &= A(x+1)^2 + B(x+2)(x+1) + C(x+2) \end{aligned}$$

$$\begin{aligned} \text{Substituting } x=-1, \quad (-1)^2 + (-1) - 5 &= C \\ \therefore C &= -5 \end{aligned}$$

$$\begin{aligned} \text{Substituting } x=-2, \quad (-2)^2 + (-2) - 5 &= A(-1)^2 \\ \therefore A &= -3 \end{aligned}$$

$$\begin{aligned} \text{Substituting } x=0, \quad 0^2 + 0 - 5 &= -3(1)^2 + B(2)(1) - 5(2) \\ \therefore -5 &= -3 + 2B - 10 \\ \therefore 2B &= 8 \\ \therefore B &= 4 \end{aligned}$$

$$\therefore \frac{x^2+x-5}{(x+2)(x+1)^2} = \frac{-3}{x+2} + \frac{4}{x+1} - \frac{5}{(x+1)^2}$$

$$\begin{aligned} \text{b } \int \frac{x^2+x-5}{(x+2)(x+1)^2} dx &= \int \left(\frac{-3}{x+2} + \frac{4}{x+1} - \frac{5}{(x+1)^2} \right) dx \quad \{\text{using a}\} \\ &= \int \left(\frac{-3}{x+2} + \frac{4}{x+1} - 5(x+1)^{-2} \right) dx \\ &= -3 \ln|x+2| + 4 \ln|x+1| + 5(x+1)^{-1} + c \\ &= -3 \ln|x+2| + 4 \ln|x+1| + \frac{5}{x+1} + c \end{aligned}$$

$$\begin{aligned} \text{101 a } \int x \sin 4x dx &= -\frac{1}{4}x \cos 4x - \int -\frac{1}{4} \cos 4x dx \quad \begin{cases} u = x & v' = \sin 4x \\ u' = 1 & v = -\frac{1}{4} \cos 4x \end{cases} \\ &= -\frac{1}{4}x \cos 4x + \frac{1}{4} \int \cos 4x dx \\ &= -\frac{1}{4}x \cos 4x + \frac{1}{16} \sin 4x + c \end{aligned}$$

$$\begin{aligned} \text{b } \int (2x+1) \ln x dx &= \ln x(x^2+x) - \int \frac{1}{x}(x^2+x) dx \quad \begin{cases} u = \ln x & v' = 2x+1 \\ u' = \frac{1}{x} & v = x^2+x \end{cases} \\ &= \ln x(x^2+x) - \int (x+1) dx \\ &= \ln x(x^2+x) - \left(\frac{1}{2}x^2 + x \right) + c \\ &= \ln x(x^2+x) - \frac{1}{2}x^2 - x + c \end{aligned}$$

$$\begin{aligned} \text{c } \int x 2^x dx &= \frac{x 2^x}{\ln 2} - \int \frac{2^x}{\ln 2} dx \quad \begin{cases} u = x & v' = 2^x \\ u' = 1 & v = \frac{2^x}{\ln 2} \end{cases} \\ &= \frac{x 2^x}{\ln 2} - \frac{1}{\ln 2} \int 2^x dx \\ &= \frac{x 2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \sin 2x \cos 3x \, dx \\
 &= \frac{1}{3} \sin 2x \sin 3x - \int (2 \cos 2x) \left(\frac{1}{3} \sin 3x \right) dx \quad \begin{cases} u = \sin 2x & v' = \cos 3x \\ u' = 2 \cos 2x & v = \frac{1}{3} \sin 3x \end{cases} \\
 &= \frac{1}{3} \sin 2x \sin 3x - \frac{2}{3} \int \cos 2x \sin 3x \, dx \\
 &= \frac{1}{3} \sin 2x \sin 3x - \frac{2}{3} \left[-\frac{1}{3} \cos 2x \cos 3x - \int (-2 \sin 2x) \left(-\frac{1}{3} \cos 3x \right) dx \right] \quad \begin{cases} u = \cos 2x & v' = \sin 3x \\ u' = -2 \sin 2x & v = -\frac{1}{3} \cos 3x \end{cases} \\
 &= \frac{1}{3} \sin 2x \sin 3x + \frac{2}{9} \cos 2x \cos 3x + \frac{4}{9} \int \sin 2x \cos 3x \, dx \\
 \therefore \quad & \frac{5}{9} \int \sin 2x \cos 3x \, dx = \frac{1}{3} \sin 2x \sin 3x + \frac{2}{9} \cos 2x \cos 3x + c \\
 \therefore \quad & \int \sin 2x \cos 3x \, dx = \frac{3}{5} \sin 2x \sin 3x + \frac{2}{5} \cos 2x \cos 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{102} \quad \mathbf{a} \quad & \int e^x \sin 2x \, dx \\
 &= -\frac{1}{2} e^x \cos 2x - \int -\frac{1}{2} e^x \cos 2x \, dx \quad \begin{cases} u = e^x & v' = \sin 2x \\ u' = e^x & v = -\frac{1}{2} \cos 2x \end{cases} \\
 &= -\frac{1}{2} e^x \cos 2x + \frac{1}{2} \int e^x \cos 2x \, dx \\
 &= -\frac{1}{2} e^x \cos 2x + \frac{1}{2} \left[\frac{1}{2} e^x \sin 2x - \int \frac{1}{2} e^x \sin 2x \, dx \right] \quad \begin{cases} u = e^x & v' = \cos 2x \\ u' = e^x & v = \frac{1}{2} \sin 2x \end{cases} \\
 &= -\frac{1}{2} e^x \cos 2x + \frac{1}{4} e^x \sin 2x - \frac{1}{4} \int e^x \sin 2x \, dx \\
 \therefore \quad & \frac{5}{4} \int e^x \sin 2x \, dx = -\frac{1}{2} e^x \cos 2x + \frac{1}{4} e^x \sin 2x + c \\
 \therefore \quad & \int e^x \sin 2x \, dx = -\frac{2}{5} e^x \cos 2x + \frac{1}{5} e^x \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \arccos x \, dx = \int 1 \times \arccos x \, dx \\
 &= x \arccos x - \int \frac{-x}{\sqrt{1-x^2}} \, dx \quad \begin{cases} u = \arccos x & v' = 1 \\ u' = \frac{-1}{\sqrt{1-x^2}} & v = x \end{cases} \\
 &= x \arccos x - \int \frac{1}{\sqrt{u}} \left(\frac{1}{2} \frac{du}{dx} \right) dx \quad \left\{ u = 1 - x^2, \frac{du}{dx} = -2x \right\} \\
 &= x \arccos x - \frac{1}{2} \int u^{-\frac{1}{2}} \, du \\
 &= x \arccos x - \frac{1}{2} (2u^{\frac{1}{2}}) + c \\
 &= x \arccos x - \sqrt{1-x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int x \tan^2 x \, dx = \int x (\sec^2 x - 1) \, dx \\
 &= \int x \sec^2 x \, dx - \int x \, dx \\
 &= x \tan x - \int \tan x \, dx - \int x \, dx \quad \begin{cases} u = x & v' = \sec^2 x \\ u' = 1 & v = \tan x \end{cases} \\
 &= x \tan x - \int \frac{\sin x}{\cos x} \, dx - \int x \, dx \\
 &= x \tan x + \ln |\cos x| - \frac{1}{2} x^2 + c
 \end{aligned}$$

$$\begin{aligned}
\mathbf{103} \quad \int x \arctan x \, dx &= \frac{1}{2}x^2 \arctan x - \int \frac{\frac{1}{2}x^2}{1+x^2} \, dx \quad \begin{cases} u = \arctan x & v' = x \\ u' = \frac{1}{1+x^2} & v = \frac{1}{2}x^2 \end{cases} \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} \, dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) \, dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}(x - \arctan x) + c \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2} \arctan x + c
\end{aligned}$$

$$\begin{aligned}
\text{Check: } \frac{d}{dx} \left(\frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2} \arctan x + c \right) &= x \arctan x + \frac{1}{2} \frac{x^2}{1+x^2} - \frac{1}{2} + \frac{1}{2} \frac{1}{1+x^2} \\
&= x \arctan x + \frac{x^2 - (1+x^2) + 1}{2(1+x^2)} \\
&= x \arctan x + \frac{x^2 - 1 - x^2 + 1}{2(1+x^2)} \\
&= x \arctan x + \frac{0}{2(1+x^2)} \\
&= x \arctan x \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
\mathbf{104} \quad \mathbf{a} \quad \frac{d}{dx}(\sec x + \tan x) &= \sec x \tan x + \sec^2 x \\
&= \sec x(\tan x + \sec x) \\
\therefore \sec x &= \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x} \\
\therefore \int \sec x \, dx &= \int \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x} \, dx \\
&= \ln |\sec x + \tan x| + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \int \sec^3 x \, dx &= \int \sec^2 x \sec x \, dx \\
&= \sec x \tan x - \int \sec x \tan x \tan x \, dx \quad \begin{cases} u = \sec x & v' = \sec^2 x \\ u' = \sec x \tan x & v = \tan x \end{cases} \\
&= \sec x \tan x - \int \frac{\sin^2 x}{\cos^3 x} \, dx \\
&= \sec x \tan x - \int \frac{1 - \cos^2 x}{\cos^3 x} \, dx \\
&= \sec x \tan x - \int \left(\frac{1}{\cos^3 x} - \frac{1}{\cos x} \right) \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x| \quad \{\text{using } \mathbf{a}\}
\end{aligned}$$

$$\therefore 2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + c$$

$$\therefore \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + c$$

$$105 \quad \int_a^{2a} \sqrt{x} dx = 2$$

$$\therefore \int_a^{2a} x^{\frac{1}{2}} dx = 2$$

$$\therefore \left[\frac{2}{3} x^{\frac{3}{2}} \right]_a^{2a} = 2$$

$$\therefore \frac{2}{3}(2a)^{\frac{3}{2}} - \frac{2}{3}a^{\frac{3}{2}} = 2$$

$$\therefore 2^{\frac{3}{2}} \times a^{\frac{3}{2}} - a^{\frac{3}{2}} = 3$$

$$\therefore a^{\frac{3}{2}}(2^{\frac{3}{2}} - 1) = 3$$

$$\therefore a^{\frac{3}{2}} = \frac{3}{2^{\frac{3}{2}} - 1}$$

$$\therefore a = \left(\frac{3}{2^{\frac{3}{2}} - 1} \right)^{\frac{2}{3}}$$

$$106 \quad y = x\sqrt{4-x} = x(4-x)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = (1)(4-x)^{\frac{1}{2}} + x\left(\frac{1}{2}(4-x)^{-\frac{1}{2}}(-1)\right) \quad \{\text{product rule}\}$$

$$= \sqrt{4-x} - \frac{x}{2\sqrt{4-x}}$$

$$= \frac{2(4-x) - x}{2\sqrt{4-x}}$$

$$= \frac{8-2x-x}{2\sqrt{4-x}}$$

$$= \frac{8-3x}{2\sqrt{4-x}} \quad \dots (*)$$

$$\therefore \int_0^2 \frac{8-3x}{\sqrt{4-x}} dx = [x\sqrt{4-x}]_0^2 \quad \{\text{using (*)}\}$$

$$= 2\sqrt{4-2} - 0\sqrt{4-0}$$

$$= 2\sqrt{2}$$

$$107 \quad \begin{aligned} \mathbf{a} \quad & \int_1^5 \frac{2x^3+1}{x^2} dx \\ &= \int_1^5 (2x + x^{-2}) dx \\ &= \left[x^2 - \frac{1}{x} \right]_1^5 \\ &= \left(25 - \frac{1}{5} \right) - (1 - 1) \\ &= \frac{124}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int_{-1}^1 e^x(2-3e^{-x})^2 dx \\ &= \int_{-1}^1 e^x(4-12e^{-x}+9e^{-2x}) dx \\ &= \int_{-1}^1 (4e^x-12+9e^{-x}) dx \\ &= [4e^x-12x-9e^{-x}]_{-1}^1 \\ &= (4e^1-12-9e^{-1}) - (4e^{-1}+12-9e^1) \\ &= 13e-24-13e^{-1} \\ &= 13\left(e-\frac{1}{e}\right)-24 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int_0^2 \frac{3}{5-2x} dx \\ &= \left[\frac{3}{-2} \ln|5-2x| \right]_0^2 \\ &= -\frac{3}{2} \ln|5-4| + \frac{3}{2} \ln|5-0| \\ &= -\frac{3}{2} \ln 1 + \frac{3}{2} \ln 5 \\ &= \frac{3}{2} \ln 5 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int_{-2}^{\frac{\pi}{4}-2} \sin^2(x+2) dx \\ &= \int_{-2}^{\frac{\pi}{4}-2} \left(\frac{1}{2} - \frac{1}{2} \cos(2(x+2)) \right) dx \\ &= \int_{-2}^{\frac{\pi}{4}-2} \left(\frac{1}{2} - \frac{1}{2} \cos(2x+4) \right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{4} \sin(2x+4) \right]_{-2}^{\frac{\pi}{4}-2} \\ &= \left(\frac{1}{2} \left(\frac{\pi}{4} - 2 \right) - \frac{1}{4} \sin \left(2 \left(\frac{\pi}{4} - 2 \right) + 4 \right) \right) - \left(\frac{-2}{2} - \frac{1}{4} \sin(-4+4) \right) \\ &= \frac{\pi}{8} - 1 - \frac{1}{4} \sin \left(\frac{\pi}{2} - 4 + 4 \right) + 1 + \frac{1}{4} \sin 0 \\ &= \frac{\pi}{8} - \frac{1}{4} \sin \frac{\pi}{2} \\ &= \frac{\pi}{8} - \frac{1}{4} \end{aligned}$$

e Let $u = x^2 + 3x + 4 \quad \therefore \frac{du}{dx} = 2x + 3$

When $x = 0$, $u = 4$

When $x = 1$, $u = 8$

$$\begin{aligned} \therefore \int_0^1 (2x+3)(x^2+3x+4)^3 dx &= \int_4^8 u^3 \frac{du}{dx} du \\ &= \int_4^8 u^3 du \\ &= \left[\frac{1}{4} u^4 \right]_4^8 \\ &= \frac{1}{4} (8)^4 - \frac{1}{4} (4)^4 \\ &= 1024 - 64 \\ &= 960 \end{aligned}$$

f Let $u = x^2 + 1 \quad \therefore \frac{du}{dx} = 2x$

When $x = -2$, $u = 5$

When $x = 2$, $u = 5$

$$\begin{aligned} \therefore \int_{-2}^2 8xe^{x^2+1} dx &= \int_5^5 e^u \left(4 \frac{du}{dx} \right) dx \\ &= \int_5^5 4e^u du \\ &= 0 \quad \left\{ \int_a^a f(x) dx = 0 \right\} \end{aligned}$$

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$$\begin{aligned} \int_0^a \frac{x}{x^2+1} dx &= 3 \\ \therefore \frac{1}{2} \int_0^a \frac{2x}{x^2+1} dx &= 3 \\ \therefore [\ln |x^2+1|]_0^a &= 6 \\ \therefore \ln |a^2+1| - \ln |0^2+1| &= 6 \\ \therefore \ln |a^2+1| - \ln 1 &= 6 \\ \therefore \ln |a^2+1| - 0 &= 6 \\ \therefore \ln(a^2+1) &= 6 \quad \{a^2+1 > 0\} \\ \therefore a^2+1 &= e^6 \\ \therefore a^2 &= e^6 - 1 \\ \therefore a &= \sqrt{e^6 - 1} \quad \{a > 0\} \end{aligned}$$

109 a $\int_2^3 3^x dx = \left[\frac{3^x}{\ln 3} \right]_2^3$

$$\begin{aligned} &= \frac{3^3}{\ln 3} - \frac{3^2}{\ln 3} \\ &= \frac{27-9}{\ln 3} \\ &= \frac{18}{\ln 3} \end{aligned}$$

b $\int_{-\sqrt{3}}^1 \frac{5}{1+x^2} dx = [5 \arctan x]_{-\sqrt{3}}^1$

$$\begin{aligned} &= 5 \arctan 1 - 5 \arctan(-\sqrt{3}) \\ &= 5\left(\frac{\pi}{4}\right) - 5\left(-\frac{\pi}{3}\right) \\ &= \frac{5\pi}{4} + \frac{5\pi}{3} \\ &= \frac{35\pi}{12} \end{aligned}$$

c Let $x = 4 \sin \theta \quad \therefore \frac{dx}{d\theta} = 4 \cos \theta$

So, $\theta = \arcsin \frac{x}{4}$

When $x = 2$, $\theta = \arcsin \frac{1}{2} = \frac{\pi}{6}$

When $x = 2\sqrt{3}$, $\theta = \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$$\begin{aligned} \therefore \int_2^{2\sqrt{3}} \frac{1}{\sqrt{16-x^2}} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4 \cos \theta}{\sqrt{16-16 \sin^2 \theta}} d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4 \cos \theta}{\sqrt{16} \sqrt{1-\sin^2 \theta}} d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos \theta}{\sqrt{\cos^2 \theta}} d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos \theta}{\cos \theta} d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 d\theta \\ &= \left[\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{\pi}{3} - \frac{\pi}{6} \\ &= \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned}
 \mathbf{110} \quad \mathbf{a} \quad \sin(2 \arcsin x) &= 2 \sin(\arcsin x) \cos(\arcsin x) \\
 &= 2x \sqrt{1 - \sin^2(\arcsin x)} \\
 &= 2x \sqrt{1 - x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } u &= 1 - x^2 \quad \therefore \frac{du}{dx} = -2x \\
 \text{When } x &= 0, u = 1 \\
 \text{When } x &= 1, u = 0
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_0^1 \sin(2 \arcsin x) dx &= \int_0^1 2x \sqrt{1 - x^2} dx \quad \{\text{from } \mathbf{a}\} \\
 &= \int_1^0 \sqrt{u} \left(-\frac{du}{dx}\right) dx \\
 &= - \int_1^0 u^{\frac{1}{2}} du \\
 &= - \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^0 \\
 &= -(0 - \frac{2}{3}) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{111} \quad \mathbf{a} \quad \text{Let } u &= x^2 - 8 \quad \therefore \frac{du}{dx} = 2x \\
 \text{When } x &= 3, u = 1 \\
 \text{When } x &= 5, u = 17 \\
 \therefore \int_3^5 \frac{x}{x^2 - 8} dx &= \int_1^{17} \frac{1}{u} \left(\frac{1}{2} \frac{du}{dx}\right) dx \\
 &= \frac{1}{2} \int_1^{17} \frac{1}{u} du \\
 &= \frac{1}{2} [\ln |u|]_1^{17} \\
 &= \frac{1}{2} (\ln 17 - \ln 1) \\
 &= \frac{1}{2} \ln 17
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{Let } u &= \sqrt{x} \quad \therefore \frac{du}{dx} = \frac{1}{2\sqrt{x}} \\
 \text{When } x &= 1, u = 1 \\
 \text{When } x &= 4, u = 2 \\
 \therefore \int_1^4 \frac{3e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_1^2 3e^u \left(2 \frac{du}{dx}\right) dx \\
 &= 6 \int_1^2 e^u du \\
 &= 6[e^u]_1^2 \\
 &= 6(e^2 - e) \\
 &= 6e(e - 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{112} \quad \mathbf{a} \quad \text{Let } u &= 1 + \cos x \quad \therefore \frac{du}{dx} = -\sin x \\
 \text{When } x &= 0, u = 2 \\
 \text{When } x &= \frac{\pi}{2}, u = 1 \\
 \therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} dx &= \int_2^1 \frac{1}{u} \left(-\frac{du}{dx}\right) dx \\
 &= - \int_2^1 \frac{1}{u} du \\
 &= \int_1^2 \frac{1}{u} du \\
 &= [\ln |u|]_1^2 \\
 &= \ln 2 - \ln 1 \\
 &= \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } u &= x^2 + 1 \quad \therefore \frac{du}{dx} = 2x \\
 \text{When } x &= 0, u = 1 \\
 \text{When } x &= 2, u = 5 \\
 \therefore \int_0^2 x \sqrt{x^2 + 1} dx &= \int_1^5 \sqrt{u} \left(\frac{1}{2} \frac{du}{dx}\right) dx \\
 &= \frac{1}{2} \int_1^5 u^{\frac{1}{2}} du \\
 &= \frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^5 \\
 &= \frac{1}{2} \left[\frac{2}{3} (5)^{\frac{3}{2}} - \frac{2}{3} (1)^{\frac{3}{2}} \right] \\
 &= \frac{5\sqrt{5}}{3} - \frac{1}{3} \\
 &= \frac{5\sqrt{5} - 1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } u &= \cos x \quad \therefore \frac{du}{dx} = -\sin x \\
 \text{When } x &= 0, u = 1 \\
 \text{When } x &= \frac{\pi}{3}, u = \frac{1}{2} \\
 \therefore \int_0^{\frac{\pi}{3}} \cos^3 x \sin x dx &= \int_1^{\frac{1}{2}} u^3 \left(-\frac{du}{dx}\right) dx \\
 &= - \int_1^{\frac{1}{2}} u^3 du \\
 &= - \left[\frac{1}{4} u^4 \right]_1^{\frac{1}{2}} \\
 &= - \left(\frac{1}{4} \left(\frac{1}{2}\right)^4 - \frac{1}{4} (1)^4 \right) \\
 &= - \left(\frac{1}{64} - \frac{1}{4} \right) \\
 &= \frac{15}{64}
 \end{aligned}$$

c Let $u = \sin x \quad \therefore \frac{du}{dx} = \cos x$

When $x = 0$, $u = 0$

When $x = \arcsin \frac{\pi}{8}$, $u = \frac{\pi}{8}$

$$\begin{aligned} \therefore \int_0^{\arcsin \frac{\pi}{8}} \sin^2(\sin x) \cos x \, dx &= \int_0^{\frac{\pi}{8}} \sin^2 u \frac{du}{dx} \, dx \\ &= \int_0^{\frac{\pi}{8}} \sin^2 u \, du \\ &= \int_0^{\frac{\pi}{8}} \left(\frac{1}{2} - \frac{1}{2} \cos 2u \right) du \\ &= \left[\frac{1}{2}u - \frac{1}{4} \sin 2u \right]_0^{\frac{\pi}{8}} \\ &= \left[\left(\frac{\pi}{16} - \frac{1}{4} \sin \frac{\pi}{4} \right) - 0 \right] \\ &= \frac{\pi}{16} - \frac{1}{4\sqrt{2}} \end{aligned}$$

113 a Let $x = 2 \sin \theta \quad \therefore \frac{dx}{d\theta} = 2 \cos \theta$

When $x = 1$, $\theta = \frac{\pi}{6}$

When $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$$\begin{aligned} \therefore \int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} \, dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2 \cos \theta}{\sqrt{4-4 \sin^2 \theta}} \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2 \cos \theta}{\sqrt{4} \sqrt{1-\sin^2 \theta}} \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos \theta}{\sqrt{\cos^2 \theta}} \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos \theta}{\cos \theta} \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 \, d\theta \\ &= \left[\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{\pi}{3} - \frac{\pi}{6} \\ &= \frac{\pi}{6} \end{aligned}$$

b Let $u = x + 3 \quad \therefore \frac{du}{dx} = 1$

When $x = 1$, $u = 4$

When $x = 6$, $u = 9$

$$\begin{aligned} \therefore \int_1^6 x \sqrt{x+3} \, dx &= \int_4^9 (u-3) \sqrt{u} \frac{du}{dx} \, dx \\ &= \int_4^9 (u-3) u^{\frac{1}{2}} \, du \\ &= \int_4^9 \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}} \right) \, du \\ &= \left[\frac{2}{5} u^{\frac{5}{2}} - 2u^{\frac{3}{2}} \right]_4^9 \\ &= \left(\frac{2}{5} (9)^{\frac{5}{2}} - 2(9)^{\frac{3}{2}} \right) - \left(\frac{2}{5} (4)^{\frac{5}{2}} - 2(4)^{\frac{3}{2}} \right) \\ &= \left(\frac{486}{5} - 54 \right) - \left(\frac{64}{5} - 16 \right) \\ &= \frac{216}{5} - \left(-\frac{16}{5} \right) \\ &= \frac{232}{5} \end{aligned}$$

114 a $\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \left(\frac{1}{x} \right) \left(\frac{1}{2} x^2 \right) \, dx \quad \begin{cases} u = \ln x & v' = x \\ u' = \frac{1}{x} & v = \frac{1}{2} x^2 \end{cases}$

$$\begin{aligned} &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \left(\frac{1}{2} x^2 \right) + c \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + c \end{aligned}$$

$$\begin{aligned} \therefore \int_1^e x \ln x \, dx &= \left[\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right]_1^e \\ &= \left(\frac{1}{2} e^2 \ln e - \frac{1}{4} e^2 \right) - \left(\frac{1}{2} \ln 1 - \frac{1}{4} \right) \\ &= \frac{1}{2} e^2 - \frac{1}{4} e^2 + \frac{1}{4} \\ &= \frac{1}{4} (e^2 + 1) \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int x^2 \sin x \, dx &= -x^2 \cos x - \int -2x \cos x \, dx && \begin{cases} u = x^2 & v' = \sin x \\ u' = 2x & v = -\cos x \end{cases} \\
 &= -x^2 \cos x + 2 \int x \cos x \, dx \\
 &= -x^2 \cos x + 2 \left[x \sin x - \int \sin x \, dx \right] && \begin{cases} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{cases} \\
 &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c \\
 &= -x^2 \cos x + 2x \sin x + 2 \cos x + c \\
 \therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x^2 \sin x \, dx &= \left[-x^2 \cos x + 2x \sin x + 2 \cos x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= \left(-\frac{\pi^2}{9} \cos \frac{\pi}{3} + 2\left(\frac{\pi}{3}\right) \sin \frac{\pi}{3} + 2 \cos \frac{\pi}{3} \right) - \left(-\frac{\pi^2}{36} \cos \frac{\pi}{6} + 2\left(\frac{\pi}{6}\right) \sin \frac{\pi}{6} + 2 \cos \frac{\pi}{6} \right) \\
 &= \left(-\frac{\pi^2}{18} + \frac{\pi\sqrt{3}}{3} + 1 \right) - \left(-\frac{\pi^2\sqrt{3}}{72} + \frac{\pi}{6} + \sqrt{3} \right) \\
 &= -\frac{\pi^2}{18} + \frac{\pi\sqrt{3}}{3} + 1 + \frac{\pi^2\sqrt{3}}{72} - \frac{\pi}{6} - \sqrt{3} \\
 &= \frac{(\sqrt{3}-4)\pi^2}{72} + \frac{(2\sqrt{3}-1)\pi}{6} + 1 - \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int e^{2x} \cos 3x \, dx &= \frac{1}{3} e^{2x} \sin 3x - \int \frac{2}{3} e^{2x} \sin 3x \, dx && \begin{cases} u = e^{2x} & v' = \cos 3x \\ u' = 2e^{2x} & v = \frac{1}{3} \sin 3x \end{cases} \\
 &= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x \, dx \\
 &= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \left[-\frac{1}{3} e^{2x} \cos 3x - \int -\frac{2}{3} e^{2x} \cos 3x \, dx \right] && \begin{cases} u = e^{2x} & v' = \sin 3x \\ u' = 2e^{2x} & v = -\frac{1}{3} \cos 3x \end{cases} \\
 &= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cos 3x \, dx \\
 \therefore \frac{13}{9} \int e^{2x} \cos 3x \, dx &= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c \\
 \therefore \int e^{2x} \cos 3x \, dx &= \frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x + c \\
 \therefore \int_0^\pi e^{2x} \cos 3x \, dx &= \left[\frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x \right]_0^\pi \\
 &= \left(\frac{3}{13} e^{2\pi} \sin 3\pi + \frac{2}{13} e^{2\pi} \cos 3\pi \right) - \left(\frac{3}{13} e^0 \sin 0 + \frac{2}{13} e^0 \cos 0 \right) \\
 &= \left(0 - \frac{2}{13} e^{2\pi} \right) - \left(0 + \frac{2}{13} \right) \\
 &= -\frac{2}{13} (e^{2\pi} + 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{115} \quad \mathbf{a} \quad \text{Let } \frac{x+5}{(x^2+5)(1-x)} &= \frac{Ax+B}{x^2+5} + \frac{C}{x-1} \\
 \therefore x+5 &= (Ax+B)(1-x) + \frac{C(1-x)(x^2+5)}{x-1} \\
 \therefore x+5 &= (Ax+B)(1-x) - C(x^2+5)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x=1, \quad 1+5 &= -6C && \text{Substituting } x=0, \quad 5 = B - (-1)(5) \\
 \therefore C &= -1 && \therefore 5 = B + 5 \\
 &&& \therefore B = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x=-1, \quad -1+5 &= -A(1-(-1)) - (-1)(6) \\
 \therefore 4 &= -2A + 6 \\
 \therefore 2A &= 2 \\
 \therefore A &= 1
 \end{aligned}$$

$$\therefore \frac{x+5}{(x^2+5)(1-x)} = \frac{x}{x^2+5} - \frac{1}{x-1}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_2^4 \frac{x+5}{(x^2+5)(1-x)} dx &= \int_2^4 \left(\frac{x}{x^2+5} - \frac{1}{x-1} \right) dx \quad \{\text{using } \mathbf{a}\} \\
 &= \int_2^4 \left(\frac{1}{2} \frac{2x}{x^2+5} - \frac{1}{x-1} \right) dx \\
 &= \left[\frac{1}{2} \ln |x^2+5| - \ln |x-1| \right]_2^4 \\
 &= \left(\frac{1}{2} \ln 21 - \ln 3 \right) - \left(\frac{1}{2} \ln 9 - \ln 1 \right) \\
 &= \frac{1}{2} (\ln 7 + \ln 3) - \ln 3 - \ln(9^{\frac{1}{2}}) \\
 &= \frac{1}{2} \ln 7 + \frac{1}{2} \ln 3 - \ln 3 - \ln 3 \\
 &= \frac{1}{2} \ln 7 - \frac{3}{2} \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{116} \quad \mathbf{a} \quad \int_1^\infty \left(\frac{1}{x^3} - \frac{1}{x^4} \right) dx \\
 &= \lim_{b \rightarrow \infty} \int_1^b (x^{-3} - x^{-4}) dx \\
 &= \lim_{b \rightarrow \infty} \left[\frac{x^{-2}}{-2} - \frac{x^{-3}}{-3} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{3b^3} - \frac{1}{2b^2} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left[\left(\frac{1}{3b^3} - \frac{1}{2b^2} \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left(\frac{1}{3b^3} - \frac{1}{2b^2} \right) - \left(-\frac{1}{6} \right) \\
 &= 0 - 0 + \frac{1}{6} \\
 &= \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_{\frac{2}{\pi}}^\infty \frac{1}{x^2} \cos \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_{\frac{2}{\pi}}^b \frac{1}{x^2} \cos \frac{1}{x} dx \\
 \text{Let } u &= \frac{1}{x} \quad \therefore \frac{du}{dx} = -\frac{1}{x^2} \\
 \text{When } x &= \frac{2}{\pi}, \quad u = \frac{\pi}{2} \\
 \text{When } x &= b, \quad u = \frac{1}{b} \\
 \therefore \int_{\frac{2}{\pi}}^\infty \frac{1}{x^2} \cos \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_{\frac{\pi}{2}}^{\frac{1}{b}} \cos u \left(-\frac{du}{dx} \right) dx \\
 &= \lim_{b \rightarrow \infty} \int_{\frac{\pi}{2}}^{\frac{1}{b}} -\cos u \, du \\
 &= \lim_{b \rightarrow \infty} \left[-\sin u \right]_{\frac{\pi}{2}}^{\frac{1}{b}} \\
 &= \lim_{b \rightarrow \infty} \left(-\sin \frac{1}{b} - \left(-\sin \frac{\pi}{2} \right) \right) \\
 &= \lim_{b \rightarrow \infty} \left(1 - \sin \frac{1}{b} \right) \\
 &= 1 - \sin 0 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{117} \quad \mathbf{a} \quad \int e^{-x} \cos x \, dx \\
 &= e^{-x} \sin x - \int -e^{-x} \sin x \, dx \quad \begin{cases} u = e^{-x} & v' = \cos x \\ u' = -e^{-x} & v = \sin x \end{cases} \\
 &= e^{-x} \sin x + \int e^{-x} \sin x \, dx \\
 &= e^{-x} \sin x + \left(-e^{-x} \cos x - \int e^{-x} \cos x \, dx \right) \quad \begin{cases} u = e^{-x} & v' = \sin x \\ u' = -e^{-x} & v = -\cos x \end{cases} \\
 &= e^{-x} (\sin x - \cos x) - \int e^{-x} \cos x \, dx \\
 \therefore 2 \int e^{-x} \cos x \, dx &= e^{-x} (\sin x - \cos x) + c \\
 \therefore \int e^{-x} \cos x \, dx &= \frac{1}{2} e^{-x} (\sin x - \cos x) + c
 \end{aligned}$$

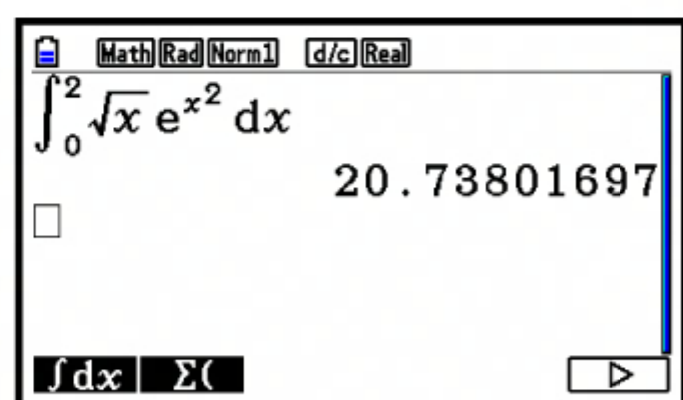
b $e^{-x} > 0$ and $\cos x + 1 \geq 0$ for all $x \geq 0$

$\therefore f(x) \geq 0$ for all $x \geq 0$.

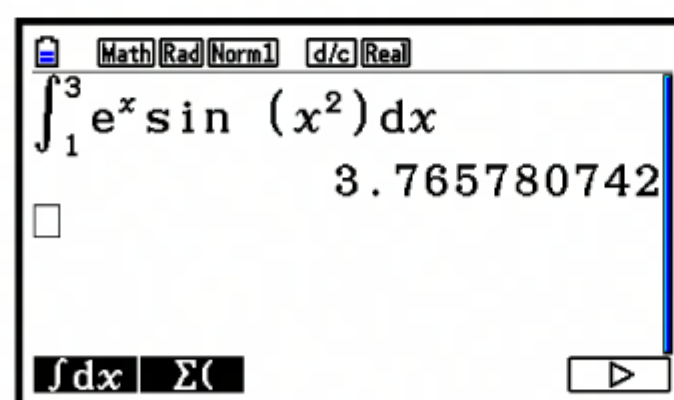
So, the area between $f(x)$ and the x -axis for $x \geq 0$ is given by

$$\begin{aligned} A &= \int_0^{\infty} e^{-x}(1 + \cos x) dx \\ &= \lim_{b \rightarrow \infty} \int_0^b (e^{-x} \cos x + e^{-x}) dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} e^{-x} (\sin x - \cos x) - e^{-x} \right]_0^b \quad \{\text{using a}\} \\ &= \lim_{b \rightarrow \infty} \left[\left(\frac{1}{2} e^{-b} (\sin b - \cos b) - e^{-b} \right) - \left(\frac{1}{2} (-1) - 1 \right) \right] \\ &= \lim_{b \rightarrow \infty} \left(\frac{1}{2} e^{-b} (\sin b - \cos b) - e^{-b} + \frac{3}{2} \right) \\ &= \frac{1}{2} (0) - 0 + \frac{3}{2} \quad \left\{ \lim_{b \rightarrow \infty} e^{-b} = 0 \right\} \\ &= \frac{3}{2} \text{ units}^2 \end{aligned}$$

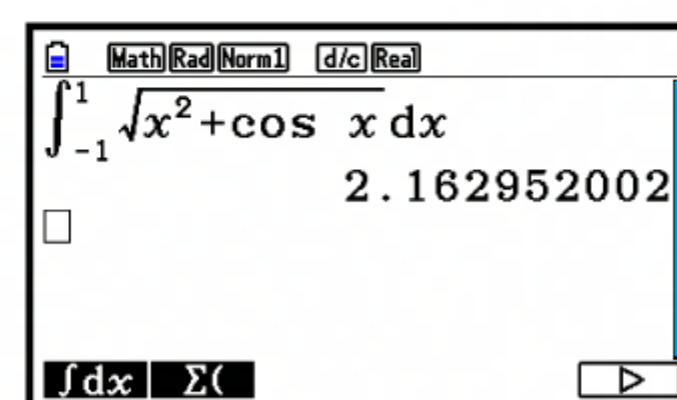
118 a $\int_0^2 \sqrt{x} e^{x^2} dx \approx 20.7$



b $\int_1^3 e^x \sin(x^2) dx \approx 3.77$

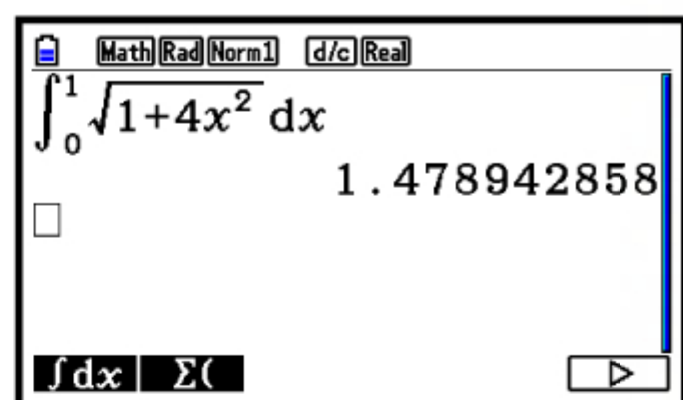


c $\int_{-1}^1 \sqrt{x^2 + \cos x} dx \approx 2.16$



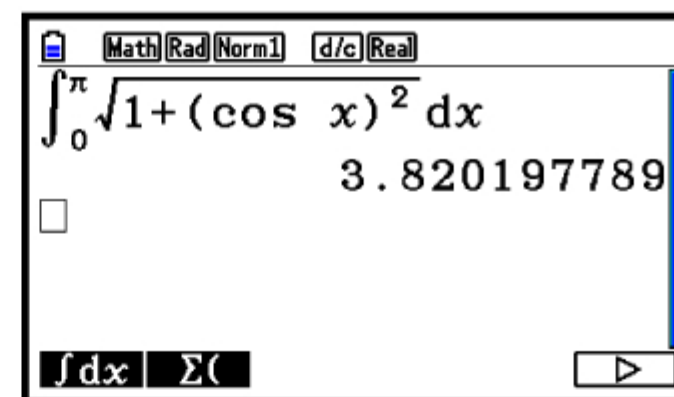
119 a $f(x) = x^2, \quad 0 \leq x \leq 1$
 $\therefore f'(x) = 2x$

$$\begin{aligned} \text{So, } L &= \int_0^1 \sqrt{1 + (2x)^2} dx \\ &= \int_0^1 \sqrt{1 + 4x^2} dx \\ &\approx 1.48 \text{ units} \end{aligned}$$



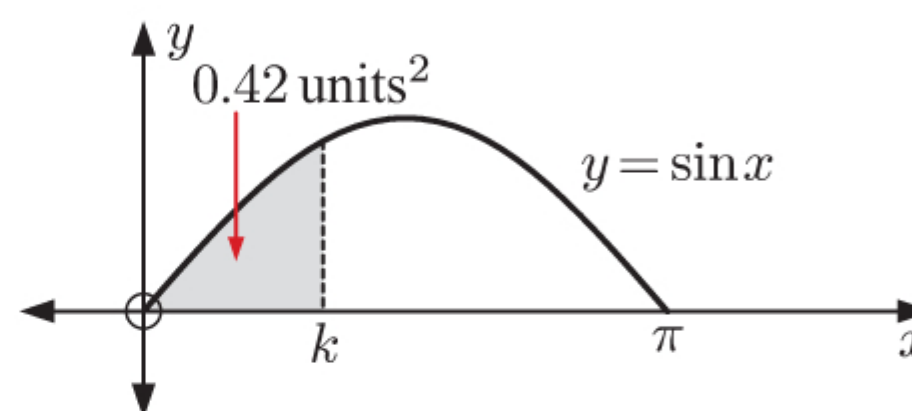
b $f(x) = \sin x, \quad 0 \leq x \leq \pi$
 $\therefore f'(x) = \cos x$

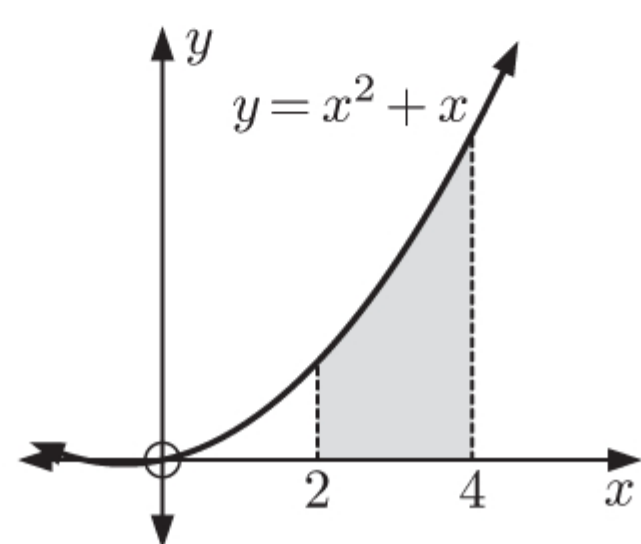
$$\begin{aligned} \text{So, } L &= \int_0^{\pi} \sqrt{1 + \cos^2 x} dx \\ &\approx 3.82 \text{ units} \end{aligned}$$



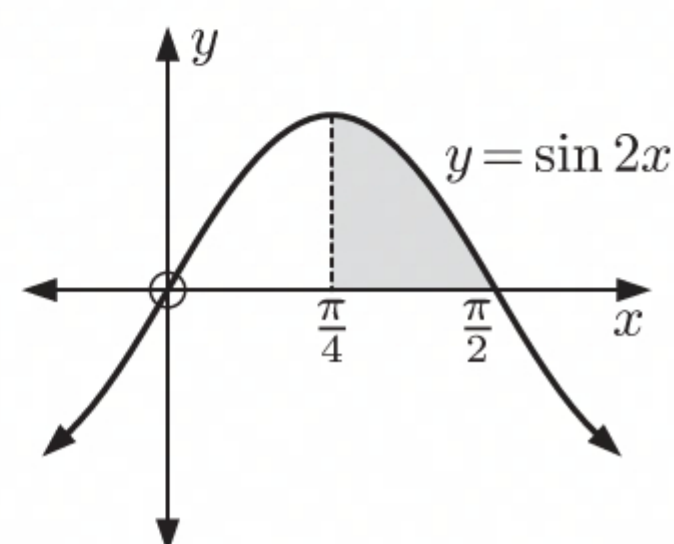
120 Shaded area = 0.42 units^2

$$\begin{aligned} \therefore \int_0^k \sin x dx &= 0.42 \\ \therefore [-\cos x]_0^k &= 0.42 \\ \therefore -\cos k + \cos 0 &= 0.42 \\ \therefore -\cos k + 1 &= 0.42 \\ \therefore -\cos k &= -0.58 \\ \therefore \cos k &= 0.58 \\ \therefore k &= \cos^{-1}(0.58) \approx 0.95 \quad \{2 \text{ d.p.}\} \end{aligned}$$

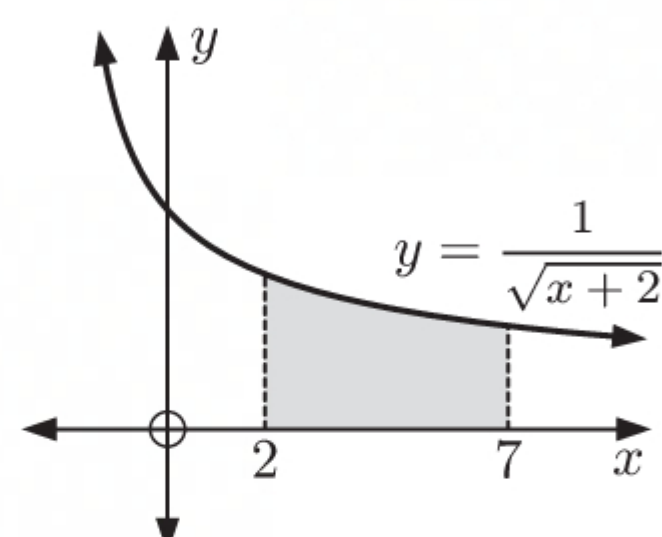


121 a

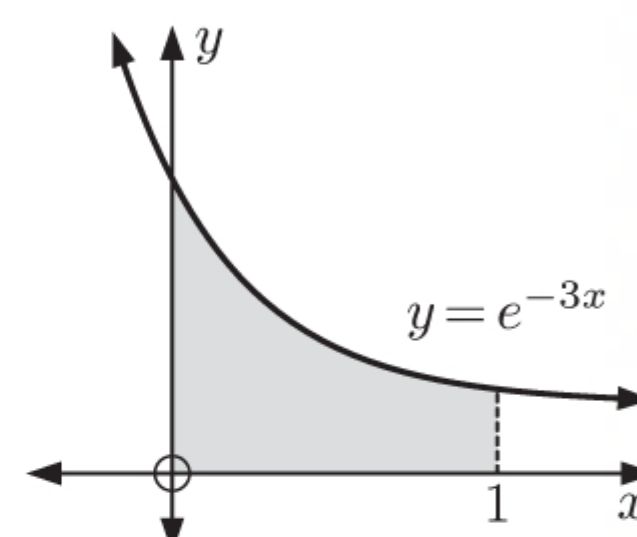
$$\begin{aligned}
 \text{Area} &= \int_2^4 (x^2 + x) dx \\
 &= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_2^4 \\
 &= \left(\frac{1}{3}(4)^3 + \frac{1}{2}(4)^2 \right) - \left(\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 \right) \\
 &= \frac{64}{3} + 8 - \frac{8}{3} - 2 \\
 &= \frac{56}{3} + 6 \\
 &= \frac{74}{3} \\
 &= 24\frac{2}{3} \text{ units}^2
 \end{aligned}$$

b

$$\begin{aligned}
 \text{Area} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2x dx \\
 &= \left[-\frac{1}{2} \cos 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= -\frac{1}{2} \cos \pi + \frac{1}{2} \cos \frac{\pi}{2} \\
 &= -\frac{1}{2}(-1) + \frac{1}{2}(0) \\
 &= \frac{1}{2} \text{ units}^2
 \end{aligned}$$

c

$$\begin{aligned}
 \text{Area} &= \int_2^7 \frac{1}{\sqrt{x+2}} dx \\
 &= \int_2^7 (x+2)^{-\frac{1}{2}} dx \\
 &= \left[2(x+2)^{\frac{1}{2}} \right]_2^7 \\
 &= 2\sqrt{7+2} - 2\sqrt{2+2} \\
 &= 2\sqrt{9} - 2\sqrt{4} \\
 &= 2(3) - 2(2) \\
 &= 2 \text{ units}^2
 \end{aligned}$$

d

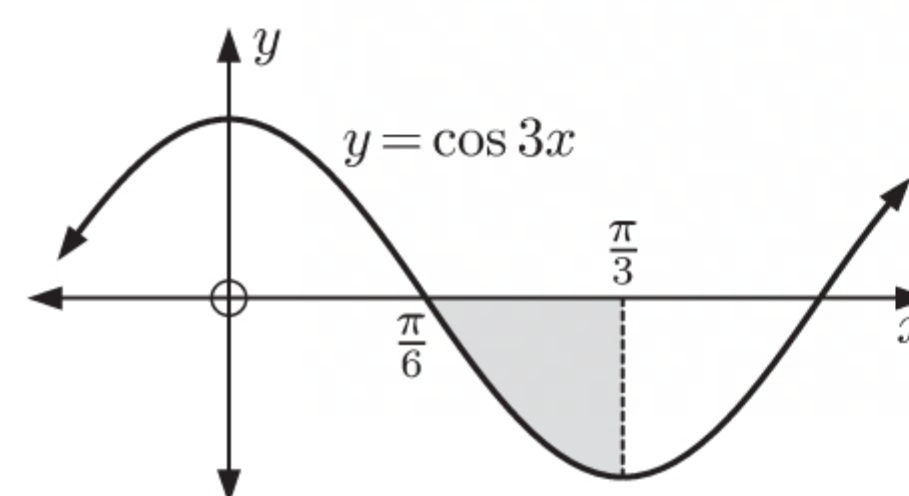
$$\begin{aligned}
 \text{Area} &= \int_0^1 e^{-3x} dx \\
 &= \left[-\frac{1}{3} e^{-3x} \right]_0^1 \\
 &= -\frac{1}{3} e^{-3} + \frac{1}{3} e^0 \\
 &= -\frac{1}{3e^3} + \frac{1}{3} \\
 &= \frac{1}{3} \left(1 - \frac{1}{e^3} \right) \text{ units}^2
 \end{aligned}$$

122 The first x -intercept of $y = \cos 3x$ is given by $\cos 3x = 0$

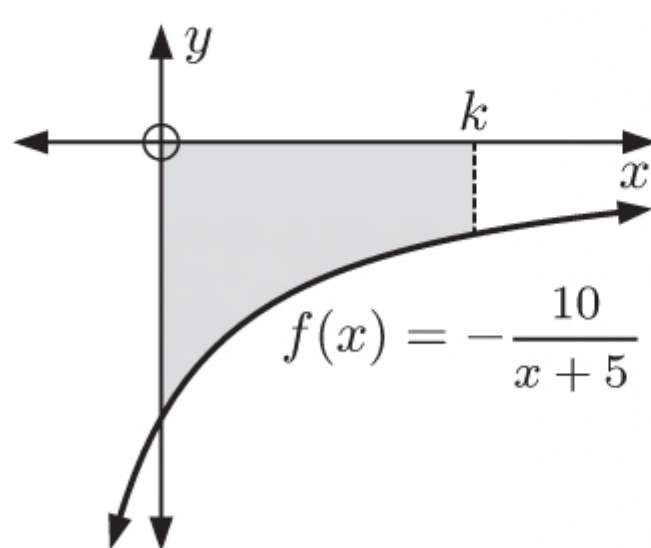
$$\therefore 3x = \frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{6}$$

$$\begin{aligned}
 \therefore \text{area} &= - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x dx \\
 &= - \left[\frac{1}{3} \sin 3x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= - \left(\frac{1}{3} \sin \pi - \frac{1}{3} \sin \frac{\pi}{2} \right) \\
 &= - \left(\frac{1}{3}(0) - \frac{1}{3}(1) \right) \\
 &= \frac{1}{3} \text{ units}^2
 \end{aligned}$$



123



$$\text{Area} = 10 \ln 3 \text{ units}^2$$

$$\begin{aligned} \therefore - \int_0^k \frac{10}{x+5} dx &= 10 \ln 3 \\ \therefore \int_0^k \frac{10}{x+5} dx &= 10 \ln 3 \\ \therefore [10 \ln |x+5|]_0^k &= 10 \ln 3 \\ \therefore 10 \ln |k+5| - 10 \ln 5 &= 10 \ln 3 \\ \therefore \ln |k+5| - \ln 5 &= \ln 3 \\ \therefore \ln \left| \frac{k+5}{5} \right| &= \ln 3 \\ \therefore \left| \frac{k+5}{5} \right| &= 3 \\ \therefore |k+5| &= 15 \\ \therefore k+5 &= \pm 15 \\ \therefore k &= 10 \text{ or } -20 \end{aligned}$$

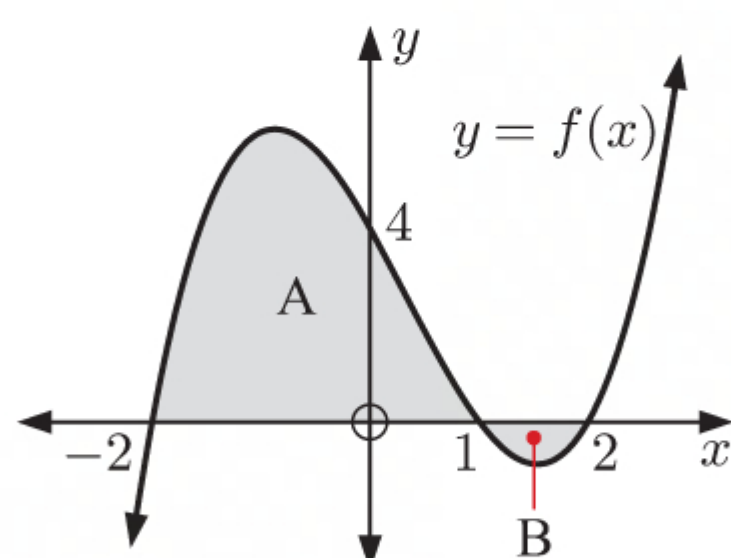
But for $k = -20$, $f(x)$ is not defined for all x such that $k \leq x \leq 0$ as $f(x)$ is not defined for $x = -5$.

So, the only possible value of k is 10.

124 $f(x) = x^3 - x^2 - 4x + 4$

$$\begin{aligned} \text{a } \int_{-2}^2 f(x) dx &= \int_{-2}^2 (x^3 - x^2 - 4x + 4) dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_{-2}^2 \\ &= \left(\frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) \right) - \left(\frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 2(-2)^2 + 4(-2) \right) \\ &= -\frac{8}{3} + 8 - \frac{8}{3} + 8 \\ &= \frac{32}{3} \end{aligned}$$

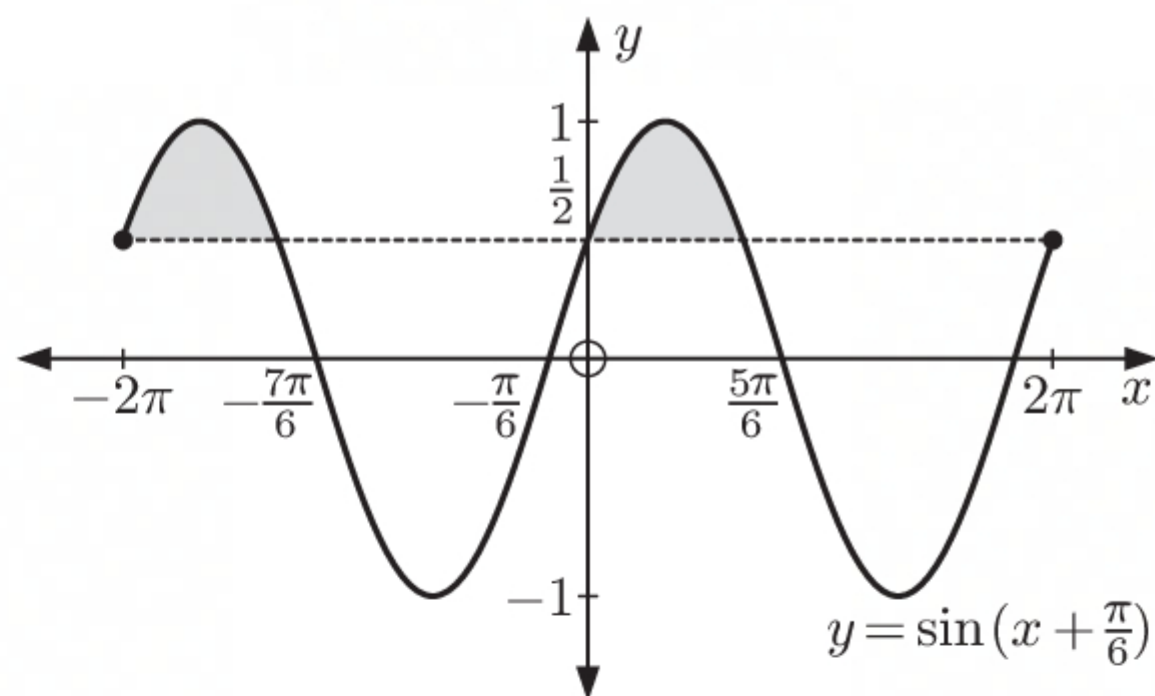
b



For $1 \leq x \leq 2$, the graph of $y = f(x)$ is below the x -axis, so $\int_1^2 f(x) dx$ is negative.

$\int_{-2}^2 f(x) dx$ is the difference of areas A and B.

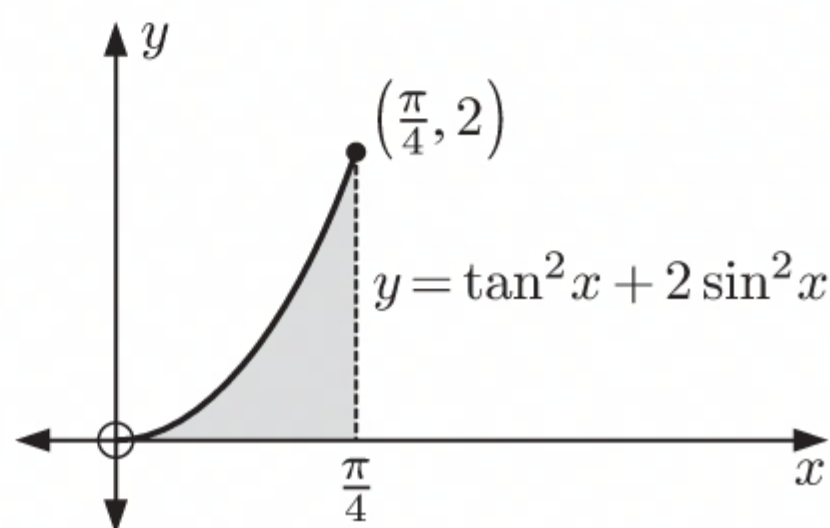
$$\begin{aligned} \text{c Shaded area} &= \int_{-2}^1 f(x) dx + \int_1^2 -f(x) dx \\ &= \int_{-2}^1 f(x) dx - \int_1^2 f(x) dx \\ &= \int_{-2}^1 (x^3 - x^2 - 4x + 4) dx - \int_1^2 (x^3 - x^2 - 4x + 4) dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_{-2}^1 - \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x \right]_1^2 \\ &= \left(\frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - 2(1)^2 + 4(1) \right) - \left(\frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 2(-2)^2 + 4(-2) \right) \\ &\quad - \left(\frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) \right) + \left(\frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - 2(1)^2 + 4(1) \right) \\ &= \left(\frac{1}{4} - \frac{1}{3} - 2 + 4 \right) - \left(\frac{16}{4} + \frac{8}{3} - 8 - 8 \right) - \left(\frac{16}{4} - \frac{8}{3} - 8 + 8 \right) + \left(\frac{1}{4} - \frac{1}{3} - 2 + 4 \right) \\ &= \left(-\frac{1}{12} + 2 \right) - (-4) - (-4) + \left(-\frac{1}{12} + 2 \right) \\ &= -\frac{1}{6} + 12 \\ &= \frac{71}{6} = 11\frac{5}{6} \text{ units}^2 \end{aligned}$$

125 a

- b** $y = \sin(x + \frac{\pi}{6})$ meets $y = \frac{1}{2}$ where $\sin(x + \frac{\pi}{6}) = \frac{1}{2}$, $-2\pi \leq x \leq 2\pi$
 $\therefore x + \frac{\pi}{6} = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$
 $\therefore x = -2\pi, -\frac{4\pi}{3}, 0, \frac{2\pi}{3}, 2\pi$

Since $\sin(x + \frac{\pi}{6}) \geq \frac{1}{2}$ on the intervals $-2\pi \leq x \leq -\frac{4\pi}{3}$ and $0 \leq x \leq \frac{2\pi}{3}$,

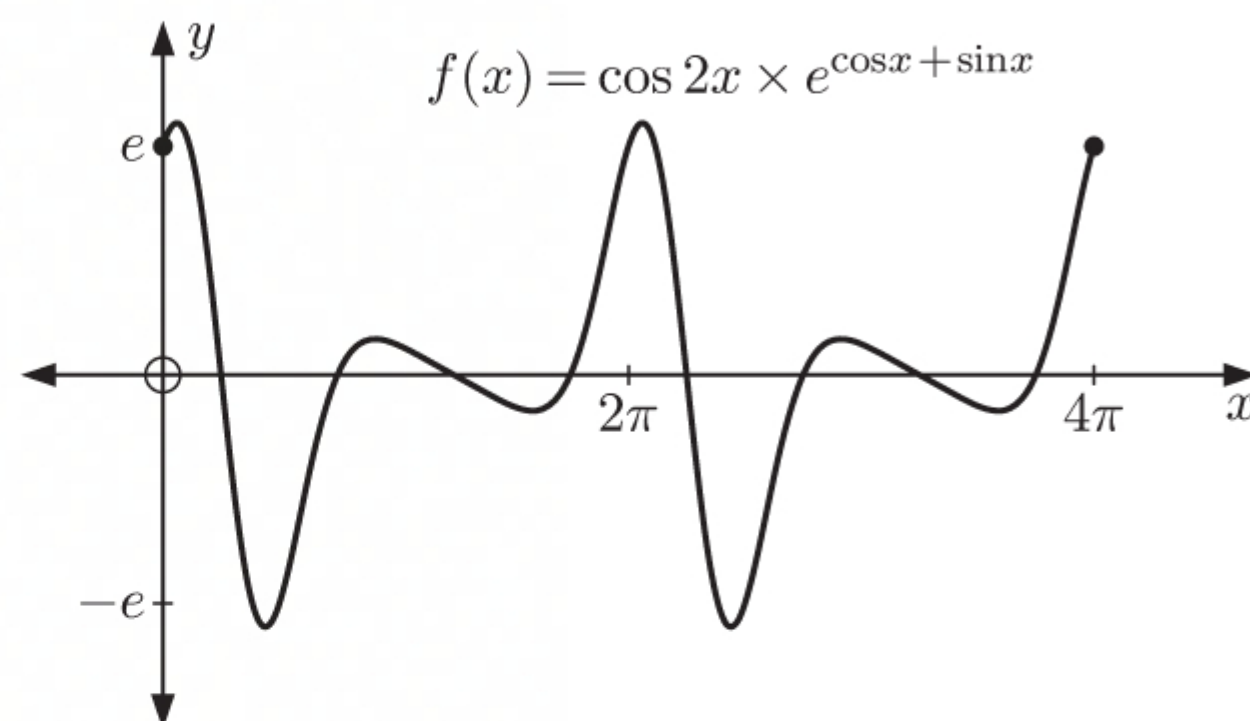
$$\begin{aligned} \text{area} &= \int_{-2\pi}^{-\frac{4\pi}{3}} (\sin(x + \frac{\pi}{6}) - \frac{1}{2}) dx + \int_0^{\frac{2\pi}{3}} (\sin(x + \frac{\pi}{6}) - \frac{1}{2}) dx \\ &= \left[-\cos(x + \frac{\pi}{6}) - \frac{1}{2}x \right]_{-2\pi}^{-\frac{4\pi}{3}} + \left[-\cos(x + \frac{\pi}{6}) - \frac{1}{2}x \right]_0^{\frac{2\pi}{3}} \\ &= (-\cos(-\frac{4\pi}{3} + \frac{\pi}{6}) - (-\frac{2\pi}{3})) - (-\cos(-2\pi + \frac{\pi}{6}) - (-\pi)) + (-\cos(\frac{2\pi}{3} + \frac{\pi}{6}) - (\frac{\pi}{3})) - (-\cos \frac{\pi}{6} - 0) \\ &= -\cos(-\frac{7\pi}{6}) + \frac{2\pi}{3} + \cos(-\frac{11\pi}{6}) - \pi - \cos \frac{5\pi}{6} - \frac{\pi}{3} + \cos \frac{\pi}{6} \\ &= -(-\frac{\sqrt{3}}{2}) + \frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \pi - (-\frac{\sqrt{3}}{2}) - \frac{\pi}{3} + \frac{\sqrt{3}}{2} \\ &= (2\sqrt{3} - \frac{2\pi}{3}) \text{ units}^2 \end{aligned}$$

126

$$\begin{aligned} \text{Shaded area} &= \int_0^{\frac{\pi}{4}} (\tan^2 x + 2 \sin^2 x) dx \\ &= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1 + 1 - \cos 2x) dx \\ &= \int_0^{\frac{\pi}{4}} (\sec^2 x - \cos 2x) dx \\ &= \left[\tan x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} \\ &= (\tan \frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2}) - (0 - 0) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \text{ units}^2 \end{aligned}$$

127 $f(x) = \cos 2x \times e^{\cos x + \sin x}$

- a** $\cos 2x$ has period $\frac{2\pi}{2} = \pi$ units.
 $\cos x$ and $\sin x$ both have period $\frac{2\pi}{1} = 2\pi$ units.
 $\therefore e^{\cos x + \sin x}$ has period 2π units.
 So, $f(x)$ has period 2π units.

b

c i Let $w = e^{\cos x + \sin x}$, so $\ln w = \cos x + \sin x$

$$\begin{aligned}\therefore \frac{dw}{dx} &= e^{\cos x + \sin x} (-\sin x + \cos x) \\ &= (\cos x - \sin x) e^{\cos x + \sin x}\end{aligned}$$

$$\begin{aligned}\text{Now } \int f(x) dx &= \int \cos 2x \times e^{\cos x + \sin x} dx \\ &= \int (\cos^2 x - \sin^2 x) e^{\cos x + \sin x} dx \\ &= \int (\cos x + \sin x)(\cos x - \sin x) e^{\cos x + \sin x} dx \\ &= \int \ln w \frac{dw}{dx} du \\ &= \int \ln w dw \\ &= w \ln w - \int 1 dw \quad \begin{cases} u = \ln w & v' = 1 \\ u' = \frac{1}{w} & v = w \end{cases} \\ &= w \ln w - w + c \\ &= w(\ln w - 1) + c \\ &= e^{\cos x + \sin x} (\cos x + \sin x - 1) + c\end{aligned}$$

ii The first positive x -intercept occurs where $f(x) = 0$

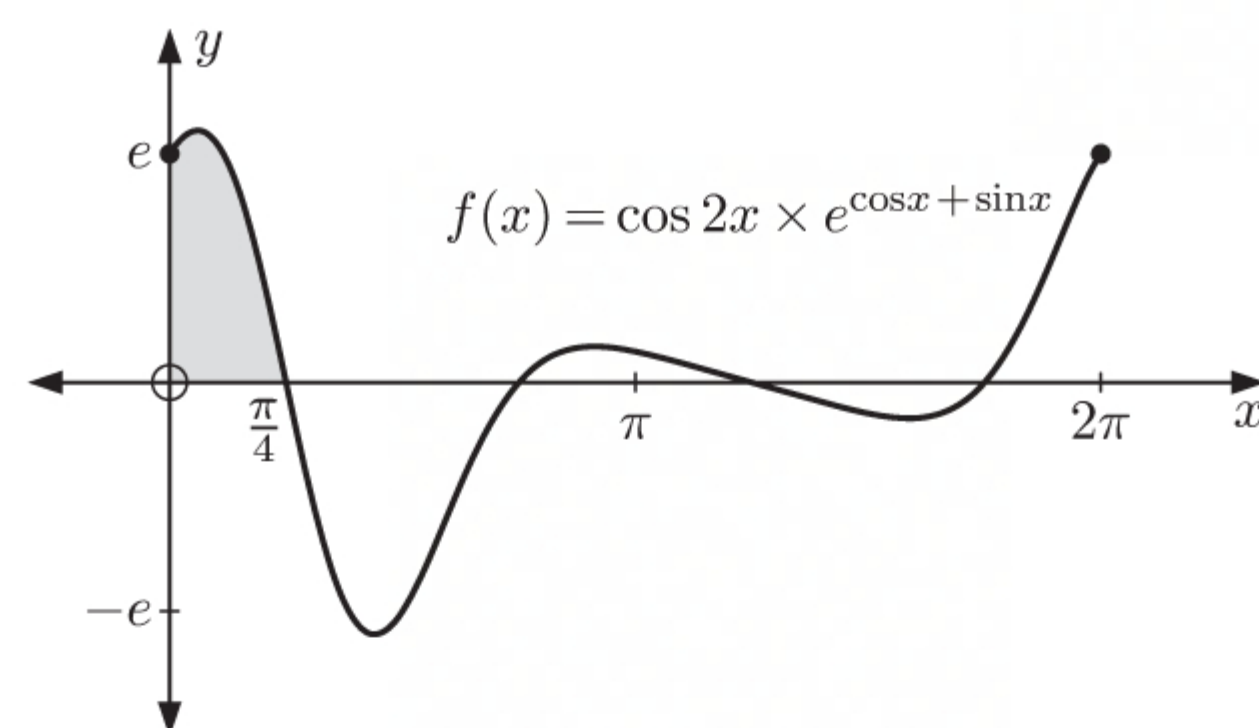
$$\therefore \cos 2x \times e^{\cos x + \sin x} = 0$$

$$\therefore \cos 2x = 0 \quad \{e^{\cos x + \sin x} > 0\}$$

$$\therefore 2x = \frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{4}$$

$$\begin{aligned}\text{d Area} &= \int_0^{\frac{\pi}{4}} \cos 2x \times e^{\cos x + \sin x} dx \\ &= \left[e^{\cos x + \sin x} (\cos x + \sin x - 1) \right]_0^{\frac{\pi}{4}} \quad \{\text{from c i}\} \\ &= e^{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right) - e^{1+0} (1 + 0 - 1) \\ &= e^{\sqrt{2}} (\sqrt{2} - 1) \text{ units}^2\end{aligned}$$



128 $f(x) = \left(2 - \frac{1}{x}\right)e^{-x}, \quad x > 0$

a $f(x) = 0$ when $\left(2 - \frac{1}{x}\right)e^{-x} = 0$

$$\therefore 2 - \frac{1}{x} = 0 \quad \{e^{-x} > 0 \text{ for all } x\}$$

$$\therefore \frac{1}{x} = 2$$

$$\therefore x = \frac{1}{2}$$

\therefore the zero of $f(x)$ is $\frac{1}{2}$.

b As $x \rightarrow 0^+$, $\frac{1}{x} \rightarrow \infty$

$$\therefore -\frac{1}{x} \rightarrow -\infty$$

$$\therefore 2 - \frac{1}{x} \rightarrow -\infty$$

$$\therefore \left(2 - \frac{1}{x}\right)e^{-x} \rightarrow -\infty \quad \{e^{-x} \approx 1 \text{ near } x = 0\}$$

$$\therefore f(x) \rightarrow -\infty$$

As $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0^+$

$$\therefore -\frac{1}{x} \rightarrow 0^-$$

$$\therefore 2 - \frac{1}{x} \rightarrow 2^-$$

$$\therefore \left(2 - \frac{1}{x}\right)e^{-x} \rightarrow 0^+$$

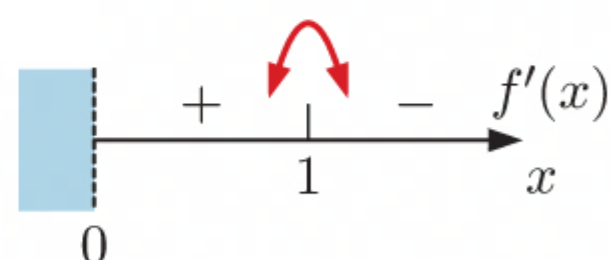
$$\therefore f(x) \rightarrow 0^+$$

$$\begin{aligned} \text{c} \quad f(x) &= \left(2 - \frac{1}{x}\right)e^{-x} \\ \therefore f'(x) &= \left(\frac{1}{x^2}\right)e^{-x} - e^{-x}\left(2 - \frac{1}{x}\right) \quad \{\text{product rule}\} \\ &= e^{-x}\left(\frac{1}{x^2} - 2 + \frac{1}{x}\right) \end{aligned}$$

Stationary point(s) occur where $f'(x) = 0$

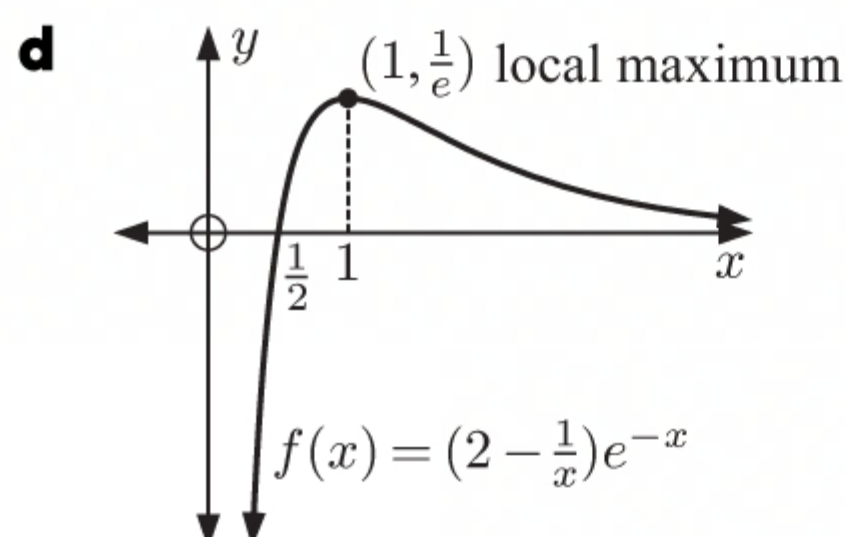
$$\begin{aligned} \therefore e^{-x}\left(\frac{1}{x^2} - 2 + \frac{1}{x}\right) &= 0 \\ \therefore \frac{1}{x^2} - 2 + \frac{1}{x} &= 0 \quad \{e^{-x} > 0 \text{ for all } x\} \\ \therefore 1 - 2x^2 + x &= 0 \\ \therefore 2x^2 - x - 1 &= 0 \\ \therefore 2x^2 - 2x + x - 1 &= 0 \\ \therefore 2x(x - 1) + (x - 1) &= 0 \\ \therefore (x - 1)(2x + 1) &= 0 \\ \therefore x = 1 \text{ or } -\frac{1}{2} \\ \therefore x = 1 \quad \{x > 0\} \end{aligned}$$

The sign diagram of $f'(x)$ is

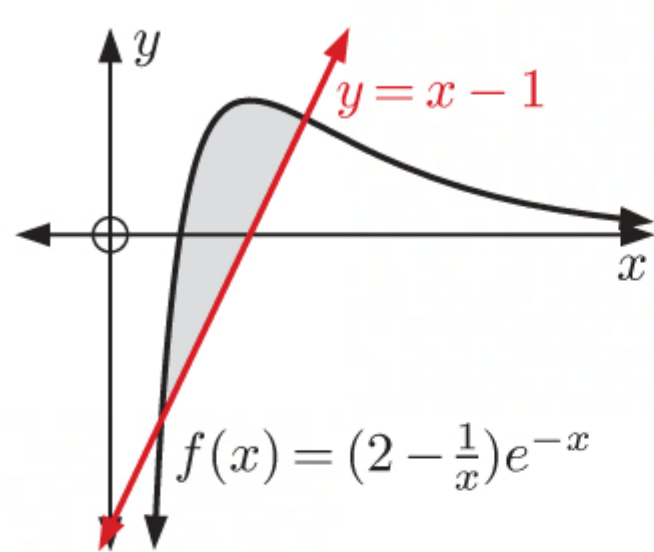


Now $f(1) = (2 - 1)e^{-1} = \frac{1}{e}$

$\therefore \left(1, \frac{1}{e}\right)$ is a local maximum.



e $y = \left(2 - \frac{1}{x}\right)e^{-x}$ meets $y = x - 1$ where $\left(2 - \frac{1}{x}\right)e^{-x} = x - 1$
 $\therefore x \approx 0.342 \text{ or } 1.33 \quad \{\text{using technology}\}$



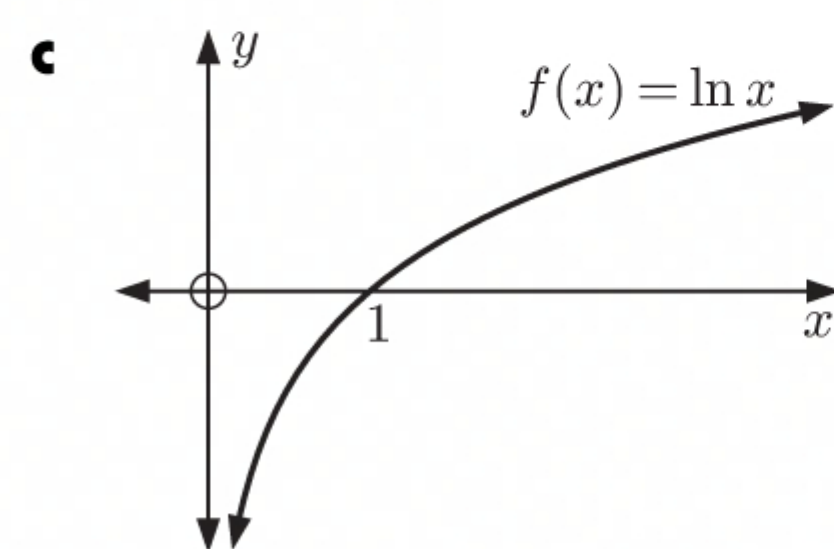
Since $\left(2 - \frac{1}{x}\right)e^{-x} \geq x - 1$ on the interval $0.342 \leq x \leq 1.33$,

$$\begin{aligned} \text{area} &\approx \int_{0.342}^{1.33} \left[\left(2 - \frac{1}{x}\right)e^{-x} - (x - 1)\right] dx \\ &\approx 0.373 \quad \{3 \text{ d.p.}\} \end{aligned}$$

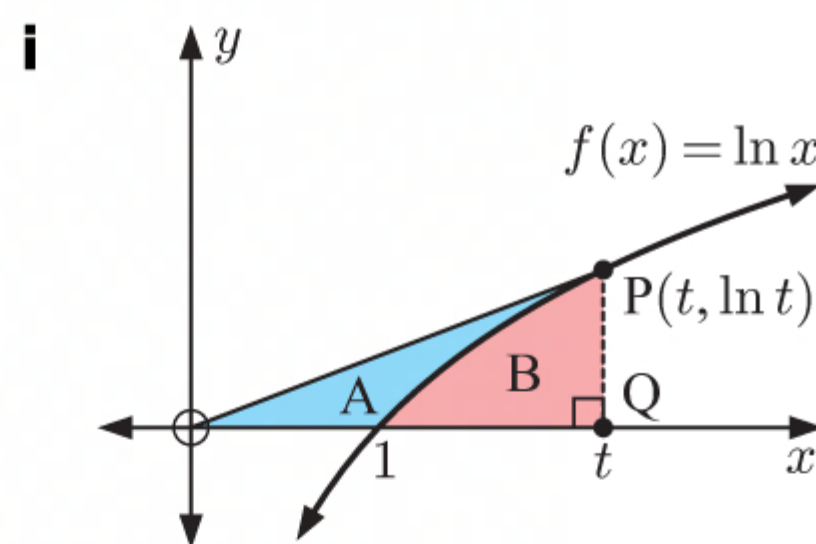
129 a i $f(x) = \ln x$
 $\therefore f'(x) = \frac{1}{x}$

ii $F(x) = x \ln x - x$
 $\therefore F'(x) = \left[(1) \ln x + x\left(\frac{1}{x}\right)\right] - 1 \quad \{\text{product rule}\}$
 $= \ln x + 1 - 1$
 $= \ln x$

b $F'(x) = f(x)$, that is, $F(x)$ is the antiderivative of $f(x)$.



d $O(0, 0)$ and $P(t, \ln t)$, $1 < t \leq e$



Area A = area of triangle – area B

$$\begin{aligned}
 &= \frac{1}{2}t \ln t - \int_1^t \ln x \, dx \\
 &= \frac{1}{2}t \ln t - [x \ln x - x]_1^t \quad \{\text{using a ii}\} \\
 &= \frac{1}{2}t \ln t - [(t \ln t - t) - (1 \ln 1 - 1)] \\
 &= \frac{1}{2}t \ln t - (t \ln t - t - 0 + 1) \\
 &= (t - \frac{1}{2}t \ln t - 1) \text{ units}^2
 \end{aligned}$$

ii The area $A = t - \frac{1}{2}t \ln t - 1$

$$\begin{aligned}
 \therefore \frac{dA}{dt} &= 1 - \frac{1}{2}(\ln t + 1) - \frac{1}{2} \ln t \quad \{\text{product rule}\} \\
 &= \frac{1}{2} - \frac{1}{2} \ln t
 \end{aligned}$$

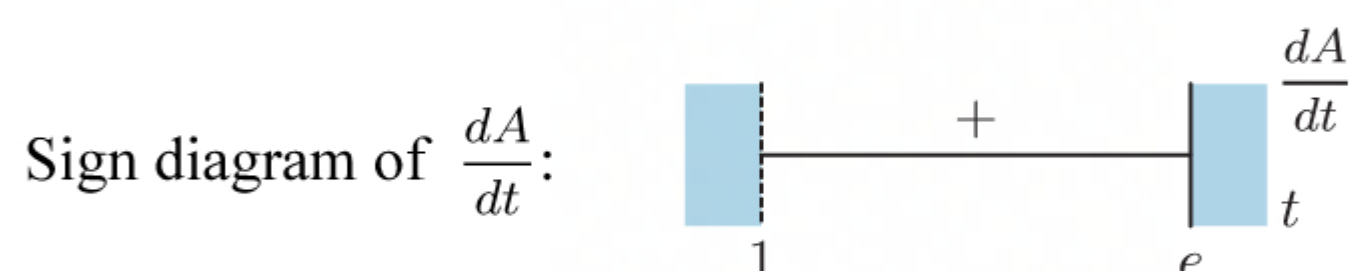
Now A is maximised when $\frac{dA}{dt} = 0$

$$\therefore \frac{1}{2} - \frac{1}{2} \ln t = 0$$

$$\therefore \frac{1}{2} \ln t = \frac{1}{2}$$

$$\therefore \ln t = 1$$

$$\therefore t = e$$



$\therefore A$ is maximised when $t = e$.

e i The tangent has equation

$$y = f'(t)(x - t) + f(t)$$

$$\therefore y = \frac{1}{t}(x - t) + \ln t$$

$$\therefore y = \frac{x}{t} - 1 + \ln t$$

ii The tangent passes through the origin when

$$-1 + \ln t = 0$$

$$\therefore \ln t = 1$$

$$\therefore t = e$$

130 a $y = x^3 - 2x^2 - 3x$ meets $y = 5x - 4x^2$ where $x^3 - 2x^2 - 3x = 5x - 4x^2$

$$\therefore x^3 + 2x^2 - 8x = 0$$

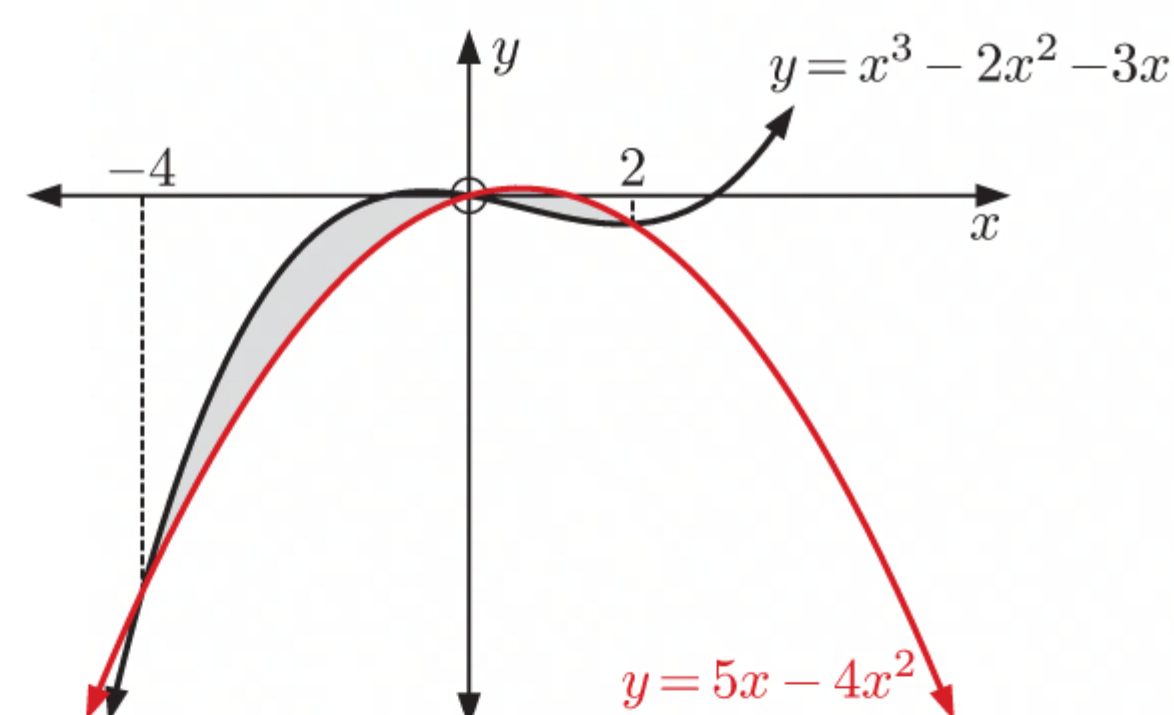
$$\therefore x(x^2 + 2x - 8) = 0$$

$$\therefore x(x + 4)(x - 2) = 0$$

$$\therefore x = -4, 0, \text{ or } 2$$

Since $x^3 - 2x^2 - 3x \geq 5x - 4x^2$ for $-4 \leq x \leq 0$, and $5x - 4x^2 \geq x^3 - 2x^2 - 3x$ for $0 \leq x \leq 2$,

$$\begin{aligned}
 \text{area} &= \int_{-4}^0 [(x^3 - 2x^2 - 3x) - (5x - 4x^2)] \, dx \\
 &\quad + \int_0^2 [(5x - 4x^2) - (x^3 - 2x^2 - 3x)] \, dx \\
 &= \int_{-4}^0 (x^3 + 2x^2 - 8x) \, dx + \int_0^2 (-x^3 - 2x^2 + 8x) \, dx \\
 &= \int_{-4}^0 (x^3 + 2x^2 - 8x) \, dx - \int_0^2 (x^3 + 2x^2 - 8x) \, dx \\
 &= \left[\frac{x^4}{4} + \frac{2x^3}{3} - 4x^2 \right]_{-4}^0 - \left[\frac{x^4}{4} + \frac{2x^3}{3} - 4x^2 \right]_0^2 \\
 &= 0 - \left(\frac{(-4)^4}{4} + \frac{2(-4)^3}{3} - 4(-4)^2 \right) - \left(\frac{2^4}{4} + \frac{2(2)^3}{3} - 4(2)^2 \right) + 0 \\
 &= -\left(64 - \frac{128}{3} - 64 \right) - \left(4 + \frac{16}{3} - 16 \right) \\
 &= \frac{128}{3} - 4 - \frac{16}{3} + 16 \\
 &= \frac{112}{3} + 12 \\
 &= \frac{148}{3} \\
 &= 49\frac{1}{3} \text{ units}^2
 \end{aligned}$$



b $y = 2x^3 - 5x + 4$ meets $y = x^3 + 2x^2 - 2$ where

$$2x^3 - 5x + 4 = x^3 + 2x^2 - 2$$

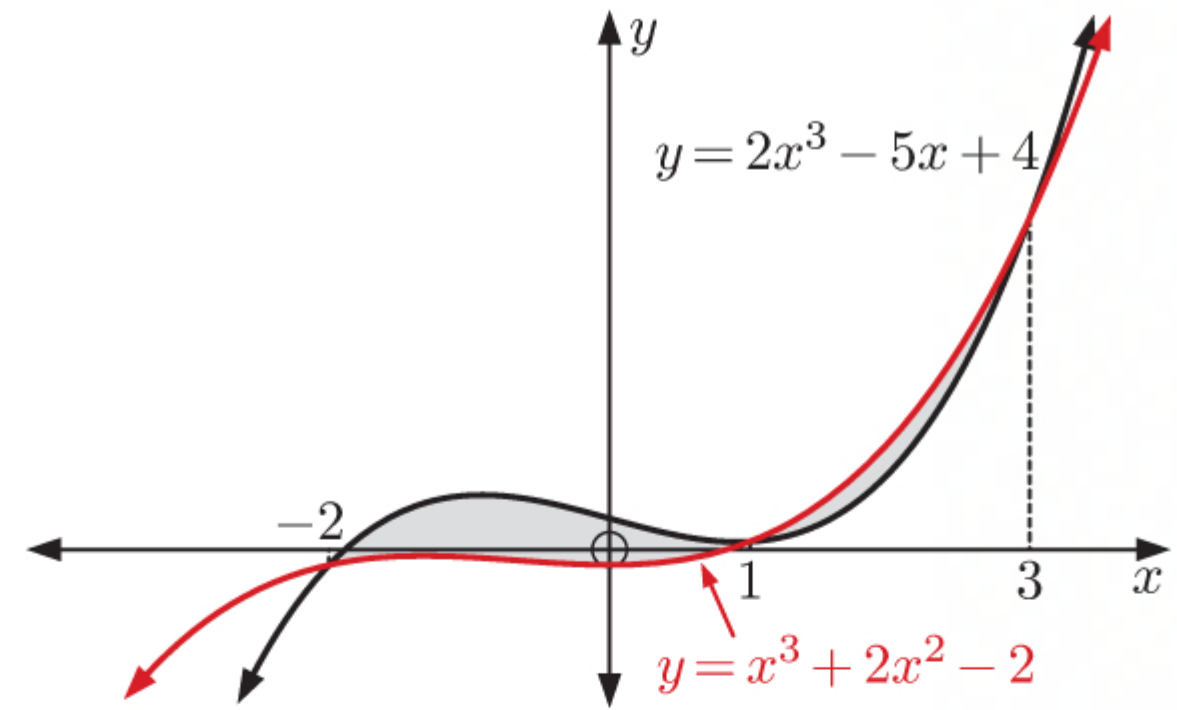
$$\therefore x^3 - 2x^2 - 5x + 6 = 0$$

$$\therefore x = -2, 1, \text{ or } 3$$

Since $2x^3 - 5x + 4 \geq x^3 + 2x^2 - 2$ for $-2 \leq x \leq 1$, and
 $x^3 + 2x^2 - 2 \geq 2x^3 - 5x + 4$ for $1 \leq x \leq 3$,

area

$$\begin{aligned} &= \int_{-2}^1 [(2x^3 - 5x + 4) - (x^3 + 2x^2 - 2)] dx \\ &\quad + \int_1^3 [(x^3 + 2x^2 - 2) - (2x^3 - 5x + 4)] dx \\ &= \int_{-2}^1 (x^3 - 2x^2 - 5x + 6) dx + \int_1^3 (-x^3 + 2x^2 + 5x - 6) dx \\ &= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_{-2}^1 + \left[-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{5x^2}{2} - 6x \right]_1^3 \\ &= \left(\frac{1^4}{4} - \frac{2(1)^3}{3} - \frac{5(1)^2}{2} + 6(1) \right) - \left(\frac{(-2)^4}{4} - \frac{2(-2)^3}{3} - \frac{5(-2)^2}{2} + 6(-2) \right) \\ &\quad + \left(-\frac{3^4}{4} + \frac{2(3)^3}{3} + \frac{5(3)^2}{2} - 6(3) \right) - \left(-\frac{1^4}{4} + \frac{2(1)^3}{3} + \frac{5(1)^2}{2} - 6(1) \right) \\ &= \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) - \left(4 + \frac{16}{3} - 10 - 12 \right) + \left(-\frac{81}{4} + 18 + \frac{45}{2} - 18 \right) - \left(-\frac{1}{4} + \frac{2}{3} + \frac{5}{2} - 6 \right) \\ &= -\frac{79}{4} - \frac{20}{3} + \frac{35}{2} + 30 \\ &= -\frac{107}{12} + 30 \\ &= \frac{253}{12} \\ &= 21\frac{1}{12} \text{ units}^2 \end{aligned}$$



131 $G(t) = \frac{2.5}{t+1}$ metres per year

a $G(t) > 0$ for all $t \geq 0$.

\therefore the rate at which the tree grows is always positive.

\therefore the tree is always growing taller.

b i $\int_0^5 G(t) dt = \int_0^5 \frac{2.5}{t+1} dt$

$$\begin{aligned} &= [2.5 \ln|t+1|]_0^5 \\ &= 2.5 \ln 6 - 2.5 \ln 1 \\ &= 2.5 \ln 6 \approx 4.48 \text{ metres} \end{aligned}$$

The tree grew about 4.48 metres in the first 5 years of its lifetime.

c $\int_0^{15} G(t) dt = \int_0^{15} \frac{2.5}{t+1} dt$

$$\begin{aligned} &= [2.5 \ln|t+1|]_0^{15} \\ &= 2.5 \ln 16 - 2.5 \ln 1 \\ &= 2.5 \ln 16 \approx 6.93 \text{ metres} \end{aligned}$$

The tree grew about 6.93 metres over its entire lifetime.

ii $\int_5^{10} G(t) dt = \int_5^{10} \frac{2.5}{t+1} dt$

$$\begin{aligned} &= [2.5 \ln|t+1|]_5^{10} \\ &= 2.5 \ln 11 - 2.5 \ln 6 \\ &= 2.5 \ln\left(\frac{11}{6}\right) \approx 1.52 \text{ metres} \end{aligned}$$

The tree grew about 1.52 metres between the 5th and 10th years of its lifetime.

132 When $x = 0$, $y = 8$

\therefore the y -intercept is 8.

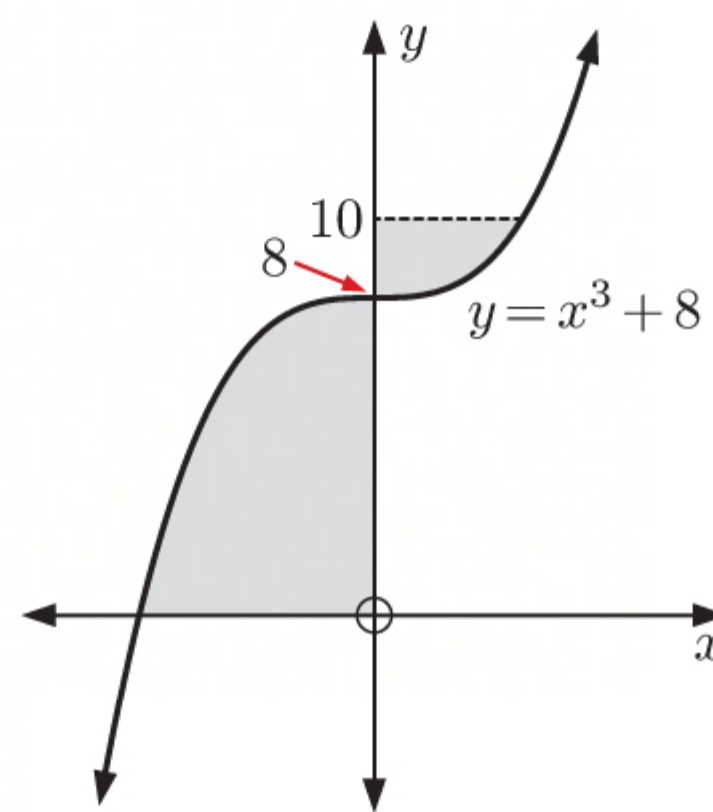
$$\text{Now } y = x^2 + 8$$

$$\therefore x^2 = y - 8$$

$$\therefore x = \sqrt{y - 8}$$

$$\therefore f^{-1}(y) = (y - 8)^{\frac{1}{2}}$$

$$\begin{aligned} \therefore \text{area} &= -\int_0^8 (y - 8)^{\frac{1}{2}} dy + \int_8^{10} (y - 8)^{\frac{1}{2}} dy \\ &= -\left[\frac{2}{3}(y - 8)^{\frac{3}{2}}\right]_0^8 + \left[\frac{2}{3}(y - 8)^{\frac{3}{2}}\right]_8^{10} \\ &= -\left(0 - \frac{2}{3}(-8)^{\frac{3}{2}}\right) + \left(\frac{2}{3}(2)^{\frac{3}{2}} - 0\right) \\ &= \left(12 + \frac{3}{\sqrt{4}}\right) \text{ units}^2 \end{aligned}$$



133 a When $x = 0$, $y = 3 - \ln 4$

\therefore the y -intercept is $3 - \ln 4$.

$$\text{Now } y = 3 - \ln(4 - x)$$

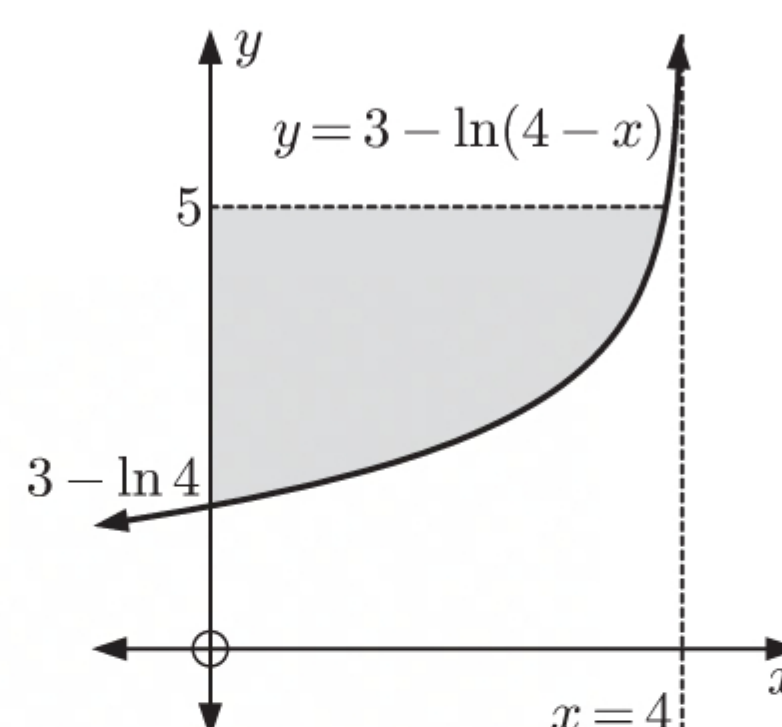
$$\therefore \ln(4 - x) = 3 - y$$

$$\therefore 4 - x = e^{3-y}$$

$$\therefore x = 4 - e^{3-y}$$

$$\therefore f^{-1}(y) = 4 - e^{3-y}$$

$$\begin{aligned} \therefore \text{area} &= \int_{3-\ln 4}^5 (4 - e^{3-y}) dy \\ &= [4y + e^{3-y}]_{3-\ln 4}^5 \\ &= (20 + e^{-2}) - (4(3 - \ln 4) + e^{3-(3-\ln 4)}) \\ &= (20 + e^{-2}) - (12 + 4 \ln 4 + e^{\ln 4}) \\ &= (20 + e^{-2}) - (16 + 8 \ln 2) \\ &= \left(4 + \frac{1}{e^2} - 8 \ln 2\right) \text{ units}^2 \end{aligned}$$



b When $x = 0$, $y = \arcsin(-1) - \frac{\pi}{2}$

$$= -\frac{\pi}{2} - \frac{\pi}{2}$$

$$= -\pi$$

\therefore the y -intercept is $-\pi$.

$$\text{Now } y = \arcsin(x - 1) - \frac{\pi}{2}$$

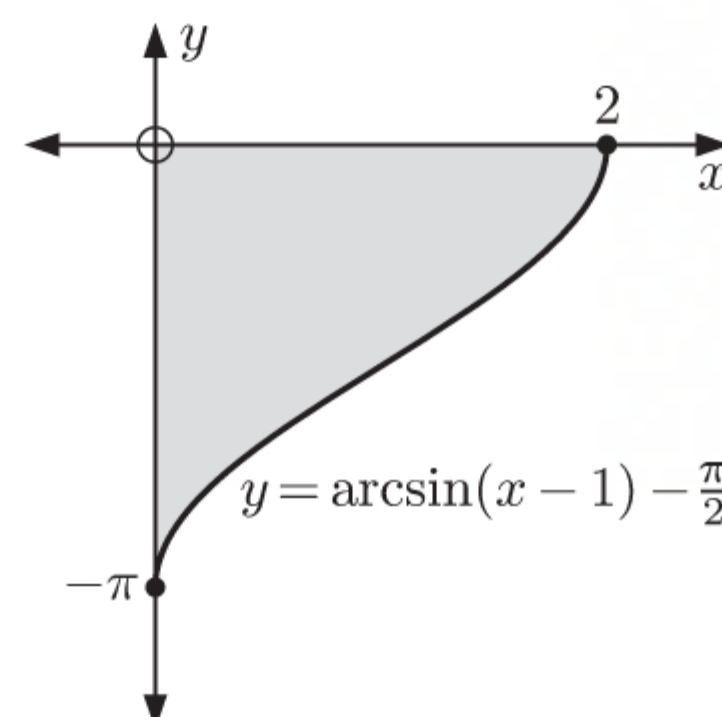
$$\therefore y + \frac{\pi}{2} = \arcsin(x - 1)$$

$$\therefore x - 1 = \sin\left(y + \frac{\pi}{2}\right)$$

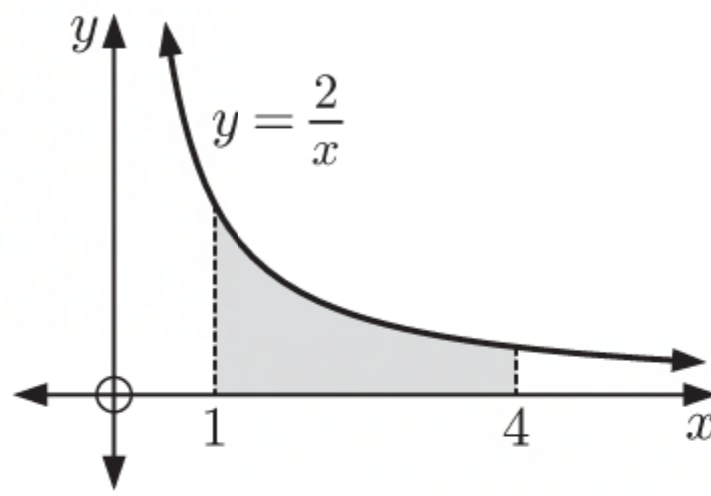
$$\therefore x = 1 + \sin\left(y + \frac{\pi}{2}\right)$$

$$\therefore f^{-1}(y) = 1 + \sin\left(y + \frac{\pi}{2}\right)$$

$$\begin{aligned} \therefore \text{area} &= \int_{-\pi}^0 (1 + \sin(y + \frac{\pi}{2})) dy \\ &= [y - \cos(y + \frac{\pi}{2})]_{-\pi}^0 \\ &= (0 - \cos \frac{\pi}{2}) - (-\pi - \cos(-\frac{\pi}{2})) \\ &= (-0) - (-\pi - 0) \\ &= \pi \text{ units}^2 \end{aligned}$$

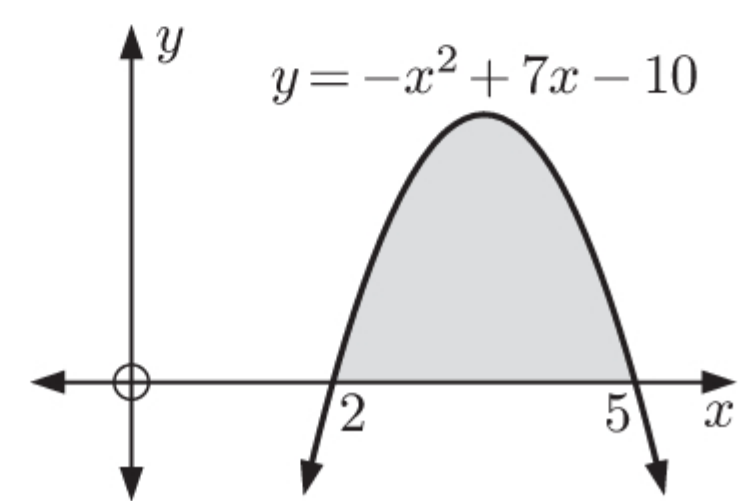


$$\begin{aligned}
 \text{134 a Volume} &= \pi \int_1^4 y^2 dx \\
 &= \pi \int_1^4 \left(\frac{2}{x}\right)^2 dx \\
 &= 4\pi \int_1^4 x^{-2} dx \\
 &= 4\pi [-x^{-1}]_1^4 \\
 &= 4\pi \left(-\frac{1}{4} - (-1)\right) \\
 &= 4\pi \left(\frac{3}{4}\right) \\
 &= 3\pi \text{ units}^3
 \end{aligned}$$

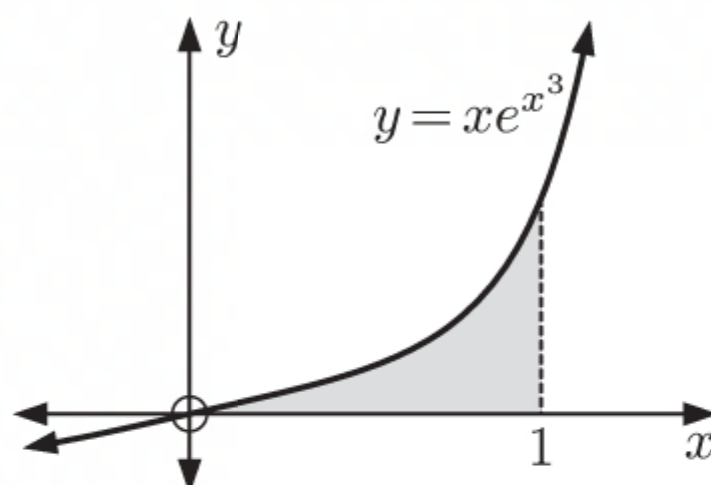


$$\begin{aligned}
 \text{b When } y = 0, \quad &-x^2 + 7x - 10 = 0 \\
 \therefore \quad &x^2 - 7x + 10 = 0 \\
 \therefore \quad &(x - 5)(x - 2) = 0 \\
 \therefore \quad &x = 5 \text{ or } 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Volume} &= \pi \int_2^5 y^2 dx \\
 &= \pi \int_2^5 (-x^2 + 7x - 10)^2 dx \\
 &= \pi \int_2^5 ((-x^2 + 7x)^2 - 20(-x^2 + 7x) + 100) dx \\
 &= \pi \int_2^5 (x^4 - 14x^3 + 49x^2 + 20x^2 - 140x + 100) dx \\
 &= \pi \int_2^5 (x^4 - 14x^3 + 69x^2 - 140x + 100) dx \\
 &= \pi \left[\frac{1}{5}x^5 - \frac{7}{2}x^4 + 23x^3 - 70x^2 + 100x \right]_2^5 \\
 &= \pi \left[\left(625 - \frac{4375}{2} + 2875 - 1750 + 500\right) - \left(\frac{32}{5} - 56 + 184 - 280 + 200\right) \right] \\
 &= \pi \left(\frac{125}{2} - \frac{272}{5} \right) \\
 &= \frac{81\pi}{10} \text{ units}^3
 \end{aligned}$$



$$\begin{aligned}
 \text{c Volume} &= \int_0^1 y^2 dx \\
 &= \int_0^1 (xe^{x^3})^2 dx \\
 &= \int_0^1 x^2 e^{2x^3} dx
 \end{aligned}$$



$$\text{Let } u = 2x^3 \quad \therefore \quad \frac{du}{dx} = 6x^2$$

$$\text{When } x = 0, \quad u = 0$$

$$\text{When } x = 1, \quad u = 2$$

$$\begin{aligned}
 \therefore \text{ volume} &= \int_0^2 e^u \left(\frac{1}{6} \frac{du}{dx} \right) dx \\
 &= \frac{1}{6} \int_0^2 e^u du \\
 &= \frac{1}{6} [e^u]_0^2 \\
 &= \frac{1}{6} (e^2 - 1) \text{ units}^3
 \end{aligned}$$

135 a Volume $= \pi \int_0^{2\pi} y^2 dx$

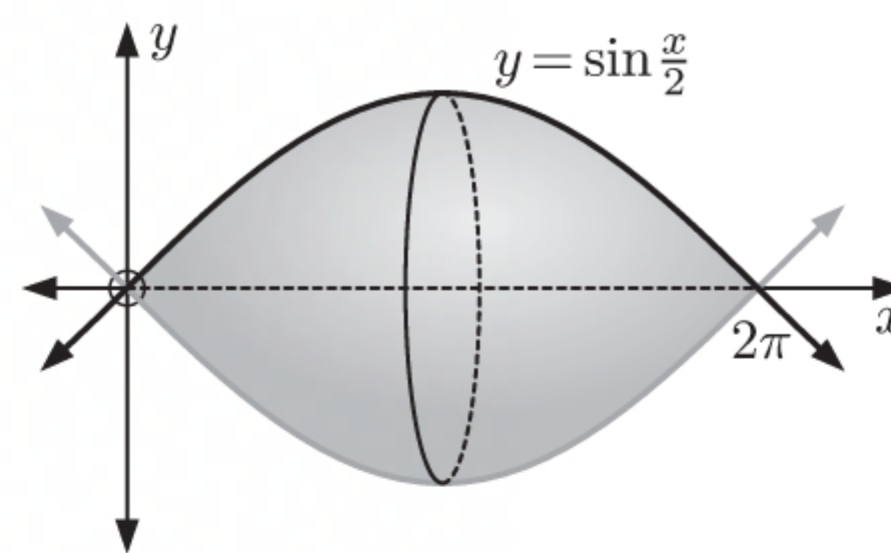
$$= \pi \int_0^{2\pi} \sin^2\left(\frac{x}{2}\right) dx$$

$$= \pi \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{2} \sin x\right]_0^{2\pi}$$

$$= \pi[(\pi - 0) - (0 - 0)]$$

$$= \pi^2 \text{ units}^3$$



b $y = 4 \sin x \cos x$
 $= 2 \sin 2x$

Volume $= \pi \int_0^{\frac{\pi}{2}} y^2 dx$

$$= \pi \int_0^{\frac{\pi}{2}} (2 \sin 2x)^2 dx$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \sin^2 2x dx$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4x) dx$$

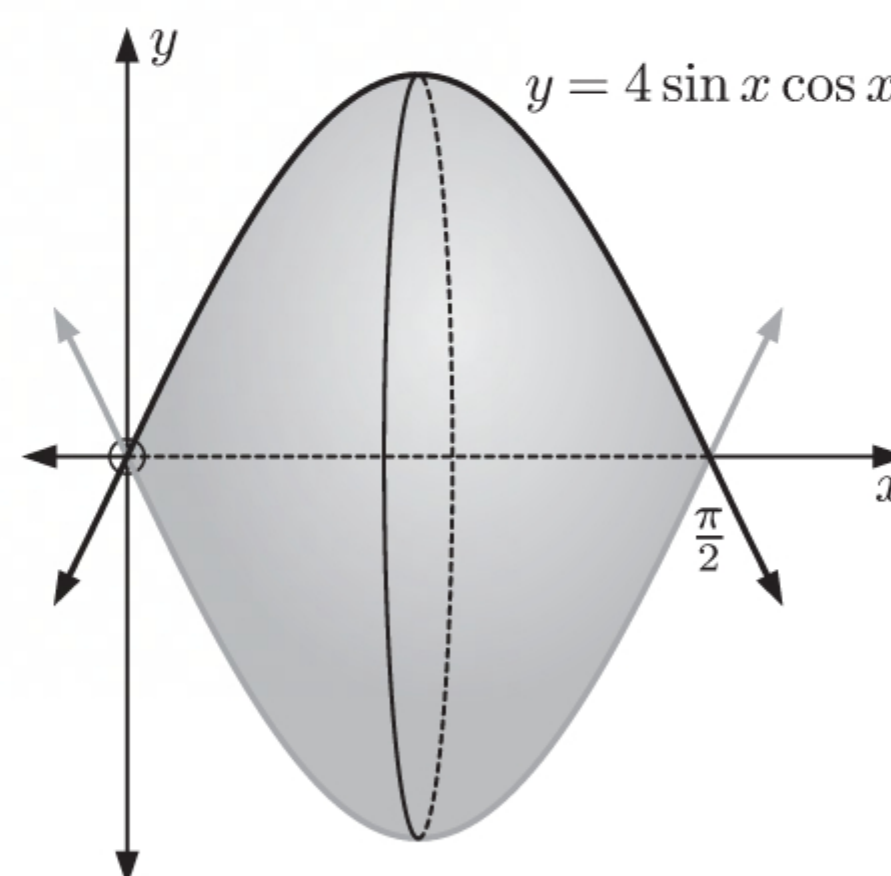
$$= 2\pi \int_0^{\frac{\pi}{2}} (1 - \cos 4x) dx$$

$$= 2\pi \left[x - \frac{1}{4} \sin 4x\right]_0^{\frac{\pi}{2}}$$

$$= 2\pi \left[\left(\frac{\pi}{2} - \frac{1}{4} \sin 2\pi\right) - (0 - 0)\right]$$

$$= 2\pi\left(\frac{\pi}{2} - 0\right)$$

$$= \pi^2 \text{ units}^3$$



c Volume $= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} y^2 dx$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + \tan x)^2 dx$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + 2 \tan x + \tan^2 x) dx$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\sec^2 x + 2 \frac{\sin x}{\cos x}\right) dx$$

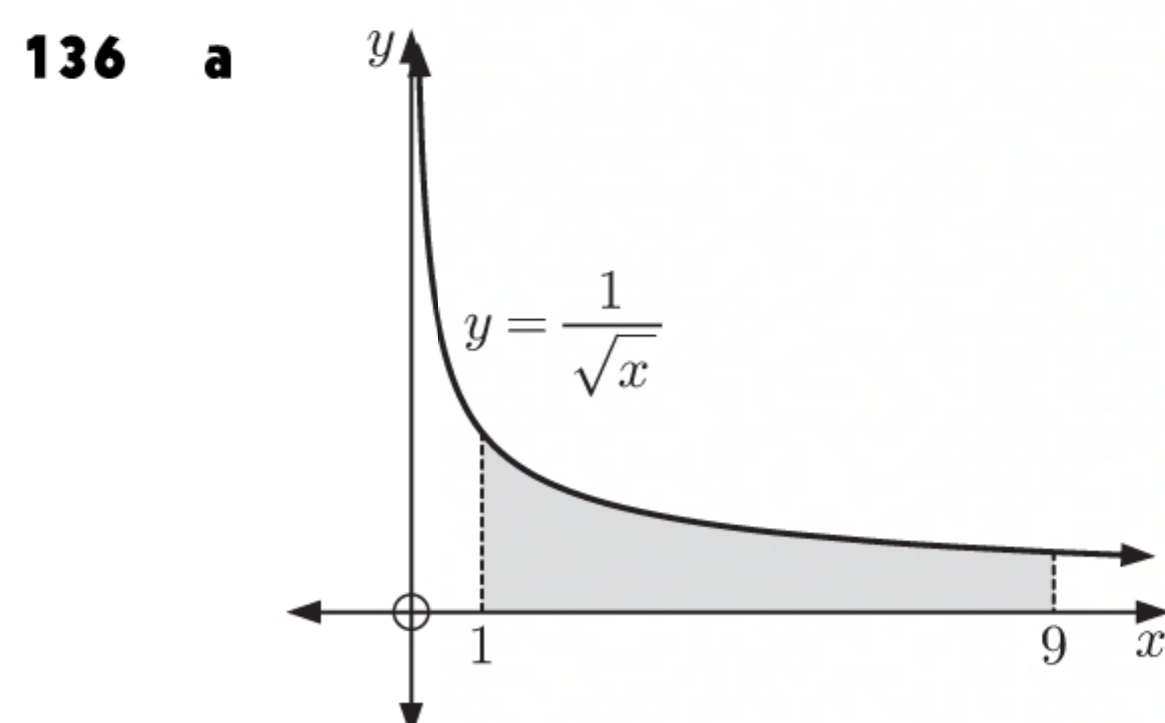
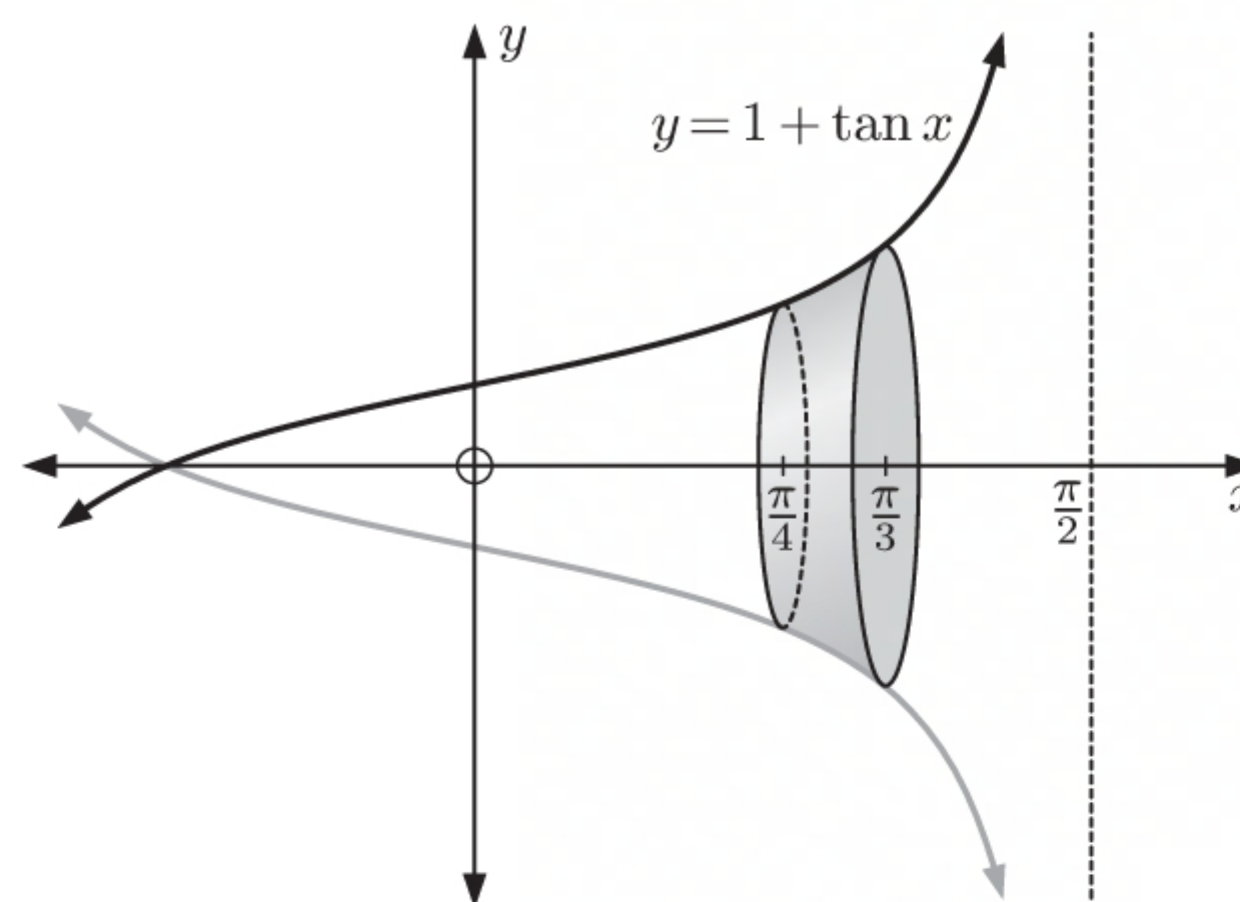
$$= \pi \left[\tan x - 2 \ln |\cos x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \pi \left[\left(\sqrt{3} - 2 \ln \frac{1}{2}\right) - \left(1 - 2 \ln \frac{1}{\sqrt{2}}\right)\right]$$

$$= \pi \left(\sqrt{3} + 2 \ln 2 - 1 + 2 \ln(2^{-\frac{1}{2}})\right)$$

$$= \pi(\sqrt{3} + 2 \ln 2 - 1 - \ln 2)$$

$$= \pi(\sqrt{3} + \ln 2 - 1) \text{ units}^3$$



$$\text{Area} = \int_1^9 x^{-\frac{1}{2}} dx$$

$$= \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9$$

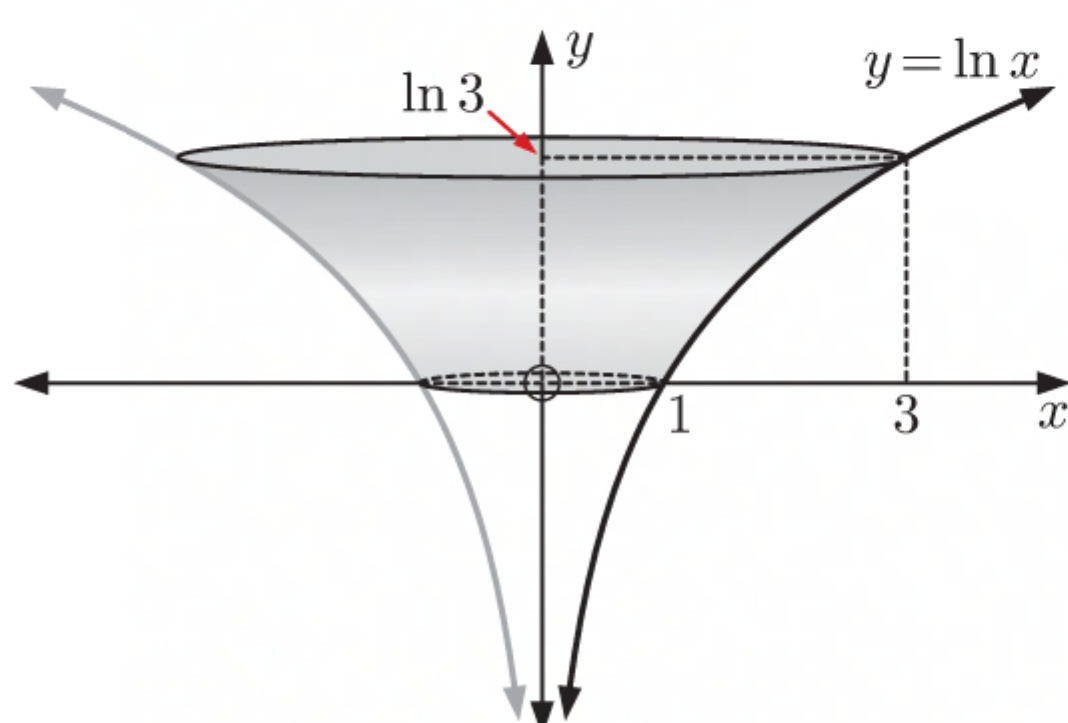
$$= \left[2\sqrt{x}\right]_1^9$$

$$= 6 - 2$$

$$= 4 \text{ units}^2$$

$$\begin{aligned}
 \text{b Volume of revolution} &= \pi \int_1^9 \left(\frac{1}{\sqrt{x}} \right)^2 dx \\
 &= \pi \int_1^9 \frac{1}{x} dx \\
 &= \pi [\ln |x|]_1^9 \\
 &= \pi (\ln 9 - \ln 1) \\
 &= 2\pi \ln 3 \text{ units}^3
 \end{aligned}$$

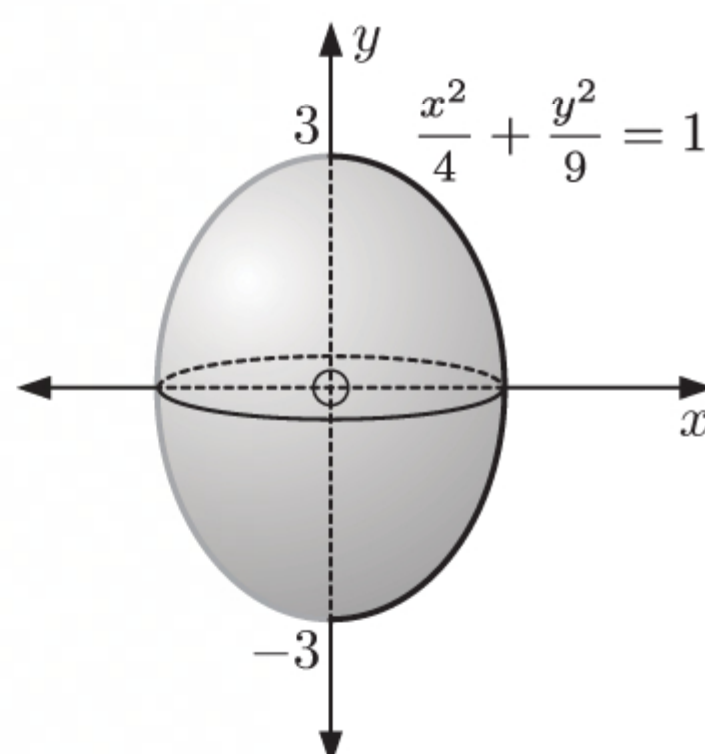
137 $y = \ln x$
 $\therefore x = e^y$



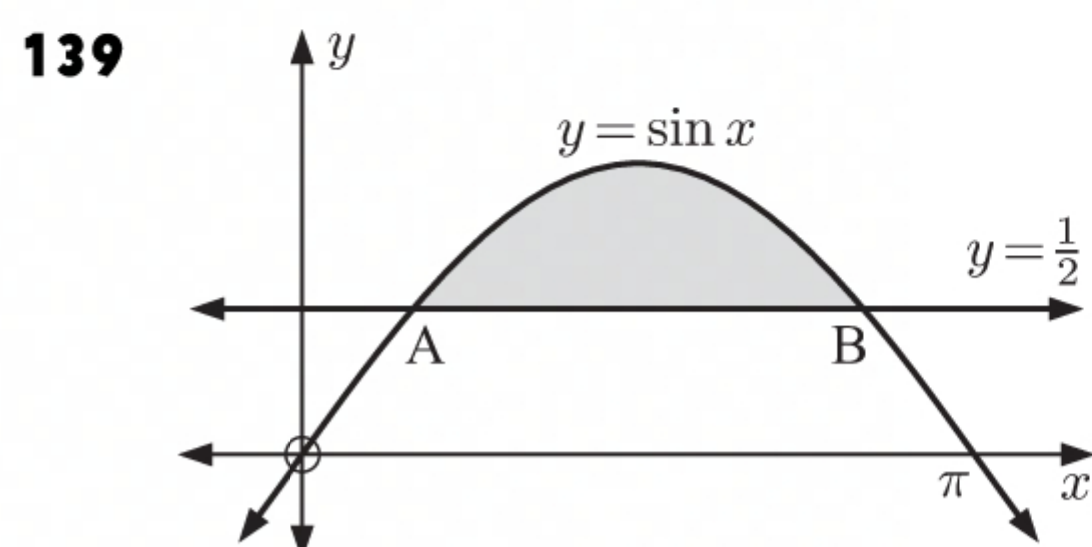
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\ln 3} x^2 dy \\
 &= \pi \int_0^{\ln 3} e^{2y} dy \\
 &= \pi \left[\frac{1}{2} e^{2y} \right]_0^{\ln 3} \\
 &= \frac{1}{2} \pi (e^{2 \ln 3} - e^0) \\
 &= \frac{1}{2} \pi (3^2 - 1) \\
 &= \frac{1}{2} \pi (9 - 1) \\
 &= 4\pi \text{ units}^3
 \end{aligned}$$

138 When $x = 0$, $0 + \frac{y^2}{9} = 1$
 $\therefore y^2 = 9$
 $\therefore y = \pm 3$

Now $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $x \geq 0$
 $\therefore \frac{x^2}{4} = 1 - \frac{y^2}{9}$
 $\therefore x^2 = 4 \left(1 - \frac{y^2}{9} \right)$



$$\begin{aligned}
 \text{Volume} &= \pi \int_{-3}^3 x^2 dy \\
 &= \pi \int_{-3}^3 4 \left(1 - \frac{y^2}{9} \right) dy \\
 &= 4\pi \int_{-3}^3 \left(1 - \frac{y^2}{9} \right) dy \\
 &= 4\pi \left[y - \frac{y^3}{27} \right]_{-3}^3 \\
 &= 4\pi [(3 - 1) - (-3 + 1)] \\
 &= 4\pi (2 - (-2)) \\
 &= 16\pi \text{ units}^3
 \end{aligned}$$



a The graphs meet where $\sin x = \frac{1}{2}$
 $\therefore x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \quad \{0 \leq x \leq \pi\}$
 $\therefore A \text{ is at } \left(\frac{\pi}{6}, \frac{1}{2} \right) \text{ and } B \text{ is at } \left(\frac{5\pi}{6}, \frac{1}{2} \right).$

$$\begin{aligned}
 \text{b Volume} &= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\sin^2 x - \frac{1}{4} \right) dx \\
 &= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\frac{1}{2} - \frac{1}{2} \cos^2 x - \frac{1}{4} \right) dx \\
 &= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\frac{1}{4} - \frac{1}{2} \cos 2x \right) dx \\
 &= \pi \left[\frac{1}{4}x - \frac{1}{4} \sin 2x \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
 &= \frac{\pi}{4} \left[x - \sin 2x \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
 &= \frac{\pi}{4} \left[\left(\frac{5\pi}{6} - \sin \frac{5\pi}{3} \right) - \left(\frac{\pi}{6} - \sin \frac{\pi}{3} \right) \right] \\
 &= \frac{\pi}{4} \left[\left(\frac{5\pi}{6} - \left(-\frac{\sqrt{3}}{2} \right) \right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2} \right) \right] \\
 &= \frac{\pi}{4} \left(\frac{2\pi}{3} + \sqrt{3} \right) \\
 &= \left(\frac{\pi^2}{6} + \frac{\pi\sqrt{3}}{4} \right) \text{ units}^3
 \end{aligned}$$

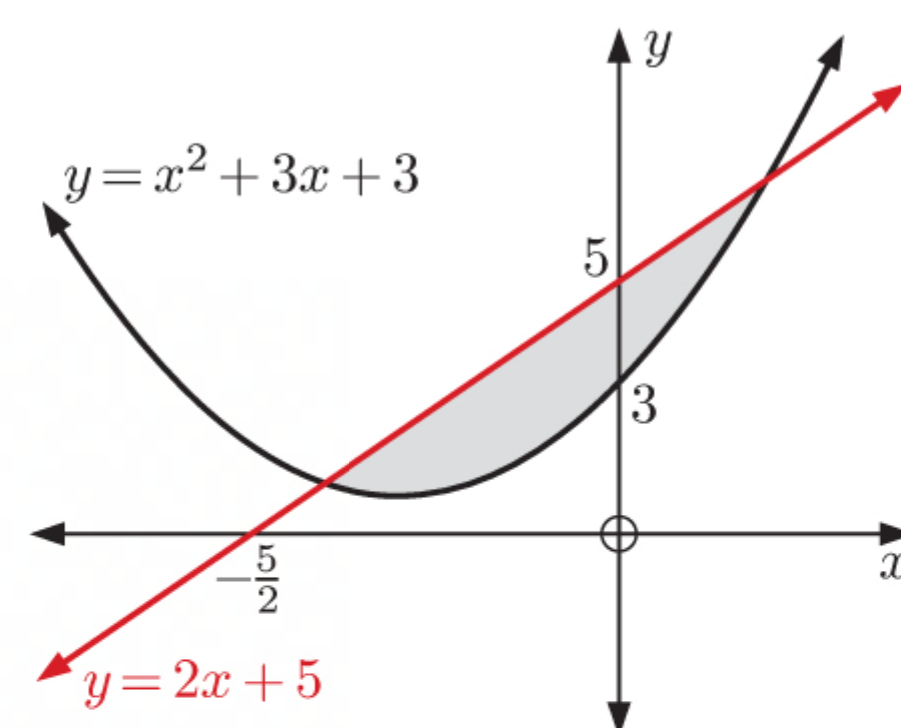
140 The graphs meet where $x^2 + 3x + 3 = 2x + 5$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x + 2)(x - 1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

$$\begin{aligned}
 \therefore \text{volume} &= \pi \int_{-2}^1 [(2x + 5)^2 - (x^2 + 3x + 3)^2] dx \\
 &= \pi \int_{-2}^1 [(4x^2 + 20x + 25) - ((x^2 + 3x)^2 + 6(x^2 + 3x) + 9)] dx \\
 &= \pi \int_{-2}^1 [(4x^2 + 20x + 25) - (x^4 + 6x^3 + 9x^2 + 6x^2 + 18x + 9)] dx \\
 &= \pi \int_{-2}^1 [(4x^2 + 20x + 25) - (x^4 + 6x^3 + 15x^2 + 18x + 9)] dx \\
 &= \pi \int_{-2}^1 (-x^4 - 6x^3 - 11x^2 + 2x + 16) dx \\
 &= \pi \left[-\frac{1}{5}x^5 - \frac{3}{2}x^4 - \frac{11}{3}x^3 + x^2 + 16x \right]_{-2}^1 \\
 &= \pi \left[\left(-\frac{1}{5} - \frac{3}{2} - \frac{11}{3} + 1 + 16 \right) - \left(\frac{32}{5} - 24 + \frac{88}{3} + 4 - 32 \right) \right] \\
 &= \pi \left(\frac{349}{30} - \left(-\frac{244}{15} \right) \right) \\
 &= \frac{279\pi}{10} \text{ units}^3
 \end{aligned}$$



141 Now $y = \log_2 x$ is $x = 2^y$

and $y = 3 - \log_4 x$ is $x = 4^{3-y}$

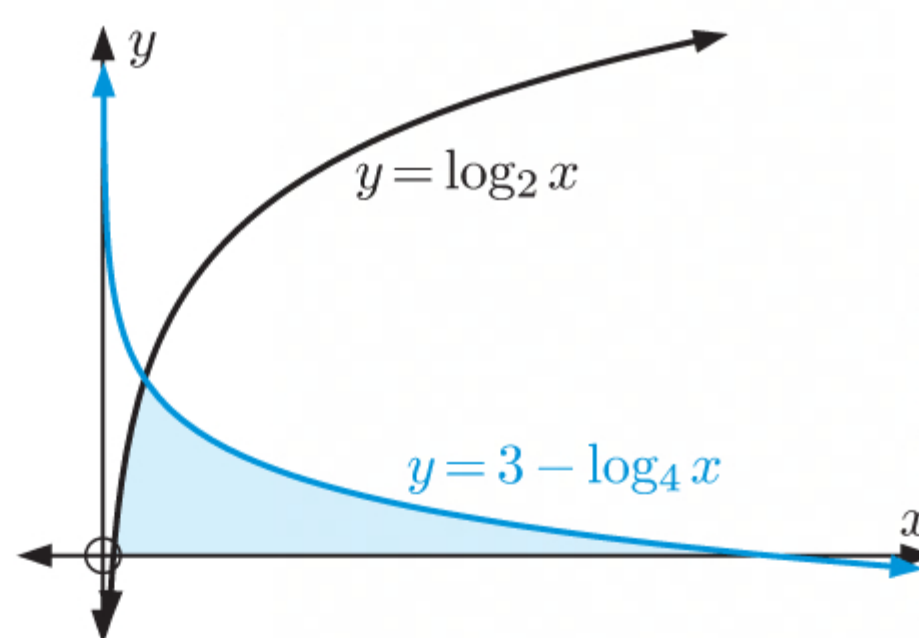
$$\text{or } x = 2^{6-2y}$$

The graphs meet where $2^y = 2^{6-2y}$

$$\therefore y = 6 - 2y$$

$$\therefore 3y = 6$$

$$\therefore y = 2$$



$$\begin{aligned}
 \therefore \text{volume} &= \pi \int_0^2 \left[(2^{6-2y})^2 - (2^y)^2 \right] dy \\
 &= \pi \int_0^2 (4^{6-2y} - 4^y) dy \\
 &= \pi \left[-\frac{1}{2} \frac{4^{6-2y}}{\ln 4} - \frac{4^y}{\ln 4} \right]_0^2 \\
 &= \pi \left[\left(-\frac{8}{\ln 4} - \frac{16}{\ln 4} \right) - \left(-\frac{2048}{\ln 4} - \frac{1}{\ln 4} \right) \right] \\
 &= \pi \left(-\frac{24}{\ln 4} - \left(-\frac{2049}{\ln 4} \right) \right) \\
 &= \frac{2025\pi}{\ln 4} \text{ units}^3
 \end{aligned}$$

142 $f(x) = x^{-\frac{2}{3}}, \quad x \geq 1$

a $\text{Area} = \int_1^\infty x^{-\frac{2}{3}} dx$

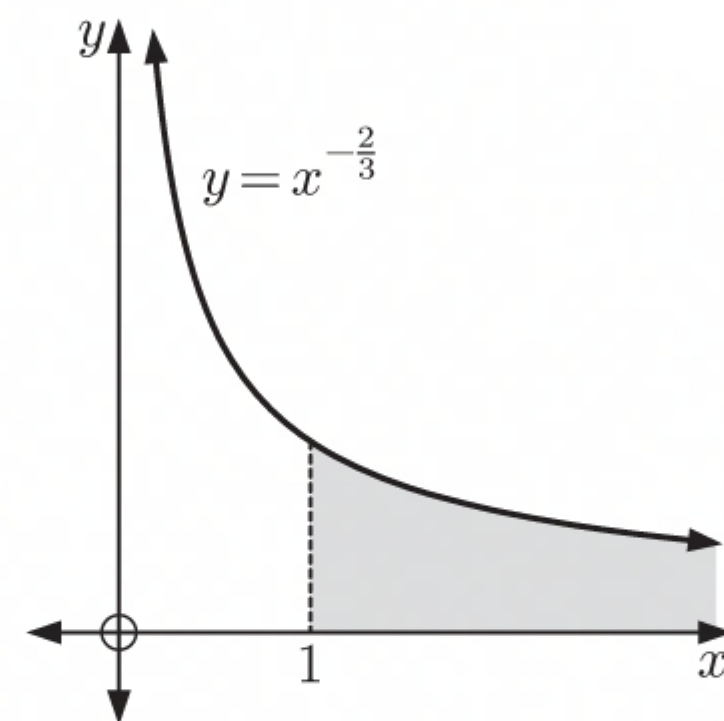
$$\begin{aligned}
 &= \lim_{b \rightarrow \infty} \int_1^b x^{-\frac{2}{3}} dx \\
 &= \lim_{b \rightarrow \infty} \left[3x^{\frac{1}{3}} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} (3b^{\frac{1}{3}} - 3)
 \end{aligned}$$

which does not exist since $b^{\frac{1}{3}} \rightarrow \infty$ as $b \rightarrow \infty$.

\therefore the area between the curve and the x -axis for $x \geq 1$ is infinite.

b $\text{Volume} = \pi \int_1^\infty (x^{-\frac{2}{3}})^2 dx$

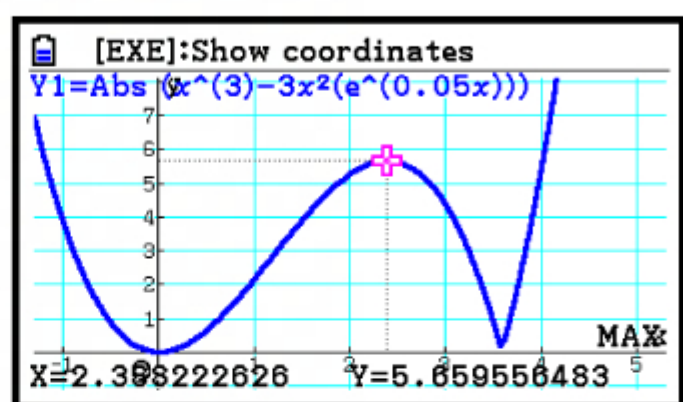
$$\begin{aligned}
 &= \pi \lim_{b \rightarrow \infty} \int_1^b x^{-\frac{4}{3}} dx \\
 &= \pi \lim_{b \rightarrow \infty} \left[-3x^{-\frac{1}{3}} \right]_1^b \\
 &= \pi \lim_{b \rightarrow \infty} (-3b^{-\frac{1}{3}} - (-3)) \\
 &= \pi \lim_{b \rightarrow \infty} \left(-\frac{3}{\sqrt[3]{b}} + 3 \right) \\
 &= \pi(0 + 3) \quad \left\{ \frac{1}{\sqrt[3]{b}} \rightarrow 0 \text{ as } b \rightarrow \infty \right\} \\
 &= 3\pi \text{ units}^3
 \end{aligned}$$



143 $v(t) = t^3 - 3t^2 e^{0.05t}, \quad t \geq 0 \text{ seconds}$

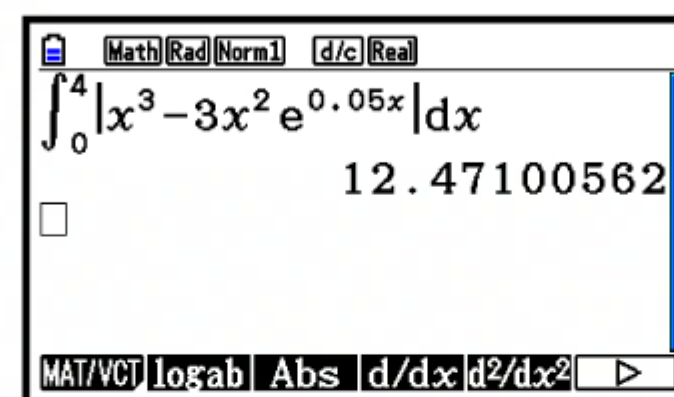
a $\text{speed} = |v(t)|$

Using technology, the maximum of $|v(t)|$ over $0 \leq t \leq 4$ is $\approx 5.66 \text{ m s}^{-1}$ when $t \approx 2.39$.



b $\text{Total distance} = \int_0^4 |v(t)| dt$

$$\begin{aligned}
 &= \int_0^4 |t^3 - 3t^2 e^{0.05t}| dt \\
 &\approx 12.5 \text{ m}
 \end{aligned}$$



144 $v = t^3 - 9t^2 + 24t \text{ m s}^{-1}, \quad 0 \leq t \leq 6$

a $a = \frac{dv}{dt}$

$$\begin{aligned}
 &= 3t^2 - 18t + 24 \text{ m s}^{-2}
 \end{aligned}$$

- b** The greatest velocity of the particle occurs when $\frac{dv}{dt} = a = 0$

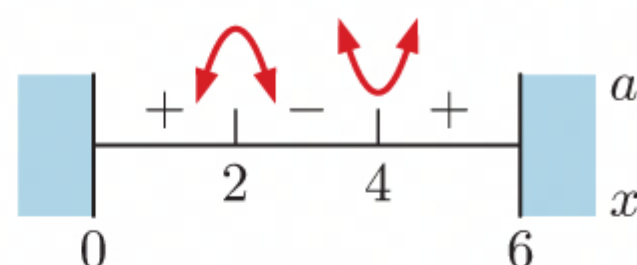
$$\therefore 3t^2 - 18t + 24 = 0$$

$$\therefore t^2 - 6t + 8 = 0$$

$$\therefore (t - 2)(t - 4) = 0$$

$$\therefore t = 2 \text{ or } 4$$

The sign diagram of a is



\therefore there is a local maximum at $t = 2$.

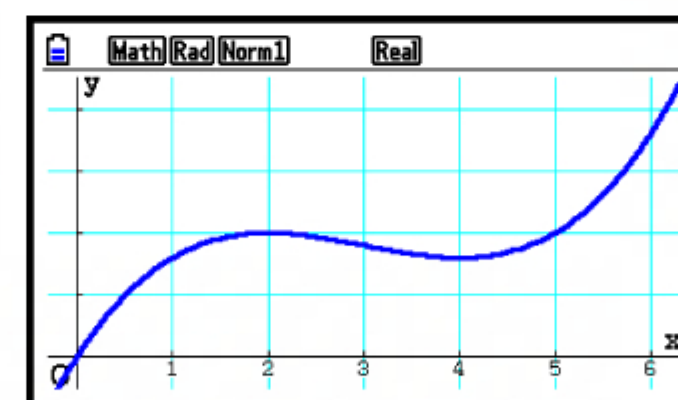
Critical value (t)	v (m s^{-1})
0 (end point)	0
2 (local maximum)	20
6 (end point)	36

\therefore the greatest velocity of the particle is 36 m s^{-1} which occurs at $t = 6$ seconds.

- c** The speed of the particle is decreasing when v and a have opposite signs.

Now $v \geq 0$ for all $0 \leq t \leq 6$.

\therefore the speed of the particle is decreasing when $a < 0$, which is when $2 < t < 4$.



- 145 a** $a(t) = 2 - 3t \text{ m s}^{-1}$, $v(1) = 0 \text{ m s}^{-1}$

$$\begin{aligned} \therefore v(t) &= \int a(t) dt \\ &= \int (2 - 3t) dt \\ &= 2t - \frac{3}{2}t^2 + c \end{aligned}$$

$$\text{Now } v(1) = 0 \quad \therefore 2(1) - \frac{3}{2}(1)^2 + c = 0$$

$$\therefore 2 - \frac{3}{2} + c = 0$$

$$\therefore \frac{1}{2} + c = 0$$

$$\therefore c = -\frac{1}{2}$$

$$\therefore v(t) = 2t - \frac{3}{2}t^2 - \frac{1}{2}$$

- c** $v(t) = 2t - \frac{3}{2}t^2 - \frac{1}{2}$, $s(0) = 3 \text{ m}$

$$\begin{aligned} \therefore s(t) &= \int v(t) dt \\ &= \int \left(2t - \frac{3}{2}t^2 - \frac{1}{2}\right) dt \\ &= t^2 - \frac{1}{2}t^3 - \frac{1}{2}t + c \end{aligned}$$

$$\text{Now } s(0) = 3 \quad \therefore 0^2 - \frac{1}{2}(0)^3 - \frac{1}{2}(0) + c = 3$$

$$\therefore c = 3$$

$$\therefore s(t) = t^2 - \frac{1}{2}t^3 - \frac{1}{2}t + 3$$

- 146** $s(t) = 12t - 3t^3 + 1 \text{ cm}$, $t \geq 0$ seconds

- a** $v(t) = s'(t)$

$$= 12 - 9t^2 \text{ cm s}^{-1}$$

$$a(t) = v'(t)$$

$$= -18t \text{ cm s}^{-2}$$

- b** The particle is momentarily at rest when

$$v(t) = 0$$

$$\therefore 2t - \frac{3}{2}t^2 - \frac{1}{2} = 0$$

$$\therefore 4t - 3t^2 - 1 = 0$$

$$\therefore 3t^2 - 4t + 1 = 0$$

$$\therefore 3t^2 - 3t - t + 1 = 0$$

$$\therefore 3t(t - 1) - (t - 1) = 0$$

$$\therefore (t - 1)(3t - 1) = 0$$

$$\therefore t = 1 \text{ or } \frac{1}{3}$$


\therefore the other time the particle is momentarily at rest is at $t = \frac{1}{3} \text{ s}$.

$$\begin{aligned}\text{b i } v(1) &= 12 - 9(1)^2 \\ &= 3 \text{ m s}^{-1}\end{aligned}$$

$$\text{speed} = |v(1)| = 3 \text{ m s}^{-1}$$

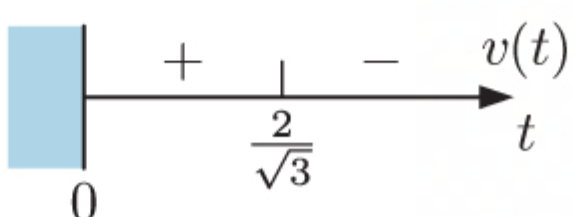
$$\begin{aligned}\text{ii } v(2) &= 12 - 9(2)^2 \\ &= 12 - 9(4) \\ &= 12 - 36 \\ &= -24 \text{ m s}^{-1}\end{aligned}$$

$$\therefore \text{speed} = |v(2)| = 24 \text{ m s}^{-1}$$

c i The sign diagram of $a(t)$ is 

\therefore the velocity of the particle is always decreasing.

$$\begin{aligned}\text{ii } v(t) = 0 \text{ when } 12 - 9t^2 &= 0 \\ \therefore 9t^2 &= 12 \\ \therefore t^2 &= \frac{4}{3} \\ \therefore t &= \frac{2}{\sqrt{3}} \quad \{t \geq 0\}\end{aligned}$$

The sign diagram of $v(t)$ is 

The speed of the particle is decreasing when $v(t)$ and $a(t)$ have opposite sign, that is when $0 \leq t \leq \frac{2}{\sqrt{3}}$.

$$\mathbf{147} \quad v = 2\sqrt{t} - t \text{ m s}^{-1}, \quad t \geq 0$$

$$\begin{aligned}\text{a } \text{When } t = 5, v &= 2\sqrt{5} - 5 \approx -0.528 \text{ m s}^{-1} \\ \therefore \text{speed} &= |v| \approx 0.528 \text{ m s}^{-1}\end{aligned}$$

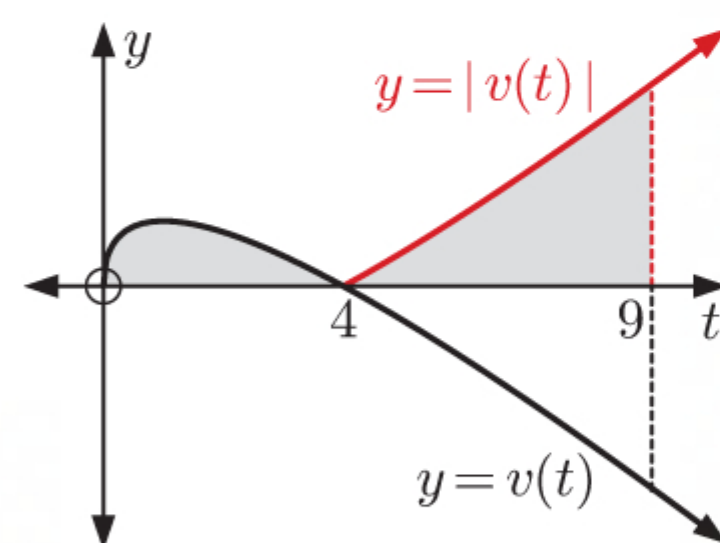
$$\begin{aligned}\text{b } a &= \frac{dv}{dt} \\ &= \frac{2}{2\sqrt{t}} - 1 \\ &= \frac{1}{\sqrt{t}} - 1 \text{ m s}^{-2}\end{aligned}$$

$$\begin{aligned}\text{c } \text{The direction of motion changes when } v &= 0 \\ \therefore 2\sqrt{t} - t &= 0 \\ \therefore 2\sqrt{t} &= t \\ \therefore 4t &= t^2 \\ \therefore t^2 - 4t &= 0 \\ \therefore t(t - 4) &= 0 \\ \therefore t &= 0 \text{ or } 4\end{aligned}$$

Now $v = 0$ when $t = 0$ means that the particle was initially stationary. So, it does not make sense to talk about the direction of motion changing when there is nothing to compare its current direction to.

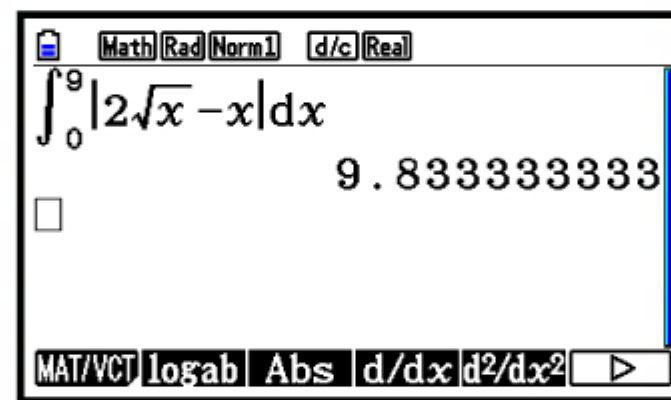
\therefore the direction of motion changes at $t = 4$ seconds.

$$\begin{aligned}\text{d } \text{Total distance} &= \int_0^9 |v| dt \\ &= \int_0^4 v dt + \int_4^9 -v dt \\ &= \int_0^4 v dt - \int_4^9 v dt \\ &= \int_0^4 (2\sqrt{t} - t) dt - \int_4^9 (2\sqrt{t} - t) dt \\ &= \left[\frac{4}{3}t^{\frac{3}{2}} - \frac{1}{2}t^2 \right]_0^4 - \left[\frac{4}{3}t^{\frac{3}{2}} - \frac{1}{2}t^2 \right]_4^9 \\ &= \left(\frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 \right) - 0 - \left(\frac{4}{3}(9)^{\frac{3}{2}} - \frac{1}{2}(9)^2 \right) + \left(\frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 \right) \\ &= \frac{32}{3} - 8 - 36 + \frac{81}{2} + \frac{32}{3} - 8 \\ &= \frac{64}{3} + \frac{81}{2} - 52 \\ &= \frac{59}{6} \\ &= 9\frac{5}{6} \text{ m}\end{aligned}$$



Alternatively, using technology:

$$\int_0^9 |v| dt \approx 9.833 \approx 9\frac{5}{6} \text{ m}$$



148 a $\frac{dv}{dt} = -kv, \quad k > 0$

$$\therefore \int \frac{1}{v} \frac{dv}{dt} dt = \int -k dt$$

$$\therefore \int \frac{1}{v} dv = - \int k dt$$

$$\therefore \ln v = -kt + c \quad \{v > 0\}$$

$$\therefore v(t) = e^{-kt+c}$$

Now $v(0) = 100$, so $\ln 100 = c$ and $v(2) = 40$, so $\ln 40 = -2k + \ln 100$

$$\therefore 2k = \ln 100 - \ln 40$$

$$\therefore 2k = \ln \frac{5}{2}$$

$$\therefore k = \frac{1}{2} \ln \left(\frac{5}{2} \right) \quad \text{as required.}$$

b Since $v > 0$ for all t , there is no change in direction.

$$\begin{aligned} \therefore \text{in the first 2 seconds, the object travels a distance} &= \int_0^2 v(t) dt \\ &= \int_0^2 e^{-kt+c} dt \\ &= \left[\frac{1}{-k} e^{-kt+c} \right]_0^2 \\ &= \frac{1}{-k} [v(t)]_0^2 \\ &= \frac{1}{-k} (v(2) - v(0)) \\ &= \frac{1}{-k} (40 - 100) \\ &= \frac{-60}{-\frac{1}{2} \ln \left(\frac{5}{2} \right)} \quad \{\text{from a}\} \\ &\approx 131 \text{ m} \end{aligned}$$

149 Let $f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2}$

$$\therefore f'(x) = -2(1+x)^{-3}$$

$$\therefore f''(x) = (-2)(-3)(1+x)^{-4}$$

\vdots

$$\therefore f^{(k)}(x) = (-1)^k (k+1)! (1+x)^{-(k+2)}$$

$$\therefore f^{(k)}(0) = (-1)^k (k+1)! \quad \text{for all } k \in \mathbb{Z}^+$$

Since $f(0) = 1$, the Maclaurin series representation for $f(x)$ is $f(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (k+1)!}{k!} x^k$

$$\therefore \frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k \quad \text{as required.}$$

150 a $3^x = e^{\ln(3^x)}$
 $= e^{x \ln 3}$

$$= \sum_{k=0}^{\infty} \frac{(x \ln 3)^k}{k!} \quad \left\{ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{x^k (\ln 3)^k}{k!}$$

$$= 1 + x \ln 3 + \frac{x^2 (\ln 3)^2}{2!} + \frac{x^3 (\ln 3)^3}{3!} + \frac{x^4 (\ln 3)^4}{4!} + \dots$$

$$= 1 + x \ln 3 + \frac{x^2 (\ln 3)^2}{2} + \frac{x^3 (\ln 3)^3}{6} + \frac{x^4 (\ln 3)^4}{24} + \dots$$

b $\sqrt{3} = 3^{\frac{1}{2}}$

$$\approx 1 + \frac{1}{2} \ln 3 + \frac{(\frac{1}{2})^2 (\ln 3)^2}{2} + \frac{(\frac{1}{2})^3 (\ln 3)^3}{6} + \frac{(\frac{1}{2})^4 (\ln 3)^4}{24} \quad \{\text{using a}\}$$

$$\approx 1 + \frac{\ln 3}{2} + \frac{(\ln 3)^2}{8} + \frac{(\ln 3)^3}{48} + \frac{(\ln 3)^4}{384}$$

$$\approx 1.73$$

151 a $e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \quad \left\{ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \right\}$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^8}{8!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \dots \right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right)$$

b When $\theta = \pi$, $e^{i\pi} = \left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} + \dots \right) + i \left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots \right) \quad \{\text{using a}\}$

$$\therefore -1 = \left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} + \dots \right) + i \left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots \right) \quad \dots (*)$$

i Equating real parts in (*) gives $-1 = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} + \dots$

$$\therefore \frac{\pi^2}{2!} - \frac{\pi^4}{4!} + \frac{\pi^6}{6!} - \frac{\pi^8}{8!} + \dots = 2$$

ii Equating imaginary parts in (*) gives $0 = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$

$$\therefore \pi + \frac{\pi^5}{5!} + \dots = \frac{\pi^3}{3!} + \frac{\pi^7}{7!} + \dots$$

152 a The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{k=0}^{\infty} x^k$ which converges when $|x| < 1$.

$$\therefore \text{for } |r| < 1, \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

b $\frac{1}{2-x} = \frac{1}{1-(x-1)}$
 $= \sum_{k=0}^{\infty} (x-1)^k \quad \{\text{using a}\}$

which converges provided $|x-1| < 1$ which is when $0 < x < 2$.

c Let $f(x) = \frac{1}{2-x} = (2-x)^{-1}$

$$\therefore f'(x) = -(2-x)^{-2}(-1) = (2-x)^{-2}$$

$$\therefore f''(x) = -2(2-x)^{-3}(-1) = 2(2-x)^{-3}$$

\vdots

$$\therefore f^{(k)}(x) = k! (2-x)^{-(k+1)}$$

$$\therefore f^{(k)}(0) = k! 2^{-(k+1)} \quad \text{for all } k \in \mathbb{Z}^+.$$

Since $f(0) = \frac{1}{2}$, the Maclaurin series representation for $f(x)$ is $f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{k! 2^{-(k+1)}}{k!} x^k$

$$\therefore \frac{1}{2-x} = \sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2} \right)^k$$

This expansion converges provided $\left| \frac{x}{2} \right| < 1$ which is when $-2 < x < 2$.

d When $x = \frac{3}{4}$, $\sum_{k=0}^{\infty} (x-1)^k = \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k$ which is a geometric series with first term $u_1 = 1$ and common ratio $r = -\frac{1}{4}$.

$$\begin{aligned} \therefore \sum_{k=0}^{\infty} (x-1)^k &= \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \\ &= \frac{1}{1 - \left(-\frac{1}{4}\right)} \quad \left\{ S = \frac{u_1}{1-r} \right\} \\ &= \frac{1}{\frac{5}{4}} \\ &= \frac{4}{5} \end{aligned}$$

Also when $x = \frac{3}{4}$, $\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2} \right)^k = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{3}{8} \right)^k$.

Now $\sum_{k=0}^{\infty} \left(\frac{3}{8}\right)^k$ is a geometric series with first term $u_1 = 1$ and common ratio $r = \frac{3}{8}$.

$$\begin{aligned}\therefore \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k &= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{3}{8}\right)^k \\ &= \frac{1}{2} \left(\frac{1}{1 - \frac{3}{8}} \right) \quad \left\{ S = \frac{u_1}{1 - r} \right\} \\ &= \frac{1}{2} \left(\frac{1}{\frac{5}{8}} \right) \\ &= \frac{1}{2} \left(\frac{8}{5} \right) \\ &= \frac{4}{5}\end{aligned}$$

So, $\sum_{k=0}^{\infty} (x-1)^k = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$ when $x = \frac{3}{4}$.

From **b**, $\sum_{k=0}^{\infty} (x-1)^k = \frac{1}{2-x}$ when $0 < x < 2$.

From **c**, $\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{2-x}$ when $-2 < x < 2$.

$\therefore \sum_{k=0}^{\infty} (x-1)^k = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$ when $0 < x < 2$.

153 a $\ln(1-x^2) = (-x^2) - \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3} - \frac{(-x^2)^4}{4} + \dots$

$$= -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \dots$$

The series is valid whenever $1 - x^2 > 0$

$$\therefore x^2 < 1$$

$$\therefore -1 < x < 1$$

b i $f(0) = \ln 1 = 0$

\therefore the y -intercept is zero.

$$f(x) = 0 \text{ when } \ln(1-x^2) = 0$$

$$\therefore 1 - x^2 = 1$$

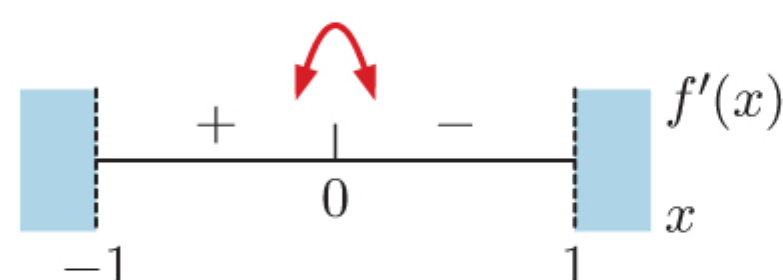
$$\therefore x^2 = 0$$

$$\therefore x = 0$$

\therefore the only x -intercept is zero.

iii $f'(x) = \frac{-2x}{1-x^2}$

$$\therefore f'(x) = 0 \text{ when } x = 0$$



\therefore there is a local maximum at $(0, 0)$.

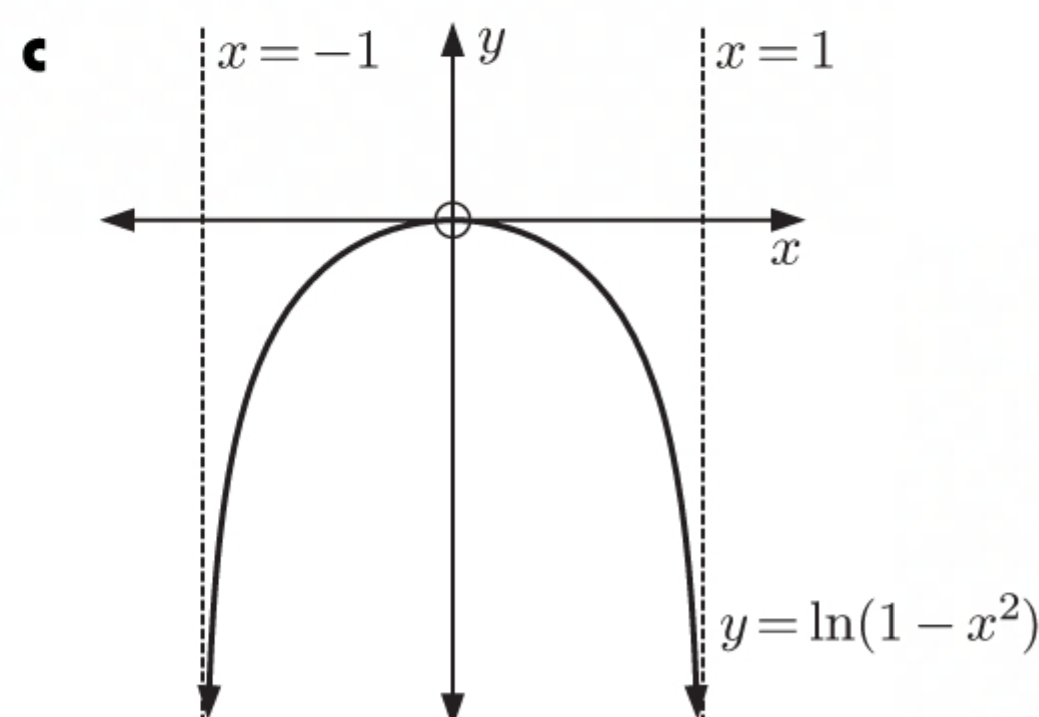
ii As $x \rightarrow -1^+$, $1 - x^2 \rightarrow 0^+$

$$\therefore \ln(1-x^2) \rightarrow -\infty$$

As $x \rightarrow 1^-$, $1 - x^2 \rightarrow 0^+$

$$\therefore \ln(1-x^2) \rightarrow -\infty$$

\therefore the two vertical asymptotes are $x = \pm 1$.



$$\begin{aligned}
 \mathbf{154} \quad \mathbf{a} \quad \cos^2 x &= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots\right) \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots\right) \\
 &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\
 &\quad - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{48}x^6 + \dots \\
 &\quad + \frac{1}{24}x^4 - \frac{1}{48}x^6 + \dots \\
 &\quad - \frac{1}{720}x^6 + \dots \\
 &= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{1}{2} + \frac{1}{2} \cos 2x &= \frac{1}{2} + \frac{1}{2} \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots\right) \\
 &= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots
 \end{aligned}$$

which matches the terms up to x^6 of the expansion of $\cos^2 x$ in **a**.

$$\mathbf{155} \quad \mathbf{a} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\text{Let } \sec x = \sum_{k=0}^{\infty} a_k x^k$$

$$\therefore \frac{1}{\cos x} = \sum_{k=0}^{\infty} a_k x^k$$

$$\begin{aligned}
 \therefore 1 &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) \\
 &= a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 - \frac{a_1}{2}\right)x^3 + \left(a_4 - \frac{a_2}{2} + \frac{a_0}{24}\right)x^4 + \dots
 \end{aligned}$$

Equating coefficients, $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 0$, $a_4 = \frac{5}{24}$

$$\therefore \sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots$$

$$\begin{aligned}
 \mathbf{b} \quad \sec^2 x &= \left(1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots\right) \left(1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots\right) \\
 &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots \\
 &\quad + \frac{1}{2}x^2 + \frac{1}{4}x^4 + \dots \\
 &\quad + \frac{5}{24}x^4 + \dots \\
 &= 1 + x^2 + \frac{2}{3}x^4 + \dots
 \end{aligned}$$

$$\mathbf{c} \quad \int \sec^2 x \, dx = \tan x + c$$

$$\begin{aligned}
 \therefore \int_0^x \sec^2 t \, dt &= [\tan t]_0^x \\
 &= \tan x - \tan 0 \\
 &= \tan x
 \end{aligned}$$

$$\begin{aligned}
 \therefore \tan x &= \int_0^x (1 + t^2 + \frac{2}{3}t^4 + \dots) \, dt \\
 &= \left[t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \dots\right]_0^x \\
 &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots
 \end{aligned}$$

$$\mathbf{156} \quad \mathbf{a} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\begin{aligned}
 \text{Now } \frac{x^2}{e^x} &= x^2 e^{-x} \\
 &= x^2 \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots\right) \\
 &= x^2 \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) \\
 &= x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \dots \\
 &= x^2 - x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad &\int_{-0.1}^{0.1} \frac{x^2}{e^x} \, dx \\
 &\approx \int_{-0.1}^{0.1} (x^2 - x^3) \, dx \quad \{\text{using a}\} \\
 &\approx \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{-0.1}^{0.1} \\
 &\approx \left(\frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4\right) - \left(\frac{1}{3}(-0.1)^3 - \frac{1}{4}(-0.1)^4\right) \\
 &\approx \frac{1}{1500} \approx 6.67 \times 10^{-4}
 \end{aligned}$$

157 Since f is continuous and can be differentiated infinitely many times, its Maclaurin series $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$ exists.

Since f is even, $f(-x) = f(x)$ for all x in the domain of f .

$$\begin{aligned}\text{Now } f(-x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)(-x)^k}{k!} \\ &= \sum_{\text{even } k} \frac{f^{(k)}(0)x^k}{k!} + \sum_{\text{odd } k} \frac{-f^{(k)}(0)x^k}{k!}\end{aligned}$$

The series for $f(x)$ and $f(-x)$ are equal polynomials, so we can equate coefficients.

$$\begin{aligned}\therefore \text{ for any odd } k, \quad \frac{f^{(k)}(0)}{k!} &= -\frac{f^{(k)}(0)}{k!} \\ \therefore f^{(k)}(0) &= 0 \quad \text{for all odd } k.\end{aligned}$$

$$\begin{aligned}\text{The } t\text{th derivative of } f \text{ is } f^{(t)}(x) &= \sum_{k=t}^{\infty} \frac{k(k-1)\dots(k-t+1)f^{(k)}(0)x^{k-t}}{k!} \\ &= \sum_{k=t}^{\infty} \frac{f^{(k)}(0)x^{k-t}}{(k-t)!} \\ &= \sum_{\text{even } k \geq t} \frac{f^{(k)}(0)x^{k-t}}{(k-t)!}\end{aligned}$$

$$\therefore f^{(t)}(-x) = \sum_{\text{even } k \geq t} \frac{f^{(k)}(0)(-x)^{k-t}}{(k-t)!}$$

$$\text{For all even } t, \quad f^{(t)}(-x) = \sum_{\text{even } k \geq t} \frac{f^{(k)}(0)x^{k-t}}{(k-t)!} = f^{(t)}(x)$$

$\therefore f^{(t)}(x)$ is an even function for all even t .

158 $\frac{dy}{dx} = e^x - 2x, \quad y(0) = 1$

$y(0) = 1$ gives us $x_0 = 0$ and $y_0 = 1$.

a i

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	1	1	0.5	1.5
2	0.5	1.5	0.6487	1	1.8244

$$\therefore y(1) \approx 1.8244$$

ii

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	1	1	0.25	1.25
2	0.25	1.25	0.7840	0.5	1.4460
3	0.5	1.4460	0.6487	0.75	1.6082
4	0.75	1.6082	0.6170	1	1.7624

$$\therefore y(1) \approx 1.7624$$

b Using the Fundamental Theorem of Calculus,

$$\begin{aligned}y(1) &= y(0) + \int_0^1 \frac{dy}{dx} dx \\ &= 1 + \int_0^1 (e^x - 2x) dx \\ &= 1 + [e^x - x^2]_0^1 \\ &= 1 + (e - 1) - (1 - 0) \\ &= e - 1 \\ &\approx 1.7183\end{aligned}$$

The accuracy of Euler's method was improved by decreasing the step size.

159 $\frac{dy}{dx} = x - y, \quad y(1) = 2$

$y(1) = 2$ gives us $x_0 = 1$ and $y_0 = 2$.

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	1	2	-1	1.2	1.8
2	1.2	1.8	-0.6	1.4	1.68
3	1.4	1.68	-0.28	1.6	1.624
4	1.6	1.624	-0.024	1.8	1.6192
5	1.8	1.6192	0.1808	2	1.65536

$\therefore y(2) \approx 1.65536$

160 a $\frac{dy}{dx} = \cos x - 2 \sin x$

$$\begin{aligned}\therefore y &= \int (\cos x - 2 \sin x) dx \\ &= \sin x + 2 \cos x + c\end{aligned}$$

But $y(3\pi) = 2$, so $2 = \sin 3\pi + 2 \cos 3\pi + c$

$$\therefore c = 2 + 2 = 4$$

So the solution is $y = \sin x + 2 \cos x + 4$.

b $\frac{d^2y}{dx^2} = 6x^2 - 15x^{\frac{1}{2}}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \int (6x^2 - 15x^{\frac{1}{2}}) dx \\ &= 2x^3 - 10x^{\frac{3}{2}} + c\end{aligned}$$

But $y'(1) = 1$, so $1 = 2 - 10 + c$

$$\therefore c = 9$$

and so $\frac{dy}{dx} = 2x^3 - 10x^{\frac{3}{2}} + 9$

$$\begin{aligned}\therefore y &= \int (2x^3 - 10x^{\frac{3}{2}} + 9) dx \\ &= \frac{1}{2}x^4 - 4x^{\frac{5}{2}} + 9x + d\end{aligned}$$

But $y(1) = 2$, so $2 = \frac{1}{2} - 4 + 9 + d$

$$\therefore d = -\frac{7}{2}$$

So the solution is $y = \frac{1}{2}x^4 - 4x^{\frac{5}{2}} + 9x - \frac{7}{2}$.

c $\sqrt{3-t^2} \frac{dR}{dt} = 4t$

$$\therefore \frac{dR}{dt} = \frac{4t}{\sqrt{3-t^2}}$$

$$\therefore R = \int \frac{4t}{\sqrt{3-t^2}} dt$$

Let $u = 3 - t^2 \quad \therefore \frac{du}{dt} = -2t$

$$\therefore R = \int \frac{1}{\sqrt{u}} \left(-2 \frac{du}{dt}\right) dt$$

$$= -2 \int u^{-\frac{1}{2}} du$$

$$= -2 \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) + c$$

$$= -4\sqrt{3-t^2} + c$$

But $R(1) = 4$, so $4 = -4\sqrt{2} + c$

$$\therefore c = 4 + 4\sqrt{2}$$

So the solution is $R = -4\sqrt{3-t^2} + 4 + 4\sqrt{2}$.

161 a $\frac{dy}{dx} = e^{2x} - \cos x$

$$\therefore y = \int (e^{2x} - \cos x) dx$$

$$= \frac{1}{2}e^{2x} - \sin x + c$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dM}{dt} &= t\sqrt{t^2 + 5} \\
 \therefore M &= \int t\sqrt{t^2 + 5} dt \\
 \text{Let } u &= t^2 + 5 \quad \therefore \frac{du}{dt} = 2t \\
 \therefore M &= \int \sqrt{u} \left(\frac{1}{2} \frac{du}{dt} \right) dt \\
 &= \frac{1}{2} \int u^{\frac{1}{2}} du \\
 &= \frac{1}{2} \left(\frac{2}{3} u^{\frac{3}{2}} \right) + c \\
 &= \frac{1}{3} (t^2 + 5)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\mathbf{162} \quad \mathbf{a} \quad \text{If } y = x \ln x, \text{ then } \frac{dy}{dx} = (1) \ln x + x \left(\frac{1}{x} \right) = \ln x + 1$$

$$\begin{aligned}
 \mathbf{b} \quad T' &= \frac{1}{20} (1 + \ln N) \\
 \therefore T &= \int \frac{1}{20} (1 + \ln N) dN \\
 &= \frac{1}{20} N \ln N + c \quad \{\text{using } \mathbf{a}\} \\
 \text{Now when } N &= 50, T = 10, \text{ so } 10 = \frac{50}{20} \ln 50 + c \\
 \therefore c &= 10 - \frac{5}{2} \ln 50 \\
 \text{So if } N &= 100, T = 5 \ln 100 + 10 - \frac{5}{2} \ln 50 \\
 &\approx 23 \text{ seconds}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{163} \quad \mathbf{a} \quad \frac{dy}{dx} &= xy^2 \\
 \therefore \frac{1}{y^2} \frac{dy}{dx} &= x \\
 \therefore \int \frac{1}{y^2} \frac{dy}{dx} dx &= \int x dx \\
 \therefore \int y^{-2} dy &= \int x dx \\
 \therefore -y^{-1} &= \frac{1}{2} x^2 + c \\
 \therefore \frac{1}{y} &= c - \frac{1}{2} x^2 \\
 \therefore y &= \frac{1}{c - \frac{1}{2} x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \frac{dy}{dx} &= \frac{xy}{x^2 + 1} \\
 \therefore \frac{1}{y} \frac{dy}{dx} &= \frac{x}{x^2 + 1} \\
 \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int \frac{x}{x^2 + 1} dx \\
 \therefore \int \frac{1}{y} dy &= \frac{1}{2} \int \frac{2x}{x^2 + 1} dx \\
 \therefore \ln |y| &= \frac{1}{2} \ln |x^2 + 1| + c \\
 &= \frac{1}{2} \ln(x^2 + 1) + c \quad \{x^2 + 1 > 0 \text{ for all } x\} \\
 \therefore |y| &= e^c \sqrt{x^2 + 1} \\
 \therefore y &= \pm e^c \sqrt{x^2 + 1} \\
 &= A \sqrt{x^2 + 1} \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \frac{dP}{dx} - \sin^2 x &= \tan x \\
 \therefore \frac{dP}{dx} &= \sin^2 x + \tan x \\
 \therefore P &= \int (\sin^2 x + \tan x) dx \\
 &= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x + \frac{\sin x}{\cos x} \right) dx \\
 &= \frac{1}{2} x - \frac{1}{4} \sin 2x - \ln |\cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dy}{dx} &= 5\sqrt{y} \\
 \therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} &= 5 \\
 \therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dx} dx &= \int 5 dx \\
 \therefore \int y^{-\frac{1}{2}} dy &= \int 5 dx \\
 \therefore 2y^{\frac{1}{2}} &= 5x + c \\
 \therefore y^{\frac{1}{2}} &= \frac{5}{2} x + c \\
 \therefore y &= \left(\frac{5}{2} x + c \right)^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{164} \quad \mathbf{a} \quad \frac{dP}{dz} &= -3P^2z \\
 \therefore \frac{1}{P^2} \frac{dP}{dz} &= -3z \\
 \therefore \int \frac{1}{P^2} \frac{dP}{dz} dz &= \int -3z dz \\
 \therefore \int P^{-2} dP &= -3 \int z dz \\
 \therefore \frac{P^{-1}}{-1} &= -3 \left(\frac{z^2}{2} \right) + c \\
 \therefore -\frac{1}{P} &= -\frac{3z^2}{2} + c
 \end{aligned}$$

But $P(2) = 1$, so $-1 = -6 + c$
 $\therefore c = 5$

$$\begin{aligned}
 \therefore -\frac{1}{P} &= -\frac{3}{2}z^2 + 5 \\
 \therefore P &= \frac{1}{\frac{3}{2}z^2 - 5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dy}{dx} &= x + \frac{1}{3}xy = \frac{1}{3}x(3 + y) \\
 \frac{1}{3+y} \frac{dy}{dx} &= \frac{1}{3}x \\
 \therefore \int \frac{1}{3+y} \frac{dy}{dx} dx &= \int \frac{1}{3}x dx \\
 \therefore \int \frac{1}{3+y} dy &= \frac{1}{3} \int x dx \\
 \therefore \ln|y+3| &= \frac{1}{3} \frac{x^2}{2} + c \\
 &= \frac{1}{6}x^2 + c \\
 \therefore |y+3| &= e^c e^{\frac{1}{6}x^2} \\
 \therefore y+3 &= \pm e^c e^{\frac{1}{6}x^2} \\
 \therefore y+3 &= A e^{\frac{1}{6}x^2} \quad \{A = \pm e^c\} \\
 \therefore y &= A e^{\frac{1}{6}x^2} - 3
 \end{aligned}$$

But when $x = 1$, $y = 2$, so $2 = A e^{\frac{1}{6}} - 3$
 $\therefore A = 5e^{-\frac{1}{6}}$

$$\begin{aligned}
 \therefore y &= 5e^{-\frac{1}{6}} e^{\frac{1}{6}x^2} - 3 \\
 &= 5e^{\frac{1}{6}(x^2-1)} - 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{165} \quad \mathbf{a} \quad x^2 + 4x + 3 &= (x+3)(x+1) \\
 \text{Let } \frac{x-5}{x^2+4x+3} &= \frac{A}{x+3} + \frac{B}{x+1} \\
 \therefore x-5 &= A(x+1) + B(x+3) \\
 \text{Substituting } x &= -1, \quad -1-5 = 2B \\
 \therefore B &= -3 \\
 \text{Substituting } x &= -3, \quad -3-5 = -2A \\
 \therefore A &= 4 \\
 \therefore \frac{x-5}{x^2+4x+3} &= \frac{4}{x+3} - \frac{3}{x+1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad (x^2 + 4x + 3) \frac{dy}{dx} &= \frac{x-5}{y^2} \\
 \therefore y^2 \frac{dy}{dx} &= \frac{x-5}{x^2+4x+3} \\
 &= \frac{4}{x+3} - \frac{3}{x+1} \quad \{\text{using } \mathbf{a}\} \\
 \therefore \int y^2 \frac{dy}{dx} dx &= \int \left(\frac{4}{x+3} - \frac{3}{x+1} \right) dx \\
 \therefore \int y^2 dy &= \int \left(\frac{4}{x+3} - \frac{3}{x+1} \right) dx \\
 \therefore \frac{1}{3}y^3 &= 4 \ln|x+3| - 3 \ln|x+1| + c \\
 \therefore y^3 &= 12 \ln|x+3| - 9 \ln|x+1| + c \\
 \therefore y &= \sqrt[3]{12 \ln|x+3| - 9 \ln|x+1| + c}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{166} \quad \frac{dy}{dx} &= \frac{2x}{\sin y} \\
 \therefore \sin y \frac{dy}{dx} &= 2x \\
 \therefore \int \sin y \frac{dy}{dx} dx &= \int 2x dx \\
 \therefore \int \sin y dy &= \int 2x dx \\
 \therefore -\cos y &= x^2 + c \\
 \therefore \cos y &= c - x^2 \\
 \therefore y &= \arccos(c - x^2)
 \end{aligned}$$

But $y(0) = \frac{\pi}{6}$, so $\frac{\pi}{6} = \arccos c$
 $\therefore c = \frac{\sqrt{3}}{2}$

So, the particular solution is $y = \arccos\left(\frac{\sqrt{3}}{2} - x^2\right)$ which is defined where

$$\begin{aligned}
 -1 &\leq \frac{\sqrt{3}}{2} - x^2 \leq 1 \\
 \therefore -1 &\leq x^2 - \frac{\sqrt{3}}{2} \leq 1 \\
 \therefore -1 + \frac{\sqrt{3}}{2} &\leq x^2 \leq 1 + \frac{\sqrt{3}}{2} \\
 \therefore 0 &\leq x^2 \leq 1 + \frac{\sqrt{3}}{2} \quad \left\{ -1 + \frac{\sqrt{3}}{2} < 0 \right\} \\
 \therefore -\sqrt{1 + \frac{\sqrt{3}}{2}} &\leq x \leq \sqrt{1 + \frac{\sqrt{3}}{2}}
 \end{aligned}$$

167 a $\frac{dP}{dt} \propto P$

$$\therefore \frac{dP}{dt} = kP, \text{ where } k \text{ is a constant}$$

$$\therefore \frac{1}{P} \frac{dP}{dt} = k$$

$$\therefore \int \frac{1}{P} \frac{dP}{dt} dt = \int k dt$$

$$\therefore \int \frac{1}{P} dP = kt + c$$

$$\therefore \ln |P| = kt + c$$

$$\therefore \ln P = kt + c \quad \{\text{as } P > 0\}$$

$$\therefore P = e^{kt+c}$$

$$\therefore P = Ae^{kt} \quad \{A = e^c\}$$

d i $\frac{dN}{dt} \propto P$

$$\therefore \frac{dN}{dt} = dP \quad \text{where } d \text{ is a constant}$$

$$\therefore \frac{dN}{dt} = de^{\frac{t}{3} \ln 2}$$

But when $t = 0$, $\frac{dN}{dt} = 0.05$

$$\therefore 0.05 = de^0 = d$$

So, $\frac{dN}{dt} = 0.05e^{\frac{t}{3} \ln 2}$ units h^{-1}

b When $t = 0$, $P = 1$ million, $\therefore 1 = Ae^0 = A$

$$\therefore P = e^{kt} \text{ million}$$

When $t = 3$, $P = 2$ million

$$\therefore 2 = e^{3k}$$

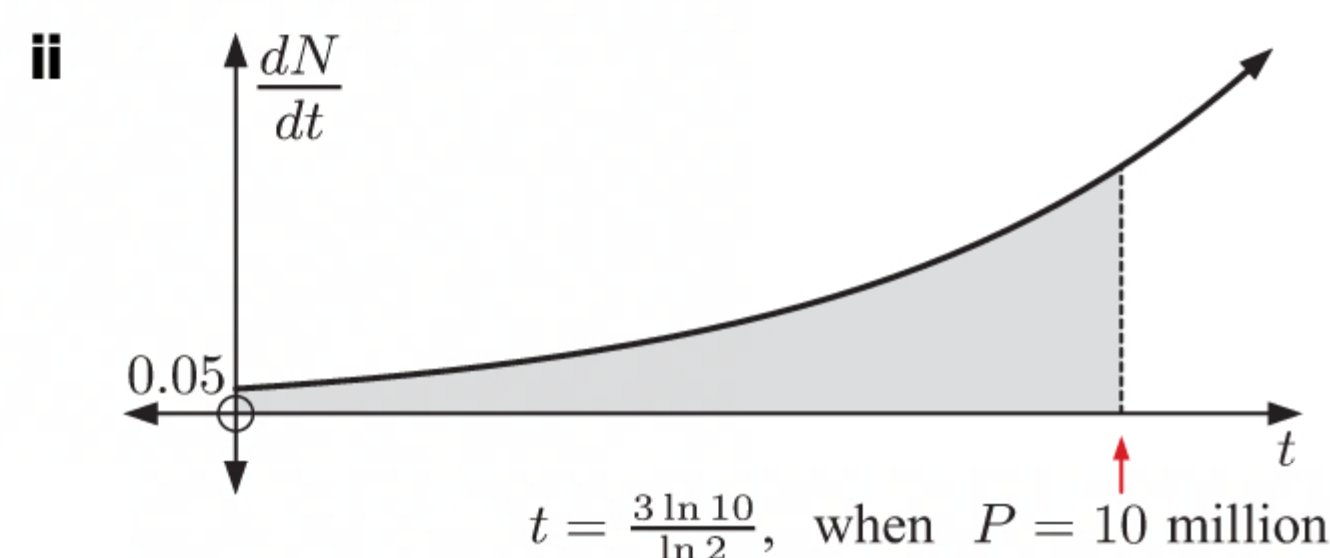
$$\therefore 3k = \ln 2$$

$$\therefore k = \frac{1}{3} \ln 2 \quad (\approx 0.2310)$$

c $P = 10$ million when $10 = e^{\frac{t}{3} \ln 2}$

$$\frac{t}{3} \ln 2 = \ln 10$$

$$t = \frac{3 \ln 10}{\ln 2} \approx 9.966 \text{ hours}$$



Amount of nutrient consumed

$$= \int_0^{\frac{3 \ln 10}{\ln 2}} 0.05e^{\frac{t}{3} \ln 2} dt$$

$$= 0.05 \left[\frac{1}{\frac{1}{3} \ln 2} e^{\frac{t}{3} \ln 2} \right]_0^{\frac{3 \ln 10}{\ln 2}}$$

$$= \frac{0.05}{\frac{1}{3} \ln 2} \left[e^{\frac{t}{3} \ln 2} \right]_0^{\frac{3 \ln 10}{\ln 2}}$$

$$= \frac{0.15}{\ln 2} [10 - 1]$$

$$= \frac{1.35}{\ln 2}$$

$$\approx 1.95 \text{ g}$$

168 a We are given that $\frac{dV}{dt} \propto \sqrt{V}$

$$\therefore \frac{dV}{dt} = -k\sqrt{V} \quad \text{where } k \text{ is a positive constant}$$

$$\therefore \frac{1}{\sqrt{V}} \frac{dV}{dt} = -k$$

$$\therefore \int \frac{1}{\sqrt{V}} \frac{dV}{dt} dt = \int -k dt$$

$$\therefore \int V^{-\frac{1}{2}} dV = \int -k dt$$

$$\therefore 2V^{\frac{1}{2}} = -kt + c$$

$$\therefore V = \left(\frac{c - kt}{2} \right)^2$$

Now when $t = 0$, $V = 100$

$$\therefore 100 = \left(\frac{c}{2} \right)^2$$

$$\therefore \frac{c}{2} = 10$$

$$\therefore c = 20$$

$$\therefore V = \left(\frac{20 - kt}{2} \right)^2$$

b When $t = 4$, $V = 100 - 19 = 81$

$$\therefore 81 = \left(\frac{20 - 4k}{2}\right)^2$$

$$\therefore 9 = 10 - 2k$$

$$\therefore 2k = 1$$

$$\therefore k = \frac{1}{2}$$

$$\text{So, } V = \left(\frac{20 - \frac{1}{2}t}{2}\right)^2 = \left(\frac{40 - t}{4}\right)^2$$

$$\text{Now } V = 0 \text{ when } \left(\frac{40 - t}{4}\right)^2 = 0$$

$$\therefore 40 - t = 0$$

$$\therefore t = 40$$

\therefore the vessel will be empty after 40 hours.

169 a $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

$$= \frac{x}{y} + \frac{y}{x}$$

$$= \frac{1}{\frac{y}{x}} + \frac{y}{x} \quad \text{so the differential equation is homogeneous.}$$

$$\text{Let } y = vx, \text{ so } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \{\text{product rule}\}$$

$$\text{Comparing with the differential equation } v + x \frac{dv}{dx} = \frac{1}{v} + v$$

$$\therefore x \frac{dv}{dx} = \frac{1}{v}$$

$$\therefore v \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int v \frac{dv}{dx} dx = \int \frac{1}{x} dx$$

$$\therefore \int v dx = \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2}v^2 = \ln|x| + c$$

$$\therefore v^2 = 2 \ln|x| + c$$

$$\therefore \left(\frac{y}{x}\right)^2 = 2 \ln|x| + c$$

$$\therefore y^2 = x^2(2 \ln|x| + c)$$

b When $x = 1$, $y = 4$

$$\therefore 16 = 1(2 \ln 1 + c)$$

$$\therefore 16 = c$$

The particular solution is $y^2 = x^2(2 \ln|x| + 16)$

170 $\frac{dy}{dx} = \frac{y}{x}(\ln y - \ln x + 1)$
 $= \frac{y}{x} \left(\ln \left(\frac{y}{x} \right) + 1 \right)$ so the differential equation is homogeneous.

Let $y = vx$, so $\frac{dy}{dx} = v + x \frac{dv}{dx}$ {product rule}

Comparing with the differential equation $v + x \frac{dv}{dx} = v(\ln v + 1)$

$$\therefore v + x \frac{dv}{dx} = v \ln v + v$$

$$\therefore x \frac{dv}{dx} = v \ln v$$

$$\therefore \frac{1}{v \ln v} \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int \frac{1}{v \ln v} \frac{dv}{dx} dx = \int \frac{1}{x} dx$$

$$\therefore \int \frac{1}{v \ln v} dv = \int \frac{1}{x} dx$$

Let $u = \ln v$ $\therefore \frac{du}{dv} = \frac{1}{v}$

$$\therefore \int \frac{1}{u} \left(\frac{du}{dv} \right) dv = \int \frac{1}{x} dx$$

$$\therefore \int \frac{1}{u} du = \int \frac{1}{x} dx$$

$$\therefore \ln |u| = \ln |x| + c$$

$$\therefore \ln \left| \frac{u}{x} \right| = c$$

$$\therefore \left| \frac{u}{x} \right| = e^c$$

$$\therefore \frac{u}{x} = \pm e^c$$

$$\therefore u = Ax \quad \{A = \pm e^c\}$$

$$\therefore \ln v = Ax$$

$$\therefore v = e^{Ax}$$

$$\therefore \frac{y}{x} = e^{Ax}$$

$$\therefore y = xe^{Ax}$$

171 $\frac{dy}{dx} + 4xy = x$

a The integrating factor is $I(x) = e^{\int 4x dx} = e^{2x^2}$.

b Multiplying both sides of the differential equation by e^{2x^2} gives $e^{2x^2} \frac{dy}{dx} + 4xe^{2x^2} y = xe^{2x^2}$

$$\therefore \frac{d}{dx} (ye^{2x^2}) = xe^{2x^2}$$

$$\therefore ye^{2x^2} = \int xe^{2x^2} dx$$

$$\therefore ye^{2x^2} = \frac{1}{4}e^{2x^2} + c$$

$$\therefore y = \frac{1}{4} + ce^{-2x^2}$$

$$172 \quad \frac{dy}{dx} = x - 2y$$

$$\therefore \frac{dy}{dx} + 2y = x$$

The integrating factor is $I(x) = e^{\int 2 dx} = e^{2x}$.

Multiplying both sides of the differential equation by e^{2x} gives

$$e^{2x} \frac{dy}{dx} + 2e^{2x}y = xe^{2x}$$

$$\therefore \frac{d}{dx}(ye^{2x}) = xe^{2x}$$

$$\therefore ye^{2x} = \int xe^{2x} dx$$

$$\therefore ye^{2x} = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \quad \begin{cases} u = x & v' = e^{2x} \\ u' = 1 & v = \frac{1}{2}e^{2x} \end{cases}$$

$$\therefore ye^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

$$\therefore y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}$$

$$173 \quad \frac{dy}{dx} + y \sin x = e^{\cos x}$$

The integrating factor is $I(x) = e^{\int \sin x dx} = e^{-\cos x}$.

Multiplying both sides of the differential equation by $e^{-\cos x}$ gives $e^{-\cos x} \frac{dy}{dx} + ye^{-\cos x} \sin x = 1$

$$\therefore \frac{d}{dx}(ye^{-\cos x}) = 1$$

$$\therefore ye^{-\cos x} = \int 1 dx$$

$$\therefore ye^{-\cos x} = x + c$$

$$\therefore y = e^{\cos x}(x + c)$$

But $y(\pi) = \frac{1}{e}$, so $\frac{1}{e} = e^{-1}(\pi + c)$

$$\therefore 1 = \pi + c$$

$$\therefore c = 1 - \pi$$

The particular solution is $y = e^{\cos x}(x + 1 - \pi)$.

$$174 \quad \frac{d^2x}{dt^2} = -k^2x$$

$$\text{Let } x = \sum_{m=0}^{\infty} a_m t^m$$

$$\therefore x'' = \sum_{m=2}^{\infty} m(m-1)a_m t^{m-2}$$

$$\therefore \sum_{m=2}^{\infty} m(m-1)a_m t^{m-2} + k^2 \sum_{m=0}^{\infty} a_m t^m = 0$$

$$\therefore \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + k^2 \sum_{m=0}^{\infty} a_m t^m = 0 \quad \{\text{let } n = m - 2 \text{ in first sum}\}$$

$$\therefore \sum_{m=0}^{\infty} [(m+2)(m+1)a_{m+2} + k^2 a_m] t^m = 0 \quad \{\text{let } m = n \text{ in first sum}\}$$

$$\therefore (m+2)(m+1)a_{m+2} + k^2 a_m = 0 \quad \text{for all } m \geq 0$$

$$\therefore a_{m+2} = -\frac{k^2}{(m+2)(m+1)} a_m \quad \text{for all } m \geq 0$$

So, we need to separate into odd and even m :


$$\text{even } m: \quad a_{2p} = \frac{(-1)^p k^{2p}}{(2p)!} a_0 \quad \text{for } p \in \mathbb{N}$$

$$\text{odd } m: \quad a_{2p+1} = \frac{(-1)^p k^{2p}}{(2p+1)!} a_1 \quad \text{for } p \in \mathbb{N}$$

$$\begin{aligned} \text{Thus, } x &= a_0 \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} k^{2p} t^{2p} + \frac{a_1}{k} \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!} k^{2p+1} t^{2p+1} \\ &= a_0 \cos(kt) + \frac{a_1}{k} \sin(kt) \end{aligned}$$

MIXED QUESTIONS

MIXED QUESTIONS SET 1

- 1 $y = 2x^2 - 9x + 3$ has $a = 2$, $b = -9$, and $c = 3$. Since $a > 0$, the shape is .

a $\frac{-b}{2a} = \frac{-(-9)}{2(2)} = \frac{9}{4}$

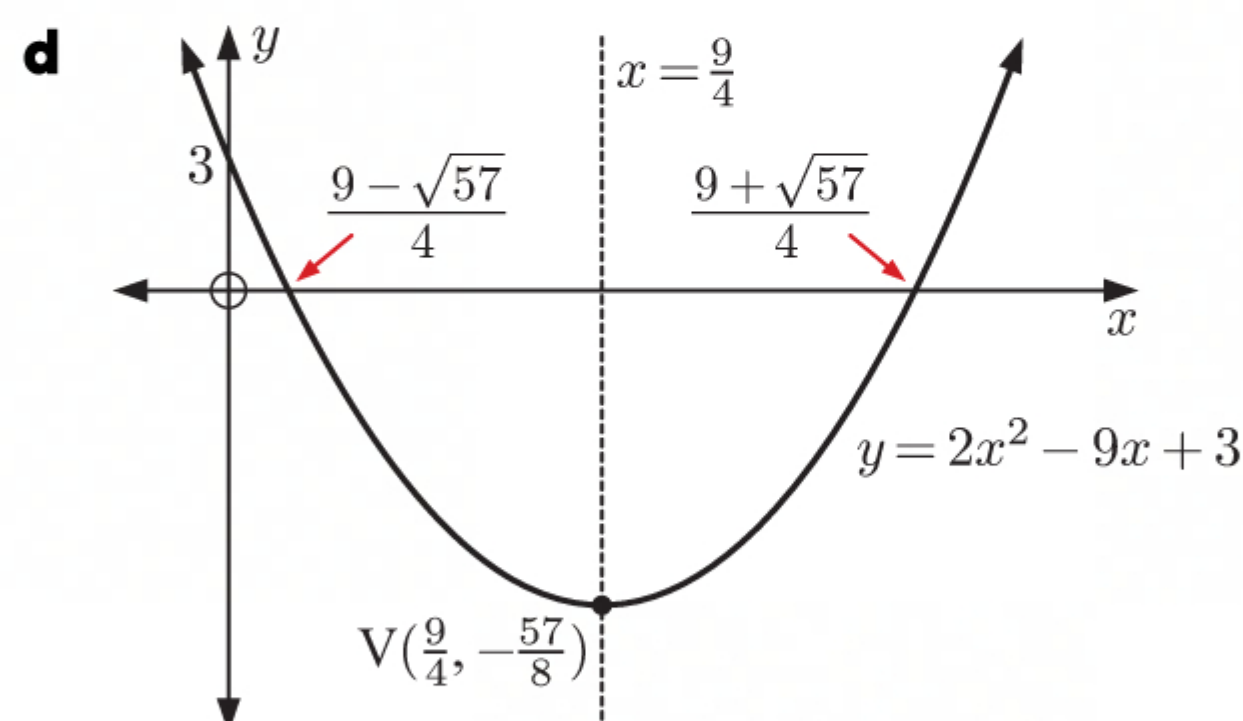
The axis of symmetry is $x = \frac{9}{4}$.

- c The y -intercept is 3.

$$\begin{aligned} \text{When } y = 0, \\ 2x^2 - 9x + 3 &= 0 \\ \therefore x^2 - \frac{9}{2}x + \frac{3}{2} &= 0 \\ \therefore x^2 - \frac{9}{2}x &= -\frac{3}{2} \\ \therefore x^2 - \frac{9}{2}x + \left(-\frac{9}{4}\right)^2 &= -\frac{3}{2} + \left(-\frac{9}{4}\right)^2 \\ \therefore \left(x - \frac{9}{4}\right)^2 &= \frac{57}{16} \\ \therefore x - \frac{9}{4} &= \pm \frac{\sqrt{57}}{4} \\ \therefore x &= \frac{9 \pm \sqrt{57}}{4} \\ \therefore \text{the } x\text{-intercepts are } \frac{9 \pm \sqrt{57}}{4}. \end{aligned}$$

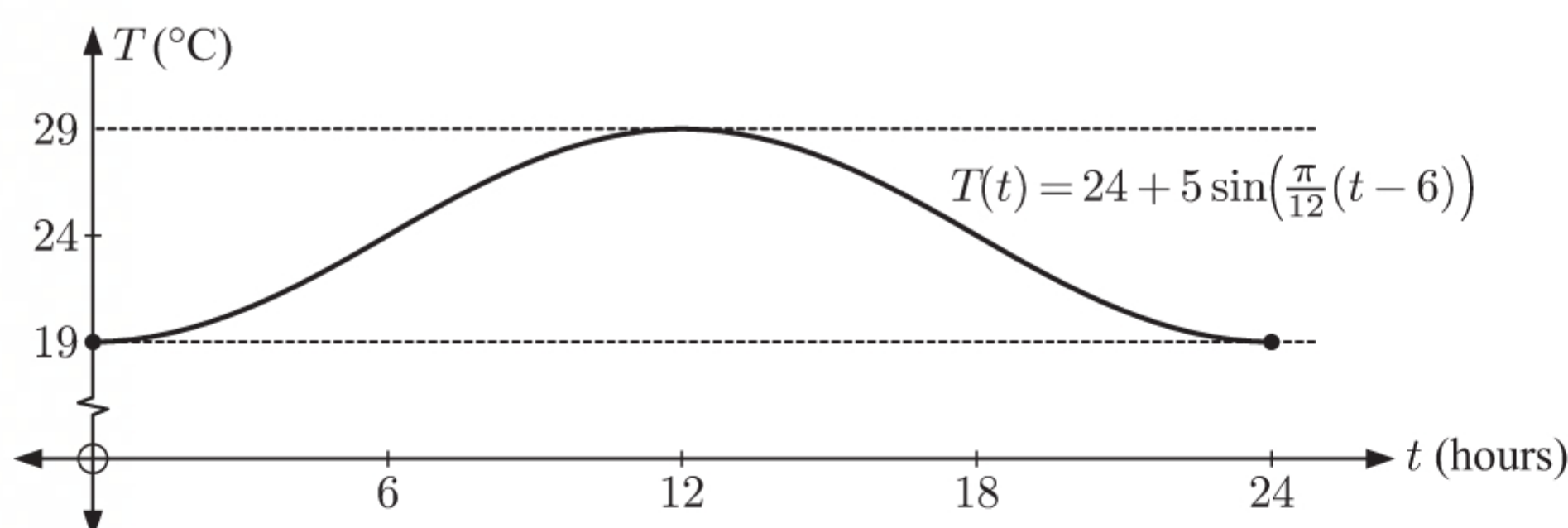
b When $x = \frac{9}{4}$, $y = 2\left(\frac{9}{4}\right)^2 - 9\left(\frac{9}{4}\right) + 3$

$$\begin{aligned} &= \frac{162}{16} - \frac{81}{4} + 3 \\ &= -\frac{57}{8} \\ \therefore \text{the vertex is } \left(\frac{9}{4}, -\frac{57}{8}\right). \end{aligned}$$



- 2 a For $T(t) = 24 + 5 \sin\left(\frac{\pi}{12}(t - 6)\right)$:

- the amplitude is 5
- the horizontal translation is 6 hours to the right
- the period is $\frac{2\pi}{(\frac{\pi}{12})} = 24$ hours
- the principal axis is $T = 24$.



- b i 2 pm is 8 hours after 6 am.

$$\begin{aligned} T(8) &= 24 + 5 \sin\left(\frac{\pi}{12}(8 - 6)\right) \\ &= 24 + 5 \sin\left(\frac{\pi}{12} \times 2\right) \\ &= 24 + 5 \sin \frac{\pi}{6} \\ &= 24 + 5 \times \frac{1}{2} \\ &= 26.5 \end{aligned}$$

\therefore at 2 pm, the temperature inside Pam's caravan is 26.5°C .

- ii 9 pm is 15 hours after 6 am.

$$\begin{aligned} T(15) &= 24 + 5 \sin\left(\frac{\pi}{12}(15 - 6)\right) \\ &= 24 + 5 \sin\left(\frac{\pi}{12} \times 9\right) \\ &= 24 + 5 \sin \frac{3\pi}{4} \\ &= 24 + 5 \times \frac{1}{\sqrt{2}} \\ &\approx 27.5 \end{aligned}$$

\therefore at 9 pm, the temperature inside Pam's caravan is about 27.5°C .

- c The maximum temperature inside Pam's caravan is $24 + 5 = 29^\circ\text{C}$, which occurs when $t = 12$. 12 hours after 6 am is 6 pm.

So, the maximum temperature inside Pam's caravan occurs at 6 pm.

3 $f(x) = \ln(x\sqrt{1-2x})$

a $\ln(x\sqrt{1-2x})$ is defined when

$$x\sqrt{1-2x} > 0$$

$$\therefore x > 0 \quad \text{and} \quad 1 - 2x > 0$$

$$\therefore x > 0 \quad \text{and} \quad 2x < 1$$

$$\therefore x > 0 \quad \text{and} \quad x < \frac{1}{2}$$

So, the domain is $\{x \mid 0 < x < \frac{1}{2}\}$.

c At the point(s) where the normal has gradient $-\frac{6}{5}$, the tangent has gradient $\frac{5}{6}$.

Now $f'(x) = \frac{5}{6}$ where $\frac{1-3x}{x(1-2x)} = \frac{5}{6}$ {from **b**}

$$\therefore 6(1-3x) = 5x(1-2x)$$

$$\therefore 6 - 18x = 5x - 10x^2$$

$$\therefore 10x^2 - 23x + 6 = 0$$

$$\therefore (10x-3)(x-2) = 0$$

$$\therefore x = \frac{3}{10} \quad \{0 < x < \frac{1}{2}\}$$

$$f\left(\frac{3}{10}\right) = \ln\left(\frac{3}{10}\sqrt{1-2\left(\frac{3}{10}\right)}\right)$$

$$= \ln\left(\frac{3}{10}\sqrt{1-\frac{3}{5}}\right)$$

$$= \ln\left(\frac{3}{10}\sqrt{\frac{2}{5}}\right)$$

$$\approx -1.66$$

\therefore the normal to $y = f(x)$ has gradient $-\frac{6}{5}$ at $\left(\frac{3}{10}, -1.66\right)$.

4 $f(x) = 5^x$, $g(x) = 2x + 1$

a $(f \circ g)(x) = f(g(x))$
 $= f(2x + 1)$
 $= 5^{2x+1}$

b $f \circ g$ is $y = 5^{2x+1}$
 $\therefore (f \circ g)^{-1}$ is $x = 5^{2y+1}$
 $\therefore \log_5 x = 2y + 1$
 $\therefore 2y = \log_5 x - 1$
 $\therefore y = \frac{1}{2} \log_5 x - \frac{1}{2}$
 $\therefore (f \circ g)^{-1}(x) = \frac{1}{2} \log_5 x - \frac{1}{2}$
 $\therefore (f \circ g)^{-1}(0.2) = \frac{1}{2} \log_5(0.2) - \frac{1}{2}$
 $= \frac{1}{2} \log_5\left(\frac{1}{5}\right) - \frac{1}{2}$
 $= \frac{1}{2} \log_5(5^{-1}) - \frac{1}{2}$
 $= -\frac{1}{2} - \frac{1}{2}$
 $= -1$

5 a Let u_0 be the original value of the car.

Since the value of the car depreciates by 10% each year, the value of the car after 3 years is
 $u_0 \times (1 - 0.1)^3 = u_0 \times (0.9)^3$.

$$\therefore u_0 \times (0.9)^3 = 26\,244$$

$$\therefore u_0 = \frac{26\,244}{(0.9)^3}$$

$$= 36\,000$$

\therefore the original value of the car is \$36 000.

b $f(x) = \ln(x(1-2x)^{\frac{1}{2}})$

$$\therefore f'(x) = \frac{(1-2x)^{\frac{1}{2}} + \frac{1}{2}x(1-2x)^{-\frac{1}{2}}(-2)}{x(1-2x)^{\frac{1}{2}}} \times \frac{(1-2x)^{\frac{1}{2}}}{(1-2x)^{\frac{1}{2}}}$$

$$= \frac{(1-2x) - x}{x(1-2x)}$$

$$= \frac{1-3x}{x(1-2x)}$$

c Under a horizontal stretch with scale factor k , $f(x)$ becomes $f\left(\frac{1}{k}x\right)$.

$$\therefore y = 5^x \quad \text{becomes} \quad y = 5^{\frac{x}{k}}$$

The resulting graph passes through $\left(\frac{1}{6}, \sqrt{5}\right)$.

$$\therefore \sqrt{5} = 5^{\frac{1}{k}\left(\frac{1}{6}\right)}$$

$$\therefore 5^{\frac{1}{2}} = 5^{\frac{1}{6k}}$$

$$\therefore \frac{1}{2} = \frac{1}{6k} \quad \{\text{equating indices}\}$$

$$\therefore 6k = 2$$

$$\therefore k = \frac{1}{3}$$

b Let u_n be the value of the car after n years.

$$\therefore u_n = u_0 \times (1-d)^n$$

$$= 36\,000 \times (0.9)^n$$

which describes a geometric sequence with
 $u_0 = 36\,000$ and $r = 0.9$.

- c** For the value of the car to fall below \$10 000, we need to find when

$$36\,000 \times (0.9)^n = 10\,000$$

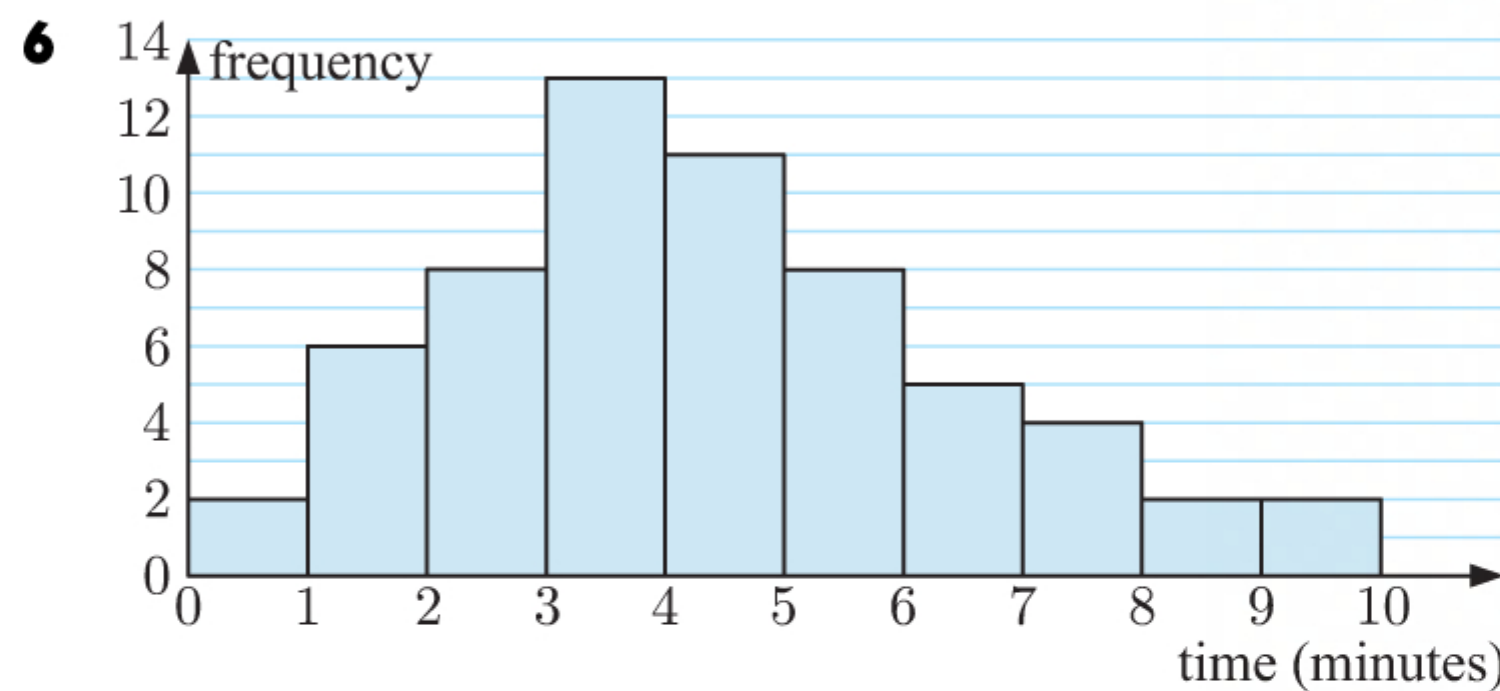
$$\therefore (0.9)^n = \frac{10\,000}{36\,000}$$

$$\therefore (0.9)^n = \frac{5}{18}$$

$$\therefore n \log(0.9) = \log\left(\frac{5}{18}\right)$$

$$\therefore n = \frac{\log\left(\frac{5}{18}\right)}{\log(0.9)} \approx 12.2$$

So, in the 13th year the value of the car falls below \$10 000.



- a** The modal class is $3 \leq t < 4$ where t is the time in minutes.

b

Duration of call (t min)	Frequency (f)	Midpoint (x)	Product (xf)
$0 \leq t < 1$	2	0.5	1
$1 \leq t < 2$	6	1.5	9
$2 \leq t < 3$	8	2.5	20
$3 \leq t < 4$	13	3.5	45.5
$4 \leq t < 5$	11	4.5	49.5
$5 \leq t < 6$	8	5.5	44
$6 \leq t < 7$	5	6.5	32.5
$7 \leq t < 8$	4	7.5	30
$8 \leq t < 9$	2	8.5	17
$9 \leq t < 10$	2	9.5	19
<i>Total</i>	$\sum f = 61$		$\sum xf = 267.5$

c
$$\bar{x} = \frac{\sum xf}{\sum f}$$

$$= \frac{267.5}{61}$$

$$\approx 4.39$$

d
$$P(\geq 6 \text{ minutes}) \approx \frac{5 + 4 + 2 + 2}{61}$$

$$\approx \frac{13}{61}$$

$$\approx 0.213$$

\therefore the mean length of a phone call is about 4.39 minutes.

7 a
$$(-1 + i\sqrt{2})^3 = (-1)^3 + 3(-1)^2 i\sqrt{2} + 3(-1)(i\sqrt{2})^2 + (i\sqrt{2})^3$$

$$= -1 + 3\sqrt{2}i + 6 - 2\sqrt{2}i$$

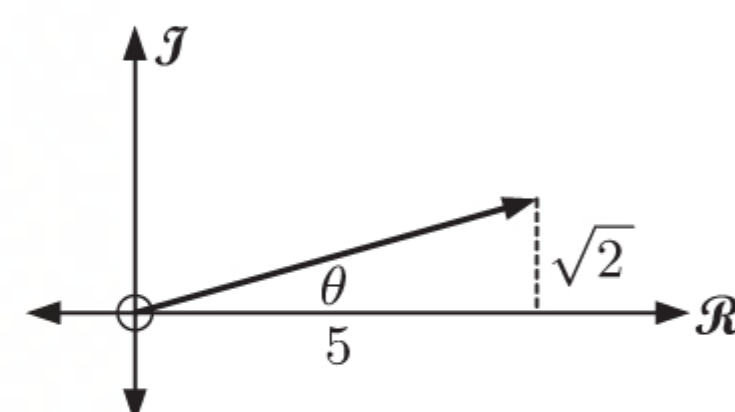
$$= 5 + i\sqrt{2}$$

b
$$|5 + i\sqrt{2}| = \sqrt{25 + 2} = \sqrt{27} = (\sqrt{3})^3$$

$$\arg(5 + i\sqrt{2}) = \theta = \arctan\left(\frac{\sqrt{2}}{5}\right)$$

$$\therefore 5 + i\sqrt{2} = (\sqrt{3})^3 \operatorname{cis}\left[\arctan\left(\frac{\sqrt{2}}{5}\right)\right]$$

$$\therefore a = \sqrt{3}, \quad \theta = \arctan\left(\frac{\sqrt{2}}{5}\right)$$



c
$$z^3 = 5 + i\sqrt{2} = (\sqrt{3})^3 \operatorname{cis}\left[\arctan\left(\frac{\sqrt{2}}{5}\right) + k2\pi\right], \quad k \in \mathbb{Z}$$

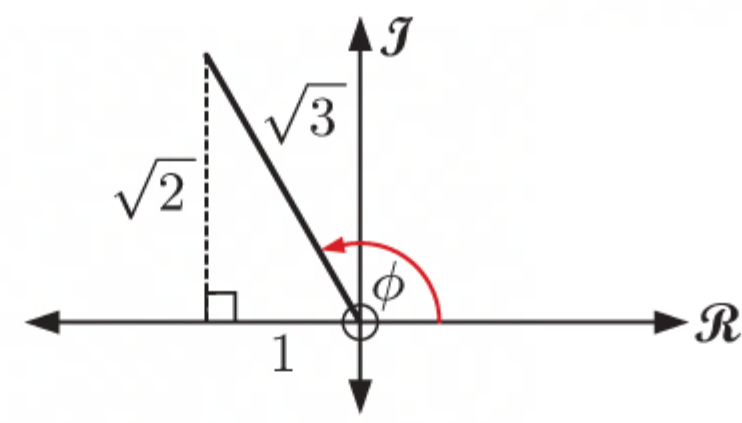
$$\therefore z = \sqrt{3} \operatorname{cis}\left[\frac{\arctan\left(\frac{\sqrt{2}}{5}\right) + k2\pi}{3}\right] \quad \text{where } k = 0, 1, 2 \quad \{\text{De Moivre}\}$$

d From **a**, one of the solutions to $z^3 = 5 + i\sqrt{2}$ is

$$z = -1 + i\sqrt{2}$$

$$\therefore z = \sqrt{3} \operatorname{cis} \phi$$

$$\therefore z = \sqrt{3} \operatorname{cis} \left[\arccos \left(-\frac{1}{\sqrt{3}} \right) \right]$$



This corresponds to the solution in **c** where $k = 1 \quad \left\{ \frac{\theta}{3} + \frac{2\pi}{3} = \phi \right\}$

Equating arguments gives: $\frac{\arctan\left(\frac{\sqrt{2}}{5}\right) + 2\pi}{3} = \arccos\left(-\frac{1}{\sqrt{3}}\right)$

$$\therefore \arctan\left(\frac{\sqrt{2}}{5}\right) + 2\pi = 3 \arccos\left(-\frac{1}{\sqrt{3}}\right)$$

8 a $\frac{1}{2}$ is a zero of $4x^3 - 8x^2 - 15x + 9$

$\therefore (2x - 1)$ is a factor of $4x^3 - 8x^2 - 15x + 9$.

$$\begin{aligned} \text{Let } 4x^3 - 8x^2 - 15x + 9 &= (2x - 1)(2x^2 + ax - 9) \\ &= 4x^3 + 2ax^2 - 18x \\ &\quad - 2x^2 - ax + 9 \\ &= 4x^3 + (2a - 2)x^2 - (18 + a)x + 9 \end{aligned}$$

Equating coefficients of x^2 : $2a - 2 = -8$

$$\therefore 2a = -6$$

$$\therefore a = -3$$

Equating coefficients of x : $18 + a = 18 - 3 = 15 \quad \checkmark$

$$\begin{aligned} \therefore \text{the quadratic factor is } 2x^2 - 3x - 9 &= 2x^2 - 6x + 3x - 9 \\ &= 2x(x - 3) + 3(x - 3) \\ &= (x - 3)(2x + 3) \end{aligned}$$

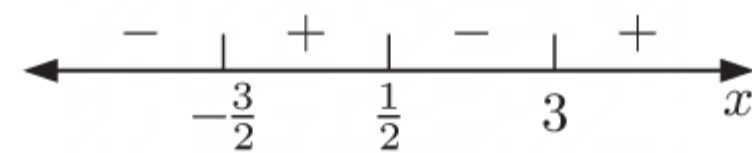
$$\therefore 4x^3 - 8x^2 - 15x + 9 = (2x - 1)(x - 3)(2x + 3)$$

b $4x^3 - 15x > 8x^2 - 9$

$$\therefore 4x^3 - 8x^2 - 15x + 9 > 0$$

$$\therefore (2x - 1)(x - 3)(2x + 3) > 0 \quad \{\text{using a}\}$$

$$\therefore -\frac{3}{2} < x < \frac{1}{2} \quad \text{or} \quad x > 3$$



9 P_n is: If $f(x) = e^{ax}(x - 3)$, then $f^{(n)}(x) = a^{n-1}e^{ax}[a(x - 3) + n]$, $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

$$\begin{aligned} (1) \text{ If } n = 1, \quad f^{(1)}(x) &= f'(x) \\ &= ae^{ax}(x - 3) + e^{ax}(1) \quad \{\text{product rule}\} \\ &= e^{ax}[a(x - 3) + 1] \\ &= a^0 e^{ax}[a(x - 3) + 1] \end{aligned}$$

$\therefore P_1$ is true.

(2) If P_k is true, then $f^{(k)}(x) = a^{k-1}e^{ax}[a(x - 3) + k]$.

$$\begin{aligned} \text{Now } f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x) \\ &= \frac{d}{dx} (a^{k-1}e^{ax}[a(x - 3) + k]) \quad \{\text{using } P_k\} \\ &= a^k e^{ax}[a(x - 3) + k] + a^{k-1}e^{ax}(a) \quad \{\text{product rule}\} \\ &= a^k e^{ax}[a(x - 3) + k + 1] \\ &= a^{(k+1)-1}e^{ax}[a(x - 3) + (k + 1)] \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. $\{\text{principle of mathematical induction}\}$

- 10** Let the shaded angle be θ , $\widehat{CXB} = \alpha$, and $\widehat{AYB} = \beta$.

Now $AB = CB$ {isosceles}

$$\therefore 4y = 2x$$

$$\therefore y = \frac{x}{2}$$

$$\therefore XB = \frac{3x}{2}$$

$$\text{In } \triangle CBX, \tan \alpha = \frac{2x}{\frac{3x}{2}} = \frac{4}{3}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

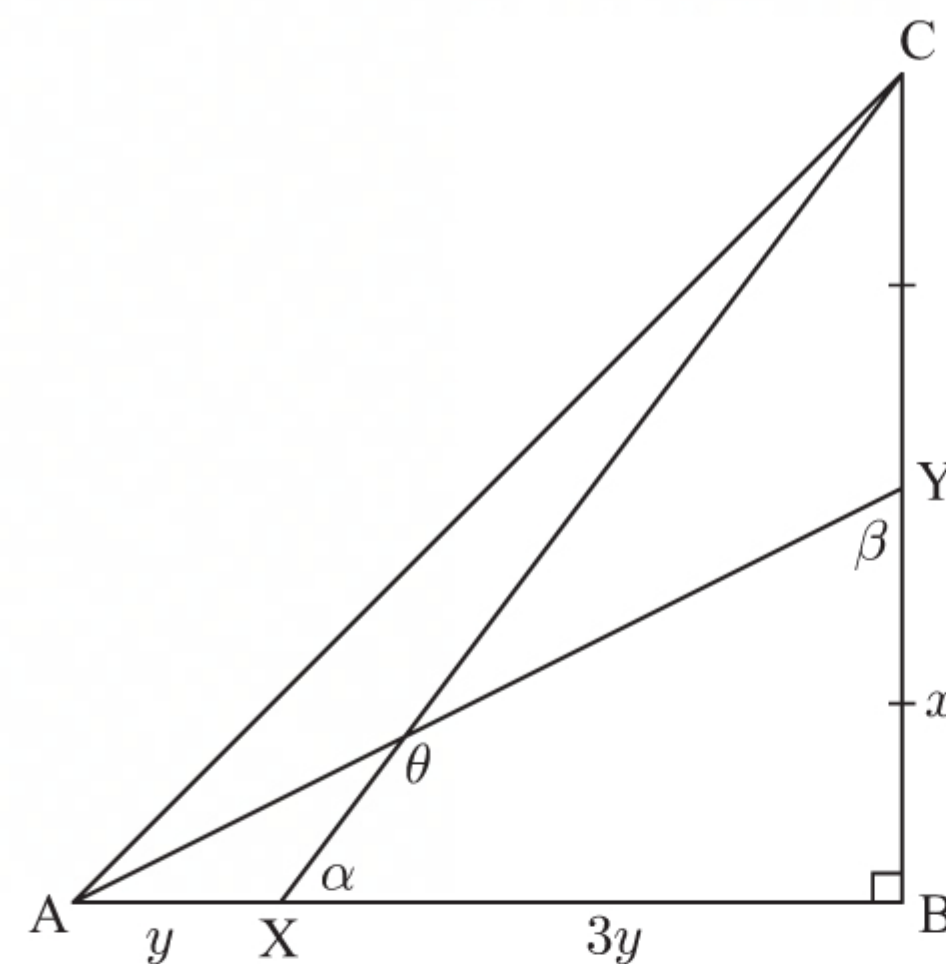
$$\text{In } \triangle AYB, \tan \beta = \frac{2x}{x} = 2$$

$$\therefore \beta = \tan^{-1}(2)$$

Now $\theta + \alpha + \beta + 90^\circ = 360^\circ$ {angles in a quadrilateral}

$$\therefore \theta + \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}(2) = 270^\circ$$

$$\therefore \theta = 270^\circ - \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(2) \\ \approx 153^\circ$$



MIXED QUESTIONS SET 2

1 a $\log_4(x^2 - x + 3)$

$$= \frac{\log_2(x^2 - x + 3)}{\log_2 4} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$= \frac{\log_2(x^2 - x + 3)}{\log_2(2^2)}$$

$$= \frac{\log_2(x^2 - x + 3)}{2}$$

$$= \frac{1}{2} \log_2(x^2 - x + 3)$$

$$= \log_2((x^2 - x + 3)^{\frac{1}{2}})$$

$$= \log_2 \sqrt{x^2 - x + 3}$$

b $\log_2(x + 2) = \log_4(x^2 - x + 3)$

$$\therefore \log_2(x + 2) = \log_2 \sqrt{x^2 - x + 3} \quad \{\text{using a}\}$$

$$\therefore x + 2 = \sqrt{x^2 - x + 3}$$

$$\therefore (x + 2)^2 = x^2 - x + 3, \quad x \geq -2$$

$$\therefore x^2 + 4x + 4 = x^2 - x + 3, \quad x \geq -2$$

$$\therefore 5x = -1, \quad x \geq -2$$

$$\therefore x = -\frac{1}{5}$$

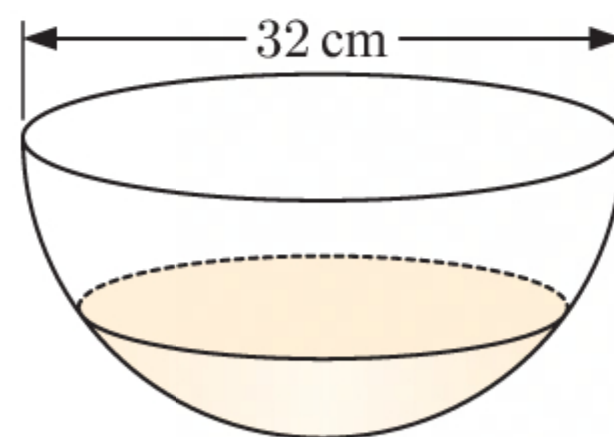
2 a $V = \frac{1}{2} \times \text{volume of sphere}$

$$= \frac{1}{2} \times \frac{4}{3} \pi r^3$$

$$= \frac{2}{3} \times \pi \times \left(\frac{32}{2}\right)^3 \text{ cm}^3$$

$$= \frac{8192}{3} \pi \text{ cm}^3$$

$$\approx 8580 \text{ cm}^3$$



The capacity of the bowl is about 8580 mL or 8.58 L.

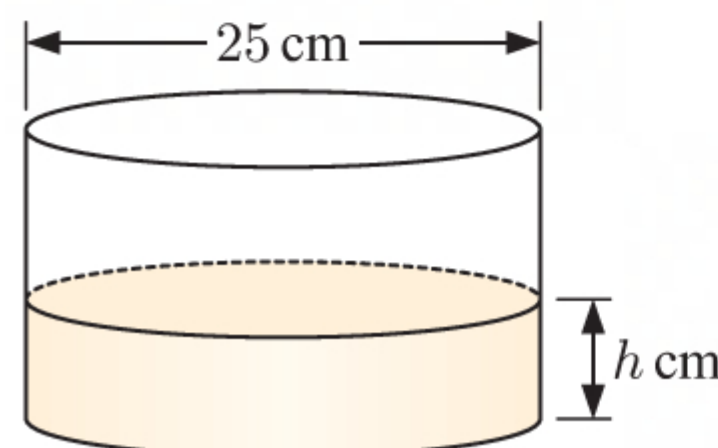
- b i** When 20% full, the bowl contains $\frac{8192}{3} \pi \times 0.2 \approx 1720 \text{ mL}$ or about 1.72 L of cake batter.

ii $V = \frac{8192}{3} \pi \times 0.2 \text{ cm}^3$

$$\therefore \pi \times \left(\frac{25}{2}\right)^2 \times h = \frac{8192}{3} \pi \times 0.2$$

$$\therefore h = \frac{8192 \times 0.2}{3 \times (12.5)^2}$$

$$\approx 3.50 \text{ cm}$$



The cake batter will reach about 3.50 cm up the tin.

3 Let W denote Wollongong, P denote Picton, and C denote Canberra.

a North-west is in the direction 315° . South-west is in the direction 225° .

$$\widehat{PWN} = 360^\circ - 315^\circ = 45^\circ$$

$$\widehat{WPN} = 180^\circ - 45^\circ = 135^\circ \quad \{\text{co-interior angles}\}$$

$$\widehat{CPW} = 225^\circ - 135^\circ = 90^\circ$$

$\therefore \triangle CPW$ is right angled at P.

$$CW^2 = 36^2 + 210^2 \quad \{\text{Pythagoras}\}$$

$$\therefore CW = \sqrt{36^2 + 210^2} \quad \{\text{as } CW > 0\}$$

$$\approx 213 \text{ km}$$

So, Canberra is about 213 km from Wollongong.

b $\widehat{CPN} = 360^\circ - 225^\circ = 135^\circ$

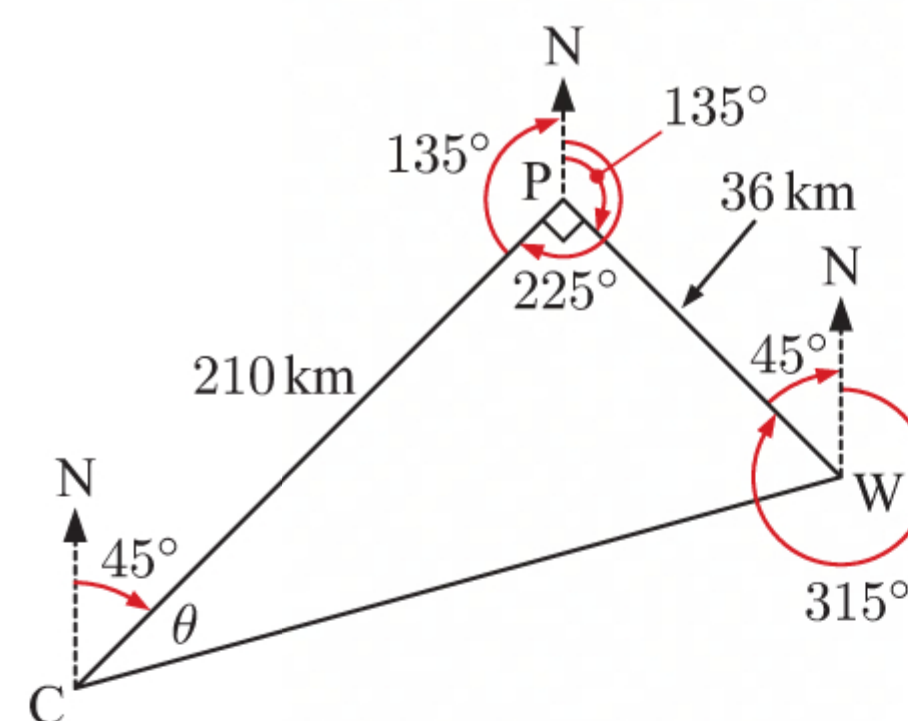
$$\widehat{NCP} = 180^\circ - 135^\circ = 45^\circ \quad \{\text{co-interior angles}\}$$

$$\tan \theta = \frac{36}{210}$$

$$\therefore \theta = \tan^{-1}\left(\frac{36}{210}\right) \approx 9.73^\circ$$

$$\therefore \text{the bearing of Wollongong from Canberra} \approx 45^\circ + 9.73^\circ$$

$$\approx 054.7^\circ$$



4 $v(t) = 30 - 20e^{-0.2t} \text{ m s}^{-1}$

a i $v(0) = 30 - 20e^0$
 $= 10$

\therefore the initial velocity of the boat is 10 m s^{-1} .

ii $v(2) = 30 - 20e^{-0.2 \times 2}$
 ≈ 16.6

\therefore the velocity of the boat after 2 seconds is about 16.6 m s^{-1} .

b $v(t) = 20$ when $30 - 20e^{-0.2t} = 20$

$$\therefore 20e^{-0.2t} = 10$$

$$\therefore e^{-0.2t} = \frac{1}{2}$$

$$\therefore e^{0.2t} = 2$$

$$\therefore 0.2t = \ln 2$$

$$\therefore t = 5 \ln 2 \approx 3.47$$

It will take about 3.47 seconds for the boat's velocity to reach 20 m s^{-1} .

c As $t \rightarrow \infty$, $e^{-0.2t} \rightarrow 0$

$$\therefore v(t) \rightarrow 30 - 20(0) = 30$$

d $v(t) = 30 - 20e^{-0.2t}$

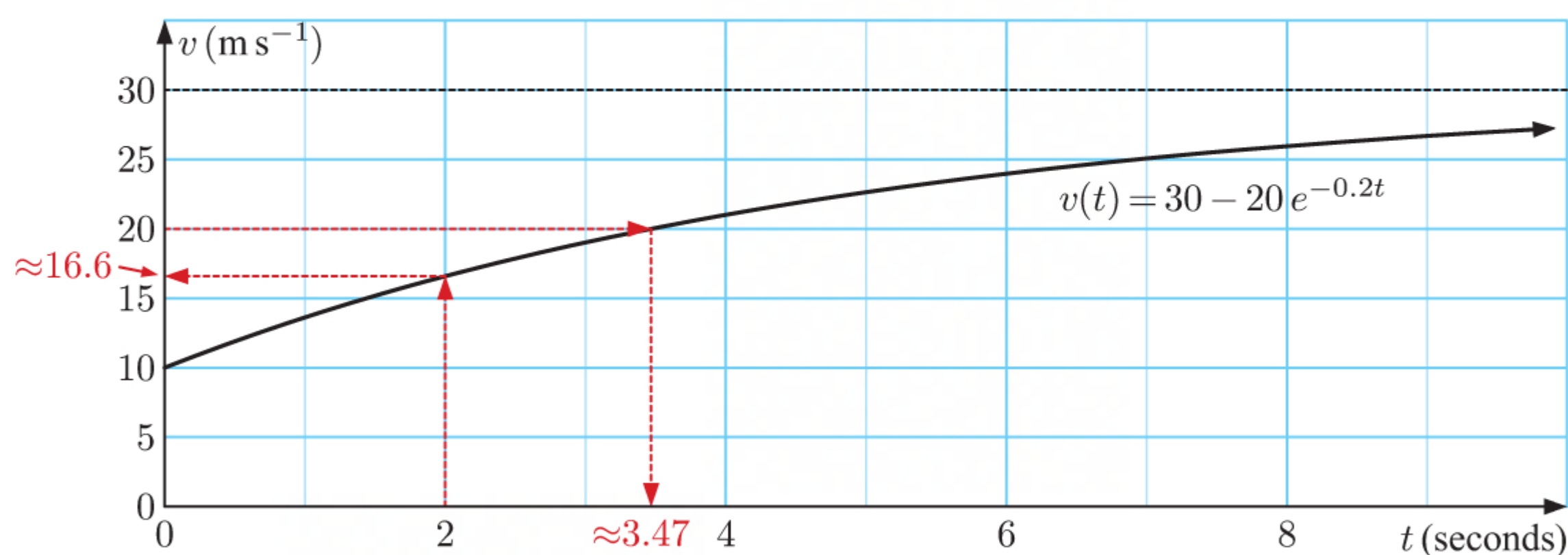
$$\therefore v'(t) = -20e^{-0.2t}(-0.2) \quad \{\text{chain rule}\}$$

$$\therefore v'(t) = 4e^{-0.2t}$$

$$\therefore v'(t) > 0 \quad \{\text{as } e^{-0.2t} > 0 \text{ for all } t\}$$

So, the acceleration $v'(t)$ is always positive.

e



- f** The boat's velocity reached 20 m s^{-1} after $5 \ln 2$ seconds. {from **b**}

$$\begin{aligned}
 \text{Distance travelled in first } 5 \ln 2 \text{ seconds} &= \int_0^{5 \ln 2} v(t) dt \\
 &= \int_0^{5 \ln 2} (30 - 20e^{-0.2t}) dt \\
 &= [30t + 100e^{-0.2t}]_0^{5 \ln 2} \\
 &= (150 \ln 2 + 100e^{-\ln 2}) - (0 + 100) \\
 &= 150 \ln 2 + 100 \times \frac{1}{2} - 100 \\
 &= 150 \ln 2 - 50 \\
 &\approx 54.0 \text{ m}
 \end{aligned}$$

\therefore the boat travelled about 54.0 m before its velocity reached 20 m s^{-1} .

5

x	-2	0	3	5
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	k	$\frac{1}{12}$

- a** X can only take the values -2, 0, 3, or 5.

$\therefore X$ is a discrete random variable.

- b** Since this is a probability distribution, $P(X = -2) + P(X = 0) + P(X = 3) + P(X = 5) = 1$

$$\therefore \frac{1}{3} + \frac{1}{6} + k + \frac{1}{12} = 1$$

$$\therefore k + \frac{7}{12} = 1$$

$$\therefore k = \frac{5}{12}$$

- c** Since $P(X = 3)$ is the greatest probability, 3 is the mode of the distribution.

$$P(X = -2) = \frac{1}{3} \approx 0.333$$

$$P(X = -2) + P(X = 0) = \frac{1}{3} + \frac{1}{6} = 0.5$$

Since $P(X = -2) + P(X = 0) \geq 0.5$, the median is 0.

- d** $E(X) = -2(\frac{1}{3}) + 0(\frac{1}{6}) + 3(\frac{5}{12}) + 5(\frac{1}{12})$

$$= -\frac{2}{3} + 0 + \frac{5}{4} + \frac{5}{12}$$

$$= 1$$

6 a $\vec{AB} = \begin{pmatrix} -1 - 3 \\ 7 - (-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$

b i $\vec{OA} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$$\therefore |\vec{OA}| = \sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ units}$$

ii $|\vec{AB}| = \sqrt{(-4)^2 + 8^2}$

$$= \sqrt{80}$$

$$= 4\sqrt{5} \text{ units}$$

c $\vec{AO} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Let $\widehat{OAB} = \theta$.

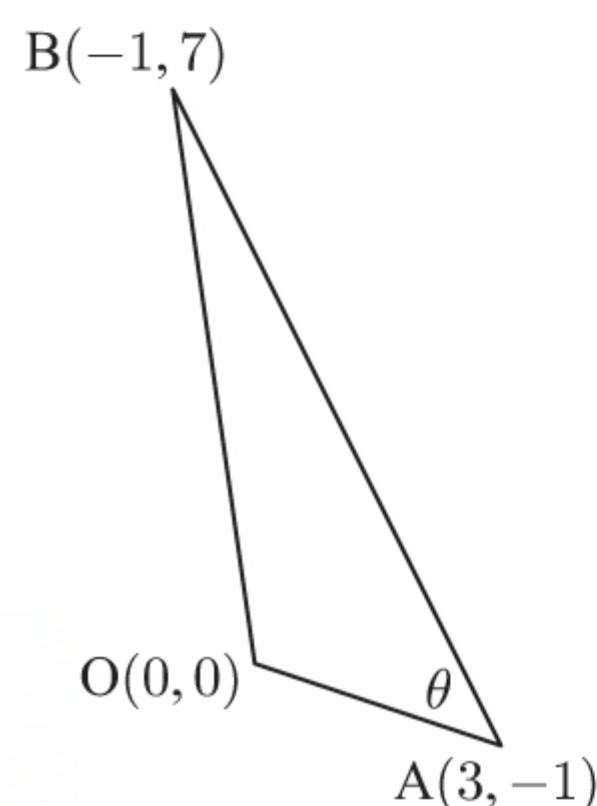
$$\begin{aligned}
 \text{Now } \cos \theta &= \frac{\vec{AO} \cdot \vec{AB}}{|\vec{AO}| |\vec{AB}|} \\
 &= \frac{\begin{pmatrix} -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 8 \end{pmatrix}}{\sqrt{10} \times 4\sqrt{5}} \\
 &= \frac{(-3)(-4) + (1)(8)}{4\sqrt{50}}
 \end{aligned}$$

$$= \frac{12 + 8}{20\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \frac{\pi}{4}$$

$$\therefore \widehat{OAB} = \frac{\pi}{4}$$



$$\begin{aligned}
 \text{d Area of } \triangle OAB &= \frac{1}{2} \times \sqrt{10} \times 4\sqrt{5} \times \sin \frac{\pi}{4} \\
 &= \frac{20\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} \\
 &= 10 \text{ units}^2
 \end{aligned}$$

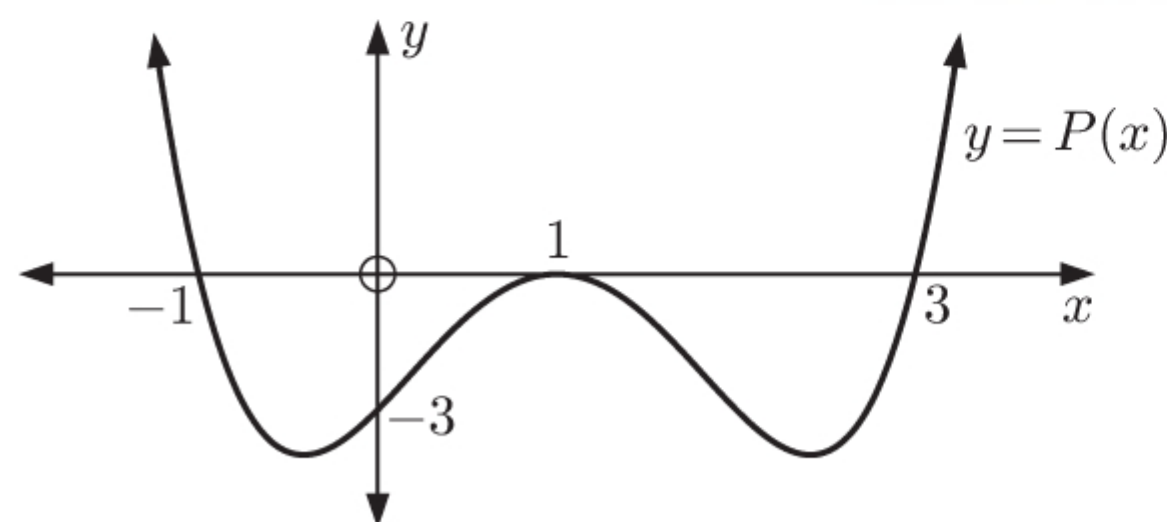
$$\begin{aligned}
 7 \text{ a Since } (x-1)^2 \text{ is a factor of } P(x), \quad & x^4 + ax^3 + 2x^2 + bx - 3 \\
 &= (x-1)^2(x^2 + cx - 3) \quad \text{for some } c \\
 &= (x^2 - 2x + 1)(x^2 + cx - 3) \\
 &= x^4 + cx^3 - 3x^2 - 2x^3 - 2cx^2 + 6x + x^2 + cx - 3 \\
 &= x^4 + (c-2)x^3 - (2+2c)x^2 + (c+6)x - 3
 \end{aligned}$$

$$\begin{aligned}
 \text{Equating coefficients of } x^2: \quad & 2 + 2c = -2 \\
 \therefore \quad & c = -2
 \end{aligned}$$

$$\begin{aligned}
 \text{Equating coefficients of } x^3: \quad & a = -2 - 2 \\
 \therefore \quad & a = -4
 \end{aligned}$$

$$\begin{aligned}
 \text{Equating coefficients of } x: \quad & b = -2 + 6 \\
 \therefore \quad & b = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{b } P(x) &= (x-1)^2(x^2 - 2x - 3) \quad \{\text{using a}\} \\
 &= (x-1)^2(x-3)(x+1)
 \end{aligned}$$



$$8 \quad f(x) = \begin{cases} x^2 + 1, & x < 0 \\ \sin x, & x \geq 0 \end{cases}$$

$$\begin{aligned}
 \text{a } \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (x^2 + 1) \quad \{x < 0\} \\
 &= 0^2 + 1 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \sin x \quad \{x > 0\} \\
 &= \sin 0 \\
 &= 0
 \end{aligned}$$

$$\text{c Since } \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x), \quad \lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$

$$\begin{aligned}
 9 \text{ a } I_0 &= \int_0^{\frac{\pi}{2}} \cos x \, dx \\
 &= \left[\sin x \right]_0^{\frac{\pi}{2}} \\
 &= 1 - 0 \\
 &= 1
 \end{aligned}$$

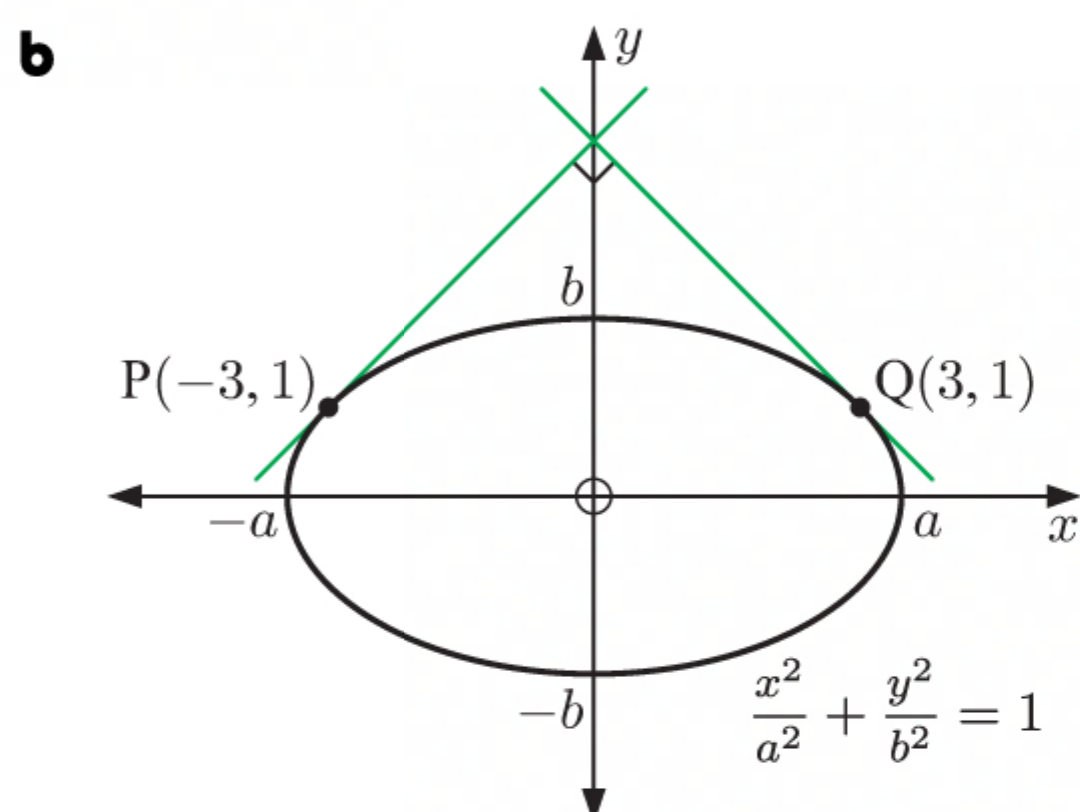
$$\begin{aligned}
 \text{b } \int x \cos x \, dx &= x \sin x - \int \sin x \, dx \quad \begin{cases} u' = \cos x & v = x \\ u = \sin x & v' = 1 \end{cases} \\
 &= x \sin x + \cos x + c \\
 \therefore I_1 &= \int_0^{\frac{\pi}{2}} x \cos x \, dx \\
 &= \frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - (0 + \cos 0) \\
 &= \frac{\pi}{2} - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int x^n \cos x \, dx &= x^n \sin x - n \int x^{n-1} \sin x \, dx \quad \begin{cases} u' = \cos x & v = x^n \\ u = \sin x & v' = nx^{n-1} \end{cases} \\
 &= x^n \sin x - n \left[-x^{n-1} \cos x + (n-1) \int x^{n-2} \cos x \, dx \right] \quad \begin{cases} u' = \sin x & v = x^{n-1} \\ u = -\cos x & v' = (n-1)x^{n-2} \end{cases} \\
 &= x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx
 \end{aligned}$$

$$\begin{aligned}
 \therefore I_n &= \left[x^n \sin x \right]_0^{\frac{\pi}{2}} + n \left[x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} - n(n-1) I_{n-2} \\
 &= \left(\frac{\pi}{2} \right)^n \sin \frac{\pi}{2} - 0 + n(0 - 0) - n(n-1) I_{n-2} \\
 &= \left(\frac{\pi}{2} \right)^n - n(n-1) I_{n-2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int_0^{\frac{\pi}{2}} x^3 \cos x \, dx &= I_3 \\
 &= \left(\frac{\pi}{2}\right)^3 - (3)(2)I_1 \quad \{\text{using } \mathbf{c}\} \\
 &= \left(\frac{\pi}{2}\right)^3 - 6\left(\frac{\pi}{2} - 1\right) \\
 &= \left(\frac{\pi}{2}\right)^3 - 3\pi + 6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\
 \therefore \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} &= 0 \quad \{\text{differentiating both sides by } x\} \\
 \therefore \frac{2y}{b^2} \frac{dy}{dx} &= -\frac{2x}{a^2} \\
 \therefore \frac{dy}{dx} &= -\frac{b^2 x}{a^2 y}
 \end{aligned}$$



At P, $x = -3$, $y = 1$

$$\therefore \frac{dy}{dx} = -\frac{b^2(-3)}{a^2(1)} = \frac{3b^2}{a^2}$$

At Q, $x = 3$, $y = 1$

$$\therefore \frac{dy}{dx} = -\frac{b^2(3)}{a^2(1)} = -\frac{3b^2}{a^2}$$

Since the tangents at P and Q are perpendicular, their gradients must be negative reciprocals of one another.

$$\begin{aligned}
 \therefore \frac{3b^2}{a^2} &= -\left(-\frac{a^2}{3b^2}\right) \\
 \therefore \frac{3b^2}{a^2} &= \frac{a^2}{3b^2} \\
 \therefore 9b^4 &= a^4 \\
 \therefore 3b^2 &= a^2 \quad \dots (*) \quad \{a^2 > 0, b^2 > 0\}
 \end{aligned}$$

Now substituting the coordinates of Q into the equation of the ellipse:

$$\begin{aligned}
 \frac{3^2}{a^2} + \frac{1^2}{b^2} &= 1 \\
 \therefore \frac{9}{a^2} + \frac{1}{b^2} &= 1 \\
 \therefore \frac{9}{3b^2} + \frac{1}{b^2} &= 1 \quad \{\text{using } (*)\} \\
 \therefore \frac{4}{b^2} &= 1 \\
 \therefore b^2 &= 4 \\
 \therefore b &= 2 \quad \{b > 0\}
 \end{aligned}$$

Substituting $b = 2$ into (*) gives $a^2 = 3(2)^2$

$$\therefore a^2 = 12$$

$$\therefore a = \sqrt{12} \quad \{a > 0\}$$

MIXED QUESTIONS SET 3

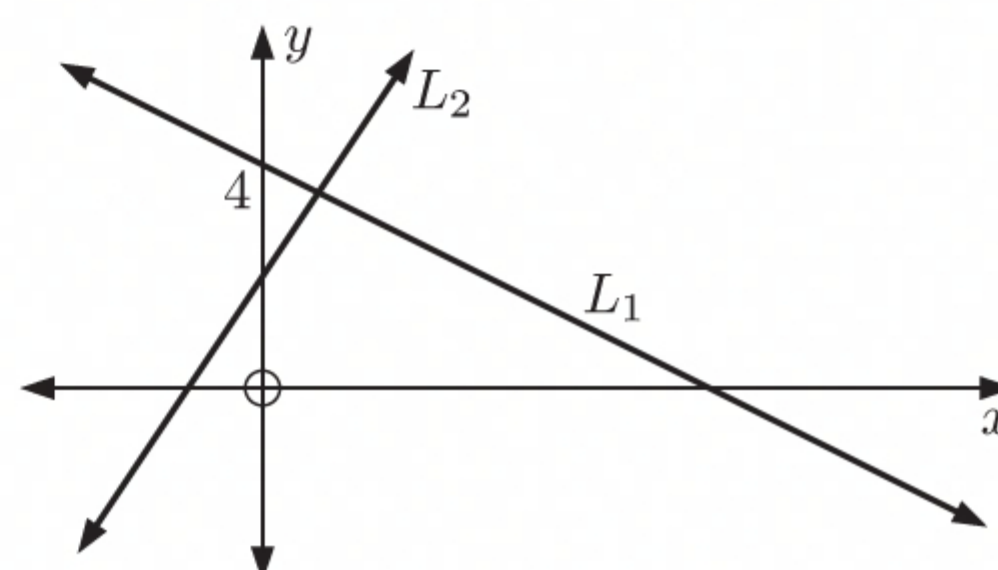
1 a L_1 has gradient $-\frac{1}{2}$ and passes through $(0, 4)$.

$$\therefore L_1 \text{ has equation } y - 4 = -\frac{1}{2}(x - 0)$$

$$\therefore y - 4 = -\frac{1}{2}x$$

$$\therefore \frac{1}{2}x + y - 4 = 0$$

$$\therefore x + 2y - 8 = 0 \quad \dots (1)$$



b L_2 has gradient $\frac{8 - (-1)}{4 - (-2)} = \frac{9}{6} = \frac{3}{2}$, and passes through $(-2, -1)$.

$$\therefore L_2 \text{ has equation } y - (-1) = \frac{3}{2}(x - (-2))$$

$$\therefore y + 1 = \frac{3}{2}(x + 2)$$

$$\therefore y + 1 = \frac{3}{2}x + 3$$

$$\therefore \frac{3}{2}x - y + 2 = 0$$

$$\therefore 3x - 2y + 4 = 0 \quad \dots (2)$$

Adding (1) and (2) gives

$$x + 2y - 8 = 0$$

$$3x - 2y + 4 = 0$$

$$\hline 4x \quad -4 = 0$$

$$\therefore 4x = 4$$

$$\therefore x = 1$$

Substituting $x = 1$ into (1) gives $1 + 2y - 8 = 0$

$$\therefore 2y = 7$$

$$\therefore y = \frac{7}{2}$$

\therefore the point of intersection of L_1 and L_2 is $(1, \frac{7}{2})$.

2 a $u_4 = u_1 r^3$

$$\therefore 8 = 27r^3$$

$$\therefore r^3 = \frac{8}{27}$$

$$\therefore r = \frac{2}{3}$$

c $u_n = u_1 r^{n-1}$

$$\therefore u_n = 27\left(\frac{2}{3}\right)^{n-1}$$

$$\begin{aligned} \text{So, } S &= \sum_{n=1}^{\infty} u_n \\ &= \sum_{n=1}^{\infty} 27\left(\frac{2}{3}\right)^{n-1} \end{aligned}$$

b $u_6 = u_1 r^5$
 $= 27\left(\frac{2}{3}\right)^5$
 $= \frac{32}{9}$

d Since $|r| = \left|\frac{2}{3}\right|$ is < 1 , the sum of the infinite series converges.

$$\begin{aligned} \therefore S &= \frac{u_1}{1-r} \\ &= \frac{27}{1-\frac{2}{3}} \\ &= \frac{27}{\frac{1}{3}} \\ &= 81 \end{aligned}$$

3

Event	Time (seconds)	μ (seconds)	σ (seconds)
100 m	9.99	10.20	0.113
200 m	17.30	18.50	0.706

a For the 100 m event, $z\text{-score} = \frac{9.99 - 10.20}{0.113} \approx -1.86$

For the 200 m event, $z\text{-score} = \frac{17.30 - 18.50}{0.706} \approx -1.70$

b A lower z -score is better as it indicates that the time is lower, and hence that Carl ran faster.

\therefore Carl performed better in the 100 m event.

4 a The tangent to the curve $y = ax^3 - bx^2$ at the point where $x = 3$ is $y = x - 6$.

\therefore the tangent has gradient 1, and the point of contact is $(3, 3 - 6)$ which is $(3, -3)$.

$$\text{Now } f(x) = ax^3 - bx^2$$

$$\therefore f'(x) = 3ax^2 - 2bx$$

$$\text{So, } f'(3) = 1$$

and

$$f(3) = -3$$

$$\therefore 3a(3)^2 - 2b(3) = 1$$

$$\therefore a(3)^3 - b(3)^2 = -3$$

$$\therefore 27a - 6b = 1 \quad \dots (1)$$

$$\therefore 27a - 9b = -3 \quad \dots (2)$$

Subtracting (2) from (1) gives $27a - 6b = 1$

$$\hline -(27a - 9b = -3)$$

$$3b = 4$$

$$\therefore b = \frac{4}{3}$$

Substituting $b = \frac{4}{3}$ into (1) gives $27a - 6\left(\frac{4}{3}\right) = 1$

$$\therefore 27a - 8 = 1$$

$$\therefore 27a = 9$$

$$\therefore a = \frac{1}{3}$$

b $f(x) = \frac{1}{3}x^3 - \frac{4}{3}x^2 = \frac{1}{3}(x^3 - 4x^2)$ {from **a**}

The tangent meets the curve where $x - 6 = \frac{1}{3}(x^3 - 4x^2)$

$$\therefore 3x - 18 = x^3 - 4x^2$$

$$\therefore x^3 - 4x^2 - 3x + 18 = 0$$

Since the tangent touches the curve at $x = 3$, there must be a repeated solution at this point.

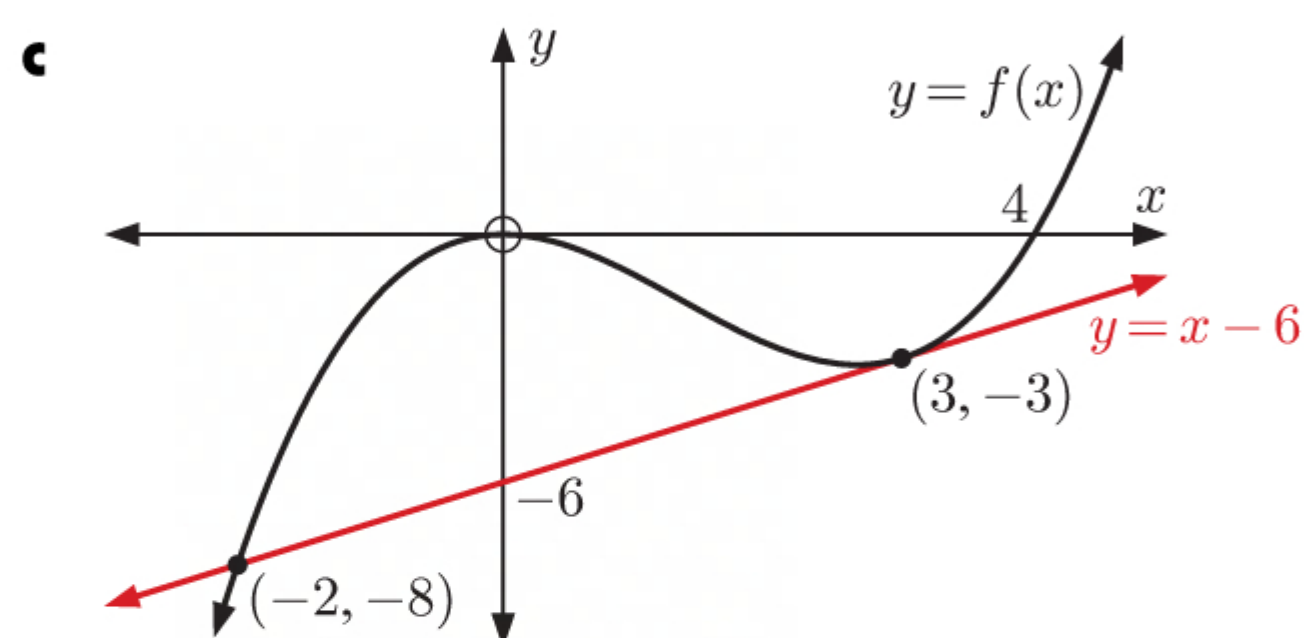
$\therefore (x - 3)^2$ must be a factor of this cubic.

$$\therefore (x - 3)^2(x + 2) = 0$$

\therefore the tangent meets the curve again when $x = -2$.

When $x = -2$, $y = -2 - 6 = -8$

\therefore the tangent meets the curve again at $(-2, -8)$.



5 a $\frac{\sin^2 \theta}{1 + \cos \theta} = \frac{1 - \cos^2 \theta}{1 + \cos \theta}$

$$= \frac{(1 + \cos \theta)(1 - \cos \theta)}{1 + \cos \theta}$$

$$= 1 - \cos \theta \quad \text{for all } \theta \text{ such that } \cos \theta \neq -1.$$

b $\frac{\sin^2 \theta}{1 + \cos \theta} = \frac{1}{2}$

$$\therefore 1 - \cos \theta = \frac{1}{2} \quad \{\text{from a as } \cos \theta \neq -1 \text{ for all } -\pi < \theta < \pi\}$$

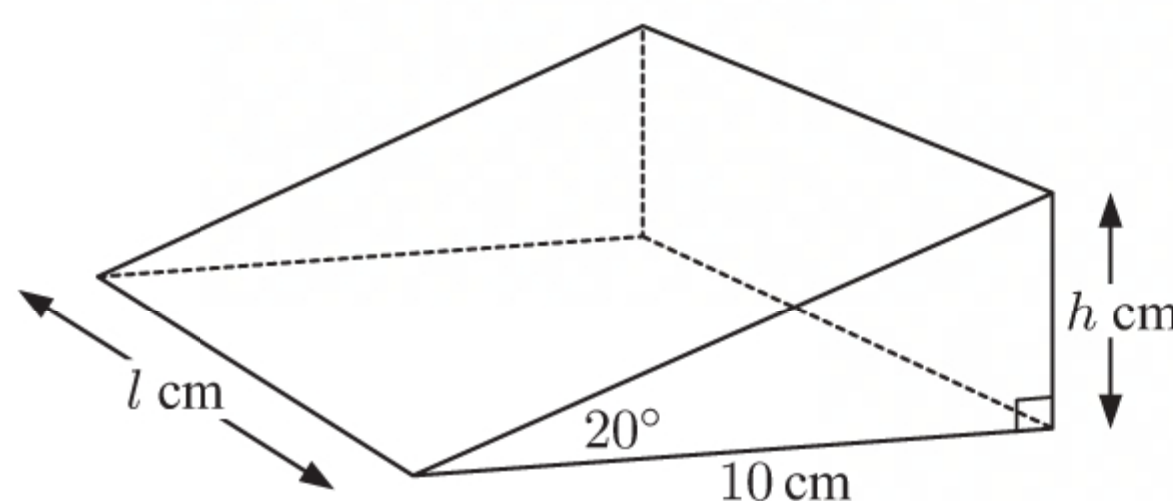
$$\therefore \cos \theta = \frac{1}{2}$$

$$\therefore \theta = -\frac{\pi}{3} \text{ or } \frac{\pi}{3} \quad \{-\pi < \theta < \pi\}$$

6 a $\tan 20^\circ = \frac{h}{10}$

$$\therefore h = 10 \tan 20^\circ$$

$$\approx 3.640$$



b Area of triangular end $= \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times 10 \times h$$

$$= 5 \times 10 \tan 20^\circ \quad \{\text{from a}\}$$

$$= 50 \tan 20^\circ \text{ cm}^2$$

$$\approx 18.2 \text{ cm}^2$$

c Volume of door-stop $= 60 \text{ cm}^3$

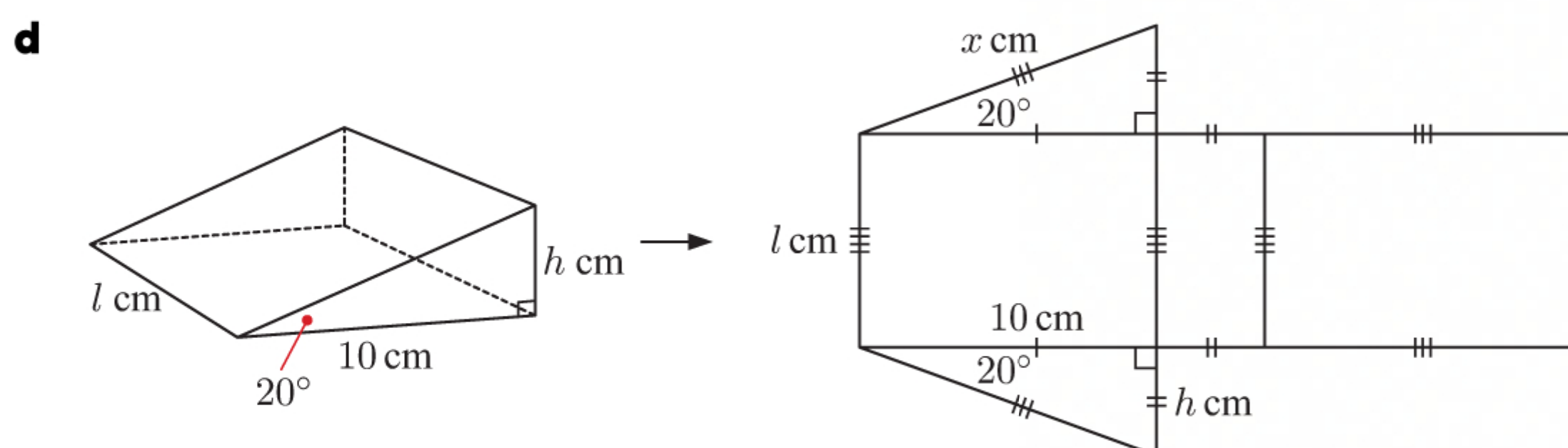
$$\therefore \text{area of triangular end} \times \text{length} = 60$$

$$\therefore 50 \tan 20^\circ \times l = 60 \quad \{\text{from b}\}$$

$$\therefore l = \frac{60}{50 \tan 20^\circ}$$

$$\therefore l = \frac{6}{5 \tan 20^\circ}$$

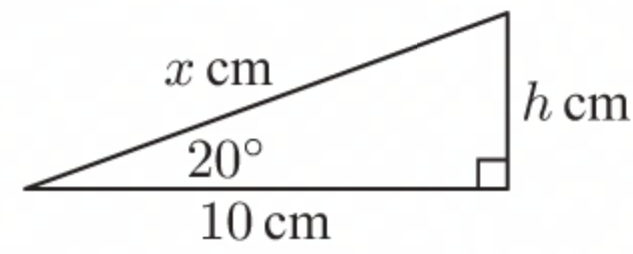
$$\therefore l \approx 3.30$$



Let the hypotenuse of the triangular end be x cm.

$$\cos 20^\circ = \frac{10}{x}$$

$$\therefore x = 10 \cos 20^\circ$$



$$\begin{aligned} \text{Surface area} &= (10 \times l) + (h \times l) + (x \times l) + 2 \times \left(\frac{1}{2} \times 10 \times h\right) \\ &= (10 + h + x) \times l + 10h \\ &= (10 + 10 \tan 20^\circ + 10 \cos 20^\circ) \times \frac{6}{5 \tan 20^\circ} + 10 \times 10 \tan 20^\circ \quad \{\text{from a and c}\} \\ &\approx 112 \text{ cm}^2 \end{aligned}$$

7 a From the triangle inequality, $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$
 $\therefore |\mathbf{a} + (\mathbf{b} - \mathbf{a})| \leq |\mathbf{a}| + |\mathbf{b} - \mathbf{a}| \quad \{\mathbf{x} = \mathbf{a}, \mathbf{y} = \mathbf{b} - \mathbf{a}\}$
 $\therefore |\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b} - \mathbf{a}|$
 $\therefore |\mathbf{b}| - |\mathbf{a}| \leq |\mathbf{b} - \mathbf{a}|$
 $\therefore |\mathbf{b} - \mathbf{a}| \geq |\mathbf{b}| - |\mathbf{a}|$

b From the triangle inequality, $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$
 $\therefore |\mathbf{b} + (\mathbf{a} - \mathbf{b})| \leq |\mathbf{b}| + |\mathbf{a} - \mathbf{b}| \quad \{\mathbf{x} = \mathbf{b}, \mathbf{y} = \mathbf{a} - \mathbf{b}\}$
 $\therefore |\mathbf{a}| \leq |\mathbf{b}| + |\mathbf{a} - \mathbf{b}|$
 $\therefore |\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} - \mathbf{b}|$
 $\therefore |\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

c From **a**, $|\mathbf{b} - \mathbf{a}| \geq |\mathbf{b}| - |\mathbf{a}|$
 $\therefore |-(\mathbf{a} - \mathbf{b})| \geq |\mathbf{b}| - |\mathbf{a}|$
 $\therefore |\mathbf{a} - \mathbf{b}| \geq |\mathbf{b}| - |\mathbf{a}|$

Since $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{b}| - |\mathbf{a}|$ and $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$, $|\mathbf{a} - \mathbf{b}|$ must be greater than or equal to the maximum of $|\mathbf{a}| - |\mathbf{b}|$ and $|\mathbf{b}| - |\mathbf{a}|$.

$$\therefore |\mathbf{a} - \mathbf{b}| \geq ||\mathbf{a}| - |\mathbf{b}||$$

8 a $(2, 4)$, $(2, -6)$, and $(-1, 3)$ lie on a circle with equation $x^2 + y^2 + ax + by + c = 0$.
 $\therefore 2^2 + 4^2 + a(2) + b(4) + c = 0$
 $2^2 + (-6)^2 + a(2) + b(-6) + c = 0$ which gives the system of equations $\begin{cases} 2a + 4b + c = -20 \\ 2a - 6b + c = -40 \\ -a + 3b + c = -10 \end{cases}$
 $(-1)^2 + 3^2 + a(-1) + b(3) + c = 0$

b The system has augmented matrix

$$\begin{pmatrix} -1 & 3 & 1 & -10 \\ 2 & 4 & 1 & -20 \\ 2 & -6 & 1 & -40 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & -1 & 10 \\ 0 & 10 & 3 & -40 \\ 0 & 0 & 3 & -60 \end{pmatrix} \begin{array}{l} -R_1 \rightarrow R_1 \\ R_2 + 2R_1 \rightarrow R_2 \\ R_3 + 2R_1 \rightarrow R_3 \end{array} \begin{pmatrix} 2 & 4 & 1 & -20 \\ -2 & 6 & 2 & -20 \\ 0 & 10 & 3 & -40 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -6 & 1 & -40 \\ -2 & 6 & 2 & -20 \\ 0 & 0 & 3 & -60 \end{pmatrix}$$

Using row 3, $3c = -60$

$$\therefore c = -20$$

Substituting into row 2, $10b + 3(-20) = -40$

$$\therefore 10b = 20$$

$$\therefore b = 2$$

Substituting into row 1, $a - 3(2) - (-20) = 10$

$$\therefore a = -4$$

So the equation of the circle is $x^2 + y^2 - 4x + 2y - 20 = 0$

$$\therefore (x - 2)^2 - 4 + (y + 1)^2 - 1 = 20$$

$$\therefore (x - 2)^2 + (y + 1)^2 = 25$$

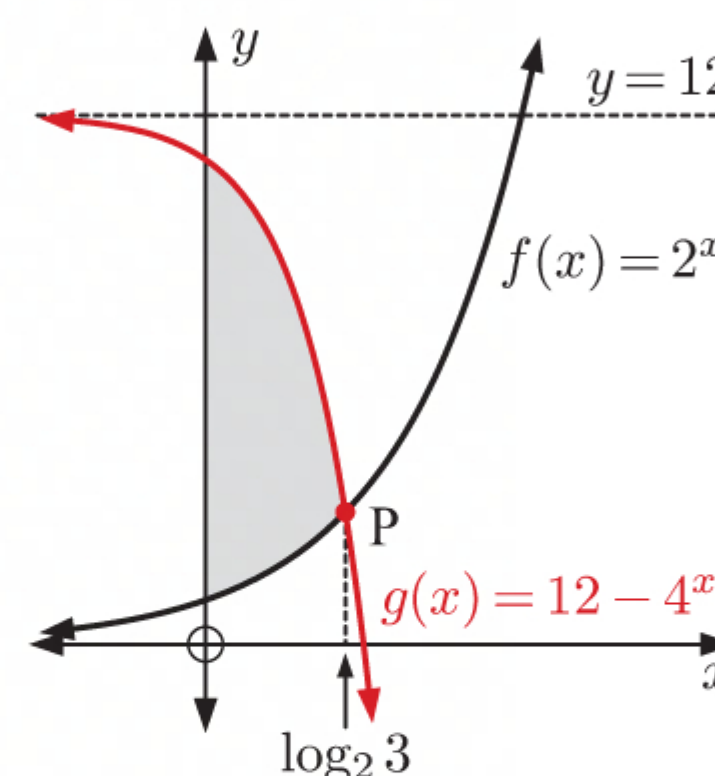
Thus the circle's centre is at $(2, -1)$.

- 9 a** The graphs of $y = f(x)$ and $y = g(x)$ meet where $2^x = 12 - 4^x$
 $\therefore 4^x + 2^x - 12 = 0$
 $\therefore (2^x)^2 + 2^x - 12 = 0$
 $\therefore (2^x - 3)(2^x + 4) = 0$
 $\therefore 2^x = 3 \quad \{2^x > 0 \text{ for all } x\}$
 $\therefore x = \log_2 3$

Now $f(\log_2 3) = 2^{\log_2 3} = 3$

\therefore P has coordinates $(\log_2 3, 3)$.

$$\begin{aligned}
 \text{b Shaded area} &= \int_0^{\log_2 3} [g(x) - f(x)] dx \\
 &= \int_0^{\log_2 3} (12 - 4^x - 2^x) dx \\
 &= \left[12x - \frac{4^x}{\ln 4} - \frac{2^x}{\ln 2} \right]_0^{\log_2 3} \\
 &= \left(12 \log_2 3 - \frac{4^{\log_2 3}}{\ln 4} - \frac{2^{\log_2 3}}{\ln 2} \right) - \left(0 - \frac{1}{\ln 4} - \frac{1}{\ln 2} \right) \\
 &= 12 \log_2 3 - \frac{3^2}{\ln 4} - \frac{3}{\ln 2} + \frac{1}{\ln 4} + \frac{1}{\ln 2} \\
 &= \frac{12 \ln 3}{\ln 2} - \frac{9}{2 \ln 2} - \frac{3}{\ln 2} + \frac{1}{2 \ln 2} + \frac{1}{\ln 2} \\
 &= \frac{12 \ln 3 - 2}{\ln 2} - \frac{8}{2 \ln 2} \\
 &= \frac{12 \ln 3 - 6}{\ln 2} \text{ units}^2
 \end{aligned}$$



10 $P_1(x) = 2x + 3$

$\therefore P_2(x) = \frac{2}{2}x^2 + \frac{3}{1}x$

$\therefore P_3(x) = \frac{2}{2 \times 3}x^3 + \frac{3}{1 \times 2}x^2$

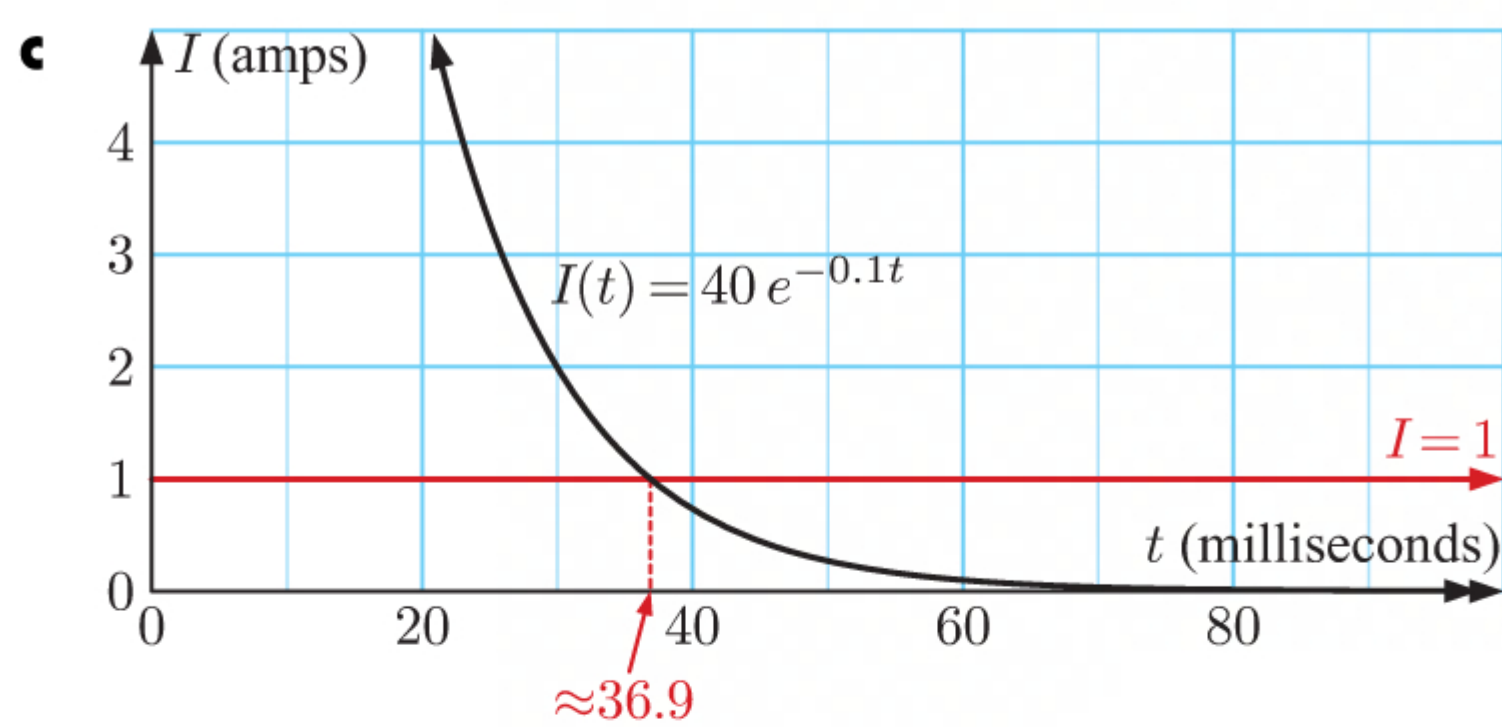
$\therefore P_4(x) = \frac{2}{2 \times 3 \times 4}x^4 + \frac{3}{1 \times 2 \times 3}x^3$

$\therefore P_n(x) = \frac{2}{n!}x^n + \frac{3}{(n-1)!}x^{n-1}$

\therefore the sum of the roots of $P_n(x) = 0$ is $-\frac{\frac{3}{(n-1)!}}{\frac{2}{n!}} = -\frac{3n!}{2(n-1)!}$
 $= -\frac{3n(n-1)!}{2(n-1)!}$
 $= -\frac{3n}{2}$

MIXED QUESTIONS SET 4

- 1 a**
 - The survey is likely to under-represent full-time weekday workers.
 - The survey was taken at a suburban shopping centre, so the people surveyed are likely to prefer suburban shopping. Therefore the sample is likely to be biased toward suburban shopping.
- b** The conclusion is unreasonable since the survey is likely to contain a coverage error, as in **a**, and so the results may not accurately represent the opinions of the whole population.
- 2** $I(t) = 40e^{-0.1t}$ amps
- a** $I(0) = 40e^0$
 $= 40$
 \therefore there was 40 amps of current flowing through the circuit initially.
- b** $I(100) = 40e^{-0.1 \times 100}$
 ≈ 0.00182
 \therefore after 100 milliseconds, there was about 0.00182 amps flowing through the circuit.



d The graphs meet where $40e^{-0.1t} = 1$
 $\therefore e^{-0.1t} = \frac{1}{40}$
 $\therefore e^{0.1t} = 40$
 $\therefore 0.1t = \ln 40$
 $\therefore t = 10 \ln 40$
 $\therefore t \approx 36.9$

\therefore it took about 36.9 milliseconds for the current to fall to 1 amp.

3 $a > b > c > 0$

a i $a > b > 0$
 $\therefore a^2 > b^2$
 $\therefore a^2 - b^2 > 0$

ii $b > c > 0$
 $\therefore b^2 > c^2$
 $\therefore b^2 - c^2 > 0$

b $(a^2 - b^2)(b^2 - c^2) > 0$ {using **a i** and **a ii**}
 $\therefore a^2b^2 - a^2c^2 - b^4 + b^2c^2 > 0$
 $\therefore (ab)^2 + (bc)^2 - (ac)^2 > b^4$

4 a $QR^2 = x^2 + 8^2$ {Pythagoras}
 $\therefore QR = \sqrt{x^2 + 64}$ {as $QR > 0$ }
 Also $QS = PS - PQ$
 $= 11 - x$

So, the length of pipeline under the sea is $\sqrt{x^2 + 64}$ km,
 and the length of pipeline overland is $(11 - x)$ km.

\therefore the cost $C(x) = 5\sqrt{x^2 + 64} + 3(11 - x)$ million dollars
 $= 5\sqrt{x^2 + 64} + 33 - 3x$ million dollars.

b $C(x) = 5(x^2 + 64)^{\frac{1}{2}} + 33 - 3x$
 $\therefore C'(x) = 5 \times \frac{1}{2}(x^2 + 64)^{-\frac{1}{2}}(2x) - 3 = \frac{5x}{\sqrt{x^2 + 64}} - 3$

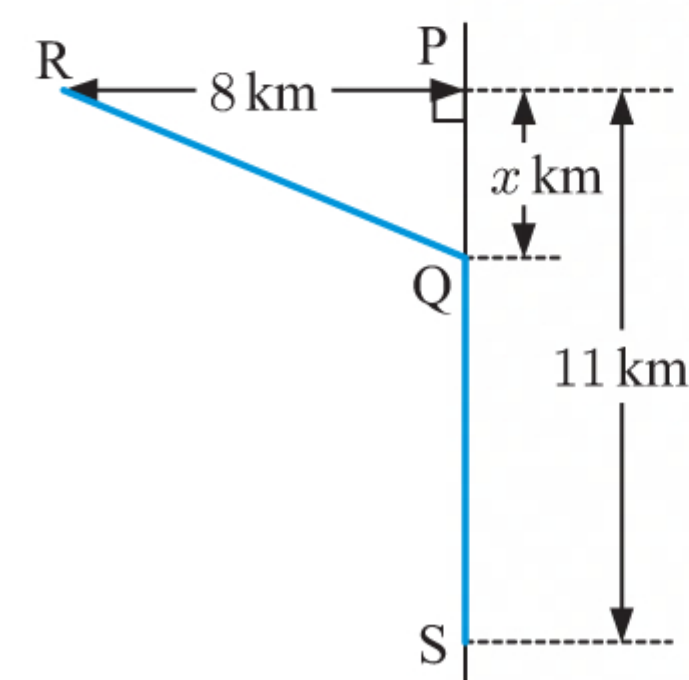
Now $C'(x) = 0$ where $\frac{5x}{\sqrt{x^2 + 64}} - 3 = 0$
 $\therefore \frac{5x}{\sqrt{x^2 + 64}} = 3$
 $\therefore 5x = 3\sqrt{x^2 + 64}$
 $\therefore (5x)^2 = 9(x^2 + 64)$
 $\therefore 25x^2 = 9x^2 + 576$
 $\therefore 16x^2 = 576$
 $\therefore x^2 = 36$
 $\therefore x = 6$ $\{0 \leq x \leq 11\}$

$C'(x)$ has sign diagram:

The minimum cost occurs when $x = 6$.

$C(6) = 5\sqrt{6^2 + 64} + 33 - 3(6)$
 $= 5\sqrt{100} + 33 - 18$
 $= 65$

\therefore the minimum cost of the pipeline is 65 million dollars.



5 We extend the table to include totals for each row and column.

	Defective	Not defective	Total
Corn	37	581	618
Pineapple	24	617	641
Total	61	1198	1259

a There were 1259 tins included in the sample.

b i 1198 of the 1259 tins were not defective.

$$\therefore P(\text{is not defective}) \approx \frac{1198}{1259} \approx 0.952$$

iii 37 of the 618 tins of corn were defective.

$$\therefore P(\text{is defective, given it is a tin of corn}) \approx \frac{37}{618} \approx 0.0599$$

ii 24 of the 1259 tins were defective tins of pineapple.

$$\begin{aligned} \therefore P(\text{is a defective tin of pineapple}) \\ \approx \frac{24}{1259} \approx 0.0191 \end{aligned}$$

6 a $y = \ln(\tan x), \quad 0 < x < \frac{\pi}{2}$

$$= \ln\left(\frac{\sin x}{\cos x}\right)$$

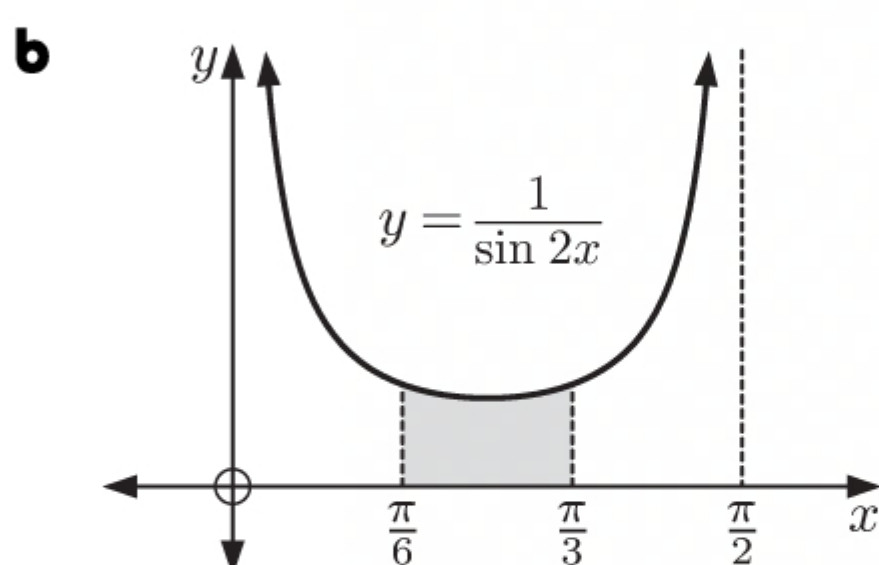
$$= \ln(\sin x) - \ln(\cos x)$$

$$\therefore \frac{dy}{dx} = \frac{\cos x}{\sin x} - \frac{-\sin x}{\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$$

$$= \frac{1}{\frac{1}{2} \sin 2x} \quad \{\cos^2 x + \sin^2 x = 1, \quad \sin 2x = 2 \sin x \cos x\}$$

$$= \frac{2}{\sin 2x}$$



$$\begin{aligned} \text{Shaded area} &= \int_{\pi/6}^{\pi/3} \frac{1}{\sin 2x} dx \\ &= \frac{1}{2} \int_{\pi/6}^{\pi/3} \frac{2}{\sin 2x} dx \\ &= \frac{1}{2} \left[\ln(\tan x) \right]_{\pi/6}^{\pi/3} \quad \{\text{using a}\} \\ &= \frac{1}{2} (\ln(\tan \frac{\pi}{3}) - \ln(\tan \frac{\pi}{6})) \\ &= \frac{1}{2} (\ln \sqrt{3} - \ln(\frac{1}{\sqrt{3}})) \\ &= \frac{1}{2} (\ln \sqrt{3} - \ln 1 + \ln \sqrt{3}) \\ &= \frac{1}{2} (2 \ln \sqrt{3}) \\ &= \frac{1}{2} \ln 3 \text{ units}^2 \end{aligned}$$

7 a A parametric equation of a plane through $(3, -1, 2)$ parallel to the vectors $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

$$\begin{aligned} \mathbf{b} \quad \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 3 \\ 3 & -1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix} \mathbf{k} \\ &= (1 - (-3)) \mathbf{i} - (-2 - 9) \mathbf{j} + (2 - 3) \mathbf{k} \\ &= 4\mathbf{i} + 11\mathbf{j} - \mathbf{k} \end{aligned}$$

So, the Cartesian equation of the plane is $4x + 11y - z = 4(3) + 11(-1) - 2$

$$\therefore 4x + 11y - z = -1$$

8 a $f(x)$ is undefined when $x^2 - 6x = 0$
 $\therefore x(x - 6) = 0$
 $\therefore x = 0$ or 6

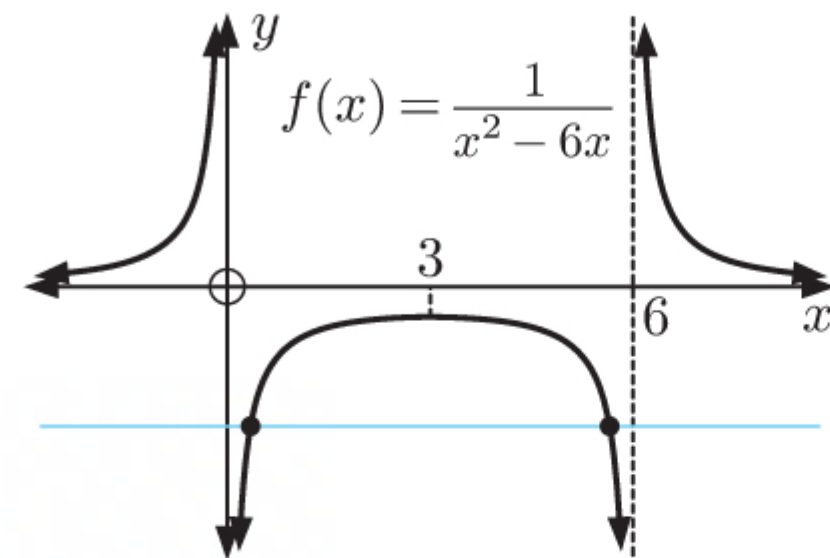
\therefore domain of $f(x)$ is $\{x \mid x \neq 0, x \neq 6\}$.

$f(x)$ has a local maximum at $x = 3$.

Now $f(3) = \frac{1}{3^2 - 6(3)} = -\frac{1}{9}$

\therefore range of $f(x)$ is $\{y \mid y > 0, y \leq -\frac{1}{9}\}$.

b $f(x)$ does not have an inverse function because it does not pass the horizontal line test. That is, it is not one-to-one.



c g is $y = \frac{1}{x^2 - 6x}$, $x \geq 3$, $x \neq 6$

$\therefore g^{-1}$ is $x = \frac{1}{y^2 - 6y}$, $y \geq 3$, $y \neq 6$

$\therefore y^2 - 6y = \frac{1}{x}$

$\therefore y^2 - 6y + (-3)^2 = \frac{1}{x} + (-3)^2$

$\therefore (y - 3)^2 = \frac{1}{x} + 9$

$\therefore y - 3 = \sqrt{\frac{1}{x} + 9} \quad \{y \geq 3\}$

$\therefore y = 3 + \sqrt{\frac{1}{x} + 9}$

$\therefore g^{-1}(x) = 3 + \sqrt{\frac{1}{x} + 9}$ which has domain $\{x \mid x > 0, x \leq -\frac{1}{9}\}$ and range $\{y \mid y \geq 3, y \neq 6\}$.

9 a $I(x) = e^{\int (-3) dx}$
 $= e^{-3x}$

Multiplying both sides of the differential equation by e^{-3x} gives

$e^{-3x} \frac{dy}{dx} - 3e^{-3x}y = 2x^2e^{-3x}$

$\therefore \frac{d}{dx}(ye^{-3x}) = 2x^2e^{-3x}$

$\therefore ye^{-3x} = \int 2x^2e^{-3x} dx$

$= 2x^2(-\frac{1}{3}e^{-3x}) - \int 4x(-\frac{1}{3}e^{-3x}) dx \quad \begin{cases} u = 2x^2 & v' = e^{-3x} \\ u' = 4x & v = -\frac{1}{3}e^{-3x} \end{cases}$

$= -\frac{2}{3}x^2e^{-3x} + \frac{4}{3} \int xe^{-3x} dx$

$= -\frac{2}{3}x^2e^{-3x} + \frac{4}{3} \left(-\frac{1}{3}xe^{-3x} - \int -\frac{1}{3}e^{-3x} dx \right) \quad \begin{cases} u = x & v' = e^{-3x} \\ u' = 1 & v = -\frac{1}{3}e^{-3x} \end{cases}$

$= -\frac{2}{3}x^2e^{-3x} - \frac{4}{9}xe^{-3x} + \frac{4}{3}(-\frac{1}{9}e^{-3x}) + c$

$= -\frac{2}{3}x^2e^{-3x} - \frac{4}{9}xe^{-3x} - \frac{4}{27}e^{-3x} + c$

But $y(0) = -\frac{4}{27}$

$\therefore 0 - 0 - \frac{4}{27}(1) + c = -\frac{4}{27}$

$\therefore c = 0$

$\therefore ye^{-3x} = e^{-3x} \left(-\frac{2}{3}x^2 - \frac{4}{9}x - \frac{4}{27} \right)$

$\therefore y = -\frac{2}{3}x^2 - \frac{4}{9}x - \frac{4}{27}$

$$\mathbf{b} \quad \frac{dy}{dx} = 2x^2 + 3y$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= \frac{d}{dx}(2x^2 + 3y) \\ &= 4x + 3 \frac{dy}{dx} \end{aligned}$$

$$\therefore \frac{d^3y}{dx^3} = 4 + 3 \frac{d^2y}{dx^2}$$

$$\therefore \frac{d^k y}{dx^k} = 3 \frac{d^{k-1} y}{dx^{k-1}} \quad \text{for all } k \geq 4$$

Now $y(0) = -\frac{4}{27}$, so at $(0, -\frac{4}{27})$:

$$\frac{dy}{dx} = 3\left(-\frac{4}{27}\right) = -\frac{4}{9}$$

$$\frac{d^2y}{dx^2} = 3\left(-\frac{4}{9}\right) = -\frac{4}{3}$$

$$\frac{d^3y}{dx^3} = 4 + 3\left(-\frac{4}{3}\right) = 0, \quad \text{and so } \frac{d^k y}{dx^k} = 0 \quad \text{for all } k \geq 4.$$

So, the Maclaurin polynomial is $y = -\frac{4}{27} - \frac{4}{9}x + \left(-\frac{4}{3}\right) \frac{x^2}{2!}$
 $= -\frac{4}{27} - \frac{4}{9}x - \frac{2}{3}x^2$, which agrees with **a**.

10 Let $z^5 = a + bi = r \operatorname{cis} \theta \dots (*)$

$z_1 = 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ is one of the roots of $a + bi$.

$$\text{Now } z_1^5 = r \operatorname{cis} \theta$$

$$\therefore \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^5 = r \operatorname{cis} \theta$$

$$\therefore (\sqrt{2})^5 \operatorname{cis} \frac{5\pi}{4} = r \operatorname{cis} \theta \quad \{\text{De Moivre}\}$$

$$\therefore r = (\sqrt{2})^5 \quad \text{and} \quad \theta = \frac{5\pi}{4}$$

Substituting into $(*)$ gives

$$z^5 = (\sqrt{2})^5 \operatorname{cis} \frac{5\pi}{4}$$

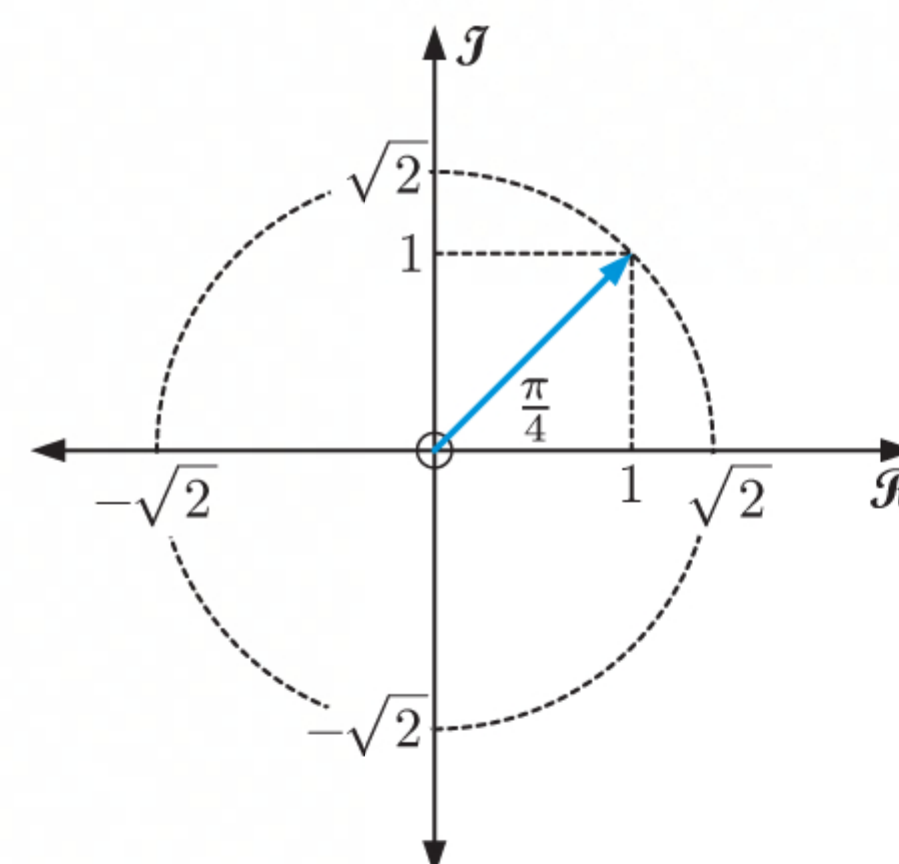
$$\therefore z^5 = (\sqrt{2})^5 \operatorname{cis} \left(\frac{5\pi}{4} + k2\pi\right), \quad k \in \mathbb{Z}$$

$$\therefore z = \left((\sqrt{2})^5 \operatorname{cis} \left(\frac{5\pi}{4} + k2\pi\right)\right)^{\frac{1}{5}}$$

$$\therefore z = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} + \frac{k2\pi}{5}\right) \quad \{\text{De Moivre}\}$$

$$\therefore z = \sqrt{2} \operatorname{cis} \frac{\pi}{4}, \sqrt{2} \operatorname{cis} \frac{13\pi}{20}, \sqrt{2} \operatorname{cis} \frac{21\pi}{20}, \sqrt{2} \operatorname{cis} \frac{29\pi}{20}, \sqrt{2} \operatorname{cis} \frac{37\pi}{20} \quad \{\text{letting } k = 0, 1, 2, 3, 4\}$$

\therefore the four other roots of $a + bi$ are $\sqrt{2} \operatorname{cis} \frac{13\pi}{20}, \sqrt{2} \operatorname{cis} \frac{21\pi}{20}, \sqrt{2} \operatorname{cis} \frac{29\pi}{20}, \sqrt{2} \operatorname{cis} \frac{37\pi}{20}$.



MIXED QUESTIONS SET 5

1 a $x^2 + 8x + k = 0$ has $a = 1$, $b = 8$, and $c = k$ $\therefore \Delta = b^2 - 4ac$

$$= 8^2 - 4(1)(k)$$

$$= 64 - 4k$$

b i For no real roots $\Delta < 0$

$$\therefore 64 - 4k < 0$$

$$\therefore 4k > 64$$

$$\therefore k > 16$$

ii For two distinct real roots $\Delta > 0$

$$\therefore 64 - 4k > 0$$

$$\therefore 4k < 64$$

$$\therefore k < 16$$

2 $f(x) = \frac{x-3}{2-x} = -\frac{x-3}{x-2} = -\left(\frac{x-2-1}{x-2}\right) = \frac{1}{x-2} - 1$

a The domain is $\{x \mid x \neq 2\}$.

The range is $\{y \mid y \neq -1\}$.

b The vertical asymptote is $x = 2$.

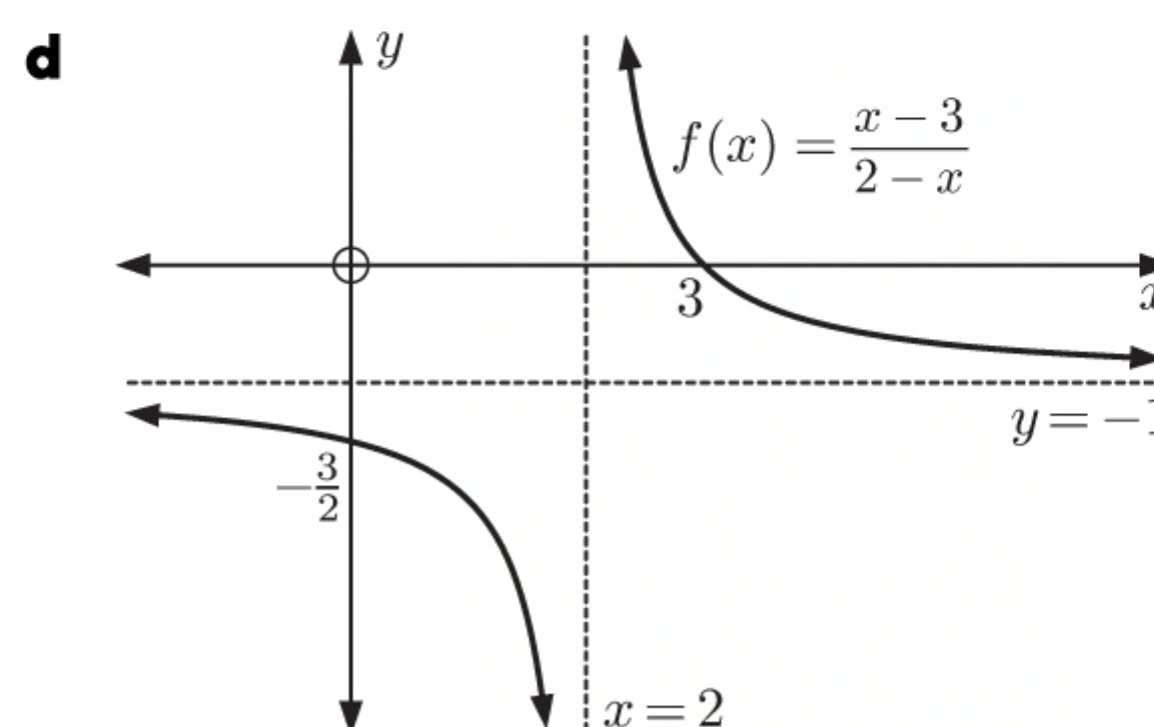
The horizontal asymptote is $y = -1$.

c $f(0) = \frac{-3}{2} = -\frac{3}{2}$, so the y -intercept is $-\frac{3}{2}$.

$$f(x) = 0 \quad \text{when} \quad x - 3 = 0$$

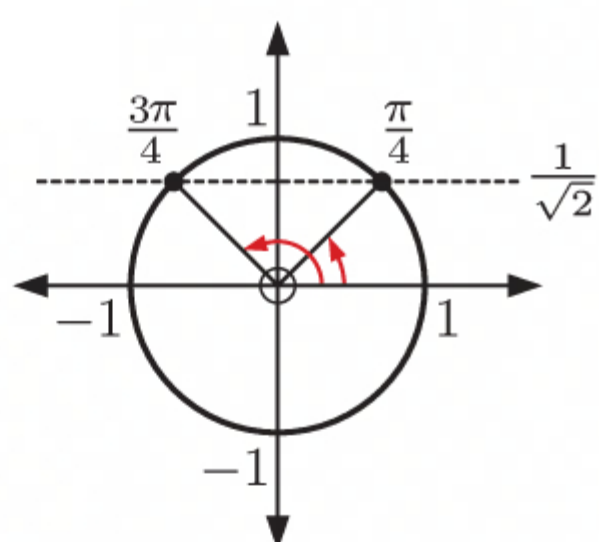
$$\therefore x = 3$$

\therefore the x -intercept is 3.



3 $\sqrt{2} \sin(2(x - \frac{\pi}{6})) = 1, \quad -\pi \leq x \leq 2\pi$

$$\therefore \sin(2(x - \frac{\pi}{6})) = \frac{1}{\sqrt{2}}$$



There are two points on the unit circle with sine $\frac{1}{\sqrt{2}}$.

They correspond to $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

Since $-\pi \leq x \leq 2\pi$

$$\therefore -\frac{7\pi}{6} \leq x - \frac{\pi}{6} \leq \frac{11\pi}{6}$$

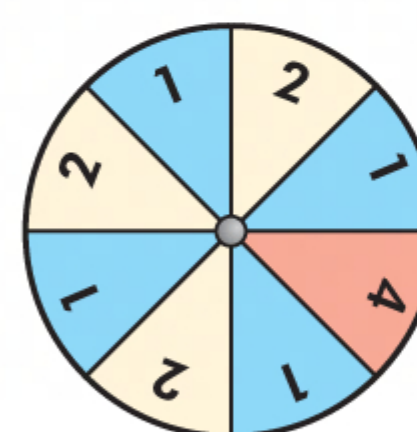
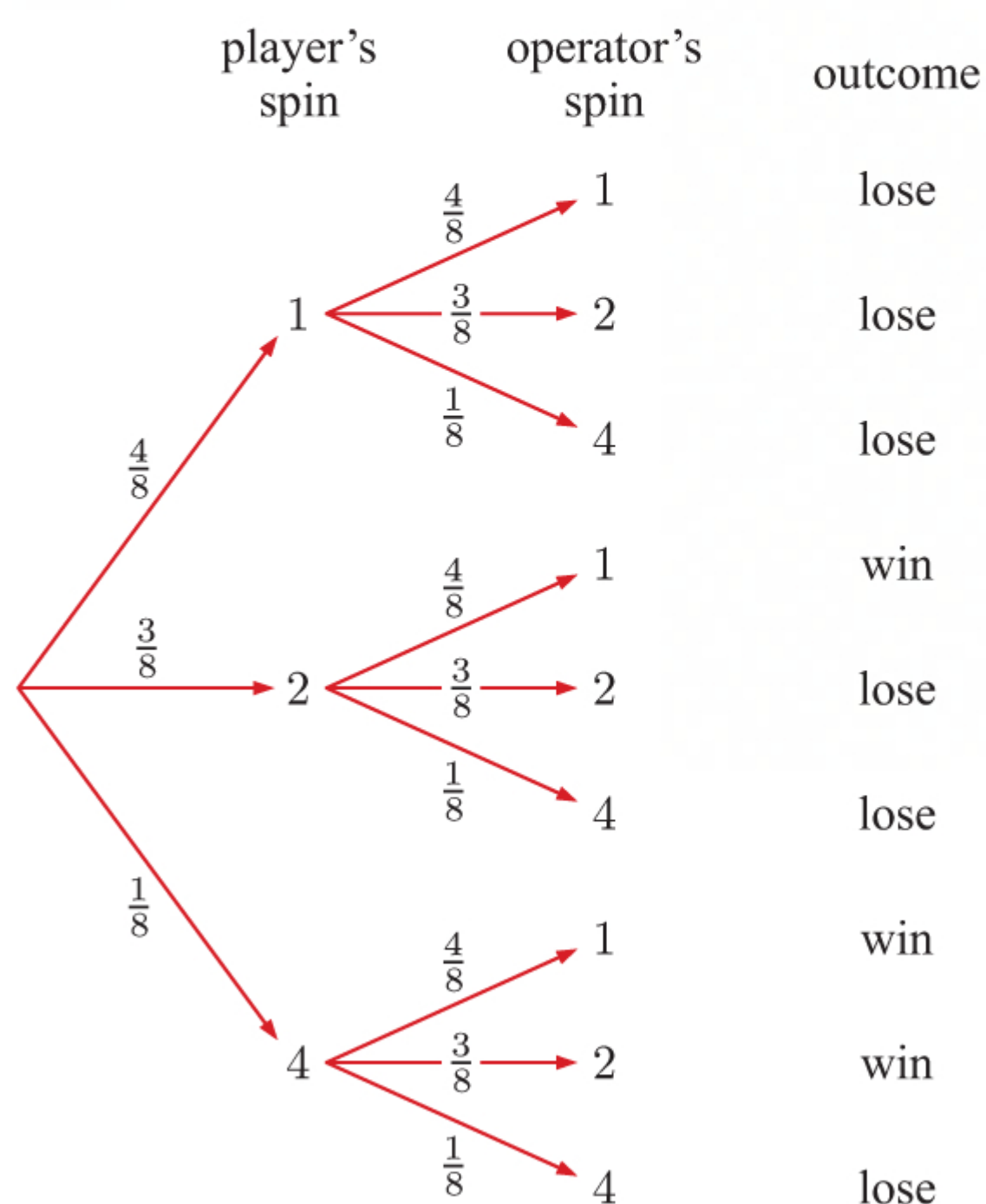
$$\therefore -\frac{7\pi}{3} \leq 2(x - \frac{\pi}{6}) \leq \frac{11\pi}{3}$$

So, $2(x - \frac{\pi}{6}) = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{11\pi}{4}$

$$\therefore x - \frac{\pi}{6} = -\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \text{ or } \frac{11\pi}{8}$$

$$\therefore x = -\frac{17\pi}{24}, -\frac{11\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}, \frac{31\pi}{24}, \text{ or } \frac{37\pi}{24}$$

4 We first construct a tree diagram of the possible outcomes.



The player wins if their spin is higher than the operator.

$$\begin{aligned} \therefore P(\text{win}) &= \left(\frac{3}{8} \times \frac{4}{8}\right) + \left(\frac{1}{8} \times \frac{4}{8}\right) + \left(\frac{1}{8} \times \frac{3}{8}\right) \\ &= \frac{12}{64} + \frac{4}{64} + \frac{3}{64} \\ &= \frac{19}{64} \end{aligned}$$

Outcome	Win	Lose
Winnings	\$a	\$0
Probability	$\frac{19}{64}$	$\frac{41}{64}$

Let X denote the return from one game.

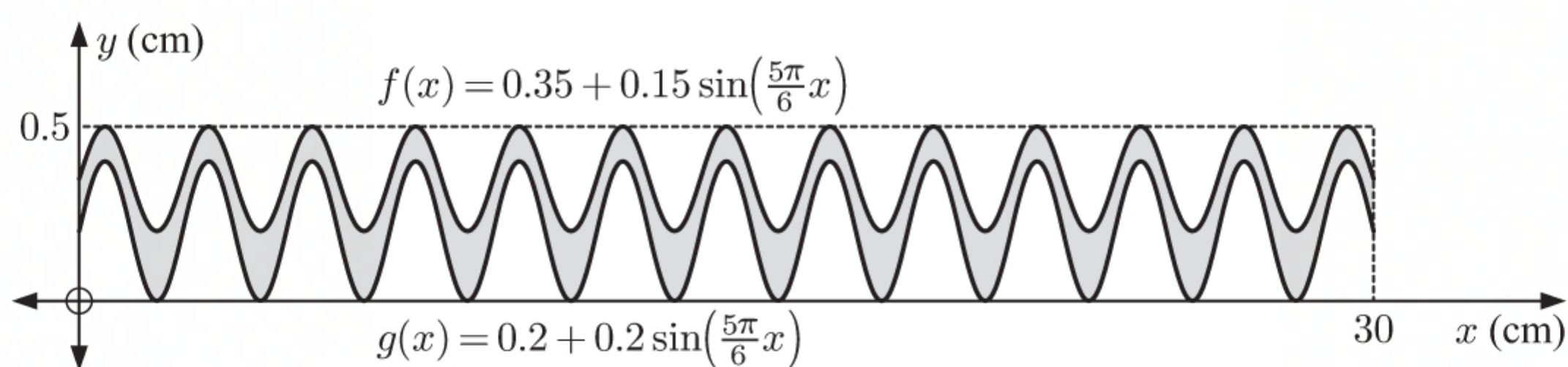
$$\begin{aligned} E(X) &= \left(a \times \frac{19}{64}\right) + \left(0 \times \frac{41}{64}\right) \\ &= \frac{19a}{64} \text{ dollars} \end{aligned}$$

It costs $\$k$ to play a game, so the expected gain $= \frac{19a}{64} - k$ dollars.

The game is fair when the expected gain is 0.

$$\therefore \frac{19a}{64} - k = 0 \quad \text{or} \quad 19a = 64k$$

5

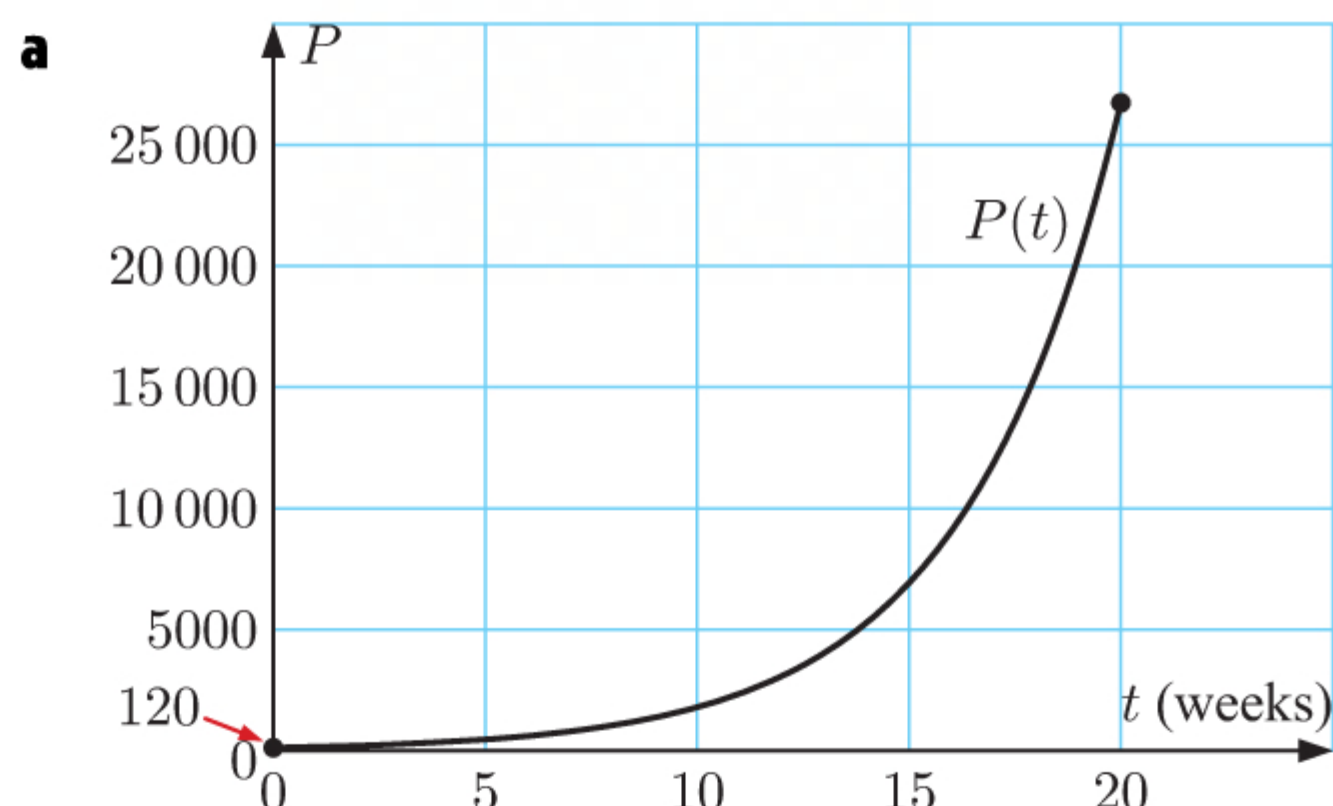


$$\begin{aligned}
 \text{Cross-sectional area} &= \int_0^{30} (f(x) - g(x)) dx \\
 &= \int_0^{30} ((0.35 + 0.15 \sin(\frac{5\pi}{6}x)) - (0.2 + 0.2 \sin(\frac{5\pi}{6}x))) dx \\
 &= \int_0^{30} (0.15 - 0.05 \sin(\frac{5\pi}{6}x)) dx \\
 &= [0.15x + 0.05(\frac{6}{5\pi}) \cos(\frac{5\pi}{6}x)]_0^{30} \\
 &= (0.15 \times 30 + \frac{3}{50\pi} \cos(\frac{5\pi}{6} \times 30)) - (0.15 \times 0 + \frac{3}{50\pi} \cos(\frac{5\pi}{6} \times 0)) \\
 &= 4.5 + \frac{3}{50\pi} \cos(25\pi) - \frac{3}{50\pi} \\
 &= 4.5 + \frac{3}{50\pi} \cos \pi - \frac{3}{50\pi} \quad \{\cos \theta = \cos(\theta + 2\pi)\} \\
 &= 4.5 + \frac{3}{50\pi}(-1) - \frac{3}{50\pi} \\
 &= 4.5 - \frac{3}{25\pi} \text{ cm}^2
 \end{aligned}$$

 So, volume = cross-sectional area \times length

$$\begin{aligned}
 &= (4.5 - \frac{3}{25\pi}) \times 100 \quad \{1 \text{ m} \equiv 100 \text{ cm}\} \\
 &\approx 446 \text{ cm}^3
 \end{aligned}$$

6 $P(t) = 120 \times (2.25)^{\frac{t}{3}}$



b $P(10) = 120 \times (2.25)^{\frac{10}{3}}$
 ≈ 1790

 \therefore the population of bees in the hive is about 1790 after 10 weeks.

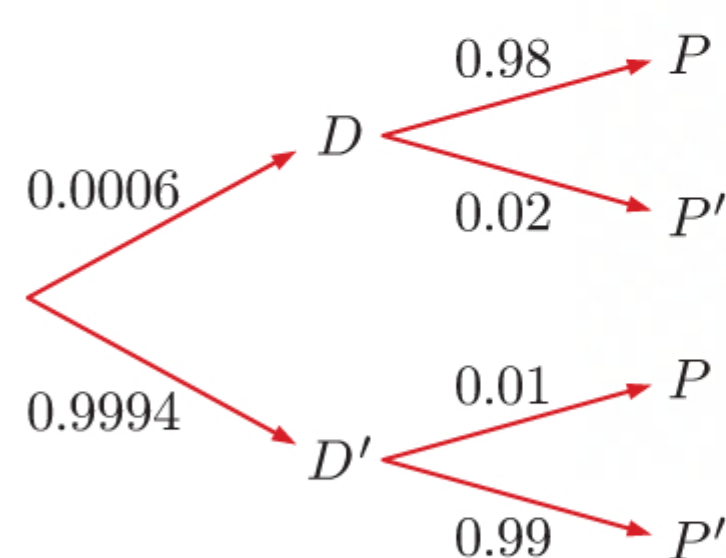
c $P = 120 \times (2.25)^{\frac{t}{3}}$
 $\therefore \frac{P}{120} = (2.25)^{\frac{t}{3}}$
 $\therefore \ln\left(\frac{P}{120}\right) = \frac{t}{3} \ln(2.25)$
 $\therefore t = \frac{3 \ln\left(\frac{P}{120}\right)}{\ln(2.25)}$

d When $P = 5000$, $t = \frac{3 \ln\left(\frac{5000}{120}\right)}{\ln(2.25)}$
 $= \frac{3 \ln\left(\frac{125}{3}\right)}{\ln(2.25)}$
 ≈ 13.8

 \therefore it will take about 13.8 weeks for the population to reach 5000.

 7 a Let D represent a person with the disease and P represent a positive blood test.

$$\begin{aligned}
 P(P) &= 0.0006(0.98) + 0.9994(0.01) \\
 &= 0.010582
 \end{aligned}$$



b $P(D | P) = \frac{P(P | D) P(D)}{P(P)}$
 $= \frac{0.98 \times 0.0006}{0.010582} \quad \{\text{using a}\}$
 ≈ 0.0556

$$\begin{aligned}
 \text{8 a radius of cone } r &= \sqrt{(4 - (-2))^2 + (2 - 5)^2 + (0 - 0)^2} \\
 &= \sqrt{36 + 9} \\
 &= \sqrt{45} \text{ units}
 \end{aligned}$$

$$\text{Volume} = 90\pi \text{ units}^3$$

$$\therefore \frac{1}{3}\pi r^2 h = 90\pi$$

$$\therefore \frac{1}{3}\pi(45)h = 90\pi$$

$$\therefore h = \frac{270\pi}{45\pi}$$

$$\therefore h = 6$$

\therefore the height of the cone is 6 units.

$$\begin{aligned}
 \text{b slant height } s &= \sqrt{h^2 + r^2} \quad \{\text{Pythagoras}\} \\
 &= \sqrt{36 + 45} \\
 &= \sqrt{81} \\
 &= 9 \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{surface area} &= \pi r s + \pi r^2 \\
 &= \pi\sqrt{45}(9) + \pi(45) \\
 &= \pi(9\sqrt{45} + 45) \text{ units}^2
 \end{aligned}$$

- c Since $(k, 7, 2)$ lies on the curved surface and has positive Z-coordinate, the apex of the cone has coordinates $(-2, 5, 6)$.

$$\begin{aligned}
 BE &= \sqrt{(k - (-2))^2 + (7 - 5)^2 + (2 - 2)^2} \\
 &= \sqrt{(k + 2)^2 + 4} \\
 &= \sqrt{k^2 + 4k + 4 + 4} \\
 &= \sqrt{k^2 + 4k + 8}
 \end{aligned}$$

Now $\triangle AEB$ and $\triangle ADC$ are similar.

$$\therefore \frac{BE}{CD} = \frac{AB}{AC}$$

$$\therefore \frac{\sqrt{k^2 + 4k + 8}}{\sqrt{45}} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \sqrt{k^2 + 4k + 8} = \frac{2}{3}\sqrt{45}$$

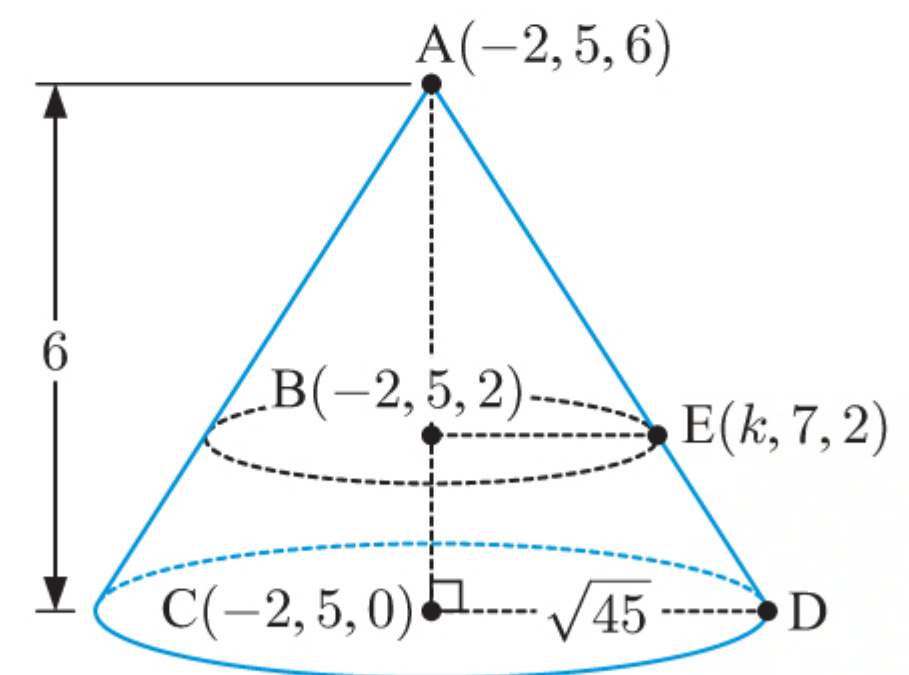
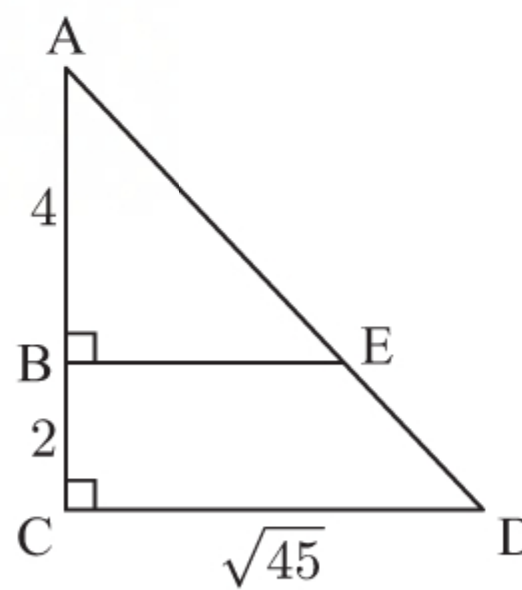
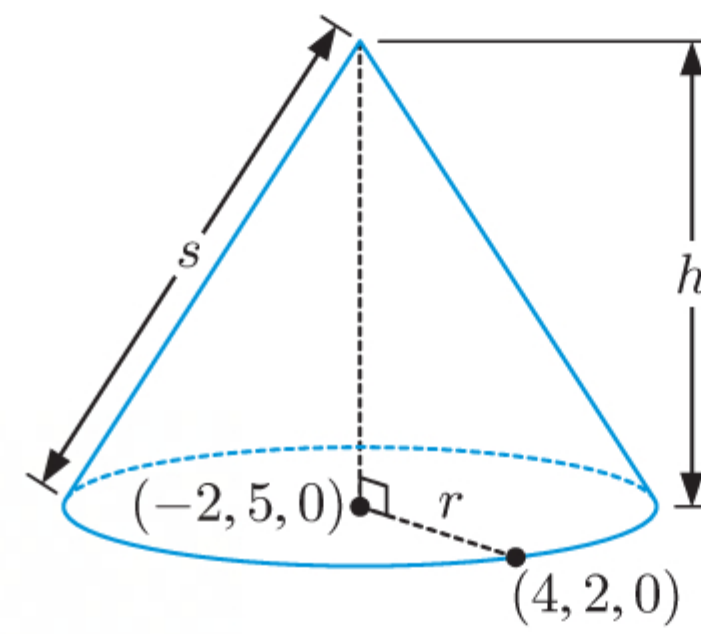
$$\therefore k^2 + 4k + 8 = \frac{4}{9}(45)$$

$$\therefore k^2 + 4k + 8 = 20$$

$$\therefore k^2 + 4k - 12 = 0$$

$$\therefore (k + 6)(k - 2) = 0$$

$$\therefore k = -6 \text{ or } 2$$



$$\text{9 a Let } y = f(x) = -2x^2 + 3$$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[-2(x+h)^2 + 3] - [-2x^2 + 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2(x^2 + 2xh + h^2) + 3 + 2x^2 - 3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2x^2 - 4xh - 2h^2 + 2x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2h(2x + h)}{h} \\
 &= \lim_{h \rightarrow 0} -2(2x + h) \quad \{\text{as } h \neq 0\} \\
 &= -4x
 \end{aligned}$$

- b** When $x = -1$, $y = -2(-1)^2 + 3 = 1$ and $\frac{dy}{dx} = -4(-1) = 4$.

So, the point of contact is $(-1, 1)$ and the tangent has gradient 4.

Since the gradient of the tangent is 4, the gradient of the normal is $-\frac{1}{4}$.

So, the equation of the normal is $\frac{y-1}{x-(-1)} = -\frac{1}{4}$

$$\therefore 4y - 4 = -(x + 1)$$

$$\therefore 4y = -x + 3$$

$$\therefore y = \frac{-x + 3}{4}$$

- c** The normal meets the curve where $\frac{-x + 3}{4} = -2x^2 + 3$

$$\therefore -x + 3 = -8x^2 + 12$$

$$\therefore 8x^2 - x - 9 = 0$$

$$\therefore (8x - 9)(x + 1) = 0$$

$$\therefore x = -1 \text{ or } \frac{9}{8}$$

$$\begin{aligned} \text{When } x = \frac{9}{8}, \quad y &= -2\left(\frac{9}{8}\right)^2 + 3 \\ &= -2\left(\frac{81}{64}\right) + 3 \\ &= \frac{15}{32} \end{aligned}$$

So, the normal meets the curve again at $\left(\frac{9}{8}, \frac{15}{32}\right)$.

- 10** P_n is: If $u_{n+2} = u_n + u_{n+1}$, $u_1 = u_2 = 1$, then $u_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$, $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

$$\begin{aligned} (1) \text{ If } n = 1, \text{ RHS} &= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\ &= \frac{1 + \sqrt{5} - 1 + \sqrt{5}}{2\sqrt{5}} \\ &= \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \end{aligned}$$

$\therefore P_1$ is true.

$$\begin{aligned} \text{If } n = 2, \text{ RHS} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\ &= \frac{(1 + 2\sqrt{5} + 5) - (1 - 2\sqrt{5} + 5)}{4\sqrt{5}} \\ &= \frac{4\sqrt{5}}{4\sqrt{5}} = 1 \end{aligned}$$

$\therefore P_2$ is true.

- (2) Now suppose P_k and P_{k+1} are true.

$$\text{So, } u_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} \quad \text{and} \quad u_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

Now $u_{k+2} = u_k + u_{k+1}$

$$\begin{aligned} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} \quad \{\text{using } P_k \text{ and } P_{k+1}\} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k \left[1 + \frac{1+\sqrt{5}}{2}\right] - \left(\frac{1-\sqrt{5}}{2}\right)^k \left[1 + \frac{1-\sqrt{5}}{2}\right]}{\sqrt{5}} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{5}} \end{aligned}$$

$$\text{We notice that } \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1 + 2\sqrt{5} + 5}{4} = \frac{6 + 2\sqrt{5}}{4} = \frac{3 + \sqrt{5}}{2}$$

$$\text{and likewise } \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3 - \sqrt{5}}{2}.$$

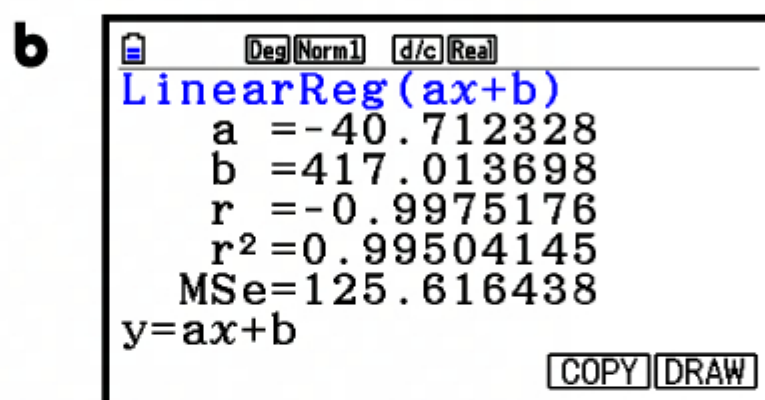
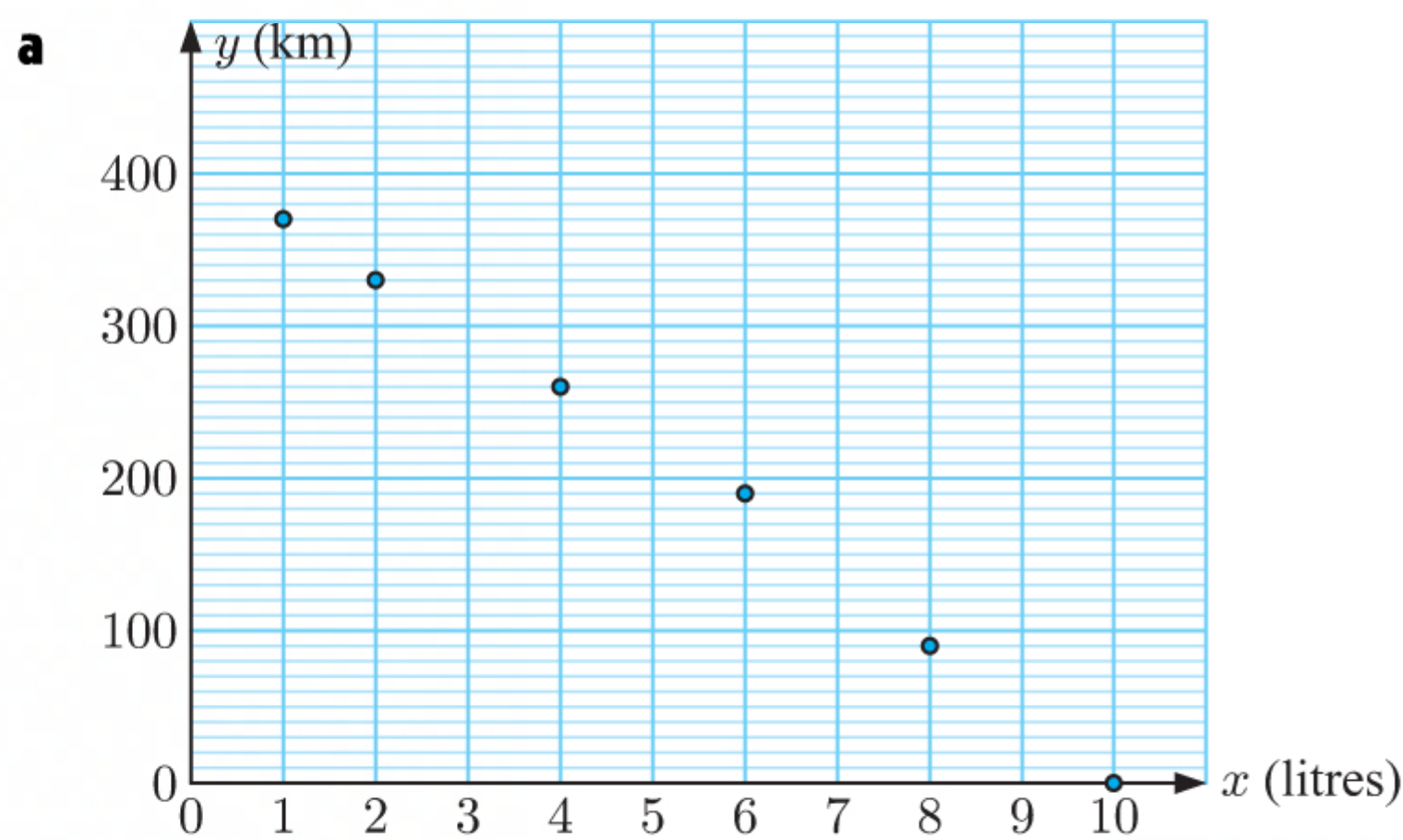
$$\begin{aligned}\text{So, } u_{k+2} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}}{\sqrt{5}}\end{aligned}$$

Since P_{k+2} is true whenever P_k and P_{k+1} are true, and P_1 and P_2 are true, P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

MIXED QUESTIONS SET 6

1

Remaining fuel (x litres)	10	8	6	4	2	1
Distance (y km)	0	90	190	260	330	370



Using technology, the regression line is $y \approx -40.7x + 417$.

c The y -intercept of the regression line ≈ 417 . This indicates that the motorbike can travel about 417 km on a full tank of petrol.

d i When $y = 220$, $220 \approx -40.7x + 417$

$$\therefore -197 \approx -40.7x$$

$$\therefore x \approx 4.84$$

\therefore there is about 4.84 litres of fuel left in the tank after the motorbike has travelled 220 km.

ii Average distance travelled per litre $\approx \frac{220}{10 - 4.84} \approx 42.6$ km per litre.

2 $a(t) = 2 - 6t \text{ m s}^{-2}$, $t \geq 0$

a
$$\begin{aligned}v(t) &= \int a(t) dt \\ &= \int (2 - 6t) dt \\ &= 2t - 3t^2 + c\end{aligned}$$

The particle is initially at rest, so $v(0) = 0$

$$\therefore 2(0) - 3(0)^2 + c = 0$$

$$\therefore c = 0$$

$$\therefore v(t) = 2t - 3t^2, \quad t \geq 0$$

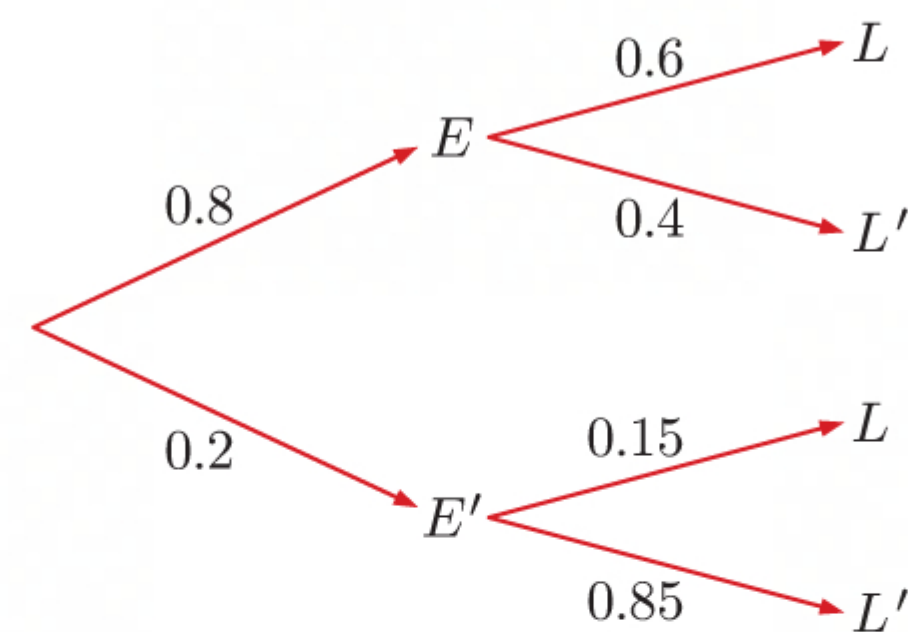
b Change in displacement in first second

$$\begin{aligned}&= \int_0^1 v(t) dt \\ &= \int_0^1 (2t - 3t^2) dt \\ &= [t^2 - t^3]_0^1 \\ &= (1 - 1) - (0 - 0) \\ &= 0 \text{ m}\end{aligned}$$

- c** The particle changes direction where $v(t) = 0$
 $\therefore 2t - 3t^2 = 0$
 $\therefore t(2 - 3t) = 0$
 $\therefore t = 0 \text{ or } t = \frac{2}{3}$

$$\begin{aligned} \text{So, total distance travelled in first second} &= \int_0^1 |v(t)| dt \\ &= \int_0^{\frac{2}{3}} |2t - 3t^2| dt + \int_{\frac{2}{3}}^1 |2t - 3t^2| dt \\ &= \int_0^{\frac{2}{3}} (2t - 3t^2) dt + \int_{\frac{2}{3}}^1 (3t^2 - 2t) dt \\ &= \left[t^2 - t^3 \right]_0^{\frac{2}{3}} + \left[t^3 - t^2 \right]_{\frac{2}{3}}^1 \\ &= \left[\left(\left(\frac{2}{3} \right)^2 - \left(\frac{2}{3} \right)^3 \right) - 0 \right] + \left[(1 - 1) - \left(\left(\frac{2}{3} \right)^3 - \left(\frac{2}{3} \right)^2 \right) \right] \\ &= \frac{4}{9} - \frac{8}{27} - \frac{8}{27} + \frac{4}{9} \\ &= \frac{8}{27} \approx 0.296 \text{ m} \end{aligned}$$

- 3 a** Let E be the event that Mark wakes up early, and L be the event that Mark packs his lunch.

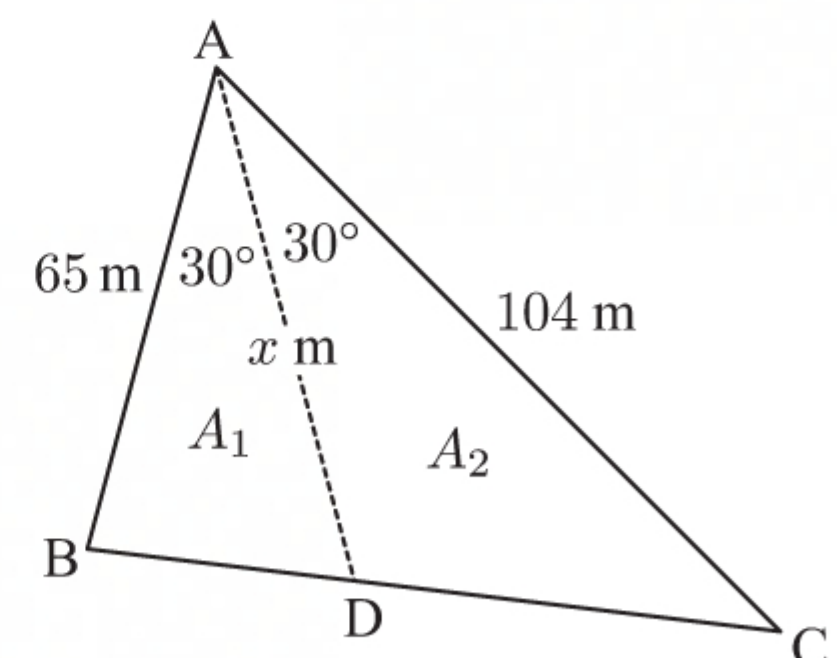


b $P(L) = P(E \cap L) + P(E' \cap L)$
 $= 0.8 \times 0.6 + 0.2 \times 0.15$
 $= 0.51$

- 4 a** In $\triangle ABC$, by the cosine rule:

$$\begin{aligned} BC^2 &= 65^2 + 104^2 - 2 \times 65 \times 104 \times \cos 60^\circ \\ \therefore BC &= \sqrt{65^2 + 104^2 - 2 \times 65 \times 104 \times \cos 60^\circ} \quad \{\text{as } BC > 0\} \\ \therefore BC &= 91 \text{ m} \end{aligned}$$

b Area of $\triangle ABC = \frac{1}{2} \times AB \times AC \times \sin \widehat{BAC}$
 $= \frac{1}{2} \times 65 \times 104 \times \sin 60^\circ$
 $= 65 \times 52 \times \frac{\sqrt{3}}{2}$
 $= 1690\sqrt{3} \text{ m}^2$
 $\approx 2930 \text{ m}^2$



\therefore the total area of the field is about 2930 m^2 .

c Area of $A_1 = \frac{1}{2} \times AB \times AD \times \sin \widehat{BAD}$
 $= \frac{1}{2} \times 65 \times x \times \sin 30^\circ$
 $= \frac{65x}{2} \times \frac{1}{2}$
 $= \frac{65x}{4} \text{ m}^2$

Area of $A_2 = \frac{1}{2} \times AC \times AD \times \sin \widehat{CAD}$
 $= \frac{1}{2} \times 104 \times x \times \sin 30^\circ$
 $= 52x \times \frac{1}{2}$
 $= 26x \text{ m}^2$

Now, the total area of the field $= A_1 + A_2$

$$\therefore 1690\sqrt{3} = \frac{65x}{4} + 26x \quad \{\text{from a}\}$$

$$\therefore 1690\sqrt{3} = x \left(\frac{65}{4} + 26 \right)$$

$$\therefore x = \frac{1690\sqrt{3}}{\frac{65}{4} + 26}$$

$$\therefore x \approx 69.3$$

- 5** Since p, q , and r are consecutive odd integers with $p < q < r$, $p = q - 2$ and $r = q + 2$.

$$\begin{aligned}\text{So, } 2q(p + r) &= 2q((q - 2) + (q + 2)) \\ &= 2q(2q) \\ &= (2q)^2 \text{ which is a perfect square}\end{aligned}$$

6 $y = xe^{2x}$

a $\frac{dy}{dx} = e^{2x} + 2xe^{2x}$
 $= e^{2x}(1 + 2x)$

The tangent to the curve is horizontal where $\frac{dy}{dx} = 0$

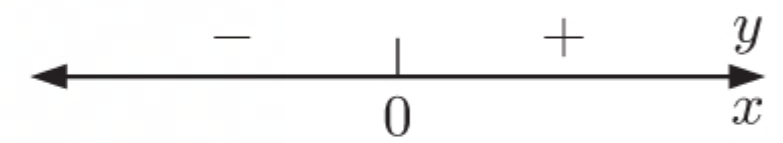
$$\begin{aligned}\therefore e^{2x}(1 + 2x) &= 0 \\ \therefore x &= -\frac{1}{2} \quad \{e^{2x} > 0\}\end{aligned}$$

When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})e^{2(-\frac{1}{2})} = -\frac{1}{2e}$.

$\therefore y = k$ is a horizontal tangent to the curve when $k = -\frac{1}{2e}$.

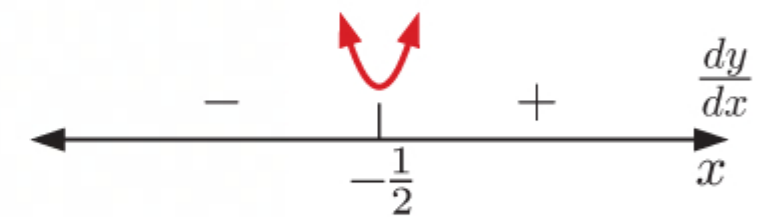
b When $y = 0$, $xe^{2x} = 0$
 $\therefore x = 0 \quad \{e^{2x} > 0\}$

So, y has sign diagram:



When $\frac{dy}{dx} = 0$, $x = -\frac{1}{2}$ {from **a**}

So, $\frac{dy}{dx}$ has sign diagram:

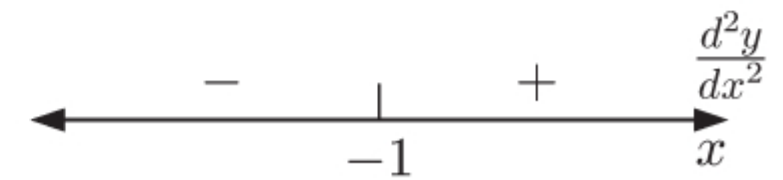


\therefore there is a local minimum at $(-\frac{1}{2}, -\frac{1}{2e})$.

$$\begin{aligned}\text{Now } \frac{d^2y}{dx^2} &= 2e^{2x}(1 + 2x) + 2e^{2x} \\ &= 2e^{2x}(2 + 2x) \\ &= 4e^{2x}(1 + x)\end{aligned}$$

When $\frac{d^2y}{dx^2} = 0$, $4e^{2x}(1 + x) = 0$
 $\therefore x = -1 \quad \{e^{2x} > 0\}$

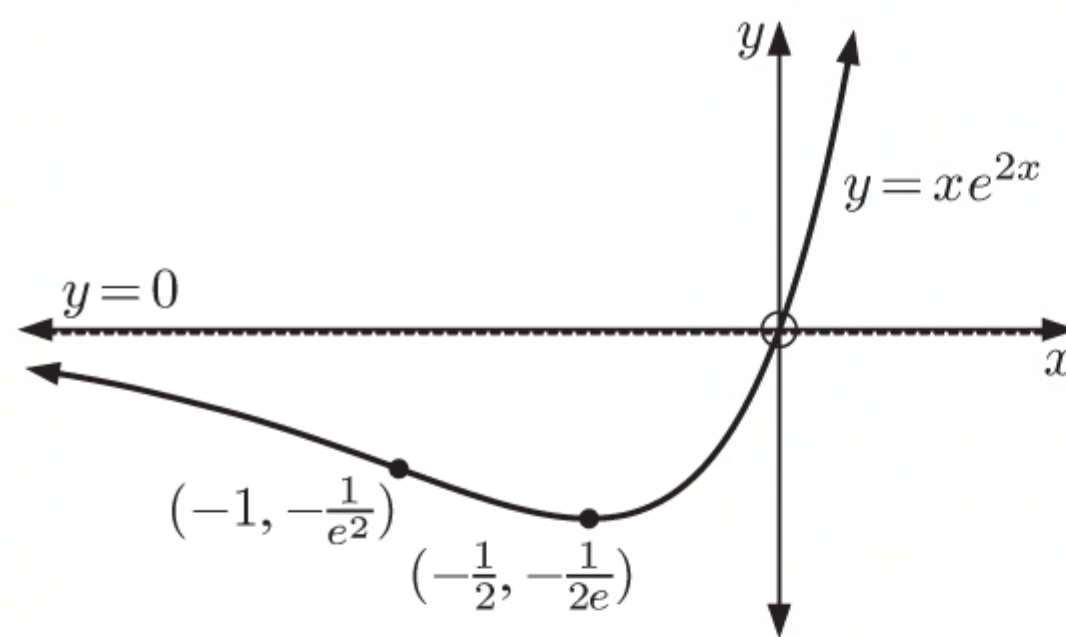
So, $\frac{d^2y}{dx^2}$ has sign diagram:



When $x = -1$, $y = (-1)e^{2(-1)} = -\frac{1}{e^2}$.

\therefore there is a non-stationary point of inflection at $(-1, -\frac{1}{e^2})$.

So, as $x \rightarrow \infty$, $y = xe^{2x} \rightarrow \infty$
 and as $x \rightarrow -\infty$, $y = xe^{2x} \rightarrow 0^-$.



- i** $y = k$ meets the curve at exactly one point for $k = -\frac{1}{2e}$ or $k \geq 0$.
- ii** $y = k$ meets the curve at two distinct points for $-\frac{1}{2e} < k < 0$.
- iii** $y = k$ meets the curve at no points for $k < -\frac{1}{2e}$.

c $y = xe^{ax}$, $a \in \mathbb{R}$, $a > 0$

i $y = x$ meets the curve $y = xe^{ax}$ where $xe^{ax} = x$

$$\therefore xe^{ax} - x = 0$$

$$\therefore x(e^{ax} - 1) = 0$$

$$\therefore x = 0 \quad \text{or} \quad e^{ax} = 1$$

$$\therefore x = 0 \quad \text{or} \quad ax = 0$$

$$\therefore x = 0 \quad \{a > 0\}$$

Now $y = xe^{ax}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= e^{ax} + axe^{ax} \\ &= e^{ax}(1 + ax) \end{aligned}$$

When $x = 0$, $y = 0e^0 = 0$

$$\text{and} \quad \frac{dy}{dx} = e^0(1 + 0) = 1$$

\therefore the tangent to the curve at $x = 0$ has gradient 1, and the point of contact is $(0, 0)$.

\therefore the equation of the tangent is $y - 0 = 1(x - 0)$

$\therefore y = x$ as required.

ii From **i**, the tangent to the curve at $y = xe^{ax}$ when $x = 0$ is $y = x$.

\therefore the normal to the curve at $x = 0$ has gradient -1 , and the point of contact is $(0, 0)$.

\therefore the equation of the normal is $y - 0 = -1(x - 0)$

$\therefore y = -x$.

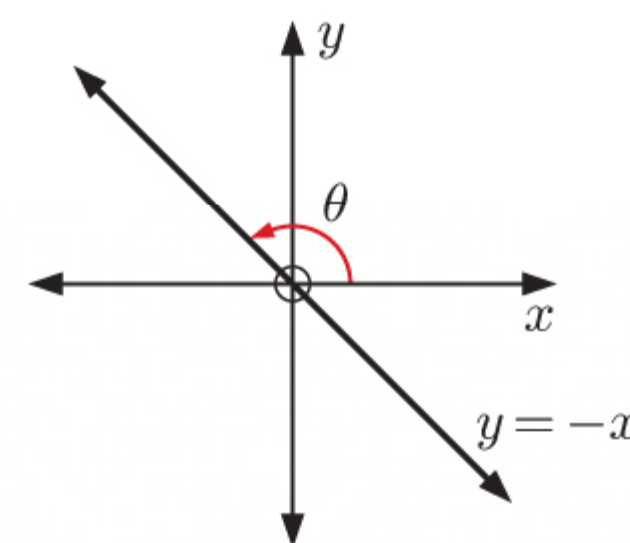
Let θ be the angle the normal makes with the positive x -axis.

The normal has gradient -1 .

$$\therefore \tan \theta = -1$$

$$\therefore \theta = \frac{3\pi}{4} \quad \{0 < \theta < \pi\}$$

So, the acute angle the normal makes with the x -axis is $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$.



7 $L_1: \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}, \quad t \in \mathbb{R}$

a L_1 meets the XY -plane where $4 - 2t = 0$

$$\therefore 2t = 4$$

$$\therefore t = 2$$

$\therefore L_1$ meets the XY -plane at $(2 + 3(2), -1 + 4(2), 0)$ which is $(8, 7, 0)$.

b L_2 has direction vector $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ and passes through $(3, 0, 0)$.

$$\therefore L_2 \text{ has vector equation } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}$$

$\therefore L_2$ has parametric equations $x = 3 + 3s$, $y = 5s$, $z = 0$, $s \in \mathbb{R}$.

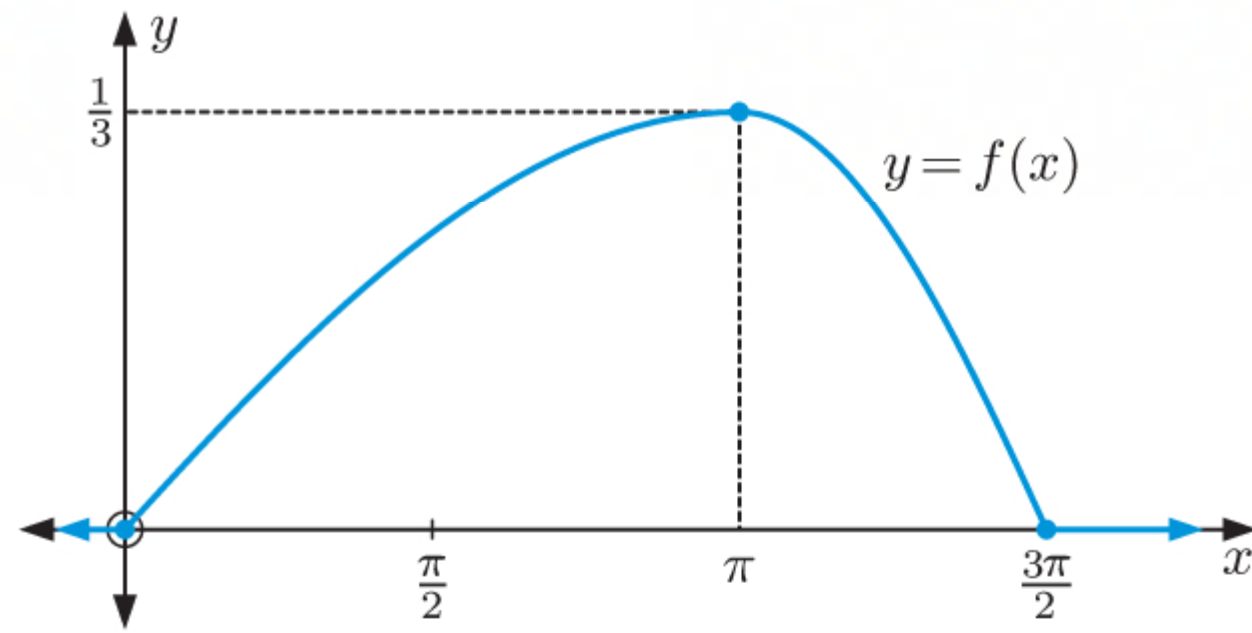
$$\begin{aligned}
 \text{c } L_3 \text{ has direction vector } & \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -2 \\ 3 & 5 & 0 \end{vmatrix} \\
 & = \begin{vmatrix} 4 & -2 \\ 5 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix} \mathbf{k} \\
 & = (0 + 10)\mathbf{i} - (0 + 6)\mathbf{j} + (15 - 12)\mathbf{k} \\
 & = 10\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} \\
 & = \begin{pmatrix} 10 \\ -6 \\ 3 \end{pmatrix}
 \end{aligned}$$

Since L_3 passes through $(-2, 5, 1)$, L_3 has vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ -6 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

$\therefore L_3$ has parametric equations $x = -2 + 10\lambda$, $y = 5 - 6\lambda$, $z = 1 + 3\lambda$, $\lambda \in \mathbb{R}$.

Equating λ values, the Cartesian equations are $\frac{x+2}{10} = \frac{y-5}{-6} = \frac{z-1}{3}$.

$$\text{8 } f(x) = \begin{cases} \frac{1}{3} \sin \frac{x}{2}, & 0 \leq x \leq \pi \\ -\frac{1}{3} \cos x, & \pi \leq x \leq \frac{3\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$



b From **a**, $f(x) \geq 0$ for all x .

$$\begin{aligned}
 \int_0^{\frac{3\pi}{2}} f(x) dx &= \int_0^{\pi} \frac{1}{3} \sin \frac{x}{2} dx + \int_{\pi}^{\frac{3\pi}{2}} -\frac{1}{3} \cos x dx \\
 &= \left[-\frac{2}{3} \cos \frac{x}{2} \right]_0^{\pi} + \left[-\frac{1}{3} \sin x \right]_{\pi}^{\frac{3\pi}{2}} \\
 &= \left(-\frac{2}{3} \cos \frac{\pi}{2} + \frac{2}{3} \cos 0 \right) + \left(-\frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{3} \sin \pi \right) \\
 &= 0 + \frac{2}{3} - \frac{1}{3}(-1) + 0 \\
 &= \frac{2}{3} + \frac{1}{3} \\
 &= 1 \quad \checkmark
 \end{aligned}$$

$\therefore f(x)$ is a valid probability density function.

$$\begin{aligned}
 \text{c } P\left(\frac{2\pi}{3} < X < \frac{7\pi}{6}\right) &= \int_{\frac{2\pi}{3}}^{\frac{7\pi}{6}} f(x) dx \\
 &= \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{3} \sin \frac{x}{2} dx + \int_{\pi}^{\frac{7\pi}{6}} -\frac{1}{3} \cos x dx \\
 &= \left[-\frac{2}{3} \cos \frac{x}{2} \right]_{\frac{2\pi}{3}}^{\pi} + \left[-\frac{1}{3} \sin x \right]_{\pi}^{\frac{7\pi}{6}} \\
 &= \left(-\frac{2}{3} \cos \frac{\pi}{2} + \frac{2}{3} \cos \frac{\pi}{3} \right) + \left(-\frac{1}{3} \sin \frac{7\pi}{6} + \frac{1}{3} \sin \pi \right) \\
 &= 0 + \frac{2}{3} \left(\frac{1}{2} \right) - \frac{1}{3} \left(-\frac{1}{2} \right) + 0 \\
 &= \frac{1}{3} + \frac{1}{6} \\
 &= \frac{1}{2}
 \end{aligned}$$

d From **a**, $P(X \leq \pi) = \int_0^\pi \frac{1}{3} \sin \frac{x}{2} dx = \frac{2}{3}$

So, $m \leq \pi$

Now $\int_0^m f(x) dx = \frac{1}{2}$

$$\therefore \int_0^m \frac{1}{3} \sin \frac{x}{2} dx = \frac{1}{2}$$

$$\therefore \left[-\frac{2}{3} \cos \frac{x}{2} \right]_0^m = \frac{1}{2}$$

$$\therefore -\frac{2}{3} \cos \frac{m}{2} + \frac{2}{3} \cos 0 = \frac{1}{2}$$

$$\therefore -\frac{2}{3} \cos \frac{m}{2} + \frac{2}{3} = \frac{1}{2}$$

$$\therefore -\frac{2}{3} \cos \frac{m}{2} = -\frac{1}{6}$$

$$\therefore \cos \frac{m}{2} = \frac{1}{4}$$

$$\therefore \frac{m}{2} = \arccos \frac{1}{4} \quad \{0 \leq m \leq \pi \quad \therefore 0 \leq \frac{m}{2} \leq \frac{\pi}{2}\}$$

$$\therefore m = 2 \arccos \frac{1}{4}$$

e Mean = $E(X)$

$$= \int_0^{\frac{3\pi}{2}} x f(x) dx$$

$$= \int_0^\pi \frac{x}{3} \sin \frac{x}{2} dx + \int_\pi^{\frac{3\pi}{2}} -\frac{x}{3} \cos x dx \quad \dots (1)$$

Now $\int \frac{x}{3} \sin \frac{x}{2} dx = -\frac{2x}{3} \cos \frac{x}{2} + \frac{2}{3} \int \cos \frac{x}{2} dx \quad \begin{cases} u = \frac{x}{2} & v' = \sin \frac{x}{2} \\ u' = \frac{1}{2} & v = -2 \cos \frac{x}{2} \end{cases}$

$$= -\frac{2x}{3} \cos \frac{x}{2} + \frac{4}{3} \sin \frac{x}{2} + c \quad \dots (2)$$

and $\int -\frac{x}{3} \cos x dx = -\frac{x}{3} \sin x + \frac{1}{3} \int \sin x dx \quad \begin{cases} u = -\frac{x}{3} & v' = \cos x \\ u' = -\frac{1}{3} & v = \sin x \end{cases}$

$$= -\frac{x}{3} \sin x - \frac{1}{3} \cos x + c \quad \dots (3)$$

Using (2) and (3) in (1) gives

$$\begin{aligned} \text{mean} &= \left[-\frac{2x}{3} \cos \frac{x}{2} + \frac{4}{3} \sin \frac{x}{2} \right]_0^\pi + \left[-\frac{x}{3} \sin x - \frac{1}{3} \cos x \right]_\pi^{\frac{3\pi}{2}} \\ &= \left(-\frac{2\pi}{3} \cos \frac{\pi}{2} + \frac{4}{3} \sin \frac{\pi}{2} + 0 \right) + \left(-\frac{\pi}{2} \sin \frac{3\pi}{2} - \frac{1}{3} \cos \frac{3\pi}{2} + \frac{\pi}{3} \sin \pi + \frac{1}{3} \cos \pi \right) \\ &= -\frac{2\pi}{3}(0) + \frac{4}{3}(1) - \frac{\pi}{2}(-1) - \frac{1}{3}(0) + \frac{\pi}{3}(0) + \frac{1}{3}(-1) \\ &= \frac{4}{3} + \frac{\pi}{2} - \frac{1}{3} \\ &= 1 + \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned}
 9 \quad \frac{dy}{dx} &= \left(\frac{5x-2y}{2x-y} \right) \times \frac{\frac{1}{x}}{\frac{1}{x}} \\
 &= \frac{5-2\left(\frac{y}{x}\right)}{2-\left(\frac{y}{x}\right)} \quad \text{so the differential equation is homogeneous}
 \end{aligned}$$

$$\text{Let } y = vx, \text{ so } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \{\text{product rule}\}$$

$$\text{Comparing with the differential equation, } v + x \frac{dv}{dx} = \frac{5-2v}{2-v}$$

$$\therefore x \frac{dv}{dx} = \frac{5-2v}{2-v} - v$$

$$\therefore x \frac{dv}{dx} = \frac{5-4v+v^2}{2-v}$$

$$\therefore \frac{2-v}{5-4v+v^2} \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int \frac{2-v}{5-4v+v^2} dv = \int \frac{1}{x} dx$$

$$\therefore \int \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dv} \right) dv = \int \frac{1}{x} dx \quad \left\{ u = 5-4v+v^2 \quad \therefore \frac{du}{dv} = -4+2v \right\}$$

$$\therefore \int -\frac{1}{2} \frac{1}{u} du = \int \frac{1}{x} dx$$

$$\therefore -\frac{1}{2} \ln|u| = \ln|x| + c$$

$$\therefore -\frac{1}{2} \ln|5-4v+v^2| = \ln|x| + c$$

$$\therefore \ln|5-4v+v^2| = -2\ln|x| + b \quad \{b = -2c\}$$

$$\therefore 5-4v+v^2 = Ax^{-2} \quad \left\{ A = \pm e^b, \quad |x|^{-2} = x^{-2} \right\}$$

$$\therefore (v-2)^2 + 1 = \frac{A}{x^2} \quad \{\text{completing the square}\}$$

$$\therefore (v-2)^2 = \frac{A}{x^2} - 1$$

$$\therefore \left(\frac{y}{x} - 2 \right)^2 = \frac{A}{x^2} - 1 \quad \left\{ v = \frac{y}{x} \right\}$$

$$\text{Now } y(1) = 3 \quad \therefore \left(\frac{3}{1} - 2 \right)^2 = \frac{A}{1^2} - 1$$

$$\therefore 1^2 = A - 1$$

$$\therefore A = 2$$

$$\text{The particular solution is } \left(\frac{y}{x} - 2 \right)^2 = \frac{2}{x^2} - 1$$

$$\therefore \frac{(y-2x)^2}{x^2} = \frac{2}{x^2} - 1$$

$$\therefore (y-2x)^2 = 2 - x^2$$

$$\therefore y-2x = \sqrt{2-x^2} \quad \left\{ y-2x = -\sqrt{2-x^2} \text{ does not satisfy } y(1) = 3 \right\}$$

$$\therefore y = \sqrt{2-x^2} + 2x$$

$$10 \quad \mathbf{a} \quad \cos 4\theta + i \sin 4\theta = \text{cis } 4\theta$$

$$= (\text{cis } \theta)^4 \quad \{\text{De Moivre}\}$$

$$= (\cos \theta + i \sin \theta)^4$$

$$= \cos^4 \theta + 4 \cos^3 \theta (i \sin \theta) + 6 \cos^2 \theta (i \sin \theta)^2 + 4 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4$$

$$= (\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) + i(4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta)$$

Equating real and imaginary parts,

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta \quad \text{and} \quad \sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta.$$

$$\text{Hence, } \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta}{\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta}$$

$$= \frac{4 \left(\frac{\sin \theta}{\cos \theta} \right) - 4 \left(\frac{\sin \theta}{\cos \theta} \right)^3}{1 - 6 \left(\frac{\sin \theta}{\cos \theta} \right)^2 + \left(\frac{\sin \theta}{\cos \theta} \right)^4} \quad \{\text{dividing all terms by } \cos^4 \theta\}$$

$$= \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

$$\mathbf{b} \quad x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

$$\text{If we let } x = \tan \theta \quad \text{then} \quad \tan^4 \theta + 4 \tan^3 \theta - 6 \tan^2 \theta - 4 \tan \theta + 1 = 0$$

$$\therefore 1 - 6 \tan^2 \theta + \tan^4 \theta = 4 \tan \theta - 4 \tan^3 \theta$$

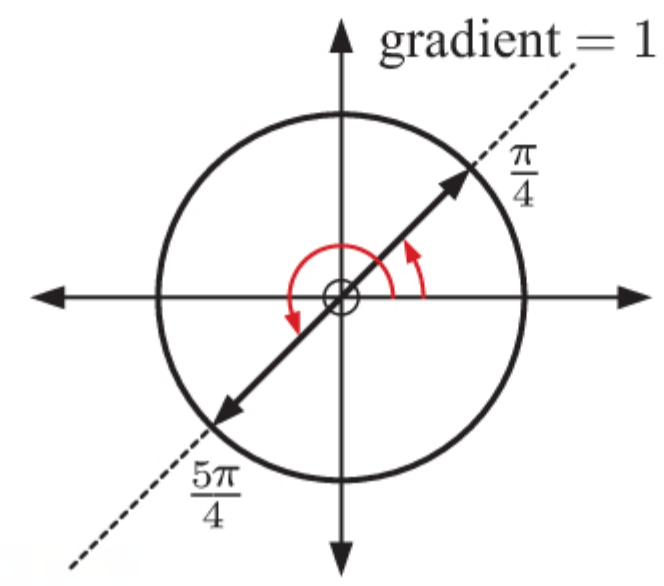
$$\therefore \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta} = 1$$

$$\therefore \tan 4\theta = 1 \quad \{\text{using a}\}$$

$$\therefore 4\theta = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \theta = \frac{\pi}{16} + \frac{k\pi}{4}$$

$$\text{Thus } x = \tan \frac{\pi}{16}, \tan \frac{5\pi}{16}, \tan \frac{9\pi}{16}, \text{ or } \tan \frac{13\pi}{16}.$$



MIXED QUESTIONS SET 7

$$\begin{array}{ll} \mathbf{1} & f \text{ is } y = 4x - 3 & g \text{ is } y = x + 2 \\ & \therefore f^{-1} \text{ is } x = 4y - 3 & \therefore g^{-1} \text{ is } x = y + 2 \\ & \therefore 4y = x + 3 & \therefore y = x - 2 \\ & \therefore y = \frac{1}{4}x + \frac{3}{4} & \therefore g^{-1}(x) = x - 2 \\ & \therefore f^{-1}(x) = \frac{1}{4}x + \frac{3}{4} \end{array}$$

$$\begin{aligned} \text{Now } (f \circ g^{-1})(x) &= f^{-1}(x) \quad \text{where } f(g^{-1}(x)) = \frac{1}{4}x + \frac{3}{4} \\ &\therefore f(x - 2) = \frac{1}{4}x + \frac{3}{4} \\ &\therefore 4(x - 2) - 3 = \frac{1}{4}x + \frac{3}{4} \\ &\therefore 4x - 11 = \frac{1}{4}x + \frac{3}{4} \\ &\therefore 16x - 44 = x + 3 \\ &\therefore 15x = 47 \\ &\therefore x = \frac{47}{15} \end{aligned}$$

$$\mathbf{2} \quad f'(x) = a\sqrt{x} + bx, \quad \text{where } a \text{ and } b \text{ are constants}$$

$$\begin{aligned} \therefore f(x) &= \int (ax^{\frac{1}{2}} + bx) dx \\ &= \frac{2}{3}ax^{\frac{3}{2}} + \frac{1}{2}bx^2 + c \\ f(0) &= -4, \text{ so } c = -4 \end{aligned}$$

$$\text{Thus } f(x) = \frac{2}{3}ax^{\frac{3}{2}} + \frac{1}{2}bx^2 - 4 \quad \text{where } f(1) = -1 \text{ and } f(2) = 4\sqrt{2}.$$

$$\begin{aligned} \text{So, } \frac{2}{3}a(1)^{\frac{3}{2}} + \frac{1}{2}b(1)^2 - 4 &= -1 & \text{and } \frac{2}{3}a(2)^{\frac{3}{2}} + \frac{1}{2}b(2)^2 - 4 &= 4\sqrt{2} \\ \therefore \frac{2}{3}a + \frac{1}{2}b &= 3 & \therefore \frac{4\sqrt{2}}{3}a + 2b &= 4\sqrt{2} + 4 \\ \therefore \frac{1}{2}b &= 3 - \frac{2}{3}a & \therefore \frac{4\sqrt{2}}{3}a + 2(6 - \frac{4}{3}a) &= 4\sqrt{2} + 4 \quad \{\text{using (*)}\} \\ \therefore b &= 6 - \frac{4}{3}a \quad \dots (*) & \therefore \frac{4\sqrt{2}}{3}a + 12 - \frac{8}{3}a &= 4\sqrt{2} + 4 \\ & & \therefore \frac{4\sqrt{2} - 8}{3}a &= 4\sqrt{2} - 8 \\ & & \therefore a &= 3 \end{aligned}$$

$$\text{Substituting } a = 3 \text{ into (*) gives } b = 6 - \frac{4}{3}(3) = 2.$$

$$\begin{aligned} \text{So, } f(x) &= \frac{2}{3}(3)x^{\frac{3}{2}} + \frac{1}{2}(2)x^2 - 4 \\ \therefore f(x) &= 2x^{\frac{3}{2}} + x^2 - 4 \end{aligned}$$

3	Neighbourhood A:	275	281	320	265	305	258	310	430	285
		290	297	345	195	230	269	300	258	273
	Neighbourhood B:	325	300	412	370	297	505	340	333	290
		428	305	520	360	410	275	320	431	410

a The sale price of a house can be counted, so it is a discrete variable.

b Neighbourhood A:

1-Variable	
n	=18
minX	=195
Q1	=265
Med	=283
Q3	=305
maxX	=430

minimum = \$195 000

 $Q_1 = \$265\,000$

median = \$283 000

 $Q_3 = \$305\,000$

maximum = \$430 000

Neighbourhood B:

1-Variable	
n	=18
minX	=275
Q1	=305
Med	=350
Q3	=412
maxX	=520

minimum = \$275 000

 $Q_1 = \$305\,000$

median = \$350 000

 $Q_3 = \$412\,000$

maximum = \$520 000

c For Neighbourhood A, $IQR = 305\,000 - 265\,000 = 40\,000$

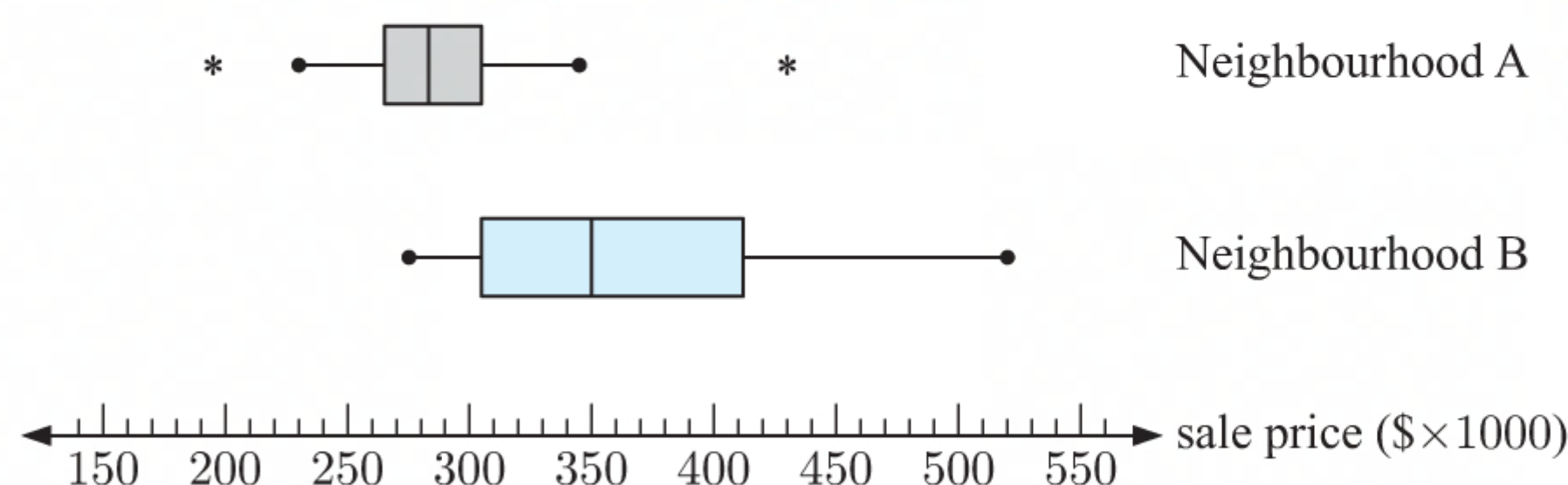
Test for outliers:	upper boundary	and	lower boundary
	$= \text{upper quartile} + 1.5 \times IQR$		$= \text{lower quartile} - 1.5 \times IQR$
	$= 305\,000 + 1.5 \times 40\,000$		$= 265\,000 - 1.5 \times 40\,000$
	$= 365\,000$		$= 205\,000$

\$430 000 is above the upper boundary, so it is an outlier.

\$195 000 is below the lower boundary, so it is an outlier.

For Neighbourhood B, $IQR = 412\,000 - 305\,000 = 107\,000$

Test for outliers:	upper boundary	and	lower boundary
	$= \text{upper quartile} + 1.5 \times IQR$		$= \text{lower quartile} - 1.5 \times IQR$
	$= 412\,000 + 1.5 \times 107\,000$		$= 305\,000 - 1.5 \times 107\,000$
	$= 572\,500$		$= 144\,500$

 \therefore there are no outliers.**d** Both sets of data are positively skewed. The sale price of houses in Neighbourhood B are generally higher than those in Neighbourhood A. With the outliers removed, there is more variation in the sale price of houses in Neighbourhood B compared to Neighbourhood A.**4 a** Using the sine rule,

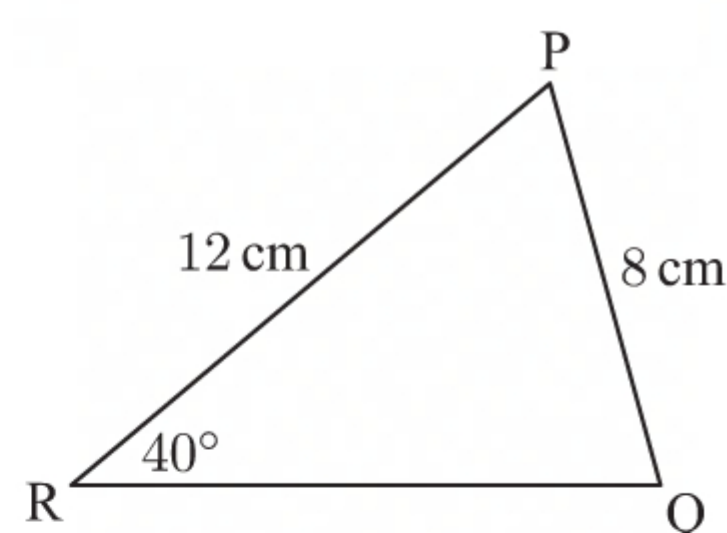
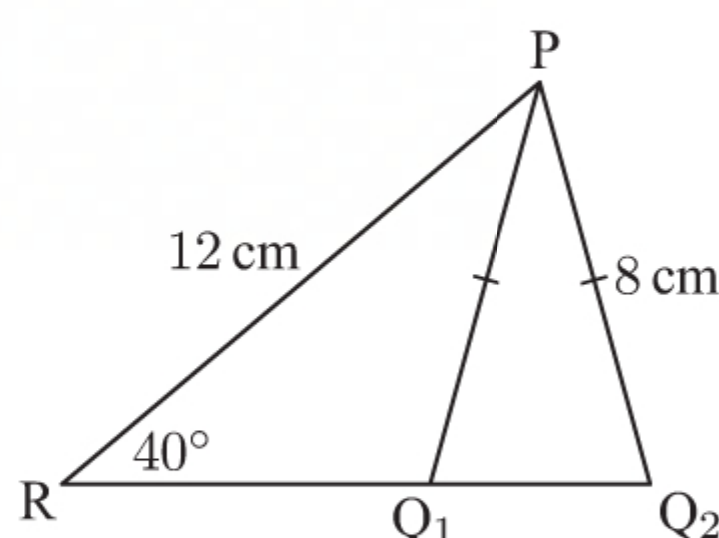
$$\frac{\sin \widehat{PQR}}{12} = \frac{\sin 40^\circ}{8}$$

$$\therefore \sin \widehat{PQR} = \frac{12 \times \sin 40^\circ}{8}$$

$$\therefore \widehat{PQR} = \sin^{-1}\left(\frac{12 \times \sin 40^\circ}{8}\right) \text{ or its supplement}$$

$$\therefore \widehat{PQR} \approx 74.6^\circ \text{ or } 180^\circ - 74.6^\circ$$

$$\therefore \widehat{PQR} \approx 74.6^\circ \text{ or } 105.4^\circ$$

**b**

c For the case in which $\widehat{PQR} \approx 74.6^\circ$:

i $\widehat{QPR} \approx 180^\circ - 40^\circ - 74.6^\circ$ {angles in a triangle}

$\therefore \widehat{QPR} \approx 65.4^\circ$

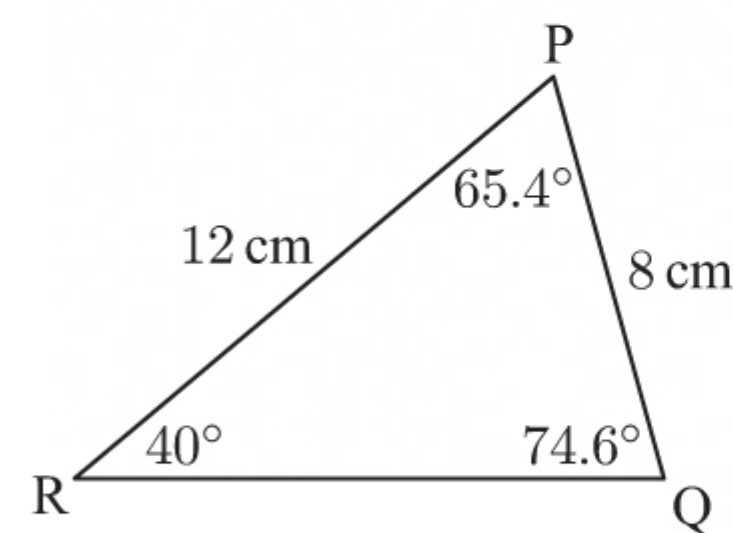
ii $\frac{QR}{\sin \widehat{QPR}} = \frac{PQ}{\sin \widehat{PRQ}}$ {sine rule}

$\therefore \frac{QR}{\sin 65.4^\circ} \approx \frac{8}{\sin 40^\circ}$

$\therefore QR \approx \frac{8 \times \sin 65.4^\circ}{\sin 40^\circ}$

$\therefore QR \approx 11.3 \text{ cm}$

So, perimeter of $\triangle PQR \approx (12 + 8 + 11.3) \text{ cm}$
 $\approx 31.3 \text{ cm}$



For the case in which $\widehat{PQR} \approx 105.4^\circ$:

i $\widehat{QPR} \approx 180^\circ - 40^\circ - 105.4^\circ$ {angles in a triangle}

$\therefore \widehat{QPR} \approx 34.6^\circ$

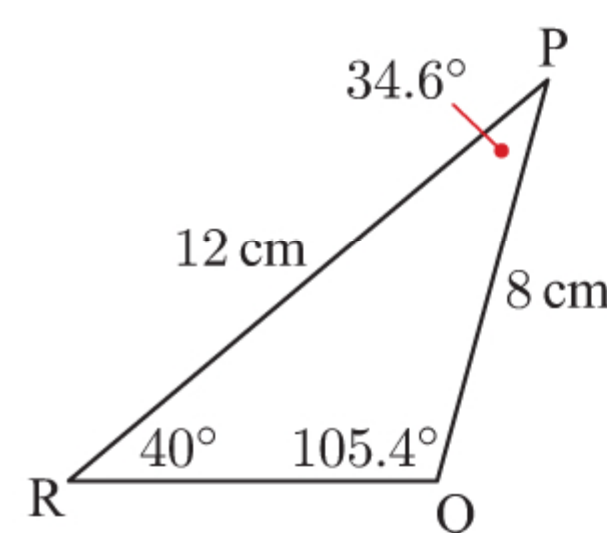
ii $\frac{QR}{\sin \widehat{QPR}} = \frac{PQ}{\sin \widehat{PRQ}}$ {sine rule}

$\therefore \frac{QR}{\sin 34.6^\circ} \approx \frac{8}{\sin 40^\circ}$

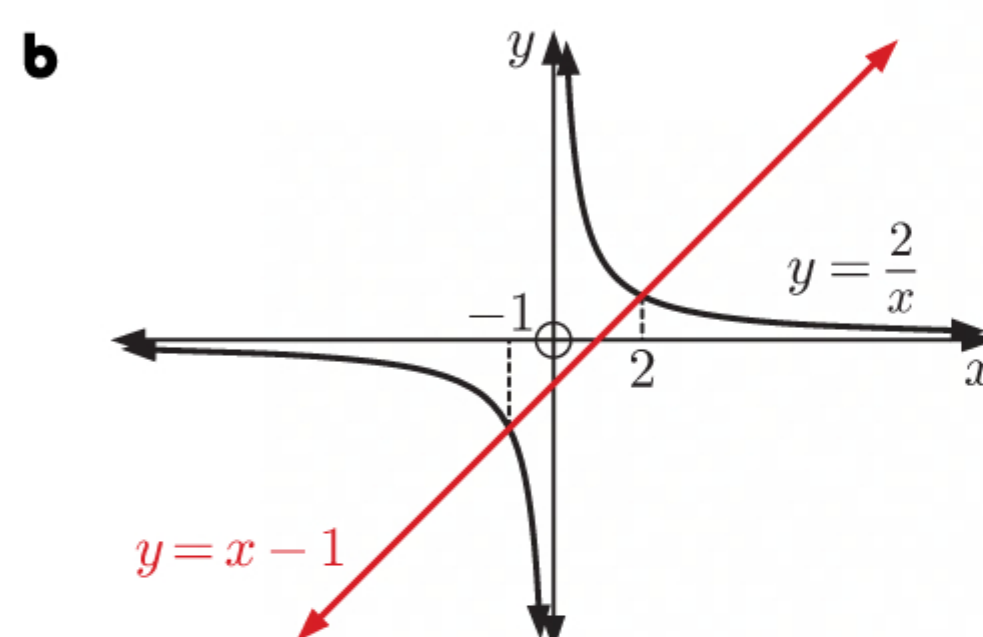
$\therefore QR \approx \frac{8 \times \sin 34.6^\circ}{\sin 40^\circ}$

$\therefore QR \approx 7.07 \text{ cm}$

So, perimeter of $\triangle PQR \approx (12 + 8 + 7.07) \text{ cm}$
 $\approx 27.1 \text{ cm}$



5 a $\frac{2}{x} = x - 1$
 $\therefore 2 = x^2 - x$
 $\therefore x^2 - x - 2 = 0$
 $\therefore (x - 2)(x + 1) = 0$
 $\therefore x = 2 \text{ or } -1$



c If $\frac{2}{x} < x - 1$, the graph of $y = \frac{2}{x}$ is below the graph of $y = x - 1$. This occurs when $-1 < x < 0$ or $x > 2$.

6 $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

So, 1, $3a$, $3a^2$ are consecutive terms in an arithmetic sequence.

Since the terms are consecutive, $3a - 1 = 3a^2 - 3a$ {equating differences}

$\therefore 3a^2 - 6a + 1 = 0$

$\therefore a = \frac{6 \pm \sqrt{36 - 4 \times 3 \times 1}}{2 \times 3}$

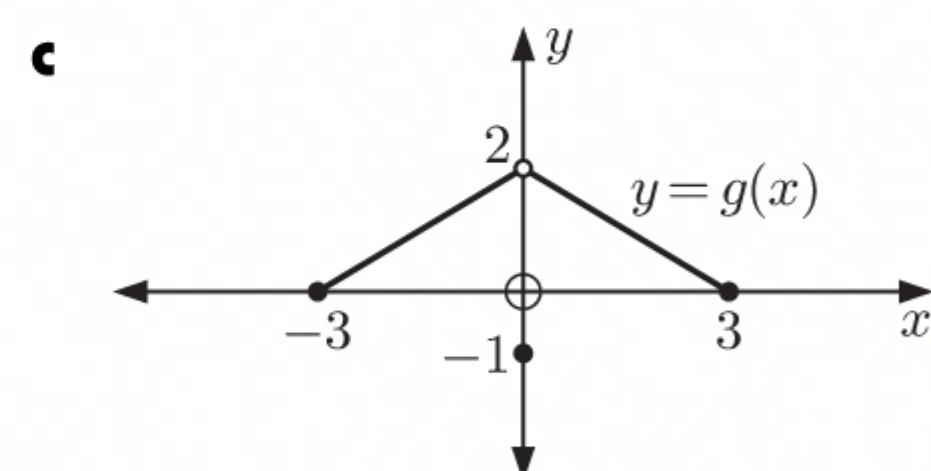
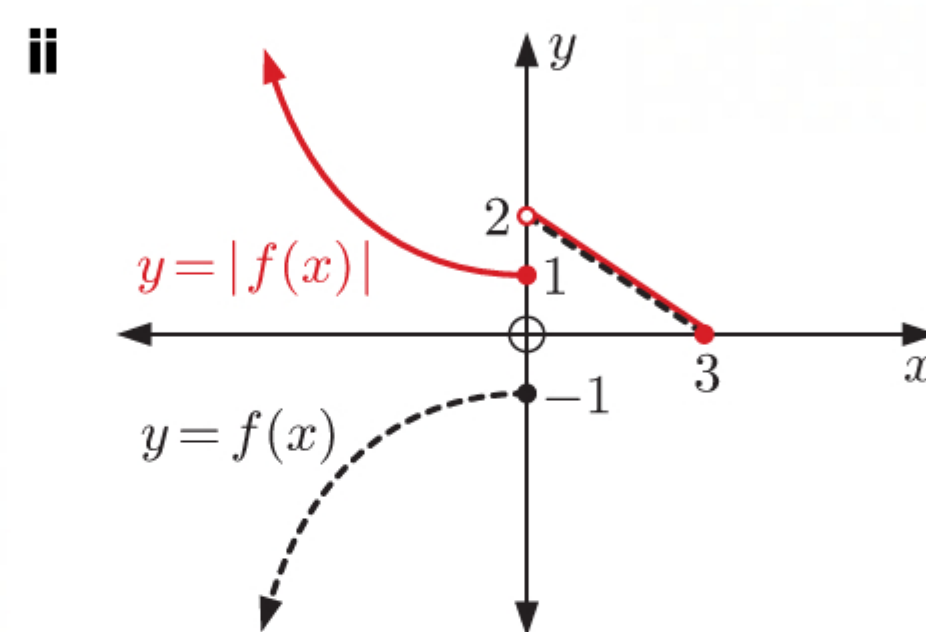
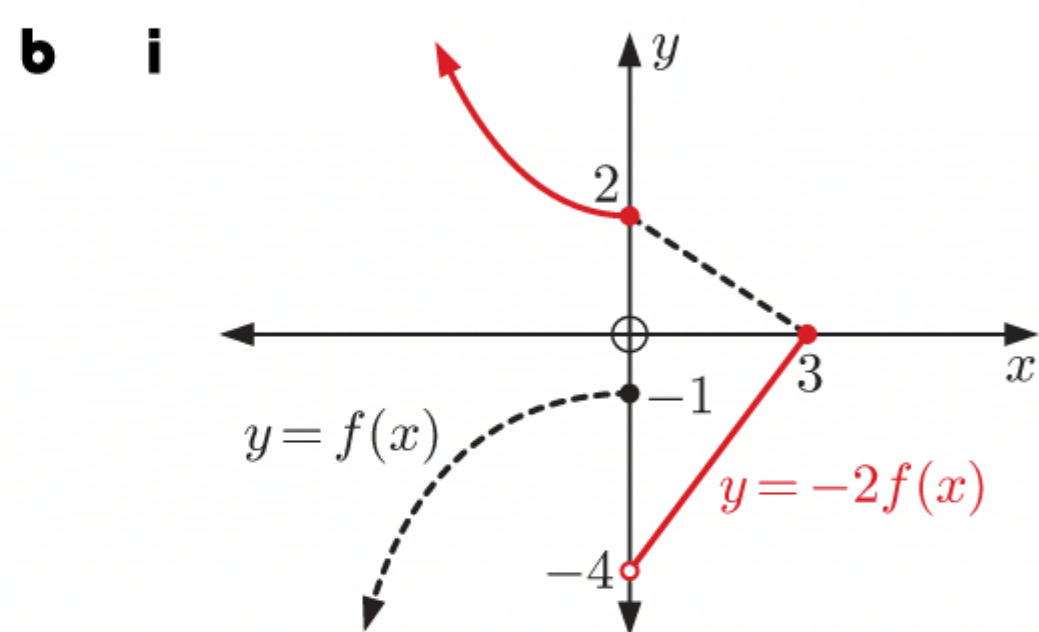
$\therefore a = \frac{6 \pm \sqrt{36 - 12}}{6}$

$\therefore a = \frac{6 \pm \sqrt{24}}{6}$

$\therefore a = \frac{3 \pm \sqrt{6}}{3}$

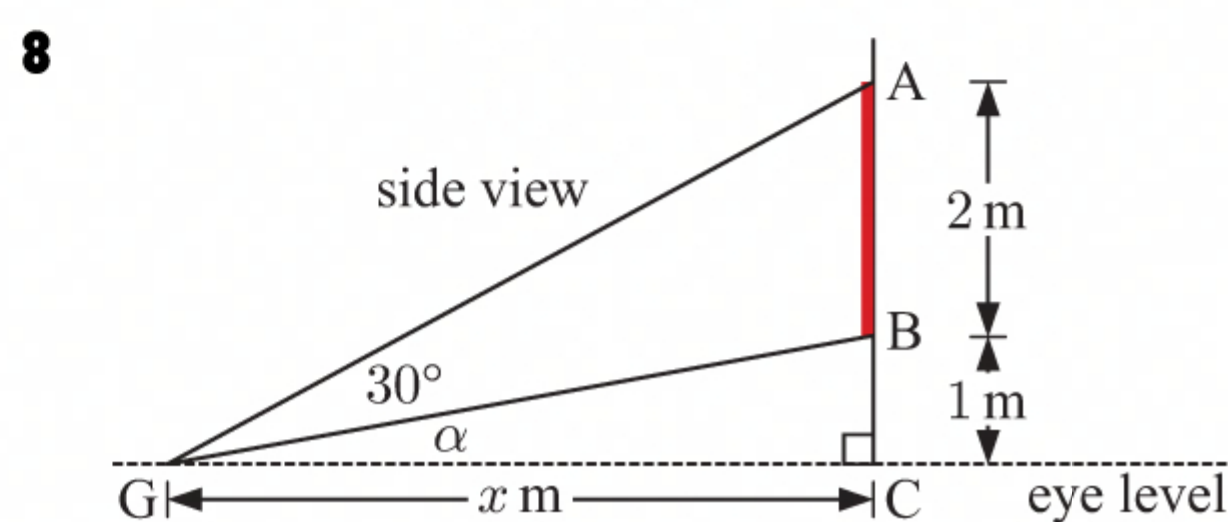
7 a Domain = $\{x \mid x \leq 3, x \in \mathbb{R}\}$

Range = $\{y \mid y \leq -1 \text{ or } 0 \leq y < 2, y \in \mathbb{R}\}$



i $g(x)$ is even as it is symmetric about the y -axis.

ii $g(x)$ fails the horizontal line test, so $g(x)$ does not have an inverse.



Let $GC = x$ m and $\widehat{BGC} = \alpha$.

$$\tan \alpha = \frac{1}{x} \quad \text{and} \quad \tan(\alpha + 30^\circ) = \frac{3}{x}$$

$$\therefore \frac{\tan \alpha + \tan 30^\circ}{1 - \tan \alpha \tan 30^\circ} = \frac{3}{x}$$

$$\therefore \frac{1}{x} + \frac{1}{\sqrt{3}} = \frac{3}{x} \left(1 - \frac{1}{x} \frac{1}{\sqrt{3}}\right) = \frac{3}{x} - \frac{\sqrt{3}}{x^2}$$

$$\therefore \frac{1}{\sqrt{3}} - \frac{2}{x} + \frac{\sqrt{3}}{x^2} = 0$$

$$\therefore x^2 - 2\sqrt{3}x + 3 = 0$$

$$\therefore (x - \sqrt{3})^2 = 0$$

$$\therefore x = \sqrt{3} \approx 1.73$$

So, the girl is $\sqrt{3}$ m or about 1.73 m from the wall.

9 a $f(x) = xe^{-2x^2}$

$$\therefore f'(x) = 1e^{-2x^2} + xe^{-2x^2}(-4x)$$

$$= e^{-2x^2}(1 - 4x^2)$$

Stationary points occur where $f'(x) = 0$

$$\therefore e^{-2x^2}(1 - 4x^2) = 0$$

$$\therefore 1 - 4x^2 = 0 \quad \left\{ e^{-2x^2} > 0 \text{ for all } x \right\}$$

$$\therefore x^2 = \frac{1}{4}$$

$$\therefore x = \pm \frac{1}{2}$$

$$\therefore x = \frac{1}{2} \quad \{0 \leq x \leq 2\}$$

Now $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)e^{-2\left(\frac{1}{2}\right)^2} = \frac{1}{2\sqrt{e}}$

\therefore the stationary point is $\left(\frac{1}{2}, \frac{1}{2\sqrt{e}}\right)$.



$\therefore \left(\frac{1}{2}, \frac{1}{2\sqrt{e}}\right)$ is a local maximum.

Critical value (x)	$f(x)$
0 (end point)	0
$\frac{1}{2}$ (local maximum)	$\frac{1}{2\sqrt{e}} \approx 0.303$
2 (end point)	$2e^{-8} \approx 0.000\,671$

The maximum value of $f(x)$ is $\frac{1}{2\sqrt{e}} \approx 0.303$.

The minimum value of $f(x)$ is 0.

$$\begin{aligned}
 \text{c Volume of revolution} &= \pi \int_0^2 [f(x)]^2 dx \\
 &= \pi \int_0^2 (xe^{-2x^2})^2 dx \\
 &= \pi \int_0^2 (x^2 e^{-4x^2}) dx \\
 &\approx 0.174 \text{ units}^3 \quad \{\text{using technology}\}
 \end{aligned}$$

$$\begin{aligned}
 10 \text{ a i } \frac{\sin a}{\sin \frac{a}{2}} &= \frac{2 \sin \frac{a}{2} \cos \frac{a}{2}}{\sin \frac{a}{2}} \\
 &= 2 \cos \frac{a}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \frac{1}{2} \sin(a+b) + \frac{1}{2} \sin(a-b) &= \frac{1}{2} [\sin a \cos b + \cos a \sin b] + \frac{1}{2} [\sin a \cos b - \cos a \sin b] \\
 &= \frac{1}{2} \sin a \cos b + \cancel{\frac{1}{2} \cos a \sin b} + \frac{1}{2} \sin a \cos b - \cancel{\frac{1}{2} \cos a \sin b} \\
 &= \sin a \cos b
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \frac{1}{2} \cos(a-b) - \frac{1}{2} \cos(a+b) &= \frac{1}{2} [\cos a \cos b + \sin a \sin b] - \frac{1}{2} [\cos a \cos b - \sin a \sin b] \\
 &= \cancel{\frac{1}{2} \cos a \cos b} + \frac{1}{2} \sin a \sin b - \cancel{\frac{1}{2} \cos a \cos b} + \frac{1}{2} \sin a \sin b \\
 &= \sin a \sin b
 \end{aligned}$$

$$\begin{aligned}
 \text{b From a iii, } \frac{1}{2} \cos(a-b) - \frac{1}{2} \cos(a+b) &= \sin a \sin b \\
 \therefore \frac{1}{2} \cos\left(\frac{P+Q}{2} - \frac{P-Q}{2}\right) - \frac{1}{2} \cos\left(\frac{P+Q}{2} + \frac{P-Q}{2}\right) &= \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right) \quad \left\{a = \frac{P+Q}{2}, b = \frac{P-Q}{2}\right\} \\
 \therefore \frac{1}{2} \cos\left(\frac{2Q}{2}\right) - \frac{1}{2} \cos\left(\frac{2P}{2}\right) &= \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right) \\
 \therefore \frac{1}{2} \cos Q - \frac{1}{2} \cos P &= \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right) \\
 \therefore \cos P - \cos Q &= -2 \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right)
 \end{aligned}$$

$$\text{c } P_n \text{ is: } \sin \theta + \sin(\theta + a) + \sin(\theta + 2a) + \dots + \sin(\theta + na) = \frac{\sin\left(\frac{(n+1)a}{2}\right) \sin\left(\theta + \frac{na}{2}\right)}{\sin \frac{a}{2}}, \quad \text{for } n \in \mathbb{Z}^+.$$

Proof: (By the principle of mathematical induction)

$$\begin{aligned}
 (1) \text{ If } n = 1, \quad \frac{\sin\left(\frac{(1+1)a}{2}\right) \sin\left(\theta + \frac{(1)a}{2}\right)}{\sin \frac{a}{2}} &= \frac{\sin a \sin\left(\theta + \frac{a}{2}\right)}{\sin \frac{a}{2}} \\
 &= 2 \cos \frac{a}{2} \sin\left(\theta + \frac{a}{2}\right) \quad \{\text{using a i}\} \\
 &= \sin\left(\theta + \frac{a}{2} + \frac{a}{2}\right) + \sin\left(\theta + \frac{a}{2} - \frac{a}{2}\right) \quad \{\text{using a ii}\} \\
 &= \sin(\theta + a) + \sin \theta
 \end{aligned}$$

$\therefore P_1$ is true.

$$(2) \text{ If } P_k \text{ is true, then } \sin \theta + \sin(\theta + a) + \sin(\theta + 2a) + \dots + \sin(\theta + ka) = \frac{\sin\left(\frac{(k+1)a}{2}\right) \sin\left(\theta + \frac{ka}{2}\right)}{\sin \frac{a}{2}}.$$

$$\begin{aligned} \text{Now } & \sin \theta + \sin(\theta + a) + \sin(\theta + 2a) + \dots + \sin(\theta + ka) + \sin(\theta + (k+1)a) \\ &= \frac{\sin\left(\frac{(k+1)a}{2}\right) \sin\left(\theta + \frac{ka}{2}\right)}{\sin \frac{a}{2}} + \sin(\theta + (k+1)a) \quad \{\text{using } P_k\} \\ &= \frac{\sin\left(\frac{(k+1)a}{2}\right) \sin\left(\theta + \frac{ka}{2}\right) + \sin \frac{a}{2} \sin(\theta + (k+1)a)}{\sin \frac{a}{2}} \\ &= \frac{\frac{1}{2} \cos\left(\frac{(k+1)a}{2} - \theta - \frac{ka}{2}\right) - \frac{1}{2} \cos\left(\frac{(k+1)a}{2} + \theta + \frac{ka}{2}\right) + \frac{1}{2} \cos\left(\frac{a}{2} - \theta - (k+1)a\right) - \frac{1}{2} \cos\left(\frac{a}{2} + \theta + (k+1)a\right)}{\sin \frac{a}{2}} \\ & \quad \{\text{using a iii}\} \\ &= \frac{\frac{1}{2} \cos\left(\frac{a}{2} - \theta\right) - \frac{1}{2} \cos\left(\theta + \left(k + \frac{1}{2}\right)a\right) + \frac{1}{2} \cos\left(-\theta - \left(k + \frac{1}{2}\right)a\right) - \frac{1}{2} \cos\left(\theta + \left(k + \frac{3}{2}\right)a\right)}{\sin \frac{a}{2}} \\ &= \frac{\frac{1}{2} \cos\left(\frac{a}{2} - \theta\right) - \frac{1}{2} \cos\left(\theta + \left(k + \frac{1}{2}\right)a\right) + \frac{1}{2} \cos\left(\theta + \left(k + \frac{1}{2}\right)a\right) - \frac{1}{2} \cos\left(\theta + \left(k + \frac{3}{2}\right)a\right)}{\sin \frac{a}{2}} \quad \{\cos(-x) = \cos x\} \\ &= \frac{\frac{1}{2} \cos\left(\frac{a}{2} - \theta\right) - \frac{1}{2} \cos\left(\theta + \left(k + \frac{3}{2}\right)a\right)}{\sin \frac{a}{2}} \\ &= \frac{-\sin\left(\frac{\frac{a}{2} - \theta + \left(k + \frac{3}{2}\right)a}{2}\right) \sin\left(\frac{\frac{a}{2} - \theta - \theta - \left(k + \frac{3}{2}\right)a}{2}\right)}{\sin \frac{a}{2}} \quad \{\text{using b}\} \\ &= \frac{-\sin\left(\frac{(k+2)a}{2}\right) \sin\left(\frac{-2\theta - (k+1)a}{2}\right)}{\sin \frac{a}{2}} \\ &= \frac{\sin\left(\frac{(k+2)a}{2}\right) \sin\left(\theta + \frac{(k+1)a}{2}\right)}{\sin \frac{a}{2}} \quad \{\sin(-x) = -\sin x\} \\ &= \frac{\sin\left(\frac{[(k+1)+1]a}{2}\right) \sin\left(\theta + \frac{(k+1)a}{2}\right)}{\sin \frac{a}{2}} \end{aligned}$$

$\therefore P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

MIXED QUESTIONS SET 8

1 a i $X \sim N(\mu, (6.8)^2)$

$$P(X < 45) = 0.75$$

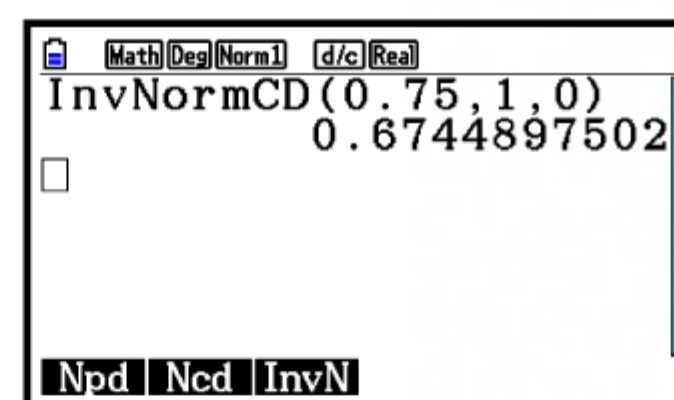
$$\therefore P\left(\frac{X - \mu}{6.8} < \frac{45 - \mu}{6.8}\right) = 0.75$$

$$\therefore P\left(Z < \frac{45 - \mu}{6.8}\right) = 0.75 \quad \left\{Z = \frac{X - \mu}{6.8}\right\}$$

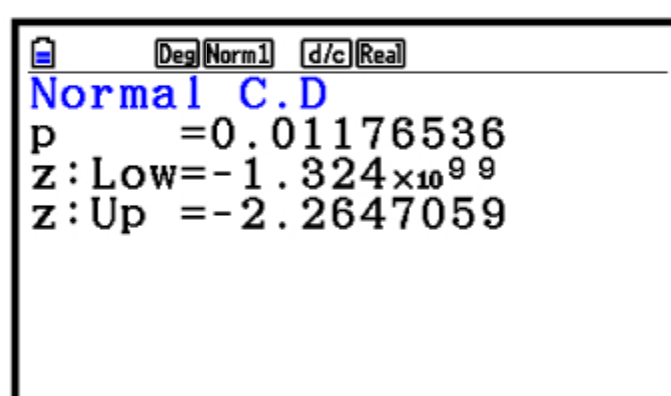
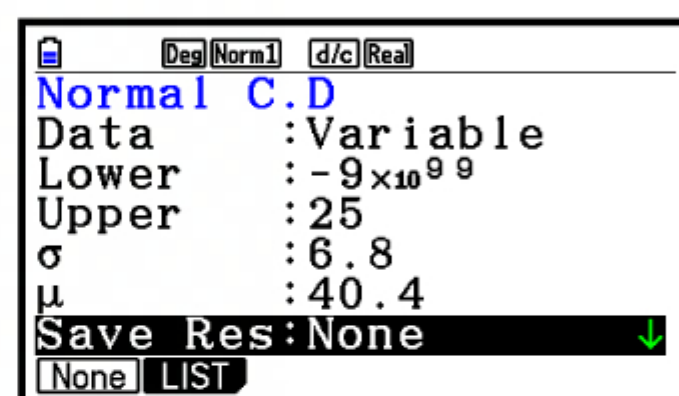
$$\therefore \frac{45 - \mu}{6.8} \approx 0.674 \quad \{Z \sim N(0, 1^2)\}$$

$$\therefore 45 - \mu \approx 4.58$$

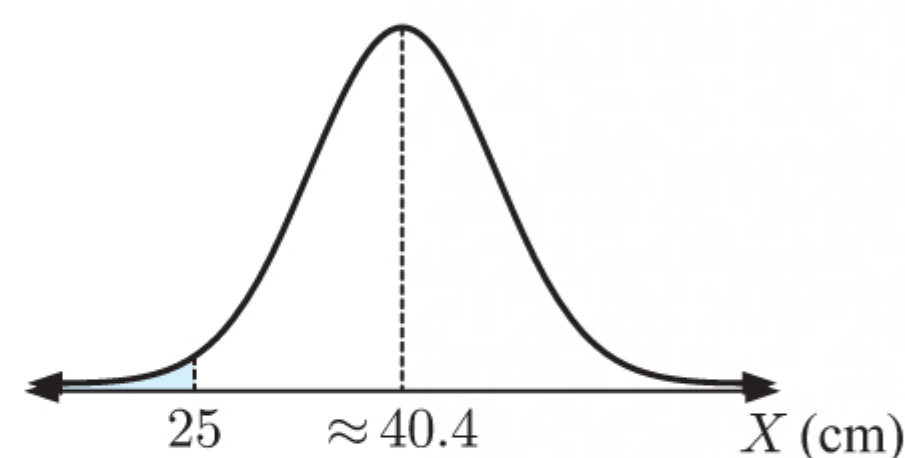
$$\therefore \mu \approx 40.4$$



ii

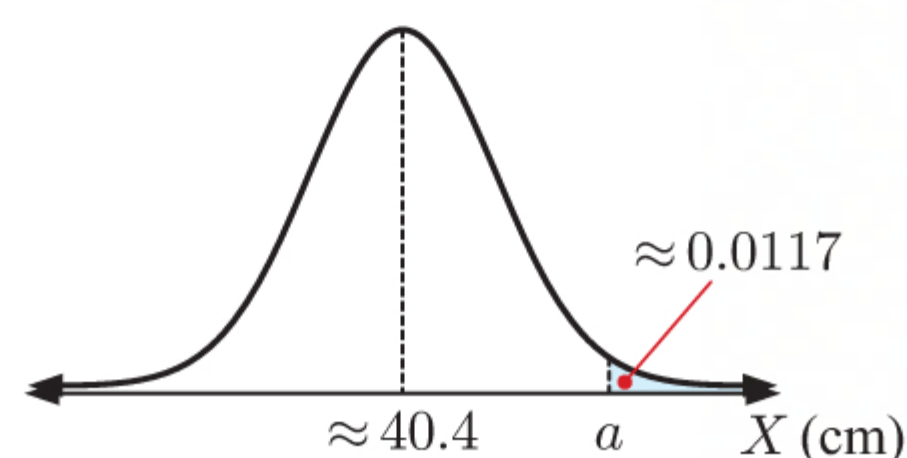


$$P(X < 25) \approx 0.0118$$



iii	<div> <div> DesNorm1 d/cReal </div> <div> Inverse Normal Data : Variable Tail : Left Area : 0.988 σ : 6.8 μ : 40.4 Save Res: None None LIST </div> </div>	<div> <div> DesNorm1 d/cReal </div> <div> Inverse Normal xInv=55.7484789 </div> </div>
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If $P(X < 25) = P(X > a)$
 then $P(X > a) \approx 0.0117$ {from ii}
 $\therefore P(X < a) \approx 0.988$
 $\therefore a \approx 55.7$



b	<div> <div> DesNorm1 d/cReal </div> <div> Normal C.D Data : Variable Lower : 35 Upper : 9×10^9 σ : 6.8 μ : 40.4 Save Res: None None LIST </div> </div>	<div> <div> DesNorm1 d/cReal </div> <div> Normal C.D p = 0.78643652 z: Low = -0.7941176 z: Up = 1.3235×10^9 </div> </div>
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$$P(X > 35) \approx 0.786$$

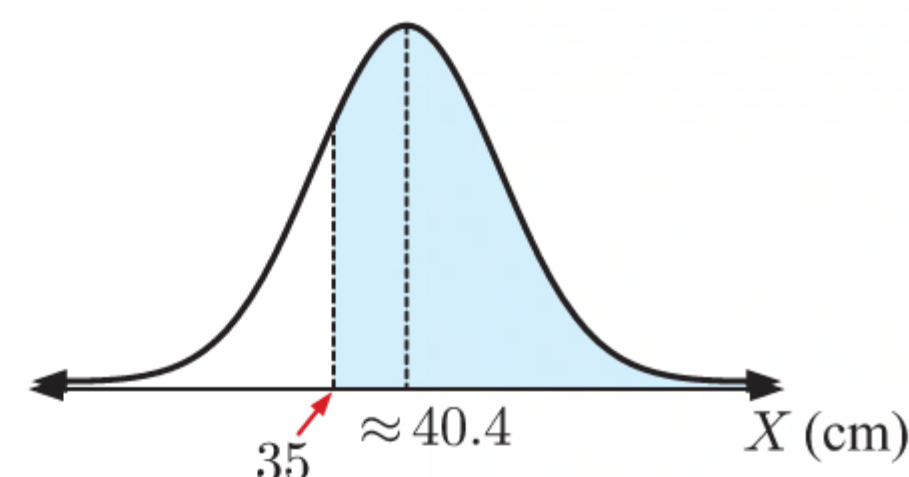
Let Y be the number of maize plants more than 35 cm high.

$n = 6$, so $Y = 0, 1, 2, 3, 4, 5$, or 6 and $p \approx 0.786$.

$$Y \sim B(6, 0.786)$$

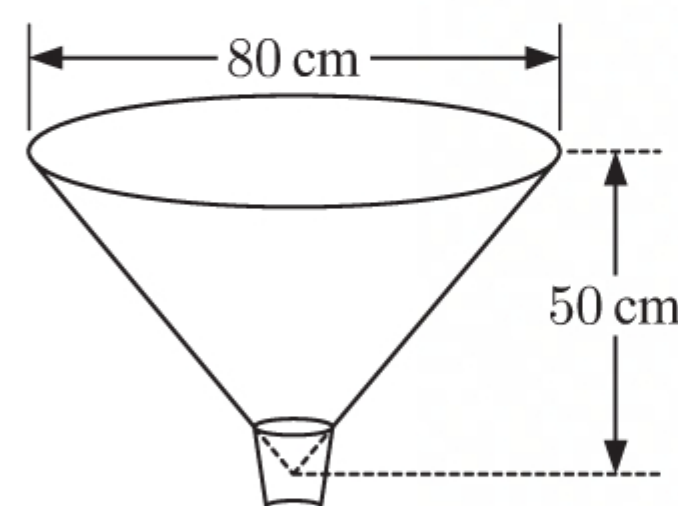
$$\text{So, } P(Y = 4) \approx \binom{6}{4} (0.786)^4 (1 - 0.786)^2 \approx 0.262$$

<div> <div> MathDesNorm1 d/cReal </div> <div> BinomialPD(4,6,0.786) 0.2621856933 </div> </div>
<div> Bpd Bcd InvB </div>



2 a $V \approx$ volume of cone

$$\begin{aligned} &\approx \frac{1}{3} \pi r^2 h \\ &\approx \frac{1}{3} \times \pi \times \left(\frac{80}{2}\right)^2 \times 50 \text{ cm}^3 \\ &\approx \frac{80\,000}{3} \pi \text{ cm}^3 \\ &\approx 83\,800 \text{ cm}^3 \end{aligned}$$

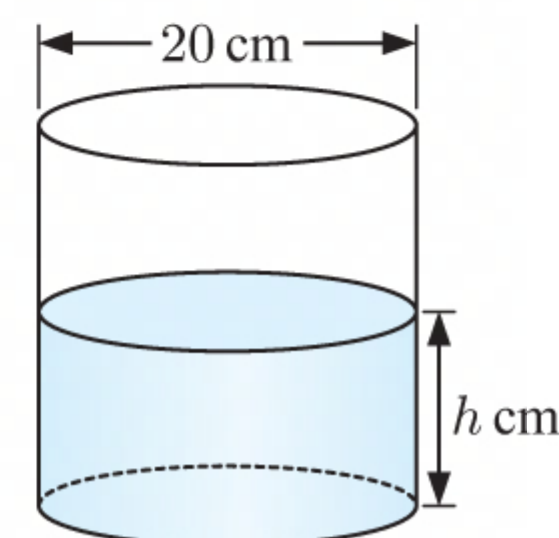


The capacity of the funnel is about 83 800 mL or 8.38×10^4 mL.

b When half full, the funnel contains about $\frac{80\,000}{3} \pi \times 0.5 \approx \frac{40\,000}{3} \pi$ mL of liquid.

$$\begin{aligned} V &\approx \frac{40\,000}{3} \pi \text{ cm}^3 \\ \therefore \pi \times \left(\frac{20}{2}\right)^2 \times h &\approx \frac{40\,000}{3} \pi \\ \therefore h &\approx \frac{40\,000}{3 \times 10^2} \\ \therefore h &\approx 133 \text{ cm} \end{aligned}$$

The liquid will reach about 133 cm up the tube.



3 $a(t) = 6 \cos 2t \text{ cm s}^{-2}$

$$\begin{aligned} \text{a } v(t) &= \int 6 \cos 2t \, dt \\ &= 3 \sin 2t + c \end{aligned}$$

$$\begin{aligned} \text{But } v(0) &= 0, \text{ so } 0 = 3 \sin 0 + c \\ \therefore c &= 0 \end{aligned}$$

$$\text{Thus, } v(t) = 3 \sin 2t \text{ cm s}^{-1}.$$

$$\begin{aligned} \text{Now } v(4) &= 3 \sin(2 \times 4) \\ &\approx 2.97 \end{aligned}$$

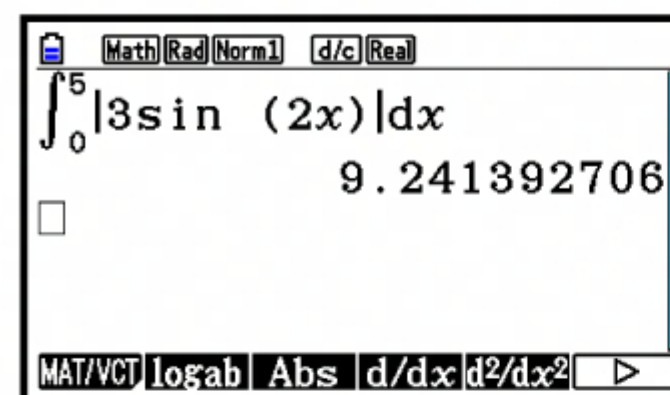
\therefore the speed of the tip of the pendulum after 4 seconds is about 2.97 cm s^{-1} .

- b** Total distance travelled in first 5 seconds

$$= \int_0^5 |v(t)| dt$$

$$= \int_0^5 |3 \sin 2t| dt$$

$$\approx 9.24 \text{ cm} \quad \{\text{using technology}\}$$



- 4 a** Let the cosine model be $H(t) = a \cos(b(t - c)) + d$.

$$\text{Low tide} = 4.7 - 2.4 = 2.3 \text{ m}$$

$$\therefore \text{the mean height} = \frac{2.3 + 4.7}{2} = 3.5 \text{ m, so } d = 3.5.$$

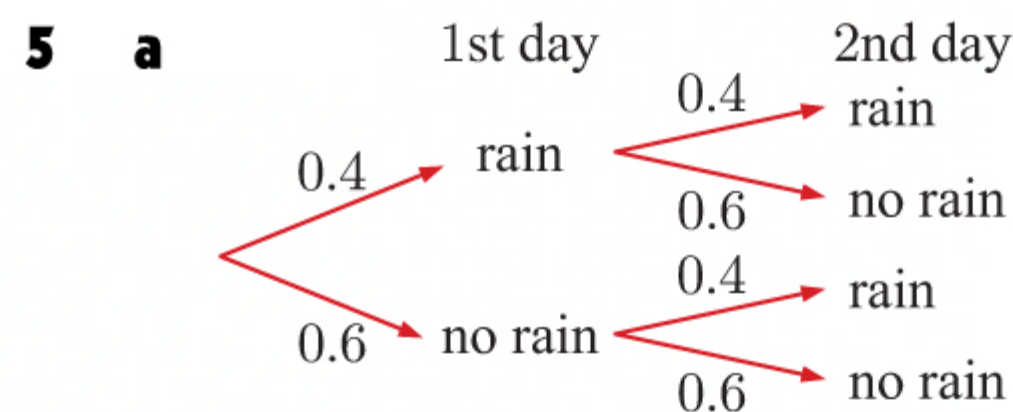
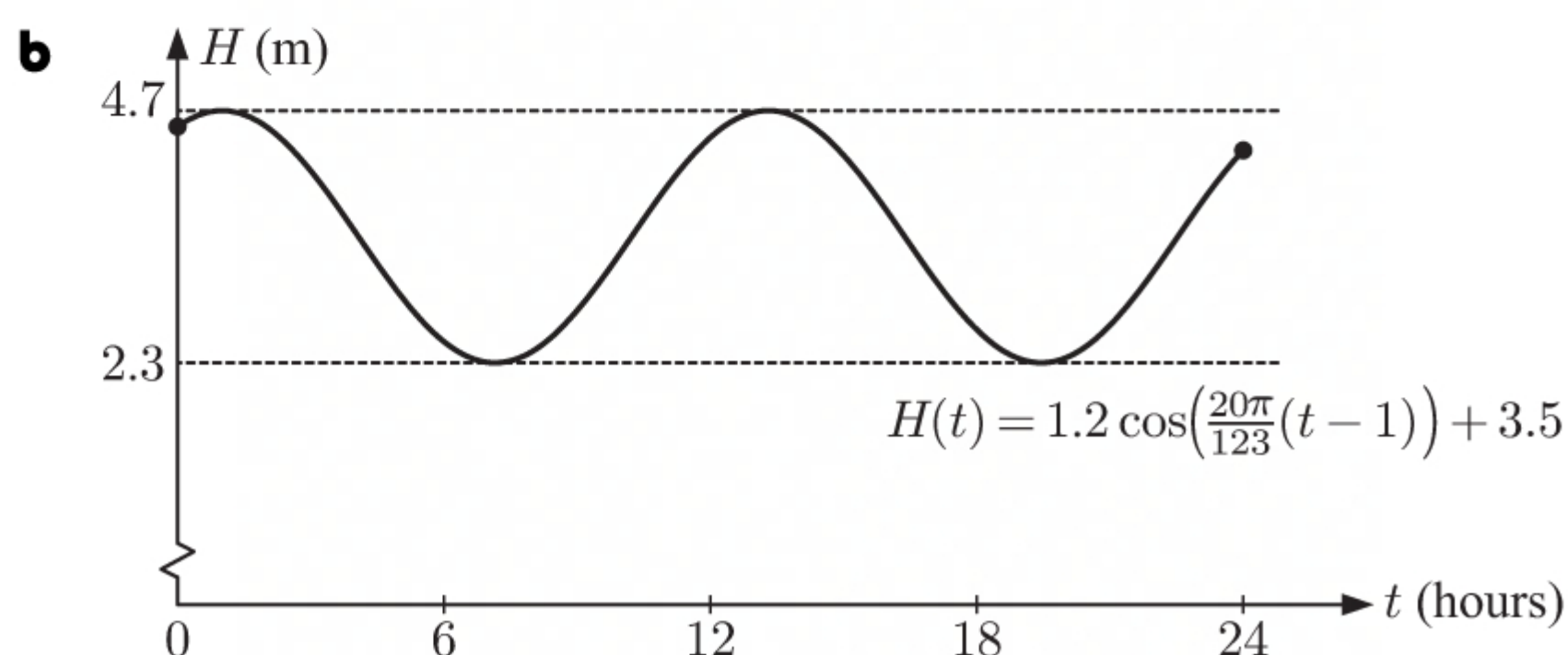
$$\text{The amplitude} = \frac{2.4}{2} = 1.2 \text{ m, so } a = 1.2.$$

$$\text{The period} = 12.3 \text{ hours, so } b = \frac{2\pi}{12.3} = \frac{20\pi}{123}.$$

High tide occurs at 1 am, so the function is shifted 1 hour to the right, thus $c = 1$.

If t is the number of hours after midnight, the height H is modelled by

$$H(t) = 1.2 \cos\left(\frac{20\pi}{123}(t - 1)\right) + 3.5 \text{ m.}$$



b i $P(\text{rain on both days}) = P(\text{rain} \cap \text{rain})$
 $= 0.4 \times 0.4$
 $= 0.16$

ii $P(\text{no rain on one day}) = P(\text{rain} \cap \text{no rain}) + P(\text{no rain} \cap \text{rain})$
 $= 0.4 \times 0.6 + 0.6 \times 0.4$
 $= 0.48$

c $P(\text{no rain on both days}) = P(\text{no rain} \cap \text{no rain})$
 $= 0.6 \times 0.6$
 $= 0.36$

$$\therefore P(\text{rain on at least one day}) = 1 - 0.36$$

$$= 0.64$$

$$\text{So, } P(\text{rain on 2nd day} \mid \text{rain on at least one day}) = \frac{P(\text{rain on 2nd day} \cap \text{rain on at least one day})}{P(\text{rain on at least one day})}$$

$$= \frac{P(\text{rain on 2nd day})}{P(\text{rain on at least one day})}$$

$$= \frac{0.4}{0.64}$$

$$= 0.625$$

6 a $f(0) = (0)e^{-0} = 0$

The y -intercept is 0.

b $f(x) = xe^{-x}$
 $\therefore f'(x) = e^{-x} - xe^{-x}$

Let a be the x -coordinate of A, so A is $(a, f(a))$.

Since A is a stationary point, $f'(a) = 0$
 $\therefore 0 = e^{-a} - ae^{-a}$
 $\therefore e^{-a}(1 - a) = 0$
 $\therefore a = 1 \quad \{e^{-a} > 0\}$

Now $f(1) = (1)e^{-1} = \frac{1}{e}$, so A is $(1, \frac{1}{e})$.

c $f'(x) = e^{-x} - xe^{-x}$ {from **b**}
 $\therefore f''(x) = -e^{-x} - e^{-x} + xe^{-x}$
 $= xe^{-x} - 2e^{-x}$

Let b be the x -coordinate of B.

Since B is a point of inflection, $f''(b) = 0$
 $\therefore 0 = be^{-b} - 2e^{-b}$
 $\therefore e^{-b}(b - 2) = 0$
 $\therefore b = 2 \quad \{e^{-b} > 0\}$

\therefore the x -coordinate of the point of inflection B is 2.

d Area of shaded region $= \int_1^2 xe^{-x} dx$ {using **b** and **c**}

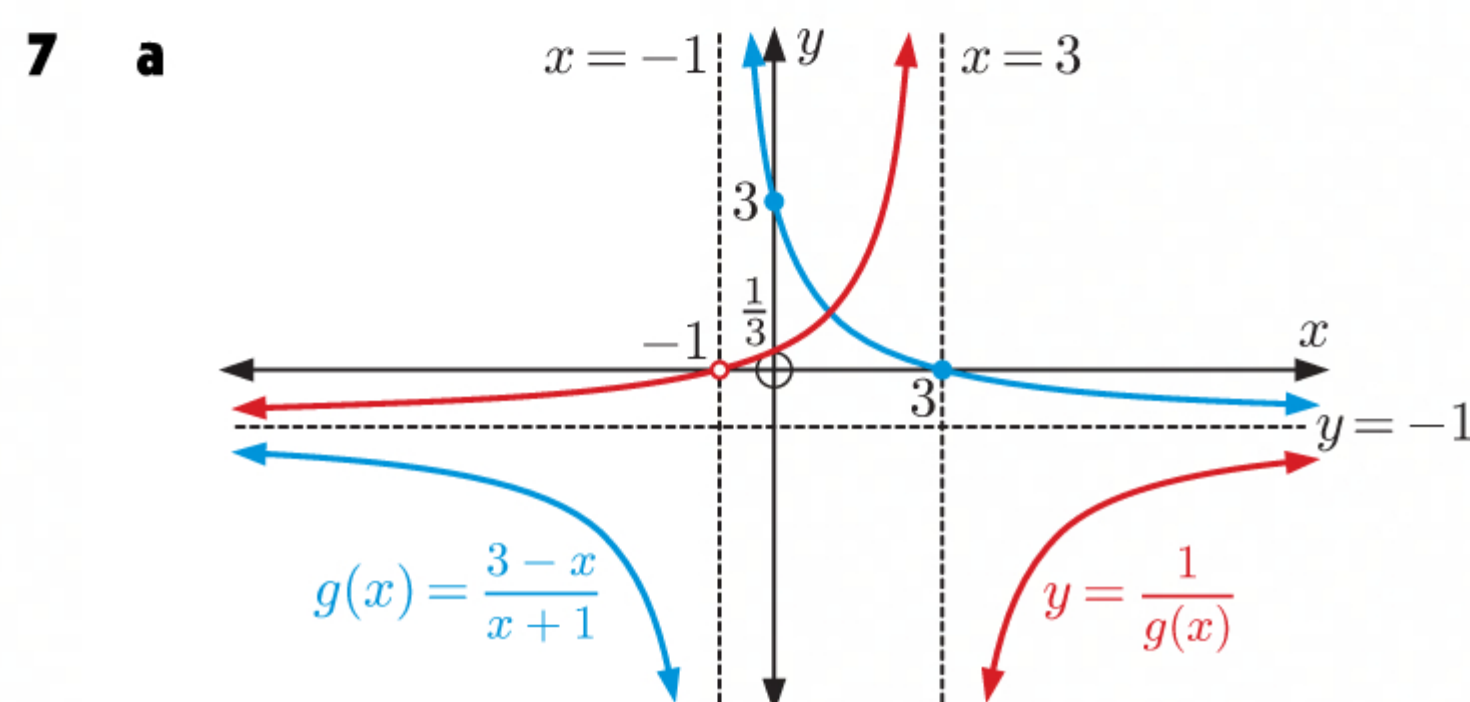
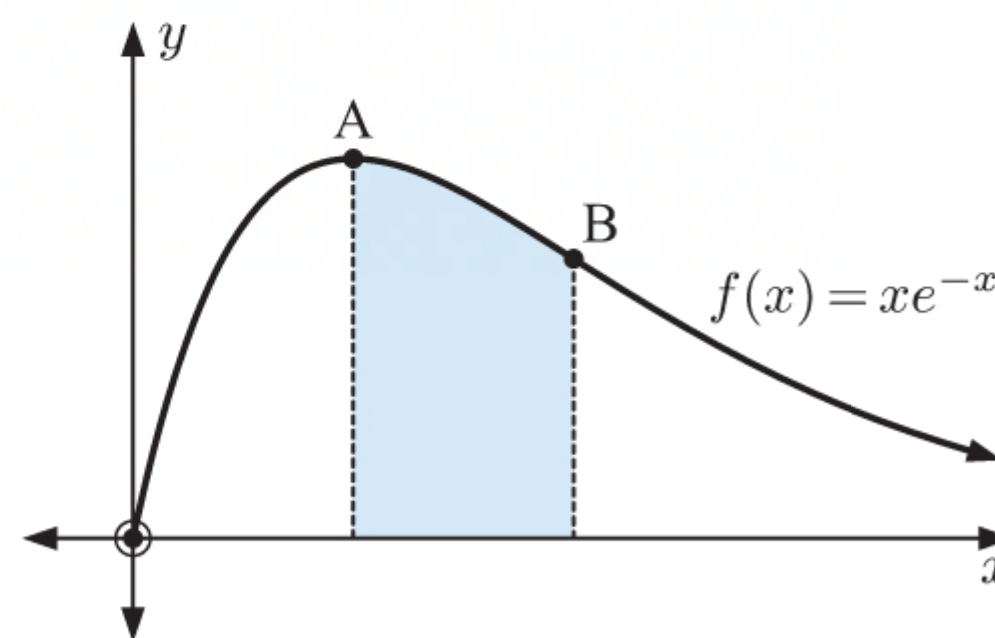
$$= [-xe^{-x}]_1^2 - \int_1^2 -e^{-x} dx \quad \begin{cases} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{cases}$$

$$= -2e^{-2} + e^{-1} - [e^{-x}]_1^2$$

$$= -\frac{2}{e^2} + \frac{1}{e} - \frac{1}{e^2} + \frac{1}{e}$$

$$= \frac{2}{e} - \frac{3}{e^2} \text{ units}^2$$

$$\approx 0.330 \text{ units}^2$$



- b** Since $x = -1$ is an asymptote of $y = g(x)$, the point where $x = -1$ is not included on the graph of $y = \frac{1}{g(x)}$. However, the graph approaches the x -axis either side of this point.

c Invariant points occur when $\frac{1}{g(x)} = g(x)$

$$\therefore [g(x)]^2 = 1$$

$$\therefore \left(\frac{3-x}{x+1}\right)^2 = 1$$

$$\therefore (3-x)^2 = (x+1)^2$$

$$\therefore 3-x = \pm(x+1)$$

$$\therefore 3-x = x+1 \quad \text{or} \quad 3-x = -x-1$$

$$\therefore 2x = 2$$

$$\therefore x = 1$$

$$g(1) = \frac{3-1}{1+1} = 1$$

\therefore the only invariant point is $(1, 1)$.

d $g(x) = \frac{3-x}{x+1} = \frac{-(x+1)+4}{x+1} = -1 + \frac{4}{x+1}$

So, if $y = g(x)$, then $y = \frac{4}{x+1} - 1$.

This is a translation through $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ from $y = \frac{4}{x}$.

To transform $y = \frac{1}{x}$ to $y = g(x)$, we vertically stretch $y = \frac{1}{x}$ with scale factor 4, then translate the resultant curve through $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

8 Let $z = a + bi$, so $z^* = a - bi$, $a, b \in \mathbb{R}$.

$$\text{Now, } z^2 = (z^*)^2$$

$$\text{So } (a + bi)^2 = (a - bi)^2$$

$$\therefore a^2 + 2abi + b^2i^2 = a^2 - 2abi + b^2i^2$$

$$\therefore (a^2 - b^2) + 2abi = (a^2 - b^2) - 2abi$$

Equating imaginary parts gives $2ab = -2ab$

$$\therefore 4ab = 0$$

$$\therefore a = 0 \quad \text{or} \quad b = 0$$

So, $z = a$ or $z = bi$, $a, b \in \mathbb{R}$.

$\therefore z$ is either real or purely imaginary.

9 a $\sqrt{3} \sin x - \cos x = A \sin(x + \alpha)$
 $= A(\sin x \cos \alpha + \cos x \sin \alpha)$
 $= A \sin x \cos \alpha + A \cos x \sin \alpha$

Equating the coefficients of $\sin x$ and $\cos x$:

$$A \cos \alpha = \sqrt{3} \quad \text{and} \quad A \sin \alpha = -1$$

$$\therefore \cos \alpha = \frac{\sqrt{3}}{A} \quad \text{and} \quad \sin \alpha = -\frac{1}{A}$$

$$\text{Now } \sin^2 \alpha + \cos^2 \alpha = 1$$

$$\therefore \left(-\frac{1}{A}\right)^2 + \left(\frac{\sqrt{3}}{A}\right)^2 = 1$$

$$\therefore \frac{1}{A^2} + \frac{3}{A^2} = 1$$

$$\therefore \frac{4}{A^2} = 1$$

$$\therefore A^2 = 4$$

$$\therefore A = 2 \quad \{A > 0\}$$

$$\text{So } \cos \alpha = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin \alpha = -\frac{1}{2}$$

$$\therefore \alpha = \frac{11\pi}{6} \quad \{0 < \alpha < 2\pi\}$$

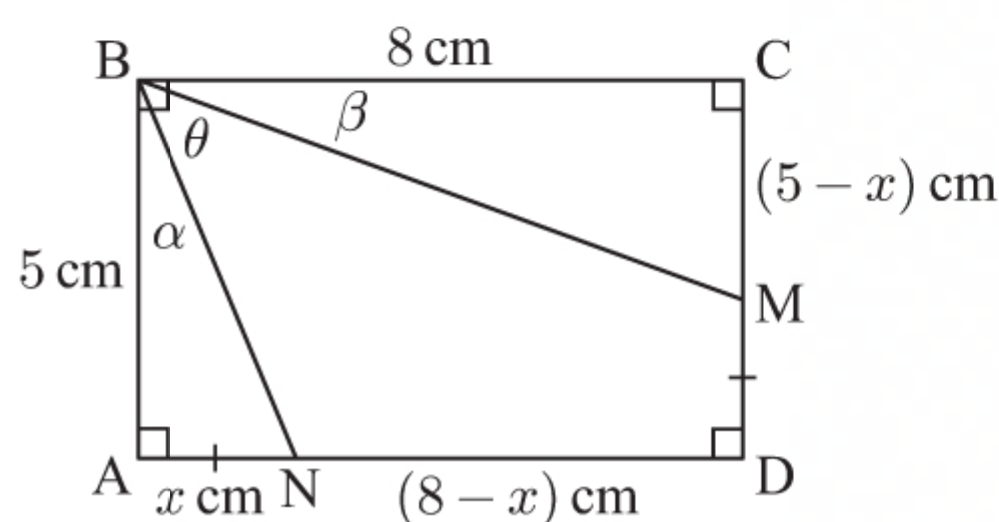
$$\therefore \sqrt{3} \sin x - \cos x = 2 \sin\left(x + \frac{11\pi}{6}\right)$$

$$\begin{aligned}
 \mathbf{b} \quad & \sqrt{3} \sin x - \cos x = 1, \quad 0 \leq x \leq 2\pi \\
 \therefore & 2 \sin\left(x + \frac{11\pi}{6}\right) = 1 \quad \{\text{using a}\} \\
 \therefore & \sin\left(x + \frac{11\pi}{6}\right) = \frac{1}{2} \\
 \therefore & x + \frac{11\pi}{6} = \frac{13\pi}{6}, \frac{17\pi}{6} \quad \left\{ \frac{11\pi}{6} \leq x + \frac{11\pi}{6} \leq \frac{23\pi}{6} \right\} \\
 \therefore & x = \frac{\pi}{3}, \pi
 \end{aligned}$$

10 Let $\alpha = \widehat{ABN}$ and $\beta = \widehat{MBC}$.

$$\begin{aligned}
 \text{In } \triangle ABN, \quad & \tan \alpha = \frac{x}{5} \\
 \therefore & \alpha = \arctan \frac{x}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{In } \triangle BCM, \quad & \tan \beta = \frac{5-x}{8} \\
 \therefore & \beta = \arctan\left(\frac{5-x}{8}\right)
 \end{aligned}$$



$$\begin{aligned}
 \text{Now } \theta &= \frac{\pi}{2} - (\alpha + \beta) \\
 &= \frac{\pi}{2} - \arctan \frac{x}{5} - \arctan\left(\frac{5-x}{8}\right) \\
 \therefore \frac{d\theta}{dx} &= -\frac{1}{1 + \left(\frac{x}{5}\right)^2} \times \frac{1}{5} - \frac{1}{1 + \left(\frac{5-x}{8}\right)^2} \times \left(-\frac{1}{8}\right) \\
 &= -\frac{5}{25 + x^2} + \frac{8}{64 + (5-x)^2}
 \end{aligned}$$

$$\theta \text{ is minimised when } \frac{d\theta}{dx} = 0$$

$$\therefore -\frac{5}{25 + x^2} + \frac{8}{64 + (5-x)^2} = 0$$

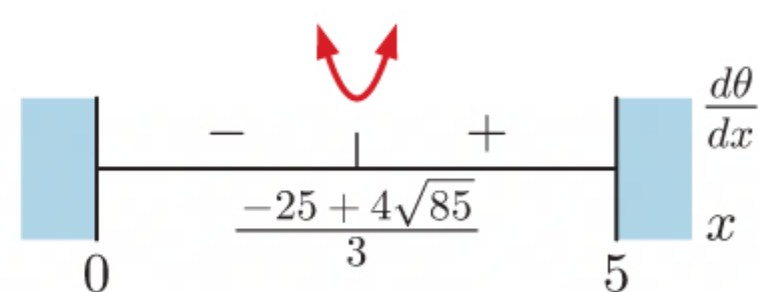
$$\therefore -320 - 5(5-x)^2 + 200 + 8x^2 = 0$$

$$\therefore -120 - 125 + 50x - 5x^2 + 8x^2 = 0$$

$$\therefore 3x^2 + 50x - 245 = 0$$

$$\begin{aligned}
 \therefore x &= \frac{-50 \pm \sqrt{2500 - 4(3)(-245)}}{6} \\
 &= \frac{-50 \pm \sqrt{5440}}{6} \\
 &= \frac{-50 \pm 8\sqrt{85}}{6} \\
 &= \frac{-25 \pm 4\sqrt{85}}{3} \\
 &= \frac{-25 + 4\sqrt{85}}{3} \quad \{0 \leq x \leq 5\}
 \end{aligned}$$

The sign diagram of $\frac{d\theta}{dx}$ is:

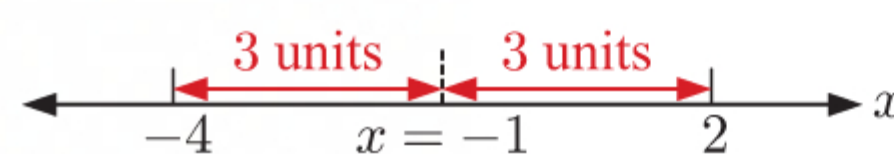


$$\therefore \theta \text{ is minimised when } x = \frac{-25 + 4\sqrt{85}}{3} \approx 3.96.$$

MIXED QUESTIONS SET 9

1 The axis of symmetry $x = -1$ lies midway between the x -intercepts.

\therefore the other x -intercept is 2.



Since the x -intercepts are -4 and 2 , the quadratic has the form $y = a(x+4)(x-2)$, $a \neq 0$.

When $x = 1$, $y = 5$

$$\therefore 5 = a(1+4)(1-2)$$

$$\therefore 5 = a(5)(-1)$$

$$\therefore a = -1$$

The quadratic has equation $y = -(x+4)(x-2)$

$$= -(x^2 + 2x - 8)$$

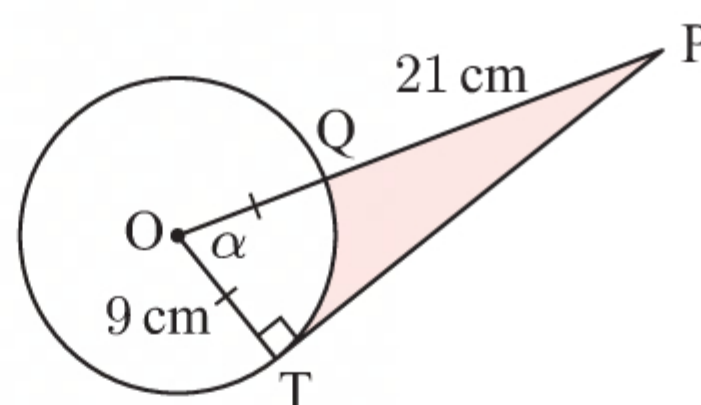
$$\therefore y = -x^2 - 2x + 8$$

2 a $\widehat{OTP} = 90^\circ$ {radius-tangent}

$\therefore \triangle OPT$ is right angled at T.

$$\therefore \cos \alpha = \frac{9}{30}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{9}{30}\right) \approx 72.5^\circ$$



b Area of $\triangle OPT = \frac{1}{2} \times 9 \times 30 \times \sin \alpha$
 $= 135 \sin \alpha \text{ cm}^2$

$$\begin{aligned} \text{Area of sector OQT} &= \frac{\theta}{360} \times \pi r^2 \\ &= \frac{\alpha}{360} \times \pi \times 9^2 \\ &= \frac{9\alpha\pi}{40} \text{ cm}^2 \end{aligned}$$

So, shaded area = area of $\triangle OPT$ – area of sector OQT

$$\begin{aligned} &= 135 \sin \alpha - \frac{9\alpha\pi}{40} \\ &= 135 \sin\left(\cos^{-1}\left(\frac{9}{30}\right)\right) - \frac{9\pi}{40} \cos^{-1}\left(\frac{9}{30}\right) \quad \{\text{using a}\} \\ &\approx 77.5 \text{ cm}^2 \end{aligned}$$

3 $2 \sin^2 x = 3 \cos x + 2, \quad -\pi \leq x \leq \pi$

$$\therefore 2(1 - \cos^2 x) = 3 \cos x + 2$$

$$\therefore 2 - 2 \cos^2 x = 3 \cos x + 2$$

$$\therefore 2 \cos^2 x + 3 \cos x = 0$$

$$\therefore \cos x(2 \cos x + 3) = 0$$

$$\therefore \cos x = 0 \quad \text{or} \quad \cos x = -\frac{3}{2}$$

$$\therefore \cos x = 0$$

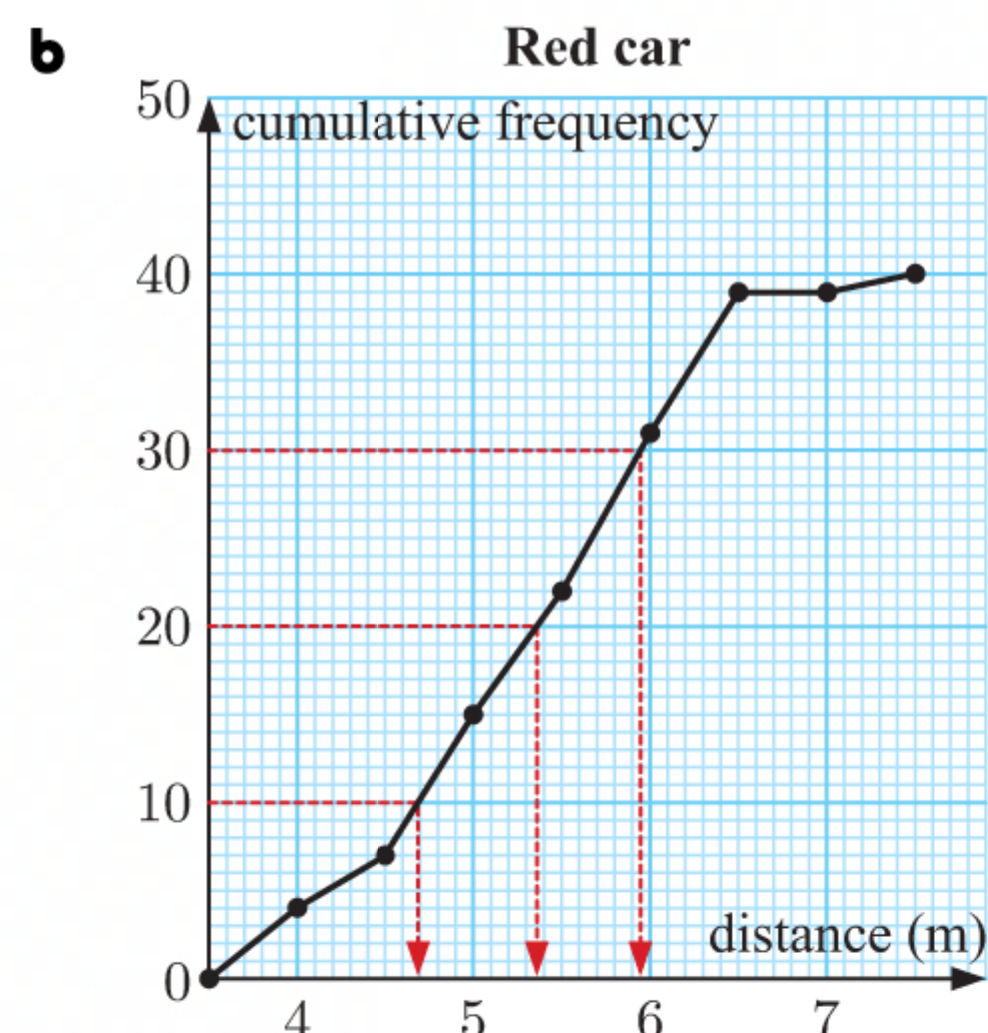
$$\therefore x = -\frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{2}$$

$$\{-1 \leq \cos x \leq 1\}$$

$$\{-\pi \leq x \leq \pi\}$$

4 a

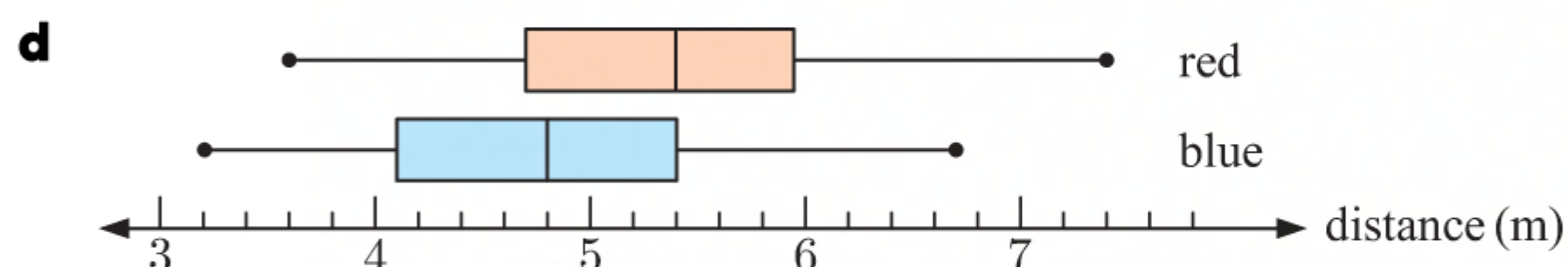
Distance (m)	Frequency	Cumulative frequency
$3.5 \leq d < 4$	4	4
$4 \leq d < 4.5$	3	7
$4.5 \leq d < 5$	8	15
$5 \leq d < 5.5$	7	22
$5.5 \leq d < 6$	9	31
$6 \leq d < 6.5$	8	39
$6.5 \leq d < 7$	0	39
$7 \leq d < 7.5$	1	40



c i Median ≈ 5.4

ii $Q_1 \approx 4.7$

iii $Q_3 \approx 5.9$



e All values of the five-number summary (min, Q_1 , median, Q_3 , and max) for the red car are higher than those for the blue car. This evidence is strongly against the view that the cars were made by the same machine.

$$\begin{aligned}
5 \quad & 4^x + 4 = 17(2^{x-1}) \\
& \therefore (2^2)^x + 4 = 17\left(\frac{2^x}{2}\right) \\
& \therefore 2(2^x)^2 + 8 = 17(2^x) \\
& \therefore 2(2^x)^2 - 17(2^x) + 8 = 0 \\
& \therefore (2(2^x) - 1)(2^x - 8) = 0 \quad \{2X^2 - 17X + 8 = (2X - 1)(X - 8)\} \\
& \therefore 2^x = \frac{1}{2} \quad \text{or} \quad 2^x = 8 \\
& \therefore 2^x = 2^{-1} \quad \text{or} \quad 2^x = 2^3 \\
& \therefore x = -1 \quad \text{or} \quad x = 3
\end{aligned}$$

6 Suppose the original arithmetic sequence has first term u_1 .

$$\text{Now } S_3 = \frac{3}{2}(2u_1 + 2d) = 3(u_1 + d)$$

$$S_6 = \frac{6}{2}(2u_1 + 5d) = 3(2u_1 + 5d)$$

$$S_8 = \frac{8}{2}(2u_1 + 7d) = 4(2u_1 + 7d)$$

S_3 , S_6 , and S_8 form an arithmetic sequence.

$$\therefore S_8 - S_6 = S_6 - S_3 \quad \{\text{equating differences}\}$$

$$\therefore 4(2u_1 + 7d) - 3(2u_1 + 5d) = 3(2u_1 + 5d) - 3(u_1 + d)$$

$$\therefore 8u_1 + 28d - 6u_1 - 15d = 6u_1 + 15d - 3u_1 - 3d$$

$$\therefore 2u_1 + 13d = 3u_1 + 12d$$

$$\therefore -u_1 = -d$$

$$\therefore u_1 = d$$

$$\begin{array}{ll}
\text{Now } S_8 - S_6 = 2u_1 + 13d & \text{Check: } S_6 - S_3 = 3u_1 + 12d \\
= 2d + 13d & = 3d + 12d \\
= 15d & = 15d \quad \checkmark
\end{array}$$

\therefore the common difference for the sequence S_3, S_6, S_8 is $15d$.

7 a There are $12! = 479\,001\,600$ possible orders.

b i There are $\binom{4}{2} \times 2! = 12$ ways to place Irena and Eva amongst the last 4, and the other 10 are ordered in $10!$ ways.
 \therefore the total number is $12 \times 10! = 43\,545\,600$ ways.

ii The 3 can be together in 2 ways (PIL or LIP) and this group together with the other 9 can be ordered in $10!$ ways.
 \therefore the total number is $2 \times 10! = 7\,257\,600$ ways.

iii Istvan will be between Paul and Laszlo in $\frac{1}{3}$ of all possible cases, since each of them will be the “middle student” $\frac{1}{3}$ of the time.
 \therefore the total number of ways = $\frac{1}{3}$ of $12! = 159\,667\,200$ ways.

iv The students can be arranged in the form

$$\left\{ \begin{array}{l}
A \square \square \square H \square \square \square \square \square \square \text{ in } 10! \text{ ways} \\
H \square \square \square A \square \square \square \square \square \square \text{ in } 10! \text{ ways} \\
\square A \square \square \square H \square \square \square \square \square \square \text{ in } 10! \text{ ways} \\
\vdots \\
\square \square \square \square \square \square H \square \square \square A \text{ in } 10! \text{ ways}
\end{array} \right\} 8 \times 2 = 16 \text{ of these}$$

\therefore the total number of ways = $16 \times 10! = 58\,060\,800$ ways.

c i There are $\binom{12}{4}$ ways to choose the first group, $\binom{8}{4}$ ways to choose the second group, and $\binom{4}{4}$ ways to choose the third group. The order of groups is not important, so we divide by $3!$.

So, there are $\frac{1}{3!} \binom{12}{4} \binom{8}{4} \binom{4}{4} = 5775$ ways.

ii There are $\binom{2}{2} \binom{10}{2}$ ways to choose the group with Ben and Marton. There are then $\binom{8}{4}$ ways to choose the second group, and $\binom{4}{4}$ ways to choose the third group. The order of groups 2 and 3 is not important, so we divide by $2!$.

So, there are $\frac{1}{2!} \binom{2}{2} \binom{10}{2} \binom{8}{4} \binom{4}{4} = 1575$ ways.

- 8 a** Since $f(x)$ is a real polynomial, $1 - i$ is also a zero.

$$(1 + i) + (1 - i) = 2$$

$$\text{and } (1 + i)(1 - i) = 1 - i^2 = 2$$

$$\therefore x^2 - 2x + 2 \text{ is a factor of } f(x)$$

$$\begin{aligned} \therefore f(x) &= (x^2 - 2x + 2)(x^2 + bx + c) \quad \text{for some } b, c \\ &= x^4 + bx^3 + cx^2 \\ &\quad - 2x^3 - 2bx^2 - 2cx \\ &\quad + 2x^2 + 2bx + 2c \\ &= x^4 + (b - 2)x^3 + (c + 2 - 2b)x^2 + (2b - 2c)x + 2c \end{aligned}$$

$$\text{Equating coefficients of } x^2: c + 2 - 2b = -2$$

$$\therefore 2b = c + 4 \quad \dots (*)$$

$$\text{Equating coefficients of } x: 2b - 2c = 10$$

$$\therefore c + 4 - 2c = 10 \quad \{\text{using } (*)\}$$

$$\therefore -c = 6$$

$$\therefore c = -6$$

$$\therefore 2b = -2$$

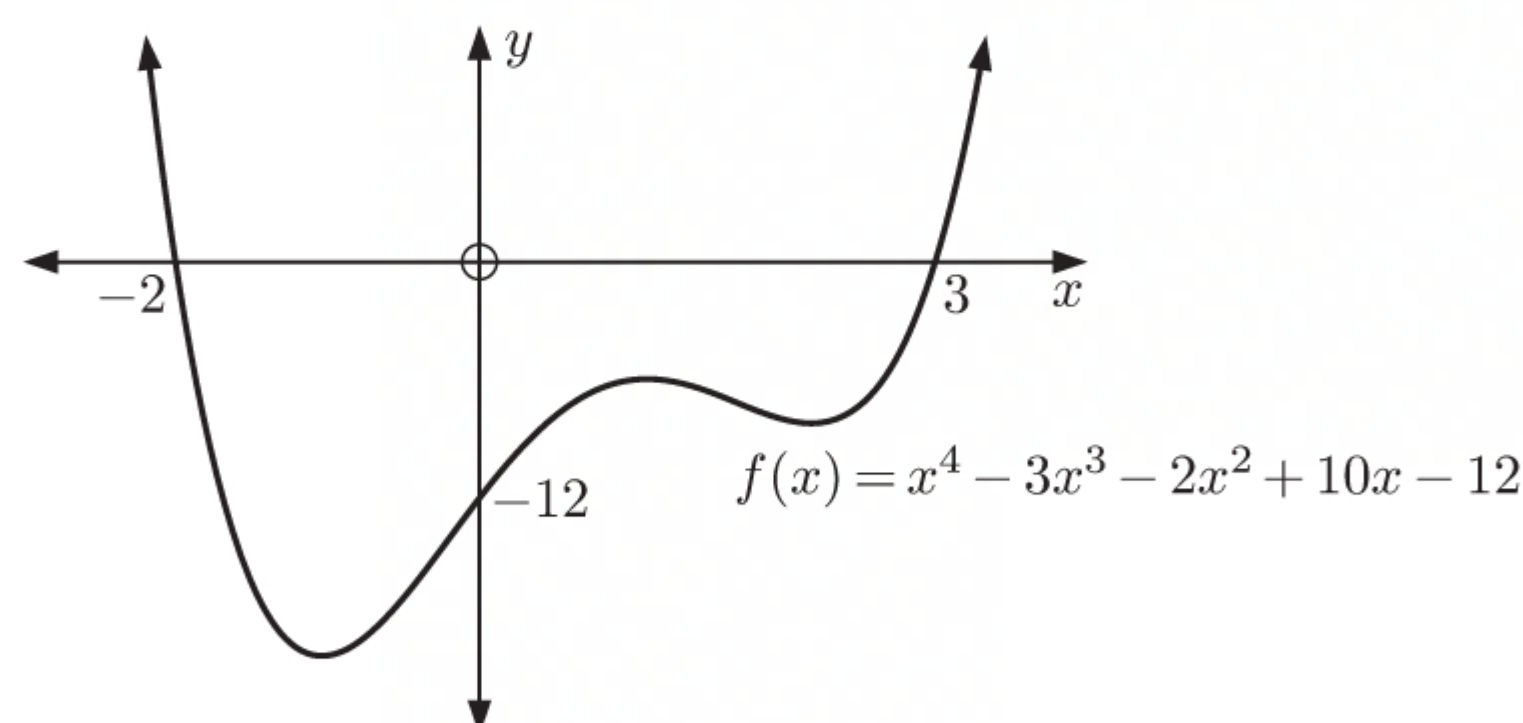
$$\therefore b = -1$$

$$\begin{aligned} \therefore f(x) &= (x^2 - 2x + 2)(x^2 - x - 6) \\ &= x^4 - 3x^3 - 2x^2 + 10x - 12 \end{aligned}$$

$$\therefore a = -3$$

b $f(0) = -12$

$$\begin{aligned} \text{and using a, } f(x) &= (x^2 - 2x + 2)(x^2 - x - 6) \\ &= (x^2 - 2x + 2)(x - 3)(x + 2) \end{aligned}$$



c The sum of the roots is $(1 + i) + (1 - i) + (-2) + 3 = 3$ and $\frac{-(-3)}{1} = 3$ ✓

The product of the roots is $(1 + i)(1 - i)(-2)(3) = -12$ and $\frac{(-1)^4(-12)}{1} = -12$ ✓

9 a $P: x + y + z = 1$ has $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\therefore \text{the parametric equations of (AN) are } x = 1 + t, y = 1 + t, z = 1 + t, t \in \mathbb{R}.$$

$$\text{This line meets the plane where } (1 + t) + (1 + t) + (1 + t) = 1$$

$$\therefore 3 + 3t = 1$$

$$\therefore 3t = -2$$

$$\therefore t = -\frac{2}{3}$$

$$\therefore \text{the foot of the normal N is } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

$$\mathbf{b} \quad \overrightarrow{AN} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \quad \text{The shortest distance, } |\overrightarrow{AN}| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2}$$

$$= \frac{2}{3}\sqrt{3}$$

$$= \frac{2}{\sqrt{3}} \text{ units}$$

c Let $A'(a, b, c)$ be the mirror image of A when reflected in P.

$$\therefore \overrightarrow{NA'} = \overrightarrow{AN}$$

$$\therefore \begin{pmatrix} a - \frac{1}{3} \\ b - \frac{1}{3} \\ c - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$\therefore a - \frac{1}{3} = -\frac{2}{3}, \quad b - \frac{1}{3} = -\frac{2}{3}, \quad \text{and} \quad c - \frac{1}{3} = -\frac{2}{3}$$

$$\therefore a = -\frac{1}{3}, \quad b = -\frac{1}{3}, \quad \text{and} \quad c = -\frac{1}{3}$$

$$\therefore A' \text{ is } \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right).$$

10 Volume of revolution about x -axis $V_X = \pi \int_0^{\sqrt{k}} y^2 dx$

$$= \pi \int_0^{\sqrt{k}} (k - x^2)^2 dx$$

$$= \pi \int_0^{\sqrt{k}} (k^2 - 2kx^2 + x^4) dx$$

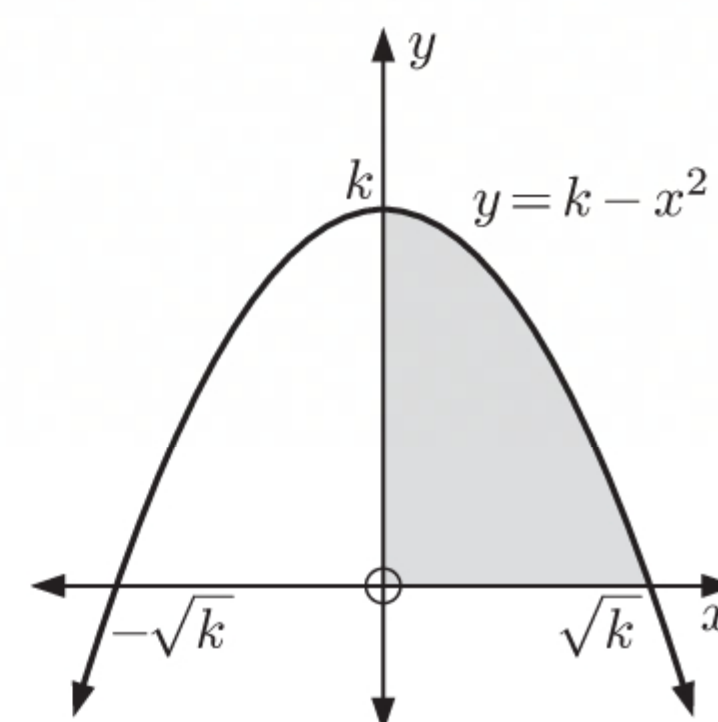
$$= \pi \left[k^2x - \frac{2}{3}kx^3 + \frac{1}{5}x^5 \right]_0^{\sqrt{k}}$$

$$= \pi \left(k^2\sqrt{k} - \frac{2}{3}k(\sqrt{k})^3 + \frac{1}{5}(\sqrt{k})^5 \right)$$

$$= \pi \left(k^2\sqrt{k} - \frac{2}{3}k^2\sqrt{k} + \frac{1}{5}k^2\sqrt{k} \right)$$

$$= \pi k^2\sqrt{k} \left(1 - \frac{2}{3} + \frac{1}{5} \right)$$

$$= \frac{8}{15} \pi k^2\sqrt{k} \text{ units}^3$$



Volume of revolution about y -axis $V_Y = \pi \int_0^k x^2 dy$

$$= \pi \int_0^k (k - y) dy$$

$$= \pi \left[ky - \frac{1}{2}y^2 \right]_0^k$$

$$= \pi \left(k^2 - \frac{1}{2}k^2 \right)$$

$$= \frac{1}{2} \pi k^2 \text{ units}^3$$

Now $V_X = V_Y$

$$\therefore \frac{8}{15} \pi k^2\sqrt{k} = \frac{1}{2} \pi k^2$$

$$\therefore \frac{8}{15} \sqrt{k} = \frac{1}{2} \quad \{k > 0\}$$

$$\therefore \sqrt{k} = \frac{15}{16}$$

$$\therefore k = \frac{225}{256}$$

MIXED QUESTIONS SET 10

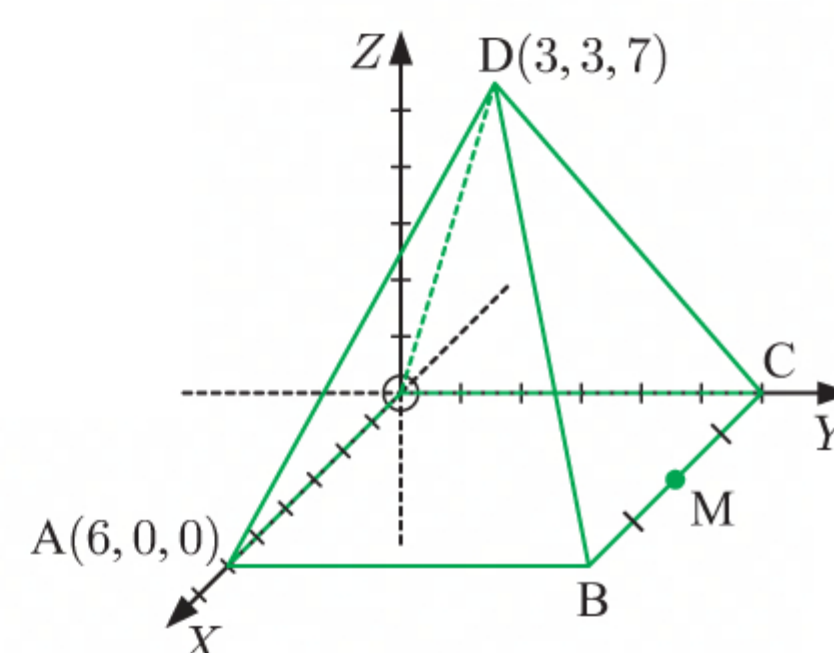
1 a B is $(6, 6, 0)$ and C is $(0, 6, 0)$.

b Volume $= \frac{1}{3} \times \text{area of base} \times \text{height}$

$$= \frac{1}{3} \times 6 \times 6 \times 7$$

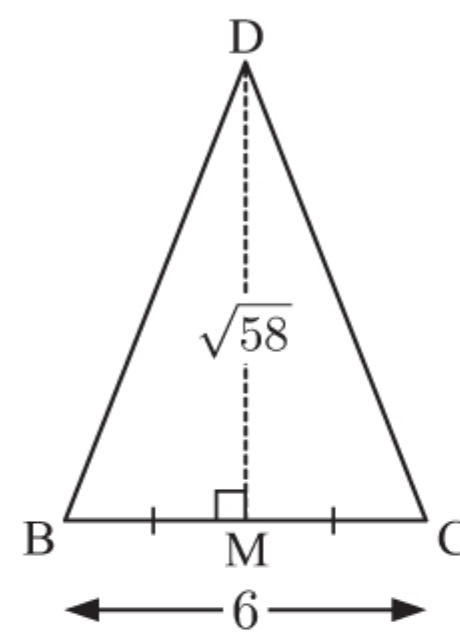
$$= 84 \text{ units}^3$$

c The midpoint M of [BC] is $\left(\frac{6+0}{2}, \frac{6+6}{2}, \frac{0+0}{2}\right)$ which is $(3, 6, 0)$.



$$\begin{aligned}
 \text{d } MD &= \sqrt{(3-3)^2 + (3-6)^2 + (7-0)^2} \\
 &= \sqrt{0^2 + (-3)^2 + 7^2} \\
 &= \sqrt{0 + 9 + 49} \\
 &= \sqrt{58} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{Area of triangle BCD} &= \frac{1}{2} \times 6 \times \sqrt{58} \\
 &= 3\sqrt{58} \text{ units}^2
 \end{aligned}$$

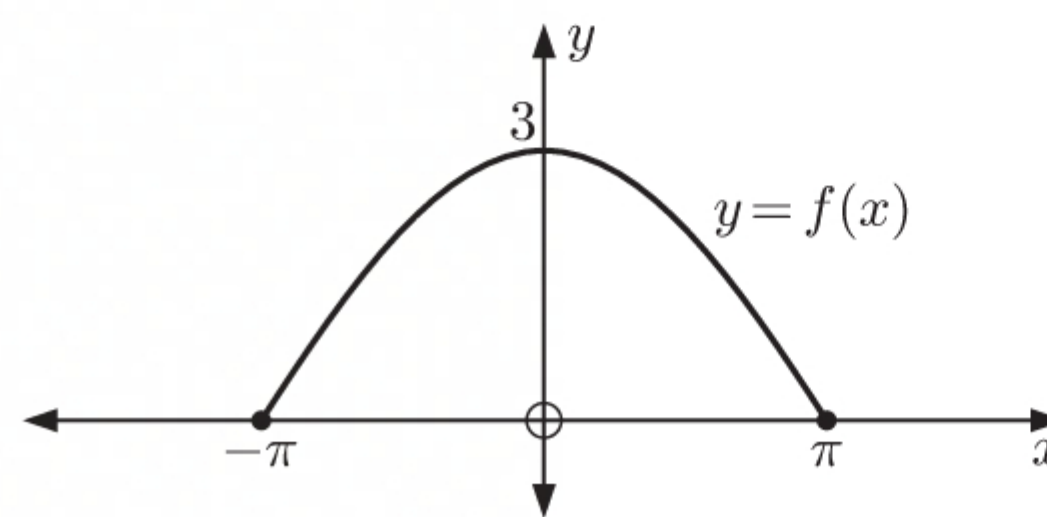


$$\begin{aligned}
 \text{Surface area of pyramid} &= \text{area of base} + \text{area of 4 triangular faces} \\
 &= 6 \times 6 + 4 \times \text{area of } \triangle BCD \\
 &= 36 + 4 \times 3\sqrt{58} \\
 &= 36 + 12\sqrt{58} \text{ units}^2 \\
 &\approx 127 \text{ units}^2
 \end{aligned}$$

2 a $f(x) = a \cos bx, \quad -\pi \leq x \leq \pi$

From the graph:

- the amplitude is 3, so $a = 3$
- the period is $2 \times 2\pi, \quad \frac{2\pi}{b} = 4\pi$
 $\therefore b = \frac{1}{2}$



b $f(x) = 3 \cos(\frac{x}{2}), \quad -\pi \leq x \leq \pi$

i $f'(x) = -\frac{3}{2} \sin(\frac{x}{2})$

Now $f(c) = 3 \cos(\frac{c}{2})$ and $f'(c) = -\frac{3}{2} \sin(\frac{c}{2})$.

\therefore the normal to $y = f(x)$ at the point where $x = c$ has gradient $\frac{2}{3 \sin(\frac{c}{2})}$ and passes through $(c, 3 \cos(\frac{c}{2}))$.

\therefore the equation of the normal is $2x - 3 \sin(\frac{c}{2})y = 2c - 3 \sin(\frac{c}{2})(3 \cos(\frac{c}{2}))$

$\therefore 2x - 3 \sin(\frac{c}{2})y = 2c - 9 \sin(\frac{c}{2}) \cos(\frac{c}{2})$

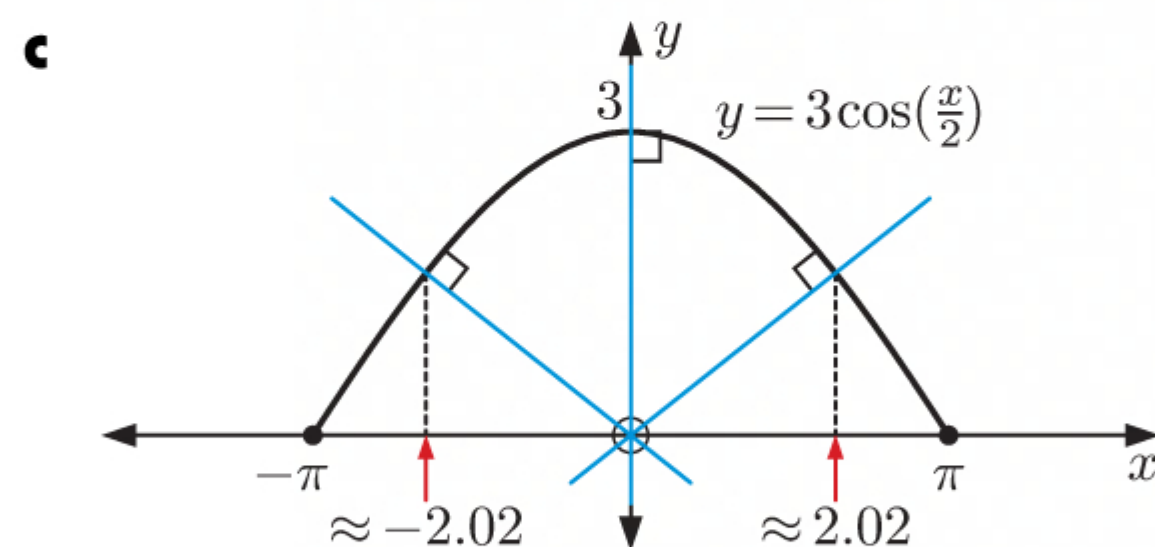
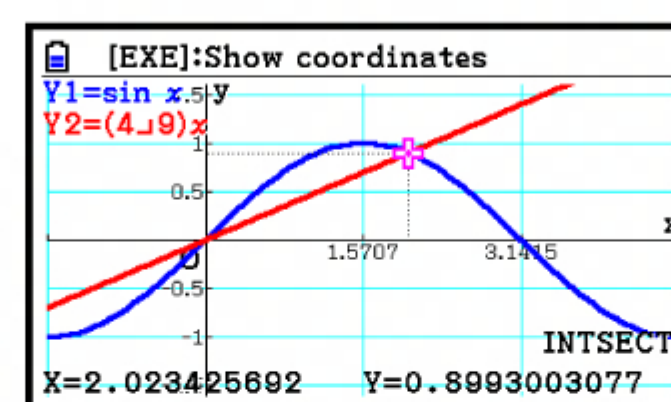
$\therefore 2x - 3 \sin(\frac{c}{2})y = 2c - \frac{9}{2} \sin c \quad \{2 \sin \theta \cos \theta = \sin 2\theta\}$

ii If the normal passes through the origin then

$$2c - \frac{9}{2} \sin c = 0$$

$$\therefore \sin c = \frac{4}{9}c$$

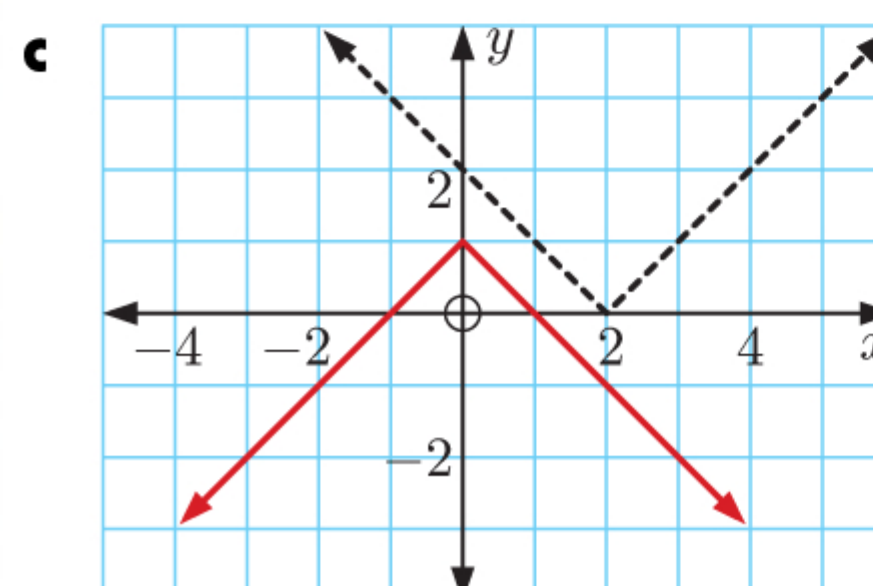
Using technology, $c = 0$ or $\approx \pm 2.02 \quad \{-\pi \leq c \leq \pi\}$



3 a Yes, as any vertical line cuts the graph no more than once.

b Domain = $\{x \mid x \in \mathbb{R}\}$

Range = $\{y \mid y \geq 0\}$



$$4 \quad R(t) = \frac{12}{\sqrt{t+1}} e^{-\sqrt{t+1}} \text{ L s}^{-1}$$

$$\begin{aligned} \mathbf{a} \quad R(10) &= \frac{12}{\sqrt{10+1}} e^{-\sqrt{10+1}} \\ &= \frac{12}{\sqrt{11}} e^{-\sqrt{11}} \\ &\approx 0.131 \text{ L s}^{-1} \end{aligned}$$

After 10 seconds, the water is still overflowing at about 0.131 L s^{-1} .

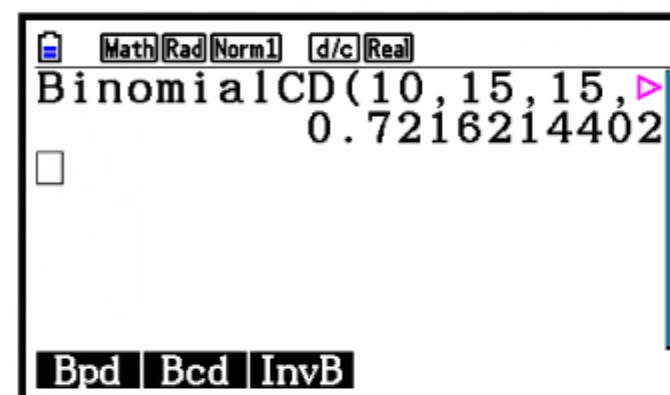
$$\begin{aligned} \mathbf{c} \quad \int_0^{60} R(t) dt &= \int_0^{60} \frac{12}{\sqrt{t+1}} e^{-\sqrt{t+1}} dt \\ &= \left[-24e^{-\sqrt{t+1}} \right]_0^{60} \quad \{\text{using } \mathbf{b}\} \\ &= -24e^{-\sqrt{61}} + 24e^{-1} \\ &= 24(e^{-1} - e^{-\sqrt{61}}) \\ &\approx 8.82 \end{aligned}$$

- 5 a** Let X be the number of committee members who attend a randomly selected meeting.

$$\therefore X \sim B(15, 0.7)$$

$$\text{Now } P(X \geq 10) \approx 0.722 \quad \{\text{using technology}\}$$

\therefore approximately 72.2% of meetings will go ahead.



- b** Suppose there are n committee members, and let Y be the number of committee members who attend a randomly selected meeting.

$$\therefore Y \sim B(n, 0.7)$$

$$\text{We require } P(Y \geq 10) \geq 0.9$$

$$\text{If } n = 16, \quad P(Y \geq 10) \approx 0.825 \quad \times$$

$$\text{If } n = 17, \quad P(Y \geq 10) \approx 0.895 \quad \times$$

$$\text{If } n = 18, \quad P(Y \geq 10) \approx 0.940 \quad \checkmark$$

So, 18 committee members are required to ensure that at least 90% of the meetings go ahead.

$$6 \quad s = \sin\left(\frac{\pi}{(t+1)^2}\right) \text{ cm}, \quad t \geq 0$$

$$\begin{aligned} \mathbf{a} \quad \text{When } t = 1, \quad s &= \sin\left(\frac{\pi}{(1+1)^2}\right) \\ &= \sin \frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} \text{ cm} \end{aligned}$$

The displacement of the object after 1 second is $\frac{1}{\sqrt{2}} \text{ cm}$.

$$\mathbf{b} \quad s = 0.5 \text{ cm} \quad \text{when} \quad \sin\left(\frac{\pi}{(t+1)^2}\right) = 0.5$$

$$\therefore \frac{\pi}{(t+1)^2} = \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}$$

$$\therefore (t+1)^2 = 6 \quad \text{or} \quad \frac{6}{5}$$

$$\therefore t+1 = \sqrt{6} \quad \text{or} \quad \sqrt{\frac{6}{5}} \quad \{t+1 > 0\}$$

$$\therefore t = \sqrt{6} - 1 \quad \text{or} \quad \sqrt{\frac{6}{5}} - 1$$

$$\therefore t \approx 1.45 \quad \text{or} \quad 0.0954 \text{ seconds}$$

The object has displacement 0.5 cm for the first time after about 0.0954 seconds.

$$\begin{aligned}
 \text{c } v &= \frac{ds}{dt} \\
 &= \cos\left(\frac{\pi}{(t+1)^2}\right) \times (-2\pi(t+1)^{-3}) \quad \{\text{chain rule}\} \\
 &= \frac{-2\pi}{(t+1)^3} \cos\left(\frac{\pi}{(t+1)^2}\right) \text{ cm s}^{-1}
 \end{aligned}$$

d The object is stationary when $v = 0$

$$\begin{aligned}
 \therefore \frac{-2\pi}{(t+1)^3} \cos\left(\frac{\pi}{(t+1)^2}\right) &= 0 \\
 \therefore \cos\left(\frac{\pi}{(t+1)^2}\right) &= 0 \\
 \therefore \frac{\pi}{(t+1)^2} &= \frac{\pi}{2} \\
 \therefore (t+1)^2 &= 2 \\
 \therefore t+1 &= \sqrt{2} \quad \{t+1 > 0\} \\
 \therefore t &= \sqrt{2} - 1 \approx 0.414 \text{ seconds}
 \end{aligned}$$

The object is stationary for the first time after about 0.414 seconds.

7 α and β are the roots of $3x^2 + 3x - 5 = 0$

$$\therefore \alpha + \beta = \frac{-3}{3} = -1 \quad \text{and} \quad \alpha\beta = \frac{-5}{3}$$

$$\begin{array}{ll}
 \text{a Sum of the roots} = -\alpha - \beta & \text{Product of the roots} = (-\alpha)(-\beta) \\
 = -(\alpha + \beta) & = \alpha\beta \\
 = 1 & = -\frac{5}{3}
 \end{array}$$

$$\begin{aligned}
 \therefore \text{a quadratic equation with roots } -\alpha \text{ and } -\beta \text{ is } x^2 - x - \frac{5}{3} &= 0 \\
 \therefore 3x^2 - 3x - 5 &= 0
 \end{aligned}$$

$$\begin{array}{ll}
 \text{b Sum of the roots} = \frac{\alpha}{2} + \frac{\beta}{2} & \text{Product of the roots} = \left(\frac{\alpha}{2}\right)\left(\frac{\beta}{2}\right) \\
 = \frac{\alpha + \beta}{2} & = \frac{\alpha\beta}{4} \\
 = -\frac{1}{2} & = -\frac{5}{12}
 \end{array}$$

$$\begin{aligned}
 \therefore \text{a quadratic equation with roots } \frac{\alpha}{2} \text{ and } \frac{\beta}{2} \text{ is } x^2 + \frac{1}{2}x - \frac{5}{12} &= 0 \\
 \therefore 12x^2 + 6x - 5 &= 0
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ a } \frac{2x-1}{x^2-x-2} &= \frac{2x-1}{(x+1)(x-2)} \\
 &= \frac{A}{x+1} + \frac{B}{x-2}
 \end{aligned}$$

$$\therefore 2x - 1 = A(x - 2) + B(x + 1)$$

$$\text{Substituting } x = -1, \quad 2(-1) - 1 = A(-1 - 2)$$

$$\therefore -3A = -3$$

$$\therefore A = 1$$

$$\text{Substituting } x = 2, \quad 2(2) - 1 = B(2 + 1)$$

$$\therefore 3B = 3$$

$$\therefore B = 1$$

$$\therefore \frac{2x-1}{x^2-x-2} = \frac{1}{x+1} + \frac{1}{x-2}$$

$$\begin{aligned}
 \text{b } \frac{1}{x+1} &= (x+1)^{-1} \\
 &= \sum_{r=0}^{\infty} \binom{-1}{r} x^r \quad \{\text{binomial theorem}\} \\
 &= 1 + (-1)x + \frac{(-1)(-2)}{2!} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \dots \\
 &= 1 - x + x^2 - x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{x-2} &= (x-2)^{-1} \\
 &= (-2)^{-1} \sum_{r=0}^{\infty} \binom{-1}{r} \left(\frac{x}{-2}\right)^r \quad \{\text{binomial theorem}\} \\
 &= -\frac{1}{2} \left(1 + (-1) \left(\frac{x}{-2}\right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{-2}\right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{-2}\right)^3 + \dots \right) \\
 &= -\frac{1}{2} \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots \right) \\
 &= -\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{2x-1}{x^2-x-2} &= \frac{1}{x+1} + \frac{1}{x-2} \quad \{\text{using a}\} \\
 &= (1-x+x^2-x^3+\dots) + \left(-\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \dots\right) \\
 &= \frac{1}{2} - \frac{5}{4}x + \frac{7}{8}x^2 - \frac{17}{16}x^3 + \dots
 \end{aligned}$$

c The binomial expansion of:

- $\frac{1}{x+1}$ converges for $|x| < 1$, which is $-1 < x < 1$
- $\frac{1}{x-2}$ converges for $\left|\frac{x}{-2}\right| < 1$, which is $-2 < x < 2$

\therefore the binomial expansion of $\frac{2x-1}{x^2-x-2}$ converges for $-1 < x < 1$.

$$\begin{aligned}
 \text{d When } x = 0.01, \quad \frac{2x-1}{x^2-x-2} &\approx \frac{1}{2} - \frac{5}{4}(0.01) + \frac{7}{8}(0.01)^2 - \frac{17}{16}(0.01)^3 \\
 &\approx 0.488
 \end{aligned}$$

9 Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AX} = \mathbf{b}$.

$$\therefore \overrightarrow{AY} = \frac{2}{3}\mathbf{b} \quad \{\text{AY : YX} = 2 : 1\}$$

$$\begin{aligned}
 \therefore \overrightarrow{BY} &= -\overrightarrow{AB} + \overrightarrow{AY} \\
 &= -\mathbf{a} + \frac{2}{3}\mathbf{b}
 \end{aligned}$$

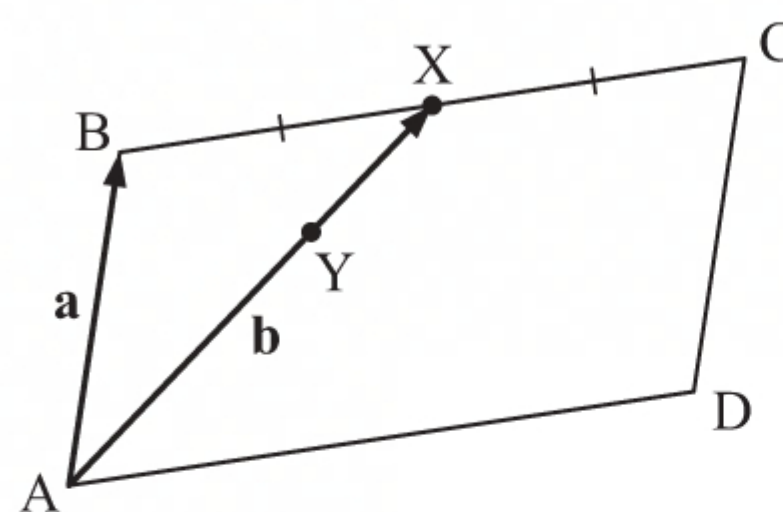
$$\begin{aligned}
 \text{Now } \overrightarrow{BC} &= 2\overrightarrow{BX} \\
 &= 2(-\overrightarrow{AB} + \overrightarrow{AX}) \\
 &= 2(-\mathbf{a} + \mathbf{b}) \\
 &= -2\mathbf{a} + 2\mathbf{b}
 \end{aligned}$$

$$\therefore \overrightarrow{AD} = \overrightarrow{BC} = -2\mathbf{a} + 2\mathbf{b} \quad \{\text{ABCD is a parallelogram}\}$$

$$\begin{aligned}
 \therefore \overrightarrow{YD} &= -\overrightarrow{AY} + \overrightarrow{AD} \\
 &= -\frac{2}{3}\mathbf{b} + (-2\mathbf{a} + 2\mathbf{b}) \\
 &= -2\mathbf{a} + \frac{4}{3}\mathbf{b} \\
 &= 2\left(-\mathbf{a} + \frac{2}{3}\mathbf{b}\right) \\
 &= 2\overrightarrow{BY}
 \end{aligned}$$

$$\therefore \overrightarrow{YD} \parallel \overrightarrow{BY}$$

\therefore B, D, and Y are collinear. {Y is a common point}



10 $\frac{dy}{dx} = \frac{1}{1+x^2}, \quad y(1) = \pi$

a

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	1	$\pi \approx 3.1416$	0.5	1.2	3.2416
2	1.2	3.2416	0.4098	1.4	3.3236
3	1.4	3.3236	0.3378	1.6	3.3911
4	1.6	3.3911	0.2809	1.8	3.4473
5	1.8	3.4473	0.2358	2	3.4945

$$\therefore y(2) \approx 3.4945$$

b Using technology with step size 0.005 for 200 steps gives $y(2) \approx 3.4641$.

c Using the Fundamental Theorem of Calculus,

$$\begin{aligned}
 y(2) &= y(1) + \int_1^2 \frac{dy}{dx} dx \\
 &= \pi + \int_1^2 \frac{1}{1+x^2} dx \\
 &= \pi + [\arctan x]_1^2 \\
 &= \pi + \arctan 2 - \arctan 1 \\
 &= \pi + \arctan 2 - \frac{\pi}{4} \\
 &= \frac{3\pi}{4} + \arctan 2 \\
 &\approx 3.4633
 \end{aligned}$$

The accuracy of Euler's method was improved by decreasing the step size.

MIXED QUESTIONS SET 11

1 $f(x) = 3 - 4^{-x}$

a $f(2) = 3 - 4^{-2} = 3 - \frac{1}{16}$
 $= 2\frac{15}{16}$
 $\therefore p = 2\frac{15}{16}$

$f(-2) = 3 - 4^2 = -13$
 $\therefore q = -13$

b i $f(0) = 3 - 4^0 = 2 \therefore$ the y -intercept is 2.

ii As $x \rightarrow \infty$, $4^{-x} \rightarrow 0$ and so $y \rightarrow 3$
 $\therefore y = 3$ is the horizontal asymptote.

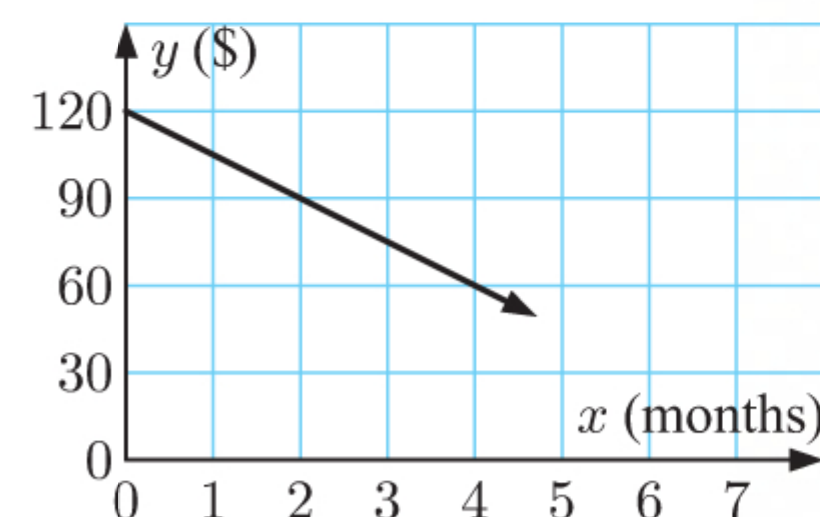
d The range is $\{y \mid y < 3\}$.

2 a The line passes through $(0, 120)$ and $(2, 90)$, so the gradient is

$$\frac{90 - 120}{2 - 0} = \frac{-30}{2} = -15.$$

This means that the amount of money left in the subscription account decreases by \$15 each month.

The y -intercept is 120. This means that the initial balance was \$120.



b The gradient is -15 and the y -intercept is 120, so the equation of the line is $y = -15x + 120$.

c The account runs out of money when $y = 0$

$$\therefore -15x + 120 = 0$$

$$\therefore 15x = 120$$

$$\therefore x = 8$$

The account will run out of money after 8 months.

3 a $v(t) = e^{2t} - 3e^t \text{ m s}^{-1}$

$$\therefore v(0) = e^0 - 3e^0 = 1 - 3 = -2 \text{ m s}^{-1}$$

$$\therefore \text{the initial velocity is } -2 \text{ m s}^{-1}.$$

b Now $v(t) = e^t(e^t - 3)$

Since $e^t > 0$ for all t , $v(t) = 0$ when $e^t = 3$

which is when $t = \ln 3$

$$\therefore \text{the particle is stationary at } t = \ln 3 \text{ seconds.}$$

c $s(t) = \int v(t) dt$

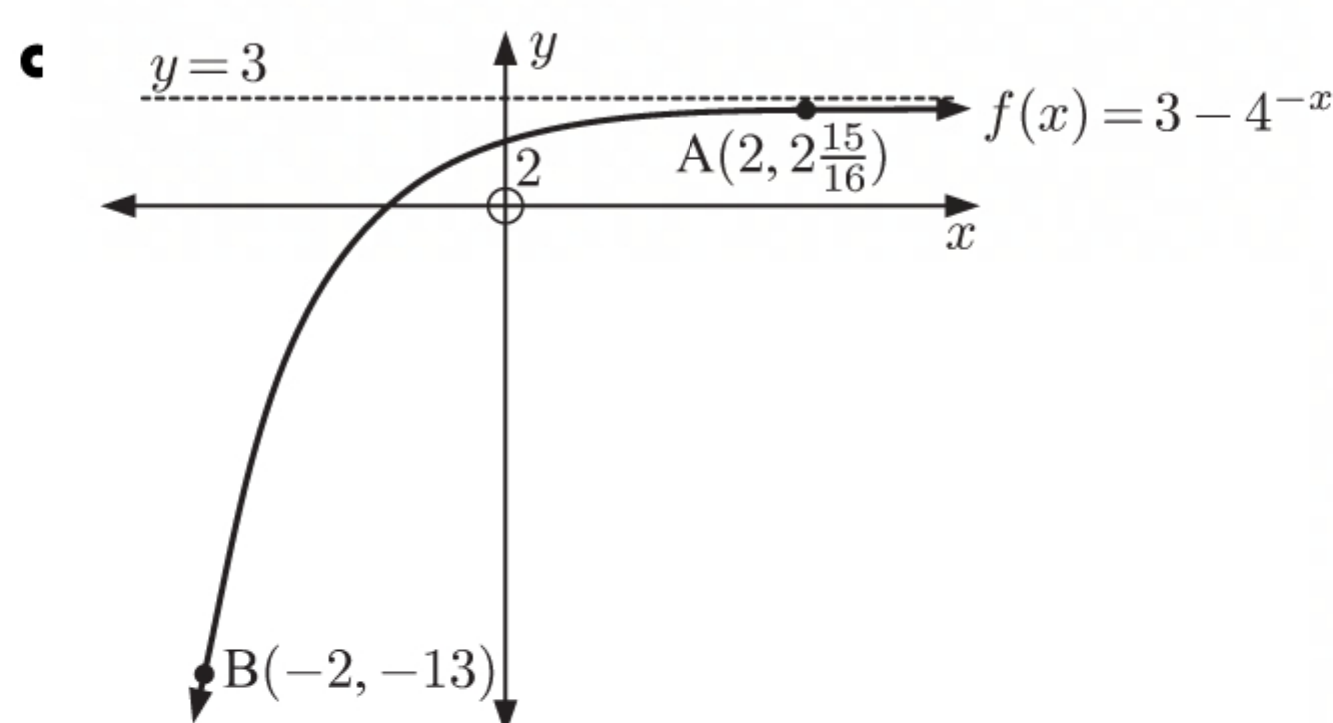
$$= \int (e^{2t} - 3e^t) dt$$

$$= \frac{1}{2}e^{2t} - 3e^t + c \text{ metres}$$

But $s(0) = 1$, so $\frac{1}{2}e^0 - 3e^0 + c = 1$

$$\therefore \frac{1}{2} - 3 + c = 1$$

$$\therefore c = 3\frac{1}{2}$$



$$\text{Thus } s(t) = \frac{1}{2}e^{2t} - 3e^t + \frac{7}{2} \text{ metres}$$

$$\begin{aligned} \therefore s(\ln 5) &= \frac{1}{2}e^{2\ln 5} - 3e^{\ln 5} + \frac{7}{2} \\ &= \frac{1}{2}e^{\ln(5^2)} - 3(5) + \frac{7}{2} \\ &= \frac{25}{2} - 15 + \frac{7}{2} \\ &= 16 - 15 \\ &= 1 \text{ m} \end{aligned}$$

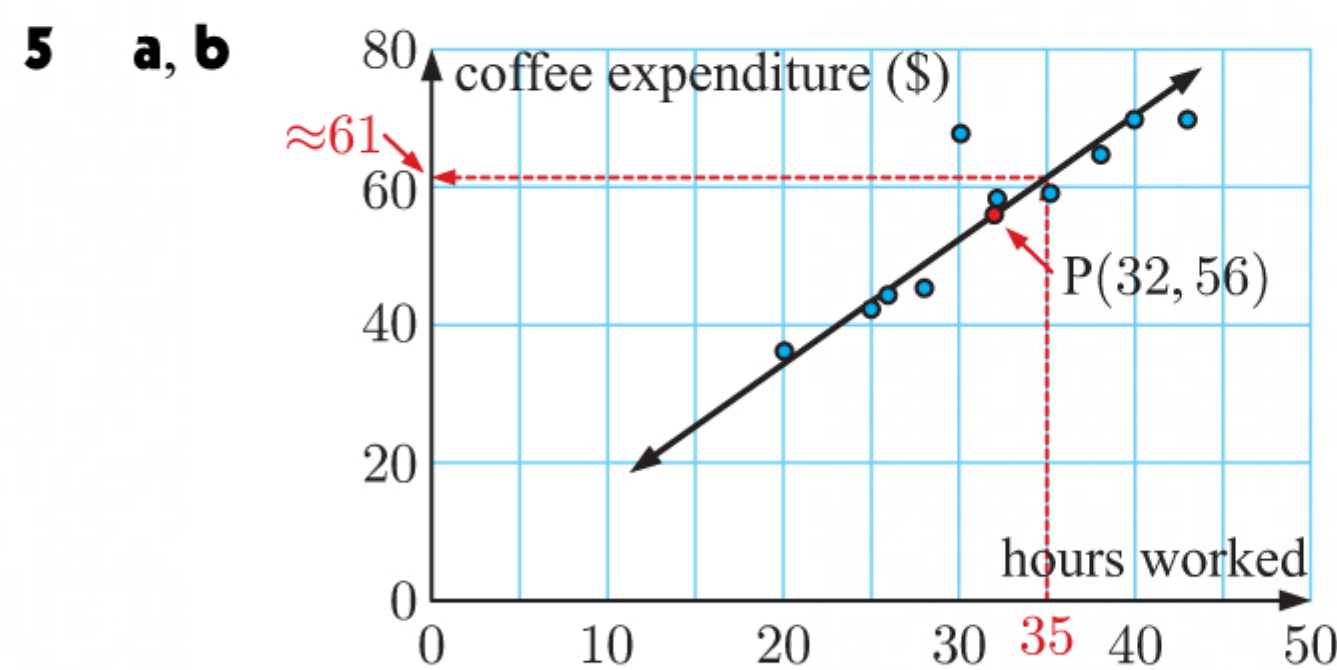
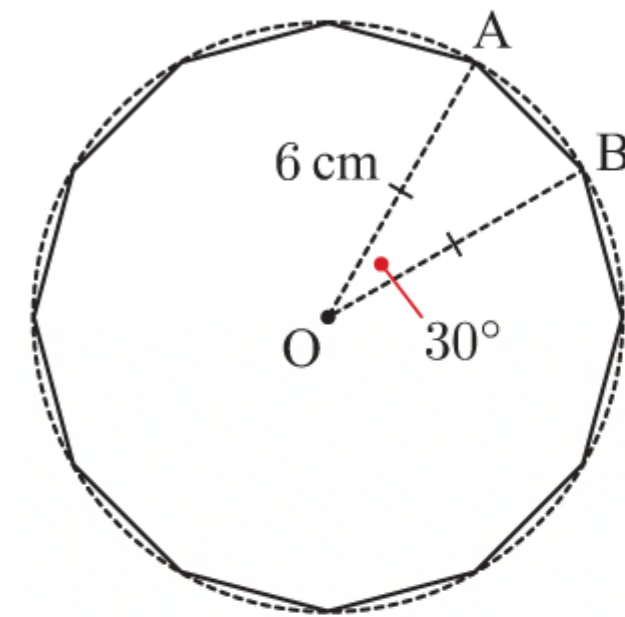
So, the particle is 1 m to the right of O after $\ln 5$ seconds.

- 4 a** There are twelve equal angles at the centre of the dodecagon.

$$\therefore \widehat{AOB} = \frac{360^\circ}{12} = 30^\circ$$

$$\begin{aligned} \text{b Area of } \triangle AOB &= \frac{1}{2} \times 6 \times 6 \times \sin 30^\circ \\ &= 9 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{c Area of dodecagon} &= 12 \times 9 \\ &= 108 \text{ cm}^2 \end{aligned}$$



- c** From the graph, if James works a 35 hour week, he spends about \$61.

- d** There is a strong positive linear relationship between the length of time James works and the amount he spends on coffee. Since the prediction in **c** was an interpolation on strongly correlated data, it is a reliable estimate.

- 6 a** $\left(kx + \frac{1}{\sqrt{x}}\right)^9$ has general term

$$\begin{aligned} T_{r+1} &= \binom{9}{r} (kx)^{9-r} \left(\frac{1}{\sqrt{x}}\right)^r \\ &= \binom{9}{r} k^{9-r} x^{9-r} \frac{1}{x^{\frac{r}{2}}} \\ &= \binom{9}{r} k^{9-r} x^{9-\frac{3r}{2}} \end{aligned}$$

- b** For the constant term, $9 - \frac{3r}{2} = 0$

$$\begin{aligned} \therefore \frac{3r}{2} &= 9 \\ \therefore r &= 6 \end{aligned}$$

$$T_7 = \binom{9}{6} k^3 x^0$$

$$\therefore 84k^3 = -10\frac{1}{2} \quad \{\text{constant term} = -10\frac{1}{2}\}$$

$$\therefore k^3 = -\frac{1}{8}$$

$$\therefore k = -\frac{1}{2}$$

- 7 a** In augmented matrix form, the system is:

$$\left(\begin{array}{ccc|c} 1 & 3 & k & 2 \\ k & -2 & 3 & k \\ 4 & -3 & 10 & 5 \end{array} \right) \quad \left\{ \begin{array}{cccc} k & -2 & 3 & k \\ -k & -3k & -k^2 & -2k \\ 0 & -2-3k & 3-k^2 & -k \end{array} \right\}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2-3k & 3-k^2 & -k \\ 0 & -15 & 10-4k & -3 \end{array} \right) \quad \begin{array}{l} R_2 - kR_1 \rightarrow R_2 \\ R_3 - 4R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 4 & -3 & 10 & 5 \\ -4 & -12 & -4k & -8 \\ 0 & -15 & 10-4k & -3 \end{array} \right\}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -15 & 10-4k & -3 \\ 0 & -2-3k & 3-k^2 & -k \end{array} \right) \quad R_2 \leftrightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -15 & 10-4k & -3 \\ 0 & 0 & 25-22k-3k^2 & 6-6k \end{array} \right) \quad 15R_3 - (2+3k)R_2 \rightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -15 & 10-4k & -3 \\ 0 & 0 & -(3k+25)(k-1) & -6(k-1) \end{array} \right)$$

$$\left\{ \begin{array}{cccc} 0 & -30-45k & 45-15k^2 & -15k \\ 0 & -30-45k & -20-22k+12k^2 & 6+9k \\ 0 & 0 & 25-22k-3k^2 & 6-6k \end{array} \right\}$$

- b** If $k = 1$, the last row is all zeros, indicating infinitely many solutions.

Letting $z = t$ in row 2 gives $-15y + 6t = -3$

$$\therefore -15y = -3 - 6t$$

$$\therefore y = \frac{1+2t}{5}$$

Using row 1, $x + 3\left(\frac{1+2t}{5}\right) + t = 2$

$$\therefore x + \frac{11t}{5} = \frac{7}{5}$$

$$\therefore x = \frac{7-11t}{5}$$

We have infinitely many solutions of the form:

$$x = \frac{7-11t}{5}, \quad y = \frac{1+2t}{5}, \quad z = t, \quad \text{where } t \in \mathbb{R}.$$

In this case we have three planes which meet in a line.

- c** If $k = -\frac{25}{3}$, we have an inconsistent system. There are no solutions.

$$\text{In this case, the system is } \begin{cases} x + 3y - \frac{25}{3}z = 2 \\ -\frac{25}{3}x - 2y + 3z = -\frac{25}{3} \\ 4x - 3y + 10z = 5 \end{cases}$$

\therefore the corresponding planes are not parallel.

\therefore the intersection of any two planes is parallel to the third plane.

- d** If $k \neq 1$ or $k \neq -\frac{25}{3}$, the system has a unique solution.

Using row 3, $-(3k+25)(k-1)z = -6(k-1)$

$$\therefore z = \frac{6}{3k+25} \quad \{k \neq 1\}$$

Using row 2, $-15y + (10-4k)\left(\frac{6}{3k+25}\right) = -3$

$$\therefore -15y = -3 - \frac{6(10-4k)}{3k+25}$$

$$\therefore -15y = \frac{-9k-75-60+24k}{3k+25}$$

$$\therefore -15y = \frac{-135+15k}{3k+25}$$

$$\therefore y = \frac{9-k}{3k+25}$$

Using row 1, $x + 3\left(\frac{9-k}{3k+25}\right) + k\left(\frac{6}{3k+25}\right) = 2$

$$\therefore x + \frac{27-3k}{3k+25} + \frac{6k}{3k+25} = 2$$

$$\therefore x + \frac{27+3k}{3k+25} = 2$$

$$\begin{aligned} \therefore x &= 2 - \left(\frac{27+3k}{3k+25}\right) \\ &= \frac{6k+50-27-3k}{3k+25} \\ &= \frac{3k+23}{3k+25} \end{aligned}$$

\therefore the unique solution is $x = \frac{3k+23}{3k+25}, \quad y = \frac{9-k}{3k+25}, \quad z = \frac{6}{3k+25}$

In this case, the planes intersect at the common point $\left(\frac{3k+23}{3k+25}, \frac{9-k}{3k+25}, \frac{6}{3k+25}\right)$.

8 $h(x) = x^3 - 6tx^2 + 11t^2x - 6t^3$

a $h(t) = t^3 - 6t^3 + 11t^3 - 6t^3 = 0$

$\therefore t$ is a zero of $h(x)$.

b By inspection, $h(x) = (x-t)(x^2 - 5tx + 6t^2)$

$$\therefore h(x) = (x-t)(x-2t)(x-3t)$$

$$\begin{aligned}
 \text{c } y = x^3 + 6x^2 \text{ meets } y = -6 - 11x \text{ where } x^3 + 6x^2 = -6 - 11x \\
 \therefore x^3 + 6x^2 + 11x + 6 = 0 \text{ which is } h(x) = 0 \text{ when } t = -1 \\
 \therefore (x+1)(x+2)(x+3) = 0 \quad \{\text{using b}\} \\
 \therefore x = -1, -2, \text{ or } -3
 \end{aligned}$$

So, the graphs meet at $(-1, 5)$, $(-2, 16)$, and $(-3, 27)$.

9

$$\begin{aligned}
 4 \operatorname{cosec} 2\theta &= \tan 2\theta + 5 \cot 2\theta \\
 \therefore \frac{4}{\sin 2\theta} &= \frac{\sin 2\theta}{\cos 2\theta} + \frac{5 \cos 2\theta}{\sin 2\theta} \\
 \therefore 4 \cos 2\theta &= \sin^2 2\theta + 5 \cos^2 2\theta \\
 \therefore 4 \cos 2\theta &= 1 + 4 \cos^2 2\theta \\
 \therefore 4 \cos^2 2\theta - 4 \cos 2\theta + 1 &= 0 \\
 \therefore (2 \cos 2\theta - 1)^2 &= 0 \\
 \therefore \cos 2\theta &= \frac{1}{2} \\
 \therefore 2\theta &= \pm \frac{\pi}{3}, \pm \frac{5\pi}{3} \quad \{-\pi \leq \theta \leq \pi \quad \therefore -2\pi \leq 2\theta \leq 2\pi\} \\
 \therefore \theta &= \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}
 \end{aligned}$$

10 a $\ln(1+x)$ has Maclaurin series expansion $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$$\begin{aligned}
 \therefore \ln(1+3x) \text{ has Maclaurin series expansion } (3x) - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \dots \\
 = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \dots
 \end{aligned}$$

$\arctan x$ has Maclaurin series expansion $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$$\begin{aligned}
 \therefore \arctan 2x \text{ has Maclaurin series expansion } (2x) - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \dots \\
 = 2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \frac{128}{7}x^7 + \dots
 \end{aligned}$$

$$\therefore \frac{\ln(1+3x)}{\arctan 2x} = \frac{3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \dots}{2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \frac{128}{7}x^7 + \dots}$$

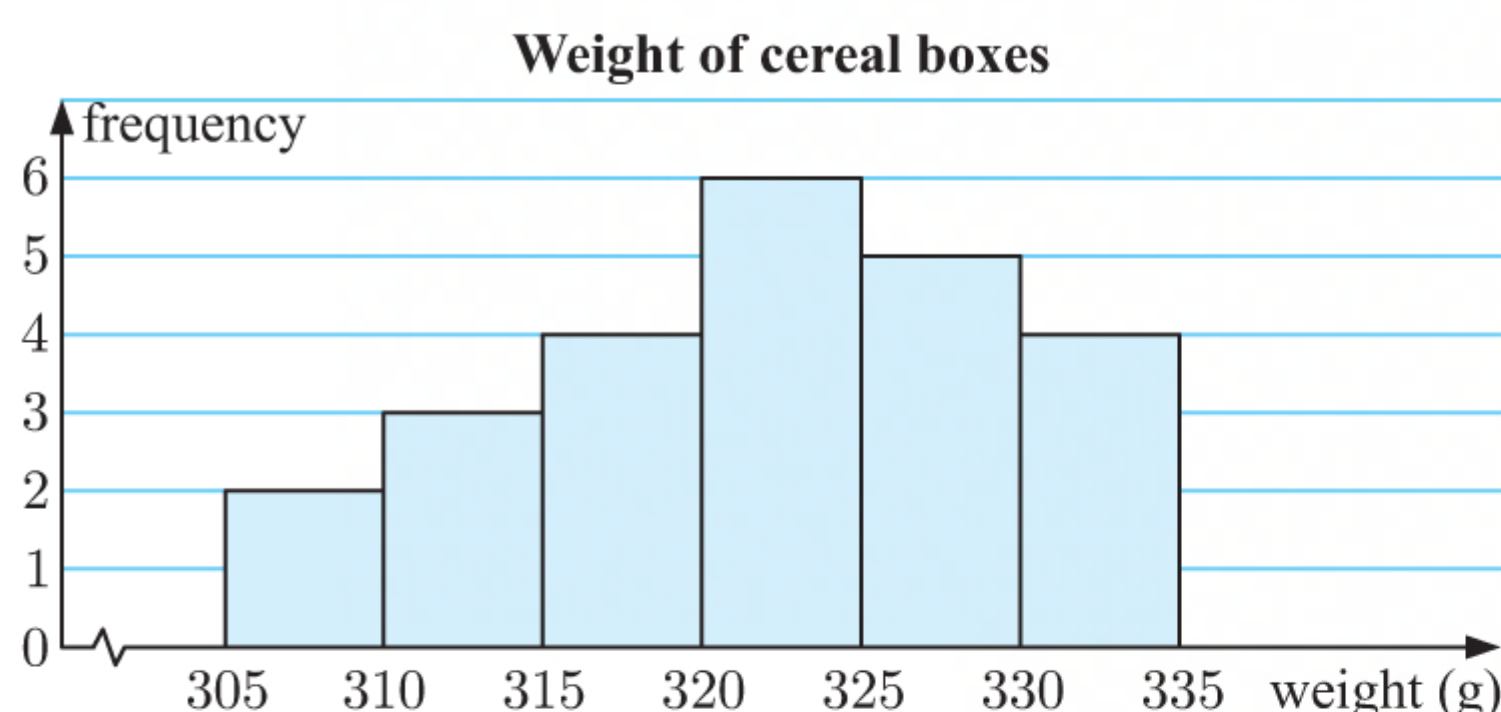
$$\begin{aligned}
 \text{b } \lim_{x \rightarrow 0} \frac{\ln(1+3x)}{\arctan 2x} &= \lim_{x \rightarrow 0} \frac{3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \dots}{2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \frac{128}{7}x^7 + \dots} \quad \{\text{using a}\} \\
 &= \lim_{x \rightarrow 0} \frac{3 - 9x + 27x^2 - 81x^3 + \dots}{2 - 8x^2 + 32x^4 - 128x^6 + \dots} \quad \{\text{l'Hôpital's rule}\} \\
 &= \frac{3}{2}
 \end{aligned}$$

MIXED QUESTIONS SET 12

1 a

Weight (w g)	Frequency
$305 \leq w < 310$	2
$310 \leq w < 315$	3
$315 \leq w < 320$	4
$320 \leq w < 325$	6
$325 \leq w < 330$	5
$330 \leq w < 335$	4

b



c The data is slightly negatively skewed.

d The modal class is the interval $320 \leq w < 325$ because it has the highest frequency.

$$\begin{aligned}
 \text{e Mean} &= \frac{312 + 320 + \dots + 324}{24} \\
 &= \frac{7696}{24} \\
 &\approx 320.67
 \end{aligned}$$

The mean of the sample is reasonably close to the average weight that the manufacturer claims.

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad y &= x(x^2 - 12x + 45) \\
 &= x^3 - 12x^2 + 45x \\
 \therefore \frac{dy}{dx} &= 3x^2 - 24x + 45
 \end{aligned}$$

$$\text{and } \frac{d^2y}{dx^2} = 6x - 24$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dy}{dx} &= 3x^2 - 24x + 45 \\
 &= 3(x^2 - 8x + 15) \\
 &= 3(x - 3)(x - 5)
 \end{aligned}$$

Sign diagram for $\frac{dy}{dx}$:

$$\begin{aligned}
 \text{When } x = 3, \quad y &= (3)^3 - 12(3)^2 + 45(3) \\
 &= 27 - 108 + 135 \\
 &= 54
 \end{aligned}$$

$$\begin{aligned}
 \text{and when } x = 5, \quad y &= (5)^3 - 12(5)^2 + 45(5) \\
 &= 125 - 300 + 225 \\
 &= 50
 \end{aligned}$$

\therefore there is a local maximum at $(3, 54)$ and a local minimum at $(5, 50)$.

$$\mathbf{c} \quad \frac{d^2y}{dx^2} = 6x - 24$$

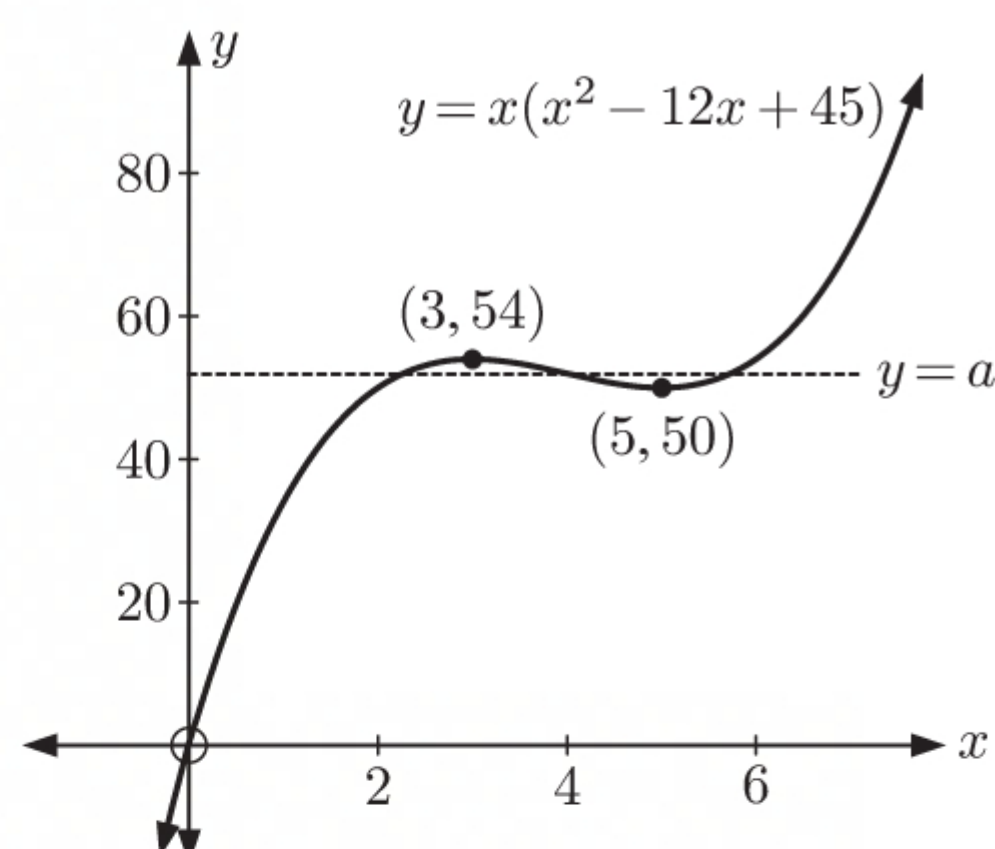
$$\therefore \frac{d^2y}{dx^2} = 0 \text{ when } x = 4$$

Sign diagram for $\frac{d^2y}{dx^2}$ is:

$$\begin{aligned}
 \text{When } x = 4, \quad y &= 4^3 - 12(4)^2 + 45(4) \\
 &= 64 - 192 + 180 \\
 &= 52
 \end{aligned}$$

So, there is a non-stationary inflection point at $(4, 52)$.

\mathbf{d} The graph cuts the x and y -axes at 0.



\mathbf{e} The equation $x^3 - 12x^2 + 45x - a = 0$ has 3 real roots if $x(x^2 - 12x + 45) = a$ has 3 real roots. This occurs provided $y = x(x^2 - 12x + 45)$ meets $y = a$ in 3 places.

$$\therefore 50 < a < 54$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad (g \circ f)(x) &= g(f(x)) \\
 &= g(25 - x^2) \\
 &= \frac{2}{\sqrt{25 - x^2}}
 \end{aligned}$$

The domain is $\{x \mid -5 < x < 5\}$.

$$\begin{aligned}
 \mathbf{b} \quad (g \circ f)(x) &= 1 \\
 \therefore \frac{2}{\sqrt{25 - x^2}} &= 1 \quad \{\text{using } \mathbf{a}\} \\
 \therefore 2 &= \sqrt{25 - x^2} \\
 \therefore 4 &= 25 - x^2 \\
 \therefore x^2 &= 21 \\
 \therefore x &= \pm\sqrt{21}
 \end{aligned}$$

\mathbf{c} $\sqrt{25 - x^2} = 0$ when $x = \pm 5$, so $x = -5$ and $x = 5$ are the vertical asymptotes.

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad \frac{\tan x}{\cos 2x + 1} &= \frac{\frac{\sin x}{\cos x}}{2 \cos^2 x - 1 + 1} \\
 &= \frac{\frac{\sin x}{\cos x}}{2 \cos^2 x} \\
 &= \frac{\sin x}{2 \cos^3 x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Shaded area} &= \int_0^{\frac{\pi}{3}} \frac{\tan x}{\cos 2x + 1} dx \\
 &= \int_0^{\frac{\pi}{3}} \frac{\sin x}{2 \cos^3 x} dx \quad \{\text{using a}\}
 \end{aligned}$$

$$= \int_1^{\frac{1}{2}} \frac{1}{2u^3} \left(-\frac{du}{dx} \right) dx \quad \left\{ \begin{array}{l} u = \cos x \quad \therefore \frac{du}{dx} = -\sin x \\ \text{When } x = 0, \quad u = 1 \\ \text{When } x = \frac{\pi}{3}, \quad u = \frac{1}{2} \end{array} \right\}$$

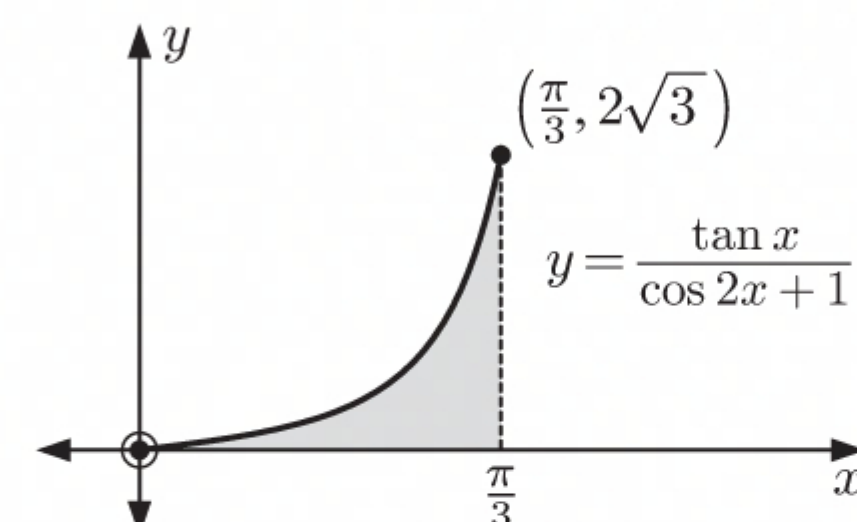
$$= \int_1^{\frac{1}{2}} -\frac{1}{2} u^{-3} du$$

$$= \left[\frac{1}{4} u^{-2} \right]_1^{\frac{1}{2}}$$

$$= \frac{1}{4(\frac{1}{2})^2} - \frac{1}{4(1)^2}$$

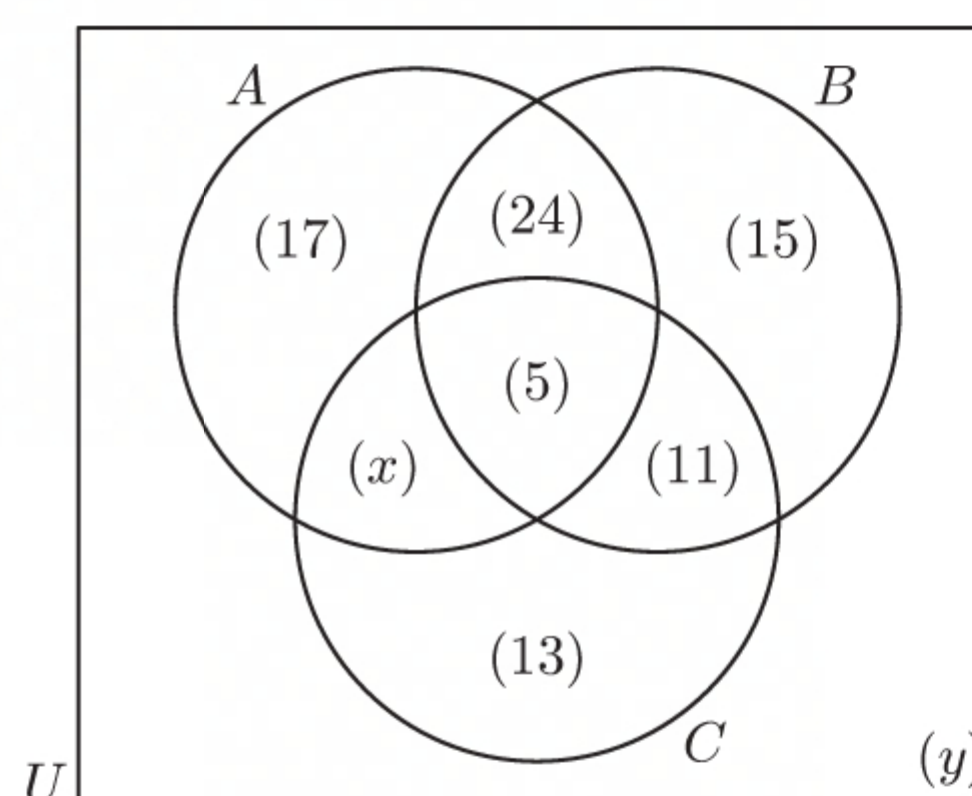
$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4} \text{ units}^2$$



$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad n(A) &= 48 \\
 \therefore 17 + 24 + 5 + x &= 48 \\
 \therefore 46 + x &= 48 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } n(U) &= 100 \\
 \therefore 17 + 24 + 5 + 2 + 15 + 11 + 13 + y &= 100 \\
 \therefore 87 + y &= 100 \\
 \therefore y &= 13
 \end{aligned}$$



$$\mathbf{b} \quad n(A) = 48, \quad n(B) = 24 + 5 + 15 + 11 = 55, \quad n(C) = 2 + 5 + 11 + 13 = 31$$

\therefore course B was the most popular.

$$\begin{aligned}
 \mathbf{c} \quad \mathbf{i} \quad P(\text{liked all the courses}) &= \frac{5}{100} \\
 &= \frac{1}{20}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad P(B \cap C') &= \frac{24 + 15}{100} \\
 &= \frac{39}{100}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iii} \quad P(\text{liked exactly 2 courses} \mid C) &= \frac{n(\text{liked } C \text{ and exactly one other course})}{n(C)} \\
 &= \frac{2 + 11}{2 + 5 + 11 + 13} \\
 &= \frac{13}{31}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iv} \quad P(\text{liked none} \mid B') &= \frac{n(\text{liked none})}{n(B')} \\
 &= \frac{13}{17 + 2 + 13 + 13} \\
 &= \frac{13}{45}
 \end{aligned}$$

6 a When $x = \sin^2 \theta$, $4x^2 - 4x + \sin^2 2\theta$

$$\begin{aligned} &= 4\sin^4 \theta - 4\sin^2 \theta + \sin^2 2\theta \\ &= 4\sin^4 \theta - 4\sin^2 \theta + 4\sin^2 \theta \cos^2 \theta \\ &= 4\sin^2 \theta (\sin^2 \theta - 1 + \cos^2 \theta) \\ &= 4\sin^2 \theta (1 - 1) \\ &= 0 \quad \checkmark \end{aligned}$$

When $x = \cos^2 \theta$, $4x^2 - 4x + \sin^2 2\theta$

$$\begin{aligned} &= 4\cos^4 \theta - 4\cos^2 \theta + \sin^2 2\theta \\ &= 4\cos^4 \theta - 4\cos^2 \theta + 4\sin^2 \theta \cos^2 \theta \\ &= 4\cos^2 \theta (\cos^2 \theta - 1 + \sin^2 \theta) \\ &= 4\cos^2 \theta (1 - 1) \\ &= 0 \quad \checkmark \end{aligned}$$

b From **a**, $x = \sin^2 \theta$ and $x = \cos^2 \theta$ are the roots of $4x^2 - 4x + \sin^2 2\theta = 0$.

Now the product of roots $= \frac{\sin^2 2\theta}{4}$

$$\therefore \sin^2 \theta \cos^2 \theta = \frac{\sin^2 2\theta}{4}$$

When $\theta = \frac{\pi}{8}$, $\sin^2\left(\frac{\pi}{8}\right) \times \cos^2\left(\frac{\pi}{8}\right) = \frac{\sin^2\left(\frac{\pi}{4}\right)}{4}$

$$\begin{aligned} &= \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{4} \\ &= \frac{1}{8} \end{aligned}$$

7 a $f(x) = \sqrt{1+2x}$

$$= (1+2x)^{\frac{1}{2}}$$

$$= \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} (2x)^r$$

$$= 1 + \left(\frac{1}{2}\right)(2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} (2x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} (2x)^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!} (2x)^4 + \dots$$

$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \dots$$

b Letting $x = -0.05$, $\sqrt{0.9} \approx 1 + (-0.05) - \frac{1}{2}(-0.05)^2 + \frac{1}{2}(-0.05)^3 - \frac{5}{8}(-0.05)^4$

$$\approx 0.948684$$

c $f'(x) = \frac{d}{dx} \left(1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \dots\right)$

$$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \dots$$

d $f(x) \times f'(x) = \left(1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \dots\right) \times \left(1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \dots\right)$

$$\begin{aligned} &= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \dots \\ &\quad + x - x^2 + \frac{3}{2}x^3 - \dots \\ &\quad - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \dots \\ &\quad + \frac{1}{2}x^3 - \dots \\ &= 1 + 0x + 0x^2 + 0x^3 + \dots \\ &= 1 \end{aligned}$$

Now using the rules of differentiation $f'(x) = \frac{1}{2}(1+2x)^{-\frac{1}{2}}(2)$

$$= \frac{1}{\sqrt{1+2x}}$$

$$\therefore f(x) \times f'(x) = \sqrt{1+2x} \times \frac{1}{\sqrt{1+2x}}$$

$$= 1 \quad \checkmark$$

8 a i The equation can be written as $\frac{x-1}{2} = \frac{3-y}{3} = z = t$, $t \in \mathbb{R}$

$$\therefore x = 2t + 1, y = -3t + 3, z = t, \quad t \in \mathbb{R}$$

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$

\therefore a vector parallel to the line is $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

ii Letting $t = 3$, the point on the line with z -coordinate 3 is $(7, -6, 3)$.

iii The point $(7, -3, 2)$ lies on the line if $7 = 2t + 1$ (1)

$$-3 = -3t + 3 \quad \dots (2)$$

$$2 = t \quad \dots (3)$$

So, from (2) and (3), $t = 2$ and from (1), $t = 3$ which is not possible.

Thus the point $(7, -3, 2)$ does not lie on the line.

b Note: There are many possible answers.

Since $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, a possible line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $s \in \mathbb{R}$ which is

$$x = 5, y = -3 + s, z = 2 + 3s, s \in \mathbb{R}.$$

c If the lines in **a** and **b** are to meet, then $5 = 2t + 1$ (1)

$$-3 + s = -3t + 3 \quad \dots (2)$$

$$2 + 3s = t \quad \dots (3)$$

and from these we see that $t = 2$ and $s = 0$.

So, the lines meet at the point $(5, -3, 2)$.

d The line in **a** has direction vector $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$. Line L has direction vector $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

$$\text{If the acute angle between the lines is } \theta, \text{ then } \left| \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right| = \sqrt{4 + 9 + 1} \sqrt{1 + 4 + 1} \cos \theta$$

$$\therefore |-2 - 6 + 1| = \sqrt{14} \sqrt{6} \cos \theta$$

$$\therefore \cos \theta = \frac{7}{\sqrt{84}} \quad \text{and so } \theta \approx 40.2^\circ$$

$$9 \quad f(x) = \arcsin x + a\sqrt{1-x^2}$$

a Domain = $\{x \mid -1 \leq x \leq 1\}$

$$\mathbf{b} \quad f(1) = \arcsin 1 + a\sqrt{1-1^2} = \frac{\pi}{2}$$

$$f(-1) = \arcsin(-1) + a\sqrt{1-(-1)^2} = -\frac{\pi}{2}$$

\therefore the coordinates of f at the endpoints of its domain are $(-1, -\frac{\pi}{2})$ and $(1, \frac{\pi}{2})$.

$$\mathbf{c} \quad f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{a}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{ax}{\sqrt{1-x^2}}$$

$$= \frac{1-ax}{\sqrt{1-x^2}}$$

f has a stationary point at $x = \frac{1}{2}$.

$$\therefore f'(\frac{1}{2}) = 0$$

$$\therefore \frac{1-\frac{a}{2}}{\sqrt{1-\frac{1}{4}}} = 0$$

$$\therefore 1 - \frac{a}{2} = 0$$

$$\therefore \frac{a}{2} = 1$$

$$\therefore a = 2$$

$$\therefore f(x) = \arcsin x + 2\sqrt{1-x^2}$$

$$\therefore f(\frac{1}{2}) = \arcsin \frac{1}{2} + 2\sqrt{1-\frac{1}{4}}$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{2}$$

$$= \frac{\pi}{6} + \sqrt{3}$$

\therefore the stationary point has coordinates $(\frac{1}{2}, \frac{\pi}{6} + \sqrt{3})$.

$$\begin{aligned}
 \text{d } f''(x) &= \frac{d}{dx} \left(\frac{1-2x}{\sqrt{1-x^2}} \right) \quad \{\text{from c}\} \\
 &= \frac{-2\sqrt{1-x^2} - \left(\frac{-x}{\sqrt{1-x^2}} \right)(1-2x)}{1-x^2} \quad \{\text{quotient rule}\} \\
 &= \frac{-2(1-x^2) + x(1-2x)}{(1-x)^{\frac{3}{2}}} \\
 &= \frac{-2 + \cancel{2x^2} + x - \cancel{2x^2}}{(1-x)^{\frac{3}{2}}} \\
 &= \frac{x-2}{(1-x)^{\frac{3}{2}}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } -1 \leq x \leq 1 & \quad \text{and} \quad -1 \leq x \leq 1 \\
 \therefore -3 \leq x-2 \leq -1 & \quad \therefore -1 \leq -x \leq 1 \\
 \therefore x-2 < 0 \text{ for all } -1 \leq x \leq 1 & \quad \therefore 0 \leq 1-x \leq 2 \\
 & \quad \therefore 1-x \geq 0 \text{ for all } -1 \leq x \leq 1 \\
 & \quad \therefore (1-x)^{\frac{3}{2}} \geq 0 \text{ for all } -1 \leq x \leq 1
 \end{aligned}$$

$$\therefore f''(x) = \frac{x-2}{(1-x)^{\frac{3}{2}}} < 0 \text{ for all } -1 \leq x \leq 1$$

$\therefore f$ is concave down over its entire domain.

$$\begin{aligned}
 \text{10 a } \tan 2x(1 - \tan^2 x) &= \frac{2 \tan x}{1 - \tan^2 x} (1 - \tan^2 x) \\
 &= 2 \tan x
 \end{aligned}$$

$$\text{b } \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

$$\begin{aligned}
 \text{i } \tan 2x &= (2x) + \frac{1}{3}(2x)^3 + \frac{2}{15}(2x)^5 + \dots \\
 &= 2x + \frac{8}{3}x^3 + \frac{64}{15}x^5 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } 1 - \tan^2 x &= 1 - \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \right) \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \right) \\
 &= 1 - \left(x^2 + \frac{1}{3}x^4 + \frac{2}{15}x^6 + \dots \right. \\
 &\quad \left. + \frac{1}{3}x^4 + \frac{1}{9}x^6 + \dots \right) \\
 &= 1 - x^2 - \frac{2}{3}x^4 - \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \tan 2x(1 - \tan^2 x) &= \left(2x + \frac{8}{3}x^3 + \frac{64}{15}x^5 + \dots \right) \left(1 - x^2 - \frac{2}{3}x^4 - \dots \right) \quad \{\text{using b}\} \\
 &= 2x - 2x^3 - \frac{4}{3}x^5 - \dots \\
 &\quad + \frac{8}{3}x^3 - \frac{8}{3}x^5 - \dots \\
 &\quad + \frac{64}{15}x^5 - \dots \\
 &= 2x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \dots \\
 &= 2 \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \right) \\
 &= 2 \tan x \quad \checkmark
 \end{aligned}$$

MIXED QUESTIONS SET 13

1 a There are $n = 5 \times 4 = 20$ time periods.

Each time period the investment increases by $i = \frac{4.4\%}{4} = 1.1\%$.

$$\begin{aligned}
 \therefore \text{the amount after 5 years is } u_{20} &= u_0 \times (1+i)^{20} \\
 &= 2000 \times (1.011)^{20} \quad \{1.1\% = 0.011\} \\
 &\approx 2489.16
 \end{aligned}$$

The final value of the investment is \$2489.16.

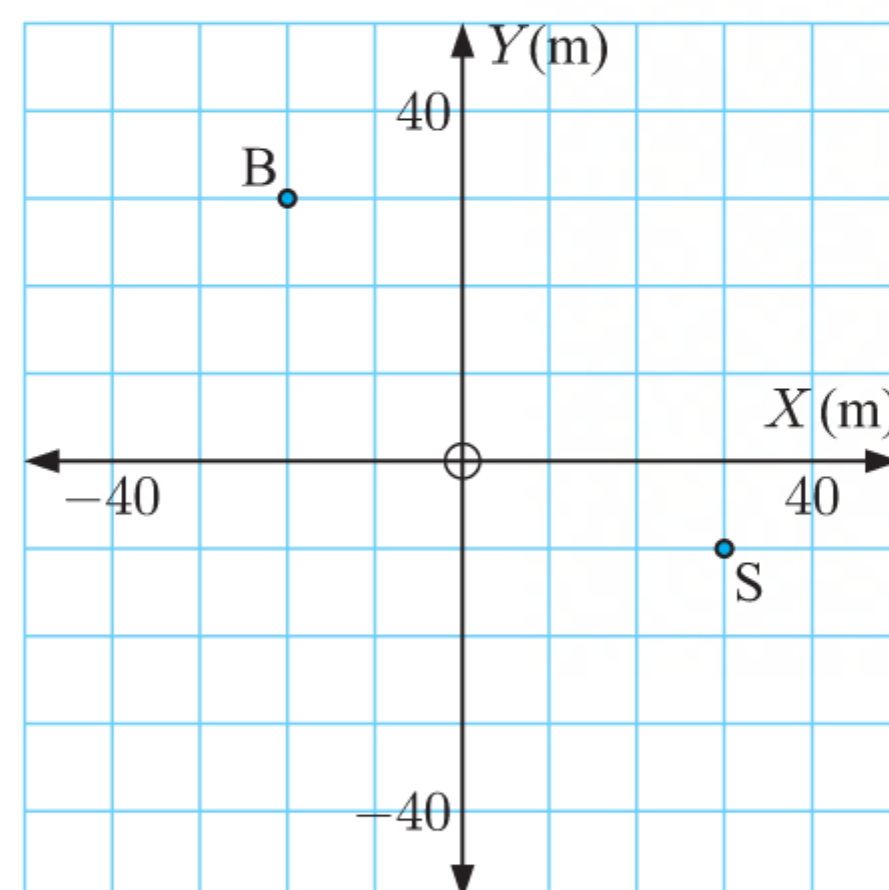
$$\begin{aligned}
 \text{b } \text{Interest} &= \$2489.16 - \$2000 \\
 &= \$489.16
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \text{real value} \times (1.025)^5 &= \$2489.16 \\
 \therefore \text{real value} &= \frac{\$2489.16}{(1.025)^5} \\
 &= \$2200.05
 \end{aligned}$$

- 2 a i** The anchor has coordinates $A(-20, 30, -50)$.
ii The shipwreck has coordinates $S(30, -10, -40)$.

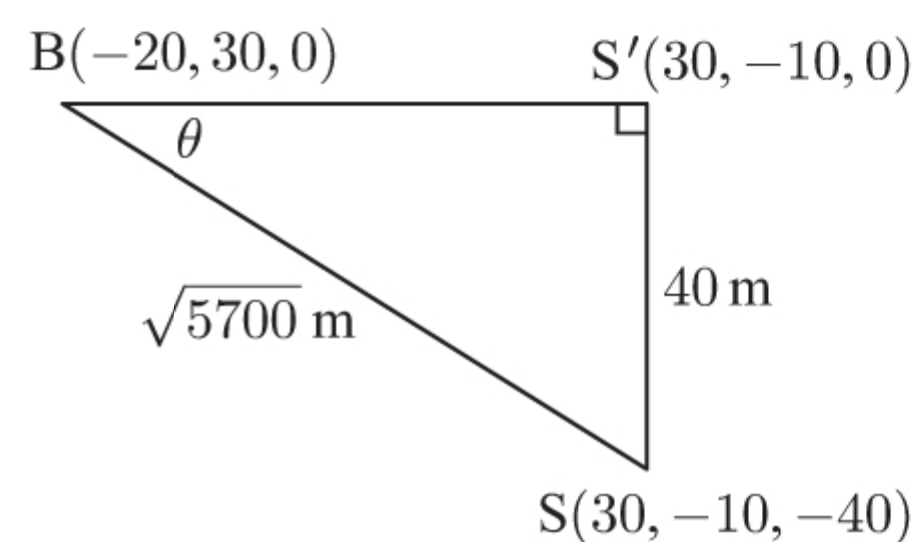
$$\begin{aligned} \mathbf{b} \quad BS &= \sqrt{(30 - (-20))^2 + (-10 - 30)^2 + (-40 - 0)^2} \\ &= \sqrt{50^2 + (-40)^2 + (-40)^2} \\ &= \sqrt{5700} \\ &\approx 75.5 \text{ m} \end{aligned}$$

\therefore the diver has to swim about 75.5 m.



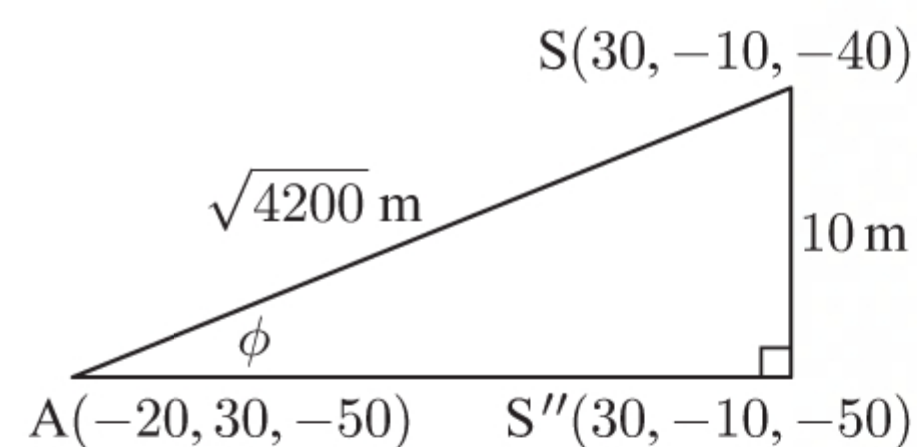
$$\begin{aligned} \mathbf{c} \quad \mathbf{i} \quad \sin \theta &= \frac{40}{\sqrt{5700}} \\ \therefore \theta &= \sin^{-1}\left(\frac{40}{\sqrt{5700}}\right) \\ \therefore \theta &\approx 32.0^\circ \end{aligned}$$

The angle of depression from the boat to the shipwreck is about 32.0° .



$$\begin{aligned} \mathbf{ii} \quad AS &= \sqrt{(-20 - 30)^2 + (30 - (-10))^2 + (-50 - (-40))^2} \\ &= \sqrt{(-50)^2 + 40^2 + (-10)^2} \\ &= \sqrt{4200} \text{ m} \\ \therefore \sin \phi &= \frac{10}{\sqrt{4200}} \\ \therefore \phi &= \sin^{-1}\left(\frac{10}{\sqrt{4200}}\right) \\ \therefore \phi &\approx 8.88^\circ \end{aligned}$$

The angle of elevation from the anchor to the shipwreck is about 8.88° .

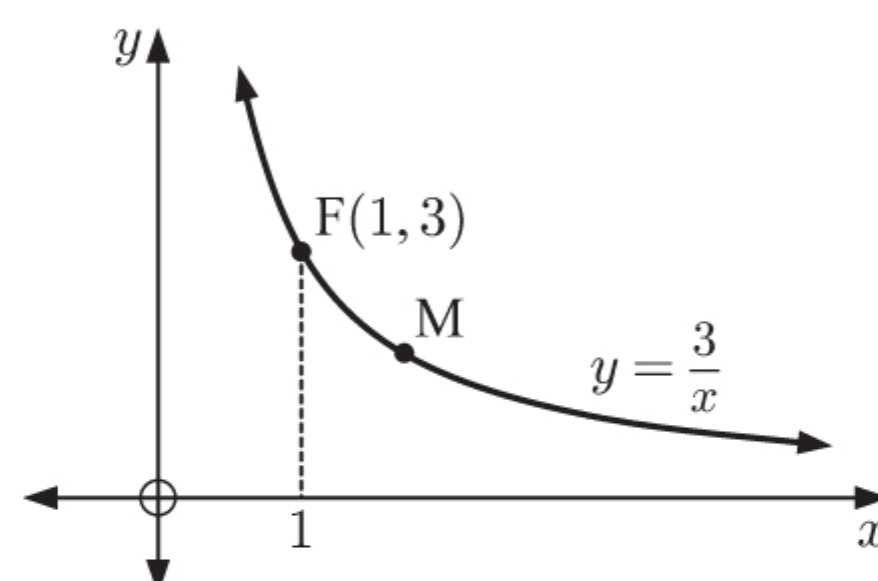


- 3 a** M has x -coordinate $1 + h$.
 \therefore M has y -coordinate $\frac{3}{1+h}$.

$$\begin{aligned} \mathbf{b} \quad 3 - \frac{3}{h+1} &= 3\left(\frac{h+1}{h+1}\right) - \frac{3}{h+1} \\ &= \frac{3h+3-3}{h+1} \\ &= \frac{3h}{h+1} \end{aligned}$$

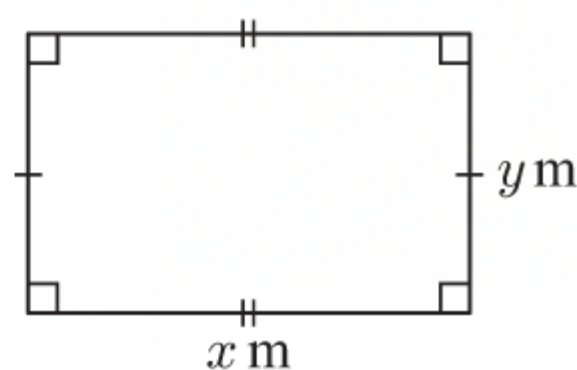
$$\begin{aligned} \mathbf{c} \quad \text{gradient of [FM]} &= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{3 - \frac{3}{1+h}}{1 - (1+h)} \\ &= \frac{3h}{h+1} \times \frac{1}{-h} \quad \{\text{using b}\} \\ &= \frac{-3}{h+1} \quad \{h \neq 0\} \end{aligned}$$

- d** As $h \rightarrow 0$, gradient of [FM] \rightarrow gradient of tangent at F
 \therefore gradient of tangent at F $= \frac{-3}{1} = -3$.



- 4 a** Let the other side of the field be y m.

$$\begin{aligned} 2x + 2y &= 160 \\ \therefore 2y &= 160 - 2x \\ \therefore y &= 80 - x \\ \therefore \text{area } A &= x(80 - x) \text{ m}^2 \end{aligned}$$



b $A = 80x - x^2$

$$\therefore \frac{dA}{dx} = 80 - 2x$$

Now $\frac{dA}{dx} = 0$ when $80 - 2x = 0$

$$\therefore 2x = 80$$

$$\therefore x = 40$$

The sign diagram of $\frac{dA}{dx}$ is:

$\therefore A$ is maximised when $x = 40$, and $y = 80 - 40 = 40$.

\therefore the maximum area occurs when the field is a $40 \text{ m} \times 40 \text{ m}$ square.

c i $A = 1200$

$$\therefore x(80 - x) = 1200$$

$$\therefore x^2 - 80x + 1200 = 0$$

$$\therefore (x - 60)(x - 20) = 0$$

$$\therefore x = 60 \text{ or } 20$$

When $x = 60$, $y = 80 - 60 = 20$.

When $x = 20$, $y = 80 - 20 = 60$.

\therefore the field is $60 \text{ m} \times 20 \text{ m}$.

ii The maximum area is 1600 m^2 .

We lose $1600 - 1200 = 400 \text{ m}^2$ of productive land

$$\therefore \text{the lost production} = 400 \times 6.5 = 2600 \text{ kg}$$

5

Height (h cm)	Mid-interval value (x)	Frequency (f)	xf
$80 \leq h < 90$	85	8	680
$90 \leq h < 100$	95	12	1140
$100 \leq h < 110$	105	17	1785
$110 \leq h < 120$	115	30	3450
$120 \leq h < 130$	125	13	1625
Total		$\sum f = 80$	$\sum xf = 8680$

a $\bar{x} = \frac{\sum xf}{\sum f}$

$$= \frac{8680}{80}$$

$$= 108.5$$

We estimate the mean height to be about 108.5 cm .

6 P_n is: $2n^3 + 3n^2 + n$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $2(1)^3 + 3(1)^2 + 1 = 6$

$\therefore P_1$ is true.

(2) If P_k is true, then $2k^3 + 3k^2 + k = 6A$ where $A \in \mathbb{Z}$.

$$\begin{aligned} \text{Now } 2(k+1)^3 + 3(k+1)^2 + (k+1) &= 2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) + k + 1 \\ &= 2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3 + k + 1 \\ &= (2k^3 + 3k^2 + k) + 6k^2 + 12k + 6 \\ &= 6A + 6(k^2 + 2k + 2) \quad \{\text{using } P_k\} \\ &= 6(A + k^2 + 2k + 2) \end{aligned}$$

where $A + k^2 + 2k + 2$ is an integer as A is an integer and $k \geq 1$ is an integer.

Thus $2(k+1)^3 + 3(k+1)^2 + (k+1)$ is divisible by 6 whenever $2k^3 + 3k^2 + k$ is divisible by 6.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

b Using technology, $s \approx 12.1 \text{ cm}$.

- 7 a** If there are no restrictions, there are $\binom{16}{6} = 8008$ possible choices.
- b** $\binom{7}{2}\binom{6}{2}\binom{3}{2} + \binom{7}{2}\binom{6}{3}\binom{3}{1} + \binom{7}{2}\binom{6}{4}\binom{3}{0} + \binom{7}{3}\binom{6}{2}\binom{3}{1} + \binom{7}{3}\binom{6}{3}\binom{3}{0} + \binom{7}{4}\binom{6}{2}\binom{3}{0} = 5320$ possible choices
- c** $\binom{1}{1}\binom{3}{1}\binom{12}{4} + \binom{1}{1}\binom{3}{2}\binom{12}{3} + \binom{1}{1}\binom{3}{3}\binom{12}{2} = 2211$ possible choices

- 8** Since the polynomial is real, $1 \pm 2i$ and $\pm ai$ are zeros, $a \neq 0$.

$1 \pm 2i$ have sum 2 and product $1 + 4 = 5$ and so come from the quadratic factor $z^2 - 2z + 5$

$\pm ai$ have sum 0 and product a^2 and so come from the quadratic factor $z^2 + a^2$

$$\therefore P(z) = k(z^2 - 2z + 5)(z^2 + a^2), \quad k \neq 0$$

$$\text{But } k = 1 \text{ and } P(0) = 1(5)(a^2) = 10 \text{ so } a^2 = 2$$

$$\therefore P(z) = (z^2 - 2z + 5)(z^2 + 2)$$

- 9 a** If $x = 1 + \lambda$, $y = -1 + a\lambda$, $z = 2 - \lambda$ lies on $3x - ky + z = 3$, then

$$3(1 + \lambda) - k(-1 + a\lambda) + 2 - \lambda = 3 \quad \text{for all } \lambda$$

$$\therefore 3 + 3\lambda + k - ak\lambda + 2 - \lambda = 3$$

$$\therefore (3 + k + 2) + \lambda(3 - ak - 1) = 3$$

$$\therefore k + 5 = 3 \quad \text{and} \quad 2 - ak = 0$$

$$\therefore k = -2 \quad \text{and} \quad 2 + 2a = 0$$

$$\therefore k = -2 \quad \text{and} \quad a = -1$$

- b** P_1 has normal vector $\mathbf{n}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and P_2 has normal vector $\mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$.

$$\text{Now } \mathbf{n}_1 \bullet \mathbf{n}_2 = 6 - 2 - 4 = 0$$

$$\therefore \mathbf{n}_1 \perp \mathbf{n}_2 \quad \text{and so} \quad P_1 \perp P_2.$$

- c** We need to solve $\begin{cases} 2x - y - 4z = 9 \\ 3x + 2y + z = 3 \end{cases}$

$$\begin{pmatrix} 2 & -1 & -4 & | & 9 \\ 3 & 2 & 1 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -4 & | & 9 \\ 0 & 7 & 14 & | & -21 \end{pmatrix} \quad 2R_2 - 3R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc|c} 6 & 4 & 2 & 6 \\ -6 & 3 & 12 & -27 \\ 0 & 7 & 14 & -21 \end{array} \right\}$$

The second equation simplifies to $y + 2z = -3$

$$\text{If } z = t \text{ then } y = -3 - 2t \text{ and } 2x + 3 + 2t - 4t = 9$$

$$\therefore 2x = 6 + 2t$$

$$\therefore x = 3 + t$$

So, L_2 has equation $x = 3 + t$, $y = -3 - 2t$, $z = t$, $t \in \mathbb{R}$.

- d** L_1 and L_2 meet when $1 + \lambda = 3 + t$, $-1 - \lambda = -3 - 2t$, and $2 - \lambda = t$
- $$\therefore \lambda = 2 + t, \quad \lambda = 2 + 2t, \quad \lambda = 2 - t$$
- $$\therefore 2 + t = 2 + 2t = 2 - t$$

The common solution is $t = 0$, so L_1 and L_2 meet at $(3, -3, 0)$.

- e** L_1 has direction vector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and L_2 has direction vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$$\cos \theta = \frac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{|1 + 2 - 1|}{\sqrt{3}\sqrt{6}} = \frac{2}{\sqrt{18}}$$

$$\therefore \theta = \arccos\left(\frac{2}{\sqrt{18}}\right) \approx 61.9^\circ$$

- 10 a** A stable (equilibrium) population occurs when $\frac{dP}{dt} = 0$.

$$\text{At this time, } 1 - \frac{P}{5160} = 0 \quad \therefore P = 5160.$$

Therefore the equilibrium number of snakes (the carrying capacity) is 5160.

$$\begin{aligned}
 \mathbf{b} \quad & \frac{1}{P} \frac{dP}{dt} = \frac{1}{100} \left(1 - \frac{P}{5160} \right) \\
 \therefore & \frac{1}{P} \frac{dP}{dt} = \frac{1}{100} \left(\frac{5160 - P}{5160} \right) \\
 \therefore & \frac{5160}{P(5160 - P)} \frac{dP}{dt} = \frac{1}{100} \\
 \therefore & \int \left(\frac{1}{P} + \frac{1}{5160 - P} \right) \frac{dP}{dt} dt = \int \frac{1}{100} dt \\
 \therefore & \int \left(\frac{1}{P} + \frac{1}{5160 - P} \right) dP = \frac{1}{100} \int dt \\
 \therefore & \ln |P| - \ln |5160 - P| = \frac{1}{100} t + c \\
 \therefore & \ln \left| \frac{P}{5160 - P} \right| = \frac{1}{100} t + c
 \end{aligned}$$

But when $t = 0$, $P = 2280$,

$$\begin{aligned}
 \text{so } c &= \ln \left| \frac{2280}{5160 - 2280} \right| \\
 &= \ln \left(\frac{19}{24} \right) \\
 \therefore \ln \left| \frac{P}{5160 - P} \right| &= \frac{1}{100} t + \ln \left(\frac{19}{24} \right) \\
 \therefore \frac{P}{5160 - P} &= e^{\frac{1}{100} t + \ln(\frac{19}{24})} = \frac{19}{24} e^{\frac{1}{100} t} \\
 \therefore \frac{5160 - P}{P} &= \frac{24}{19} e^{-\frac{1}{100} t} \\
 \therefore \frac{5160}{P} - 1 &= \frac{24}{19} e^{-\frac{1}{100} t} \\
 \therefore \frac{5160}{P} &= 1 + \frac{24}{19} e^{-\frac{1}{100} t} \\
 \therefore P &= \frac{5160}{1 + \frac{24}{19} e^{-\frac{1}{100} t}}
 \end{aligned}$$

\mathbf{c} Using the equation obtained in \mathbf{b} , when $P = 4000$,

$$\begin{aligned}
 \ln \left| \frac{4000}{5160 - 4000} \right| &= \frac{1}{100} t + \ln \left(\frac{19}{24} \right) \\
 \therefore \frac{1}{100} t &= \ln \left(\frac{100}{29} \right) - \ln \left(\frac{19}{24} \right) \\
 \therefore t &= 100 \ln \left(\frac{100}{29} \times \frac{24}{19} \right) \\
 \therefore t &\approx 147 \text{ years}
 \end{aligned}$$

MIXED QUESTIONS SET 14

$\mathbf{1}$ When $x = 0$, $y = 0$

$$\begin{aligned}
 \therefore 0 &= ke^0 + b \\
 \therefore k &= -b \quad \dots (*)
 \end{aligned}$$

Now shaded area = $\frac{3}{e^6}$ units²

$$\therefore - \int_{-3}^0 (ke^{2x} + b) dx = \frac{3}{e^6}$$

$$\therefore \int_{-3}^0 (ke^{2x} + b) dx = -3e^{-6}$$

$$\therefore \left[\frac{k}{2} e^{2x} + bx \right]_{-3}^0 = -3e^{-6}$$

$$\therefore \left(\frac{k}{2} e^0 + 0 \right) - \left(\frac{k}{2} e^{-6} - 3b \right) = -3e^{-6}$$

$$\therefore \frac{k}{2} - \frac{k}{2} e^{-6} + 3b = -3e^{-6}$$

$$\therefore -\frac{k}{2} e^{-6} + \left(\frac{k}{2} + 3b \right) = -3e^{-6}$$

$$\therefore \frac{b}{2} e^{-6} + \left(-\frac{b}{2} + 3b \right) = -3e^{-6} \quad \{\text{using } (*)\}$$

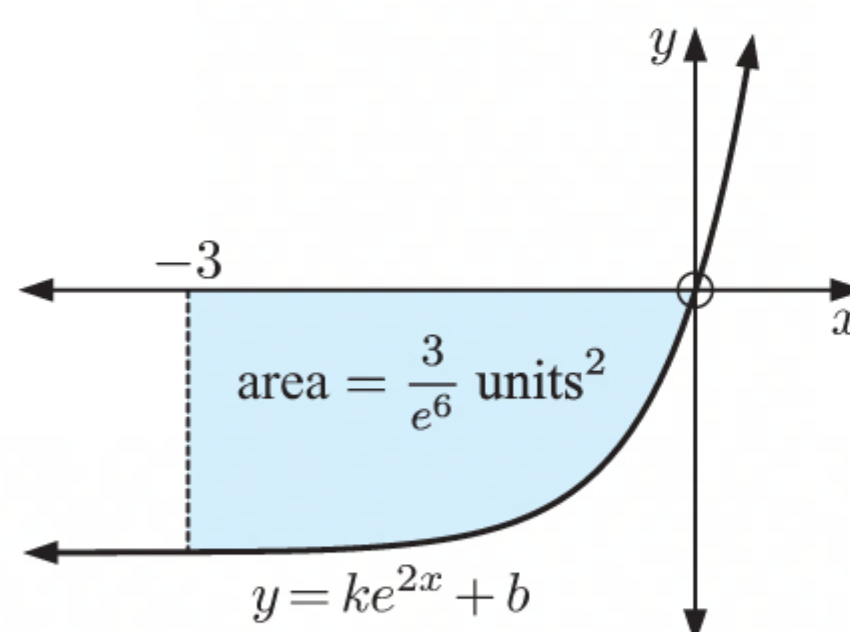
$$\therefore \frac{b}{2} e^{-6} + \frac{5}{2} b = -3e^{-6}$$

$$\therefore b + 5be^6 = -6$$

$$\therefore b(1 + 5e^6) = -6$$

$$\therefore b = \frac{-6}{1 + 5e^6}$$

$$\therefore k = \frac{6}{1 + 5e^6} \quad \{\text{using } (*)\}$$

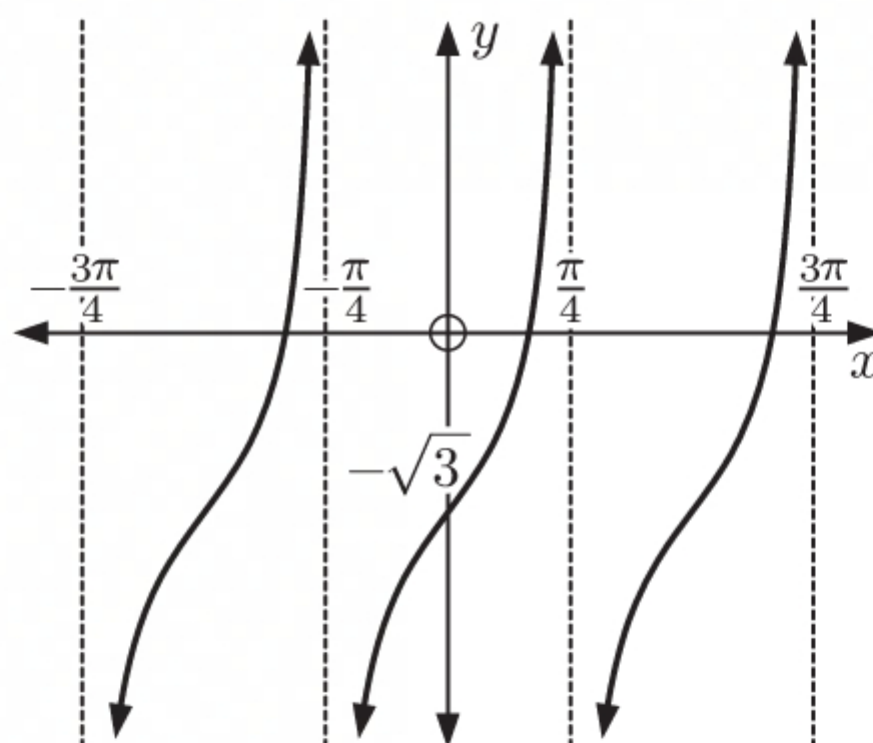


2 a The period is $\frac{\pi}{a} = \frac{\pi}{2}$
 $\therefore a = 2$

The y -intercept is $-\sqrt{3}$.

$\therefore b = -\sqrt{3}$

$\therefore a = 2, b = -\sqrt{3}$



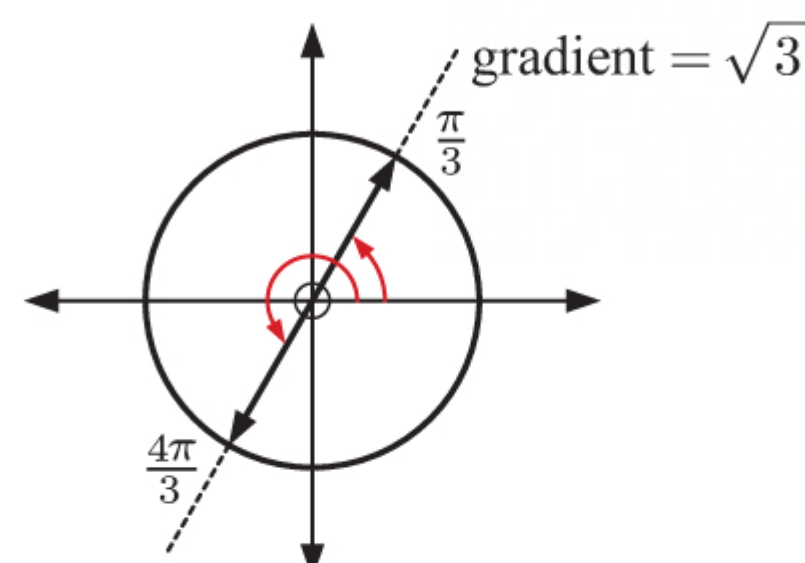
b x -intercepts occur when $\tan 2x - \sqrt{3} = 0$
 $\therefore \tan 2x = \sqrt{3}$

Now, if $-\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4}$,

then $-\frac{3\pi}{2} \leq 2x \leq \frac{3\pi}{2}$

$\therefore 2x = -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}$

$\therefore x = -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$



3 $P(B) = 0.3$ and $P(A \cup B) = 0.55$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ {addition law of probability}

$\therefore P(A \cup B) = P(A) + P(B)$ { A and B are mutually exclusive}

$\therefore 0.55 = P(A) + 0.3$

$\therefore P(A) = 0.25$

4 $3x^2 + 2x \xrightarrow[\text{scale factor 2}]{\text{vertical stretch}} 2(3x^2 + 2x) \xrightarrow[\text{translation } \begin{pmatrix} 3 \\ -1 \end{pmatrix}]{\text{translation}} 2(3(x-3)^2 + 2(x-3)) - 1$

The image has equation

$y = 2(3(x-3)^2 + 2(x-3)) - 1$

$\therefore y = 2(3(x^2 - 6x + 9) + 2x - 6) - 1$

$\therefore y = 2(3x^2 - 18x + 27 + 2x - 6) - 1$

$\therefore y = 2(3x^2 - 16x + 21) - 1$

$\therefore y = 6x^2 - 32x + 42 - 1$

$\therefore y = 6x^2 - 32x + 41$

5 a $f(\theta) = \frac{2 - \cos \theta}{\sin \theta}, 0 < \theta \leq \frac{\pi}{2}$

$\therefore f'(\theta) = \frac{\sin \theta \times \sin \theta - (2 - \cos \theta) \cos \theta}{\sin^2 \theta}$ {quotient rule}

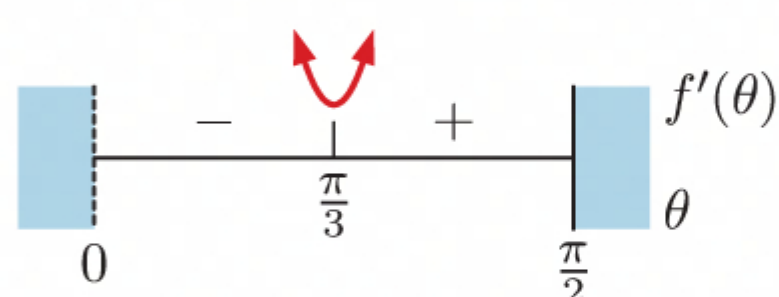
$= \frac{\sin^2 \theta - 2 \cos \theta + \cos^2 \theta}{\sin^2 \theta}$

$= \frac{1 - 2 \cos \theta}{\sin^2 \theta}$

b $f'(\theta) = 0$ when $\cos \theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3} \quad \{0 < \theta \leq \frac{\pi}{2}\}$

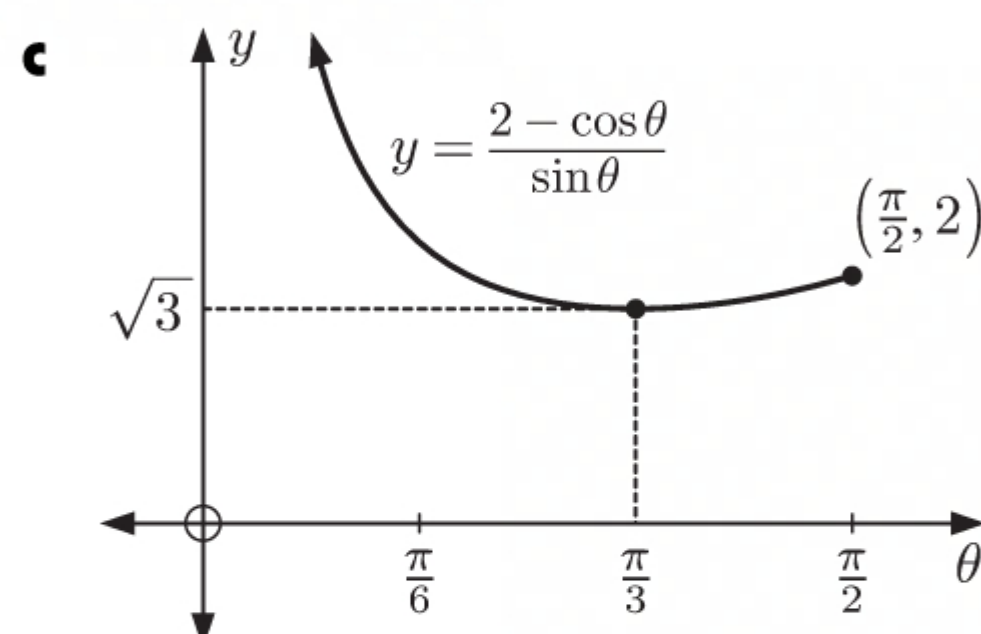
The sign diagram for $f'(\theta)$ is:



$\therefore f(\theta)$ is a minimum when $\theta = \frac{\pi}{3}$.

Now $f(\frac{\pi}{3}) = \frac{2 - \cos \frac{\pi}{3}}{\sin \frac{\pi}{3}} = \frac{2 - \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$

\therefore the minimum value is $\sqrt{3}$ when $\theta = \frac{\pi}{3}$.



- 6 a** $P(\text{win}) = P(\text{1st ball red} \cap \text{2nd ball red} \cap \text{3rd ball red})$

$$= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}$$

$$= \frac{1}{6}$$

- b** If X is the number of wins when the game is played 60 times, then $X \sim B(60, \frac{1}{6})$.

i $\mu = np$ $\sigma = \sqrt{np(1-p)}$

$$= 60(\frac{1}{6})$$

$$= 10$$

$$= \sqrt{60(\frac{1}{6})(\frac{5}{6})}$$

$$= \sqrt{\frac{25}{3}}$$

$$\approx 2.89$$

ii $P(X = \mu) = P(X = 10)$

$$= \binom{60}{10} (\frac{1}{6})^{10} (\frac{5}{6})^{50}$$

$$\approx 0.137$$

iii $P(\mu - \sigma \leq X \leq \mu + \sigma) = P(10 - \sqrt{\frac{25}{3}} \leq X \leq 10 + \sqrt{\frac{25}{3}})$

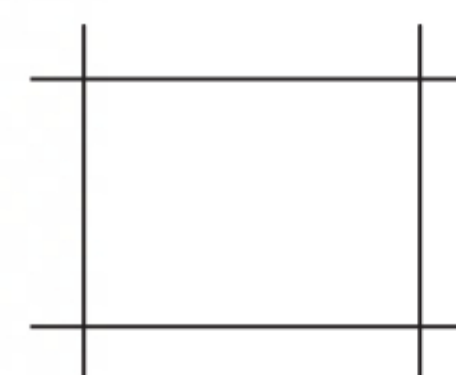
$$= P(7.11 \leq X \leq 12.9)$$

$$= P(8 \leq X \leq 12)$$

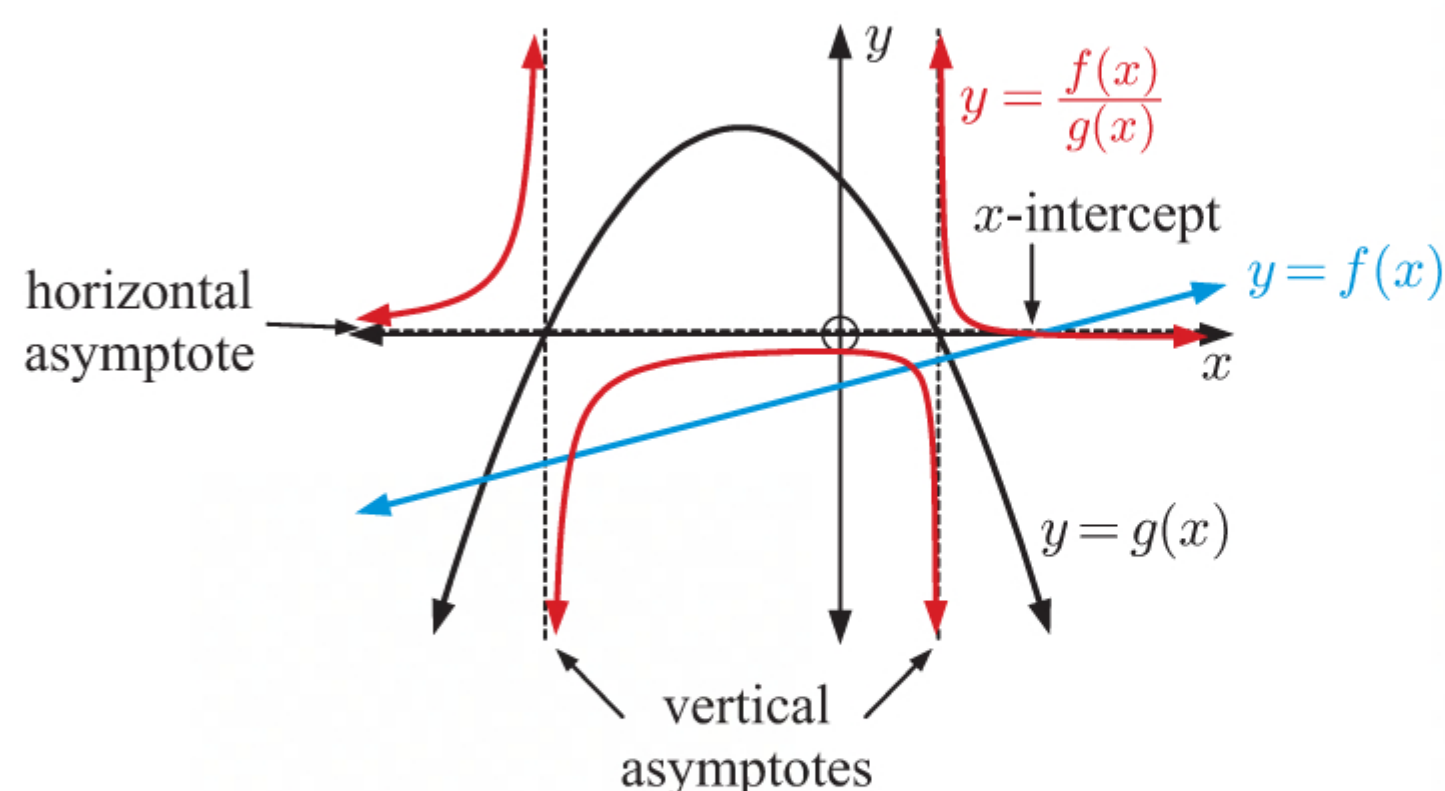
$$\approx 0.614 \quad \{\text{using technology}\}$$

- 7** Each rectangle is determined by choosing the two pairs of opposite sides.

This can be done in $\binom{m+2}{2} \times \binom{n+2}{2} = \frac{(m+2)!}{2!m!} \times \frac{(n+2)!}{2!n!} = \frac{(m+2)(m+1)(n+2)(n+1)}{4}$ ways.



8



- 9 a** The system has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ m & 1 & 0 & 1 \\ 2 & -5 & m-2 & -3 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1-3m & m & 1-2m \\ 0 & -11 & m & -7 \end{array} \right) \quad \begin{array}{l} R_2 - mR_1 \rightarrow R_2 \\ R_3 - 2R_1 \rightarrow R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -11 & m & -7 \\ 0 & 1-3m & m & 1-2m \end{array} \right) \quad R_2 \leftrightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -11 & m & -7 \\ 0 & 0 & -3m^2 + 12m & 4-m \end{array} \right) \quad 11R_3 + (1-3m)R_2 \rightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -11 & m & -7 \\ 0 & 0 & 3m(4-m) & 4-m \end{array} \right)$$

$$\left\{ \begin{array}{ccc|c} m & 1 & 0 & 1 \\ -m & -3m & m & -2m \\ 0 & 1-3m & m & 1-2m \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 2 & -5 & m-2 & -3 \\ -2 & -6 & 2 & -4 \\ 0 & -11 & m & -7 \end{array} \right\}$$

$$\left\{ \begin{array}{ccc|c} 0 & 11(1-3m) & 11m & 11(1-2m) \\ 0 & -11(1-3m) & m(1-3m) & -7(1-3m) \\ 0 & 0 & -3m^2 + 12m & 4-m \end{array} \right\}$$

The system has no solutions if $3m(4-m) = 0$ but $4-m \neq 0$

$$\therefore 3m = 0$$

$$\therefore m = 0$$

In this case, the system is
$$\begin{cases} x + 3y - z = 2 \\ y = 1 \\ 2x - 5y - 2z = -3 \end{cases}$$

\therefore none of the planes are parallel.

\therefore the intersection of any two planes is parallel to the third plane.

- b** If $m = 4$, the last row is all zeros, indicating infinitely many solutions.

$$\begin{aligned} \text{Letting } z = t \text{ in row 2 gives } -11y + 4t &= -7 & \text{Using row 1, } x + 3\left(\frac{7+4t}{11}\right) - t &= 2 \\ \therefore -11y &= -7 - 4t & \therefore x + \frac{t}{11} &= \frac{1}{11} \\ \therefore y &= \frac{7+4t}{11} & \therefore x &= \frac{1-t}{11} \end{aligned}$$

We have infinitely many solutions of the form: $x = \frac{1-t}{11}$, $y = \frac{7+4t}{11}$, $z = t$, where $t \in \mathbb{R}$.

In this case we have three planes which meet in a line.

- c i** The system has a unique solution when $m \neq 0$ and $m \neq 4$.

$$\begin{aligned} \text{Using row 3, } 3m(4-m)z &= 4-m \\ \therefore z &= \frac{4-m}{3m(4-m)} \\ \therefore z &= \frac{1}{3m} \quad \{m \neq 4\} \end{aligned}$$

$$\begin{aligned} \text{Using row 2, } -11y + m\left(\frac{1}{3m}\right) &= -7 \\ \therefore -11y + \frac{1}{3} &= -7 \\ \therefore -11y &= -\frac{22}{3} \\ \therefore y &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{Using row 1, } x + 3\left(\frac{2}{3}\right) - \frac{1}{3m} &= 2 \\ \therefore x + 2 - \frac{1}{3m} &= 2 \\ \therefore x &= \frac{1}{3m} \end{aligned}$$

In this case we have three planes which intersect at $\left(\frac{1}{3m}, \frac{2}{3}, \frac{1}{3m}\right)$.

- ii** The unique solution has the form $x = y = z = k$ for some $k \in \mathbb{R}$ if $\frac{1}{3m} = \frac{2}{3}$
 $\therefore 3m = \frac{3}{2}$
 $\therefore m = \frac{1}{2}$

- 10** Let A and B be the distances travelled by the 28 km h^{-1} and the 32 km h^{-1} roadrunners respectively.

$$\begin{aligned} \text{Using the cosine rule, } D^2 &= A^2 + B^2 - 2AB \cos 45^\circ \\ \therefore D^2 &= A^2 + B^2 - \sqrt{2}AB \end{aligned}$$

Differentiating with respect to t gives

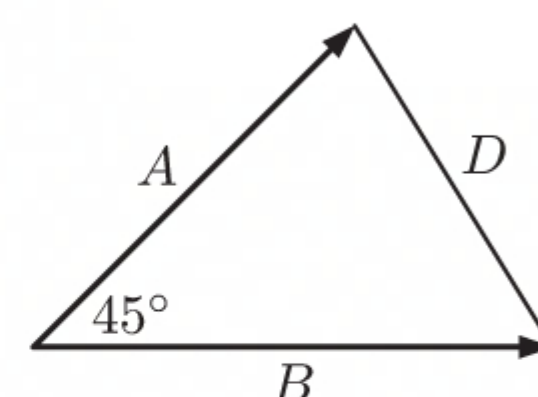
$$\begin{aligned} 2D \frac{dD}{dt} &= 2A \frac{dA}{dt} + 2B \frac{dB}{dt} - \sqrt{2} \left(\frac{dA}{dt} B + A \frac{dB}{dt} \right) \\ \therefore 2D \frac{dD}{dt} &= 2A(28) + 2B(32) - \sqrt{2}(28B + 32A) \\ &= 56A + 64B - \sqrt{2}(28B + 32A) \end{aligned}$$

Particular case:

After 15 minutes or $\frac{1}{4}$ hour, $A = 7$ and $B = 8$, and so $D^2 = 7^2 + 8^2 - \sqrt{2}(7)(8)$

$$\begin{aligned} \therefore D &= \sqrt{113 - 56\sqrt{2}} \approx 5.814 \\ \therefore 2(5.814) \frac{dD}{dt} &\approx 56(7) + 64(8) - \sqrt{2}[28(8) + 32(7)] \\ \therefore 11.628 \frac{dD}{dt} &\approx 904 - 448\sqrt{2} \\ \therefore \frac{dD}{dt} &\approx 23.3 \end{aligned}$$

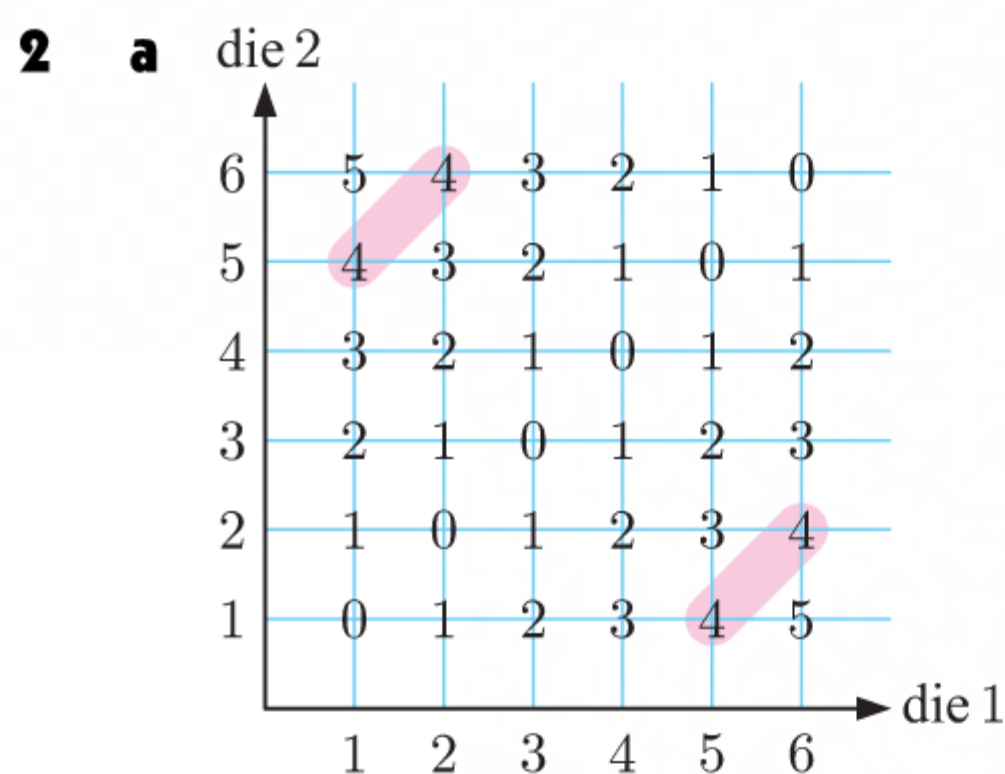
So, after 15 minutes the distance between the roadrunners is increasing at a rate of 23.3 km h^{-1} .



MIXED QUESTIONS SET 15

$$\begin{aligned}
 1 \quad & 2^{x-1} = 3^{2-x} \\
 & \therefore \log(2^{x-1}) = \log(3^{2-x}) \\
 & \therefore (x-1)\log 2 = (2-x)\log 3 \\
 & \therefore x\log 2 - \log 2 = 2\log 3 - x\log 3 \\
 & \therefore x\log 2 + x\log 3 = 2\log 3 + \log 2 \\
 & \therefore x(\log 2 + \log 3) = \log(3^2) + \log 2 \\
 & \therefore x\log 6 = \log 18 \\
 & \therefore x = \frac{\log 18}{\log 6} \\
 & \therefore x = \log_6 18 \quad \{\text{change of base rule}\}
 \end{aligned}$$

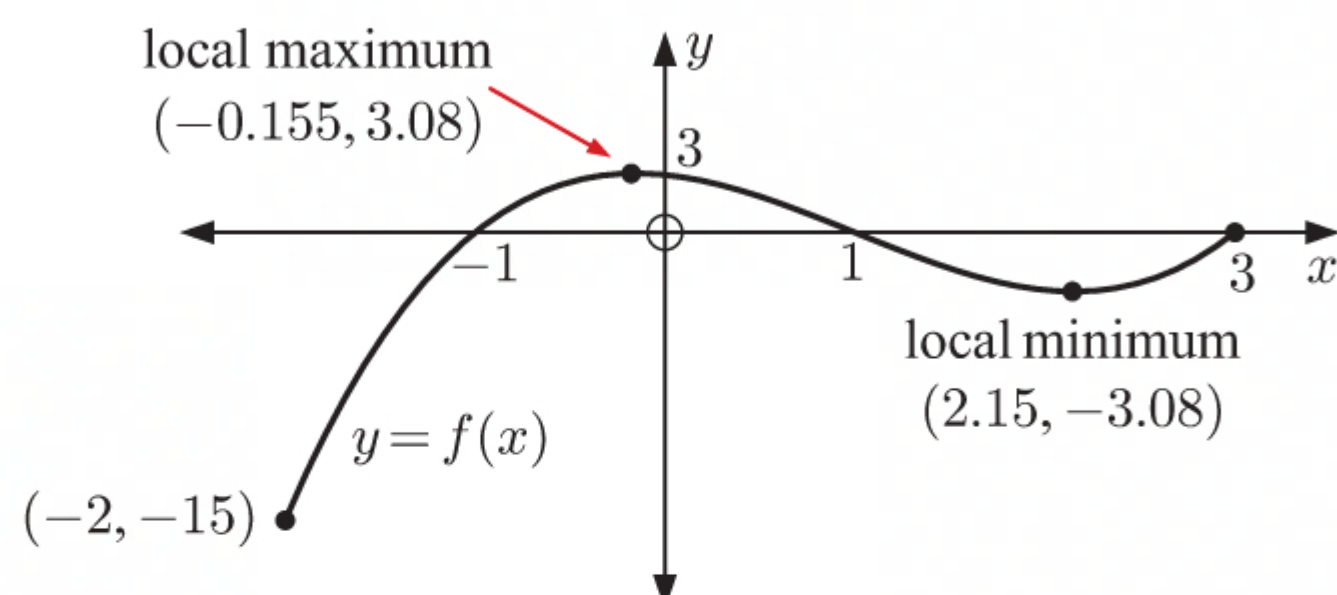
So, $a = 6$ and $b = 18$.



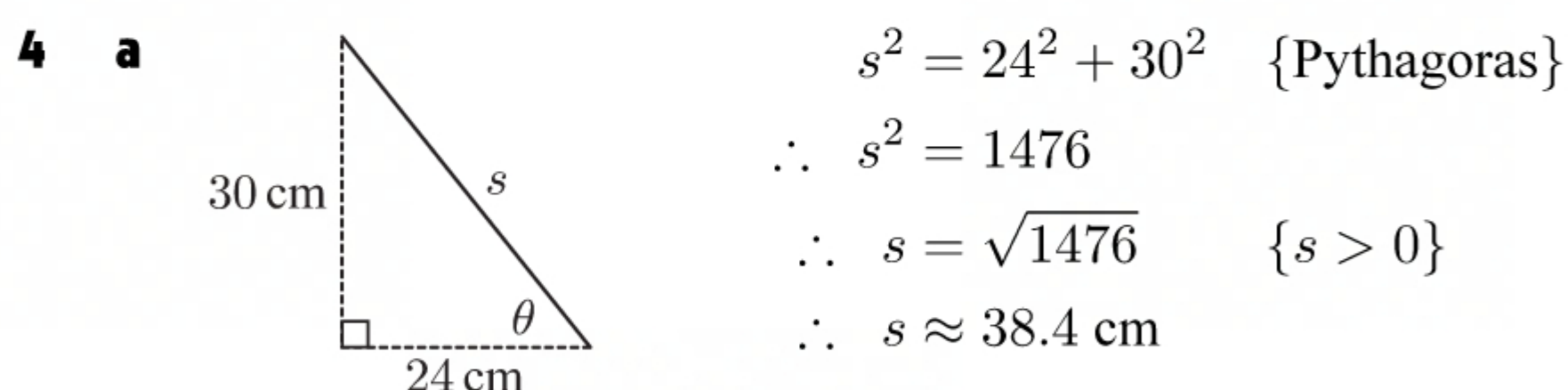
b There are 4 outcomes where the difference is 4.

As all outcomes are equally likely, the probability of the difference being 4 is $\frac{4}{36} = \frac{1}{9}$.

3 a $f(x) = x^3 - 3x^2 - x + 3, \quad -2 \leq x \leq 3$

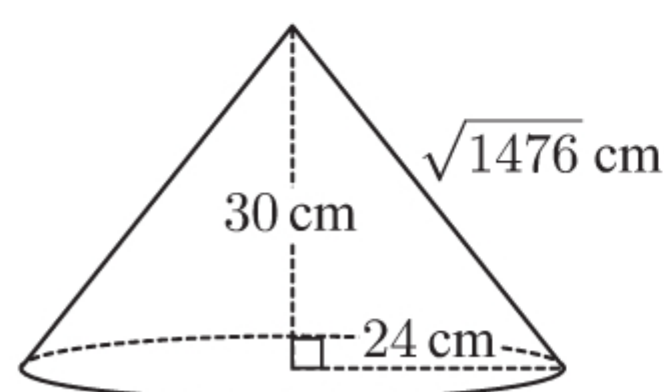


b The range of f is $\{y \mid -15 \leq y \leq 3.08\}$.



b Surface area $= \pi r s + \pi r^2$

$$\begin{aligned}
 &= \pi(24)\sqrt{1476} + \pi(24)^2 \\
 &\approx 4710 \text{ cm}^2 \\
 &\approx 4.71 \times 10^3 \text{ cm}^2
 \end{aligned}$$



5 a

Number of weeds	Frequency
0 - 4	9
5 - 9	15
10 - 14	10
15 - 19	p
20 - 24	5
25 - 29	2
Total	50

Total number of sample spots = 50

$$\therefore 9 + 15 + 10 + p + 5 + 2 = 50$$

$$\therefore p + 41 = 50$$

$$\therefore p = 9$$

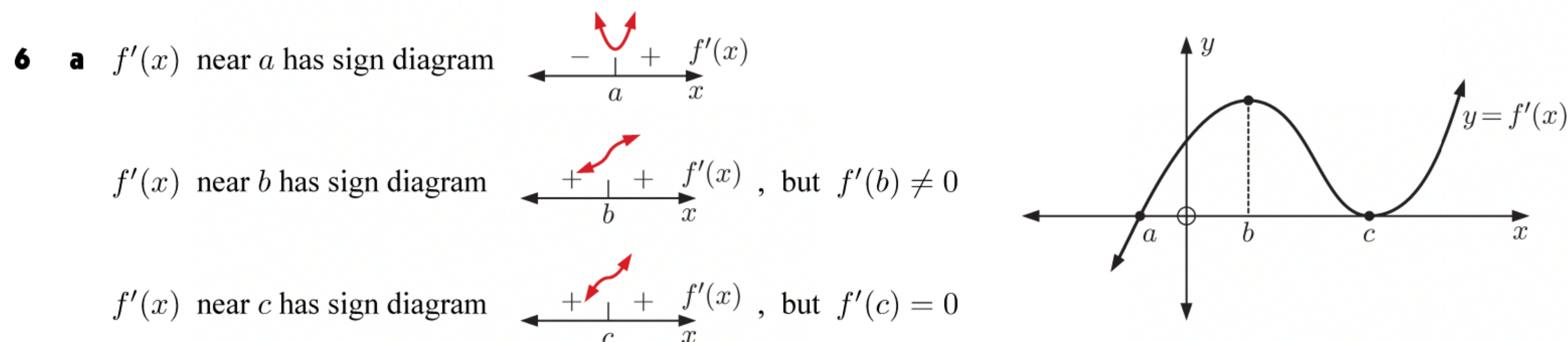
b

Number of weeds	Midpoint (x)	Frequency (f)	xf
0 - 4	2	9	18
5 - 9	7	15	105
10 - 14	12	10	120
15 - 19	17	9	153
20 - 24	22	5	110
25 - 29	27	2	54
Total		$\sum f = 50$	$\sum xf = 560$

$$\begin{aligned}\bar{x} &= \frac{\sum xf}{\sum f} \\ &= \frac{560}{50} \\ &= 11.2\end{aligned}$$

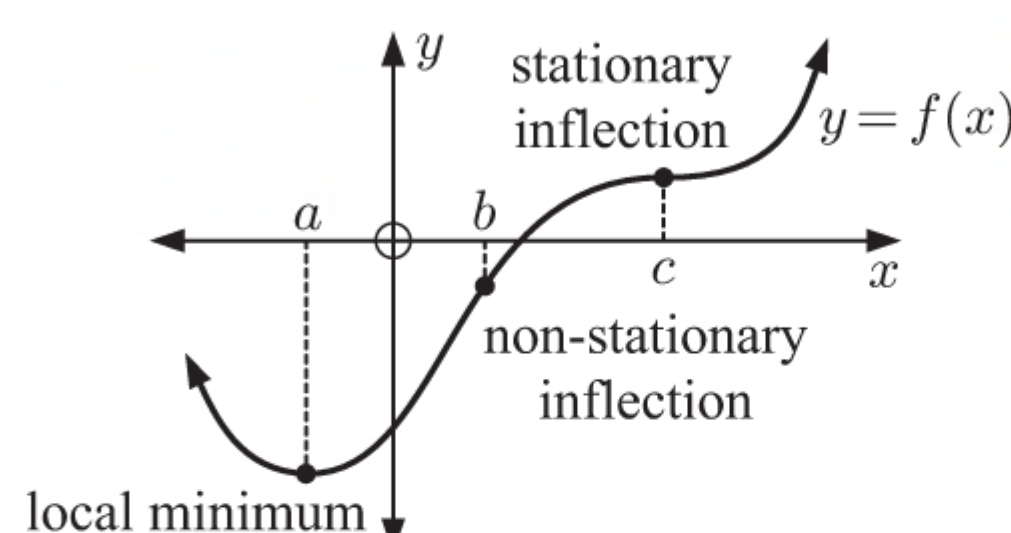
We estimate the mean number of weeds per spot to be 11.2.

c Percentage fewer than 10 weeds $= \frac{9+15}{50} \times 100\%$
 $= \frac{24}{50} \times 100\%$
 $= 48\%$



\therefore there is a local minimum at $x = a$, a non-stationary inflection at $x = b$, and a stationary inflection at $x = c$.

So a possible curve for $y = f(x)$ is:



b $f(x) = f_1(x) + k$ where k is a constant.

7 a $z^7 = 1$
 $\therefore z^7 = \text{cis}(k2\pi)$ where $k \in \mathbb{Z}$ {polar form}
 $\therefore z = (\text{cis}(k2\pi))^{\frac{1}{7}}$
 $\therefore z = \text{cis} \frac{k2\pi}{7}$ {De Moivre}
 $\therefore z = \text{cis}(-\frac{6\pi}{7}), \text{cis}(-\frac{4\pi}{7}), \text{cis}(-\frac{2\pi}{7}), \text{cis} 0, \text{cis} \frac{2\pi}{7}, \text{cis} \frac{4\pi}{7}, \text{cis} \frac{6\pi}{7}$ {letting $k = -3, -2, -1, 0, 1, 2, 3$ }
 $\therefore z = \text{cis}(-\frac{6\pi}{7}), \text{cis}(-\frac{4\pi}{7}), \text{cis}(-\frac{2\pi}{7}), 1, \text{cis} \frac{2\pi}{7}, \text{cis} \frac{4\pi}{7}, \text{cis} \frac{6\pi}{7}$

b $(3z - 1)^7 = 1$
 $\therefore 3z - 1 = \text{cis}(-\frac{6\pi}{7}), \text{cis}(-\frac{4\pi}{7}), \text{cis}(-\frac{2\pi}{7}), 1, \text{cis} \frac{2\pi}{7}, \text{cis} \frac{4\pi}{7}, \text{cis} \frac{6\pi}{7}$ {using **a**}
 $\therefore 3z = 1 + \text{cis}(-\frac{6\pi}{7}), 1 + \text{cis}(-\frac{4\pi}{7}), 1 + \text{cis}(-\frac{2\pi}{7}), 2, 1 + \text{cis} \frac{2\pi}{7}, 1 + \text{cis} \frac{4\pi}{7}, 1 + \text{cis} \frac{6\pi}{7}$
 $\therefore z = \frac{1}{3}(1 + \text{cis}(-\frac{6\pi}{7})), \frac{1}{3}(1 + \text{cis}(-\frac{4\pi}{7})), \frac{1}{3}(1 + \text{cis}(-\frac{2\pi}{7})), \frac{2}{3}, \frac{1}{3}(1 + \text{cis} \frac{2\pi}{7}), \frac{1}{3}(1 + \text{cis} \frac{4\pi}{7}), \frac{1}{3}(1 + \text{cis} \frac{6\pi}{7})$

c The sum of the roots of $z^7 = 1$ is equal to 0.

$$\therefore \text{cis}(-\frac{6\pi}{7}) + \text{cis}(-\frac{4\pi}{7}) + \text{cis}(-\frac{2\pi}{7}) + 1 + \text{cis} \frac{2\pi}{7} + \text{cis} \frac{4\pi}{7} + \text{cis} \frac{6\pi}{7} = 0$$

Equating real parts, we get

$$\cos(-\frac{6\pi}{7}) + \cos(-\frac{4\pi}{7}) + \cos(-\frac{2\pi}{7}) + 1 + \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = 0$$

$$\therefore \cos \frac{6\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{2\pi}{7} + 1 + \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = 0 \quad \{\cos(-\theta) = \cos \theta\}$$

$$\therefore 2 \cos \frac{2\pi}{7} + 2 \cos \frac{4\pi}{7} + 2 \cos \frac{6\pi}{7} = -1$$

$$\therefore \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad \operatorname{cosec} 2x - \cot 2x &= \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} \\
 &= \frac{1 - \cos 2x}{\sin 2x} \\
 &= \frac{2 \sin^2 x}{2 \sin x \cos x} \\
 &= \frac{\sin x}{\cos x} \\
 &= \tan x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \tan \frac{5\pi}{12} &= \operatorname{cosec} \frac{5\pi}{6} - \cot \frac{5\pi}{6} \quad \{\text{using } \mathbf{a}\} \\
 &= \frac{1}{\sin \frac{5\pi}{6}} - \frac{\cos \frac{5\pi}{6}}{\sin \frac{5\pi}{6}} \\
 &= \frac{1}{\frac{1}{2}} - \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
 &= 2 + \sqrt{3}
 \end{aligned}$$

$$\mathbf{9} \quad \mathbf{a} \quad \bullet \quad e - 1 > 0 \quad \text{and} \quad e^{\frac{1-x}{3}} > 0 \quad \text{for all } -2 \leq x \leq 1$$

$$\therefore f(x) \geq 0 \quad \text{for all } -2 \leq x \leq 1 \quad \checkmark$$

$$\begin{aligned}
 \bullet \quad \int_{-2}^1 f(x) dx &= \int_{-2}^1 \frac{e^{\frac{1-x}{3}}}{3(e-1)} dx \\
 &= \frac{1}{3(e-1)} \int_{-2}^1 e^{\frac{1-x}{3}} dx \\
 &= \frac{1}{3(e-1)} \left[-3e^{\frac{1-x}{3}} \right]_{-2}^1 \\
 &= \frac{1}{3(e-1)} (-3 + 3e) \\
 &= \frac{3(e-1)}{3(e-1)} \\
 &= 1 \quad \checkmark
 \end{aligned}$$

So, $f(x)$ is a valid probability density function.

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{i} \quad P(0 \leq X \leq 1) &= \int_0^1 \frac{e^{\frac{1-x}{3}}}{3(e-1)} dx \\
 &= \frac{1}{3(e-1)} \left[-3e^{\frac{1-x}{3}} \right]_0^1 \\
 &= \frac{1}{3(e-1)} (-3 + 3e^{\frac{1}{3}}) \\
 &= \frac{3(e^{\frac{1}{3}} - 1)}{3(e-1)} \\
 &= \frac{e^{\frac{1}{3}} - 1}{e-1} \approx 0.230
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad P(X \geq \tfrac{1}{4}) &= \int_{\frac{1}{4}}^1 \frac{e^{\frac{1-x}{3}}}{3(e-1)} dx \\
 &= \frac{1}{3(e-1)} \left[-3e^{\frac{1-x}{3}} \right]_{\frac{1}{4}}^1 \\
 &= \frac{1}{3(e-1)} (-3 + 3e^{\frac{3}{4}}) \\
 &= \frac{3(e^{\frac{3}{4}} - 1)}{3(e-1)} \\
 &= \frac{e^{\frac{3}{4}} - 1}{e-1} \approx 0.165
 \end{aligned}$$

$$\mathbf{10} \quad \mathbf{a} \quad \text{Let } \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2) + Bn}{n(n+2)} \quad \text{for all } n$$

$$\therefore \frac{1}{n(n+2)} = \frac{(A+B)n + 2A}{n(n+2)} \quad \text{for all } n$$

$$\therefore A + B = 0 \quad \text{and} \quad 2A = 1 \quad \therefore A = \frac{1}{2}, B = -\frac{1}{2}$$

$$\begin{aligned}
 \mathbf{b} \quad &\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \dots + \frac{1}{n(n+2)} \\
 &= \left(\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{3} \right) + \left(\frac{\frac{1}{2}}{2} - \frac{\frac{1}{2}}{4} \right) + \left(\frac{\frac{1}{2}}{3} - \frac{\frac{1}{2}}{5} \right) + \left(\frac{\frac{1}{2}}{4} - \frac{\frac{1}{2}}{6} \right) + \dots + \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1} \right) + \left(\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} \right) \quad \{\text{using } \mathbf{a}\} \\
 &= \frac{1}{2} + \frac{1}{4} - \frac{1}{2n+2} - \frac{1}{2n+4} \quad \{\text{as all other terms cancel}\} \\
 &= \frac{3}{4} - \frac{1}{2n+2} - \frac{1}{2n+4}
 \end{aligned}$$

$$\mathbf{c} \quad P_n \text{ is: } \frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \dots + \frac{1}{n(n+2)} = \frac{3}{4} - \frac{1}{2n+2} - \frac{1}{2n+4} \quad \text{for all } n \in \mathbb{Z}^+.$$

Proof: (By the principle of mathematical induction)

$$(1) \quad \text{If } n = 1, \quad \text{LHS} = \frac{1}{1 \times 3} = \frac{1}{3} \quad \text{and} \quad \text{RHS} = \frac{3}{4} - \frac{1}{4} - \frac{1}{6} = \frac{1}{3} \quad \checkmark$$

$\therefore P_1$ is true.

(2) If P_k is true, then $\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \dots + \frac{1}{k(k+2)} = \frac{3}{4} - \frac{1}{2k+2} - \frac{1}{2k+4}$.

$$\begin{aligned} \text{Now } & \frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \dots + \frac{1}{k(k+2)} + \frac{1}{(k+1)(k+3)} \\ &= \frac{3}{4} - \frac{1}{2k+2} - \frac{1}{2k+4} + \frac{1}{(k+1)(k+3)} \quad \{\text{using } P_k\} \\ &= \frac{3}{4} - \frac{1}{2k+4} + \frac{1}{(k+1)(k+3)} - \frac{1}{2(k+1)} \\ &= \frac{3}{4} - \frac{1}{2k+4} + \frac{2-(k+3)}{2(k+1)(k+3)} \\ &= \frac{3}{4} - \frac{1}{2k+4} + \frac{-(k+1)}{2(k+1)(k+3)} \\ &= \frac{3}{4} - \frac{1}{2k+4} - \frac{1}{2k+6} \\ &= \frac{3}{4} - \frac{1}{2(k+1)+2} - \frac{1}{2(k+1)+4} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

MIXED QUESTIONS SET 16

1 $W(t) = 5 \times (0.965)^t$ grams, $t \geq 0$

- a The weight of the radioactive substance at the *end* of each year forms a geometric sequence with common ratio $r = 0.965$.

$$\begin{aligned} \text{So, percentage decrease} &= (1 - r) \times 100\% \\ &= 0.035 \times 100\% \\ &= 3.5\% \end{aligned}$$

b $W(300) = 5 \times (0.965)^{300}$
 $\approx 0.000\,114$
 $\approx 1.14 \times 10^{-4}$

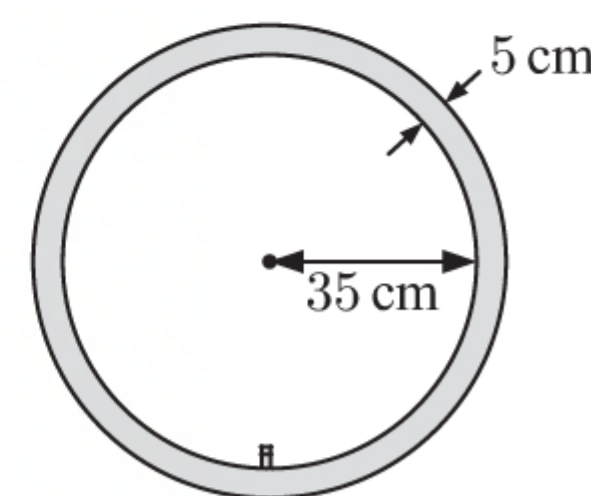
The weight of the substance after 300 years is about 1.14×10^{-4} grams.

- c We need to solve $W(t) = 1$

$$\begin{aligned} \therefore 5 \times (0.965)^t &= 1 \\ \therefore (0.965)^t &= 0.2 \\ \therefore t \log(0.965) &= \log(0.2) \\ \therefore t &= \frac{\log(0.2)}{\log(0.965)} \\ &\approx 45.2 \text{ years} \end{aligned}$$

\therefore it will take about 45.2 years for the weight of the substance to fall below 1 g.

- 2 a i At 0 seconds, the valve is at its lowest position, closest to the road, so the height of the valve above the road is 5 cm.



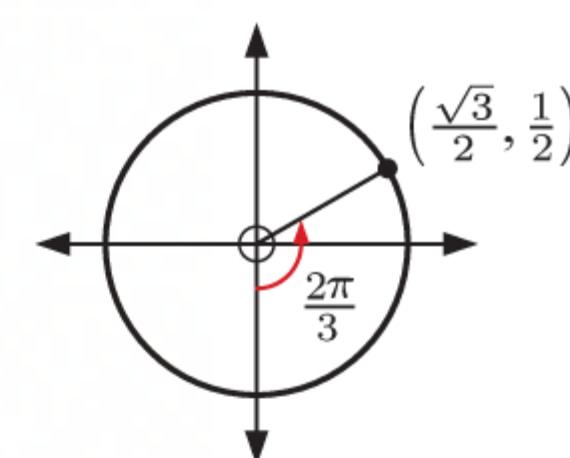
- ii The wheel rotates at a constant speed of 4 revolutions per second.

$$\therefore \text{ after } \frac{1}{12} \text{ second, the wheel has rotated } \frac{1}{12} \times 4 = \frac{1}{3} \text{ revolution.}$$

$$\therefore \text{ the valve will have moved through an angle of } \frac{2\pi}{3}.$$

So, the valve will be at a height of $1\frac{1}{2}$ times the radius of the wheel, plus 5 cm.

$$\begin{aligned} \therefore \text{ the height of the valve above the road} &= 1.5 \times 35 + 5 \\ &= 57.5 \text{ cm} \end{aligned}$$



b $H(t) = a \sin(b(t - c)) + d$ cm

i Amplitude = inner radius of the tyre
 $= 35$ cm
 $\therefore a = 35$

- ii The centre of the wheel is 40 cm above the ground, so the principal axis is at $H = 40$ cm.

$$\therefore d = 40$$

- iii There are 4 revolutions per second, so the period is $\frac{1}{4}$ second.

$$\text{Period} = \frac{2\pi}{b}$$

$$\therefore \frac{1}{4} = \frac{2\pi}{b}$$

$$\therefore b = 8\pi$$

- c From b, $H(t) = 35 \sin(8\pi(t - \frac{1}{16})) + 40$

$$\text{Now } H(t) = 60 \text{ when } 35 \sin(8\pi(t - \frac{1}{16})) + 40 = 60$$

$$\therefore \sin(8\pi(t - \frac{1}{16})) = \frac{20}{35}$$

$$\therefore t \approx 0.0867 \quad \{\text{using technology}\}$$

It takes approximately 0.0867 seconds for the valve to rise to 60 cm above the road.

- 3 The bin has capacity 500 litres = 500 000 mL

$$\therefore \pi r^2 h = 500\,000$$

$$\therefore h = \frac{500\,000}{\pi r^2}$$

$$\text{Surface area } A = 2\pi r h + \pi r^2$$

$$= 2\pi r \left(\frac{500\,000}{\pi r^2} \right) + \pi r^2$$

$$= 1\,000\,000 r^{-1} + \pi r^2$$

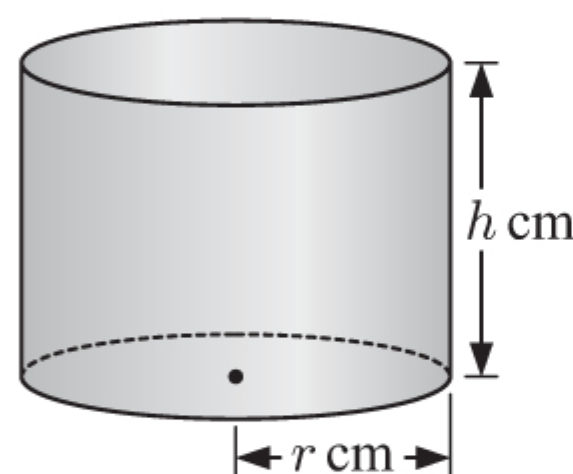
$$\therefore \frac{dA}{dr} = -\frac{1\,000\,000}{r^2} + 2\pi r$$

$$\text{Now } \frac{dA}{dr} = 0 \text{ when } 2\pi r = \frac{1\,000\,000}{r^2}$$

$$\therefore 2\pi r^3 = 1\,000\,000$$

$$\therefore r = \sqrt[3]{\frac{1\,000\,000}{2\pi}} \approx 54.2$$

$$\text{and } h \approx \frac{500\,000}{\pi(54.2)^2} \approx 54.2$$



So, the surface area of the bin is minimised when the bin has a base radius and height of about 54.2 cm.

- 4 a f is $y = x^2 + 2x$, $x \leq -1$

$$\therefore f^{-1} \text{ is } x = y^2 + 2y, \quad y \leq -1$$

$$\therefore x + 1 = y^2 + 2y + 1$$

$$\therefore x + 1 = (y + 1)^2$$

$$\therefore \pm\sqrt{x+1} = y + 1$$

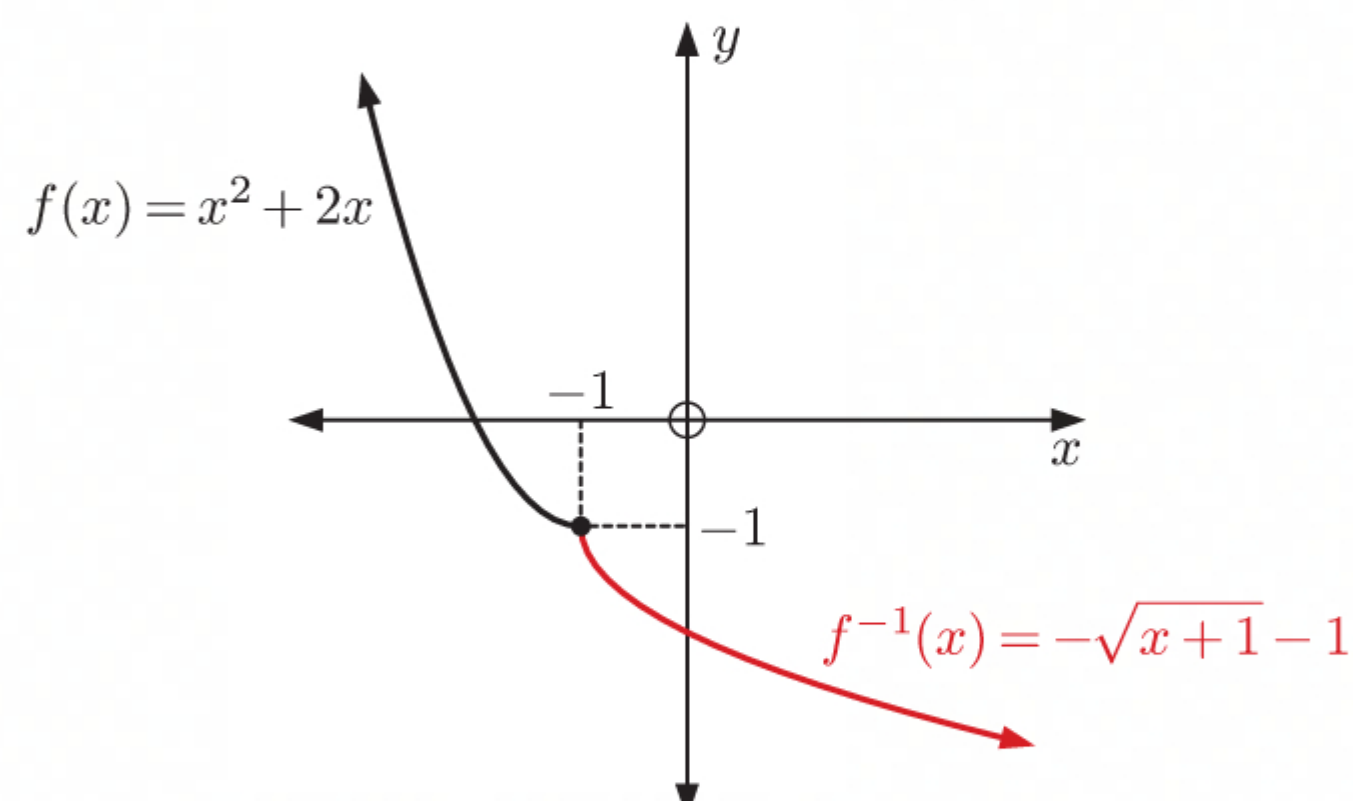
$$\therefore y = -\sqrt{x+1} - 1 \quad \{\text{as } y \leq -1\}$$

$$\text{so } f^{-1}(x) = -\sqrt{x+1} - 1$$

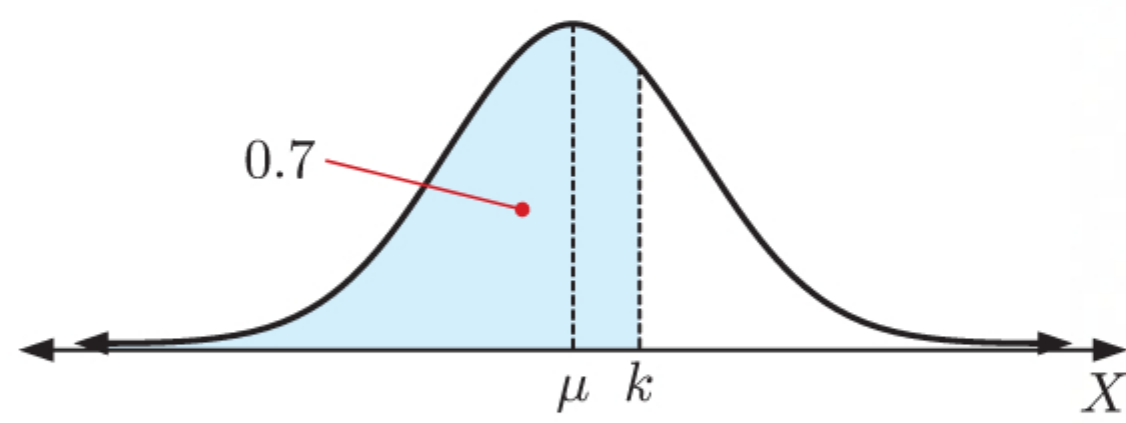
Now f has domain $\{x \mid x \leq -1\}$ and range $\{y \mid y \geq -1\}$

$\therefore f^{-1}$ has domain $\{x \mid x \geq -1\}$ and range $\{y \mid y \leq -1\}$

b



5 a



$$\begin{aligned} \text{b i } P(X > k) &= 1 - P(X < k) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

$$\begin{aligned} \text{ii } P(\mu < X < k) &= P(X < k) - P(X \leq \mu) \\ &= 0.7 - 0.5 \\ &= 0.2 \end{aligned}$$

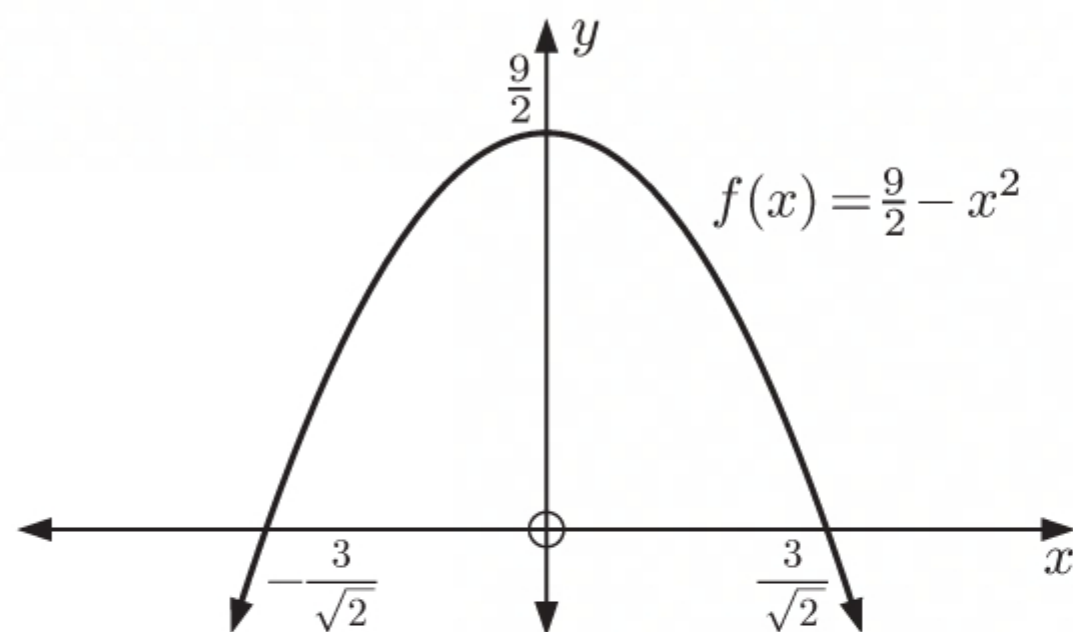
$$\begin{aligned} \text{iii } P(\mu - \sigma < X < k) &= P(\mu - \sigma < X < \mu) + P(\mu < X < k) \\ \text{Now, } P(\mu - \sigma < X < \mu) &= P(-1 < Z < 0) \approx 0.341 \\ \text{So, } P(\mu - \sigma < X < k) &\approx 0.341 + 0.2 \quad \{\text{using ii}\} \\ &\approx 0.541 \end{aligned}$$

$$\begin{aligned} \text{c } P(k \leq X \leq t) &= P(X \leq t) - P(X < k) \\ &= 1 - P(X \geq t) - P(X < k) \\ &= 1 - 0.2 - 0.7 \\ &= 0.1 \end{aligned}$$

6 a $f(x) = \frac{9}{2} - x^2$

 The vertex is $(0, \frac{9}{2})$, and the y -intercept is $\frac{9}{2}$.

$$\begin{aligned} \text{When } y = 0, \quad \frac{9}{2} - x^2 &= 0 \\ \therefore x^2 &= \frac{9}{2} \\ \therefore x &= \pm \frac{3}{\sqrt{2}} \end{aligned}$$


 c L passes through the origin.

$$\therefore \text{ when } x = 0, \quad y = 0$$

$$\therefore \frac{1}{2a}(0) + 4 - a^2 = 0$$

$$\therefore 4 - a^2 = 0$$

$$\therefore a^2 = 4$$

$$\therefore a = \pm 2$$

 When $a = 2$, L has equation $y = \frac{1}{4}x$.

 Now L intersects the graph of $y = f(x)$ where $\frac{1}{4}x = \frac{9}{2} - x^2$

$$\therefore x^2 + \frac{1}{4}x - \frac{9}{2} = 0$$

$$\therefore 4x^2 + x - 18 = 0$$

$$\therefore 4x^2 - 8x + 9x - 18 = 0$$

$$\therefore 4x(x - 2) + 9(x - 2) = 0$$

$$\therefore (x - 2)(4x + 9) = 0$$

$$\therefore x = 2 \text{ or } -\frac{9}{4}$$

b $f(x) = \frac{9}{2} - x^2$

$$\therefore f'(x) = -2x$$

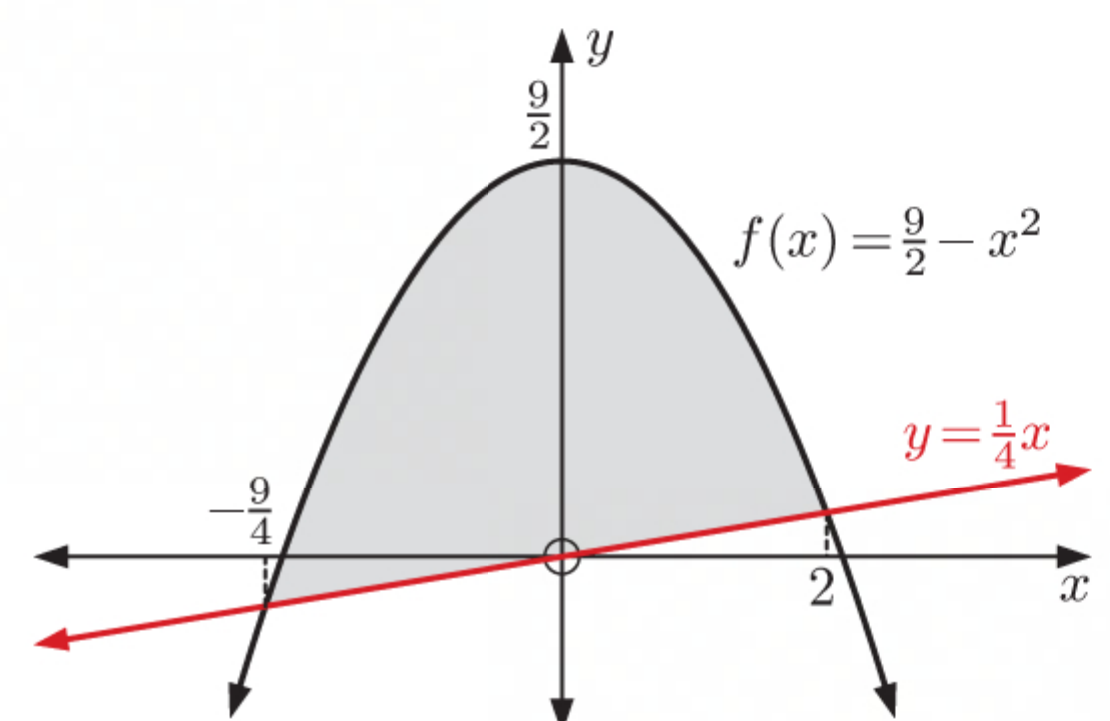
 \therefore the tangent at $P(a, f(a))$ has gradient $-2a$.

 \therefore the normal at $P(a, f(a))$ has gradient $\frac{1}{2a}$.

 The equation of normal L is $y = \frac{1}{2a}(x - a) + f(a)$

$$\therefore y = \frac{1}{2a}x - \frac{1}{2} + \frac{9}{2} - a^2$$

$$\therefore y = \frac{1}{2a}x + 4 - a^2$$



$$\begin{aligned}
\text{Since } f(x) \geq \frac{1}{4}x \text{ for } -\frac{9}{4} \leq x \leq 2, \quad \text{area} &= \int_{-\frac{9}{4}}^2 \left[\left(\frac{9}{2} - x^2 \right) - \frac{1}{4}x \right] dx \\
&= \int_{-\frac{9}{4}}^2 \left(-x^2 - \frac{1}{4}x + \frac{9}{2} \right) dx \\
&= \left[-\frac{x^3}{3} - \frac{x^2}{8} + \frac{9}{2}x \right]_{-\frac{9}{4}}^2 \\
&= \left(-\frac{8}{3} - \frac{4}{8} + 9 \right) - \left(\frac{243}{64} - \frac{81}{128} - \frac{81}{8} \right) \\
&= \frac{35}{6} + \frac{891}{128} \\
&\approx 12.8 \text{ units}^2
\end{aligned}$$

Now since $y = f(x)$ is symmetric about its axis of symmetry $x = 0$, and L passes through the origin, when $a = -2$, the area enclosed by L and $y = f(x)$ is also $\approx 12.8 \text{ units}^2$.

- 7** Suppose $\sqrt[3]{3}$ is rational, so $\sqrt[3]{3} = \frac{p}{q}$ for some positive integers p and q , $q \neq 0$.

We assume this fraction has been written in lowest terms, so p and q have no common factors.

$$\begin{aligned}
\text{Cubing both sides, } 3 &= \frac{p^3}{q^3} \\
\therefore p^3 &= 3q^3 \quad \dots (*)
\end{aligned}$$

$\therefore p^3$ is a multiple of 3, and so p must be a multiple of 3.

Thus $p = 3k$ for some $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Substituting into } (*), \quad 27k^3 &= 3q^3 \\
\therefore q^3 &= 9k^3
\end{aligned}$$

$\therefore q^3$ is a multiple of 3, and so q must be a multiple of 3.

This is a contradiction, as p and q have no common factors.

Thus our original supposition is false, and $\sqrt[3]{3}$ is irrational.

- 8 a** $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos \theta$

$$\begin{aligned}
&= |\mathbf{p}| \times 2 |\mathbf{p}| \cos \theta \\
&= 2 |\mathbf{p}|^2 \cos \theta
\end{aligned}$$

$$\begin{aligned}
\text{b } \overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD} & \overrightarrow{DB} &= \overrightarrow{DA} + \overrightarrow{AB} \\
&= \mathbf{p} + k\mathbf{q} & &= -k\mathbf{q} - \mathbf{p} + \mathbf{q} \\
& & &= -\mathbf{p} + (1-k)\mathbf{q}
\end{aligned}$$

$$\text{c } \quad \text{Since } \overrightarrow{OD} \perp \overrightarrow{DB} \text{ then } \overrightarrow{OD} \cdot \overrightarrow{DB} = 0$$

$$\therefore (\mathbf{p} + k\mathbf{q}) \cdot (-\mathbf{p} + (1-k)\mathbf{q}) = 0 \quad \{\text{from b}\}$$

$$\therefore -(\mathbf{p} \cdot \mathbf{p}) + (1-k)(\mathbf{p} \cdot \mathbf{q}) - k(\mathbf{p} \cdot \mathbf{q}) + k(1-k)(\mathbf{q} \cdot \mathbf{q}) = 0$$

$$\therefore -|\mathbf{p}|^2 + (1-2k)(\mathbf{p} \cdot \mathbf{q}) + k(1-k)|\mathbf{q}|^2 = 0$$

$$\therefore (1-2k)(\mathbf{p} \cdot \mathbf{q}) - |\mathbf{p}|^2 + k(1-k) \times 4|\mathbf{p}|^2 = 0 \quad \left\{ |\mathbf{q}| = 2|\mathbf{p}| \quad \therefore |\mathbf{q}|^2 = 4|\mathbf{p}|^2 \right\}$$

$$\therefore (1-2k)(\mathbf{p} \cdot \mathbf{q}) + (4k-4k^2-1)|\mathbf{p}|^2 = 0$$

$$\therefore (1-2k)(\mathbf{p} \cdot \mathbf{q}) - (1-4k+4k^2)|\mathbf{p}|^2 = 0$$

$$\therefore (1-2k)(\mathbf{p} \cdot \mathbf{q}) - (1-2k)^2|\mathbf{p}|^2 = 0$$

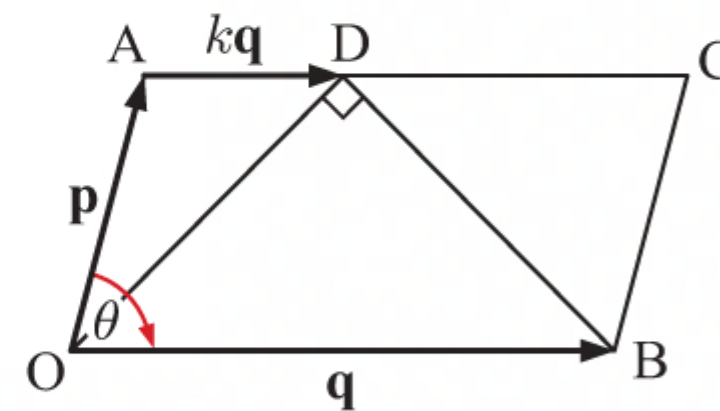
$$\text{d } \text{From c, } (1-2k) \times 2|\mathbf{p}|^2 \cos \theta - (1-2k)^2|\mathbf{p}|^2 = 0 \quad \left\{ \mathbf{p} \cdot \mathbf{q} = 2|\mathbf{p}|^2 \cos \theta \quad \text{from a} \right\}$$

$$\therefore (1-2k)(2 \cos \theta - (1-2k))|\mathbf{p}|^2 = 0$$

$$\therefore 1-2k = 0 \quad \text{or} \quad 2 \cos \theta = 1-2k \quad \text{as } |\mathbf{p}| \neq 0$$

$$\therefore k = \frac{1}{2} \quad \text{or} \quad k = \frac{1}{2} - \cos \theta$$

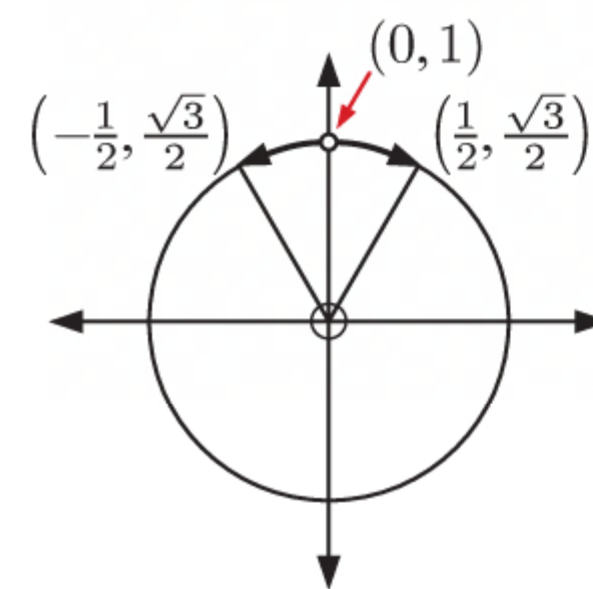
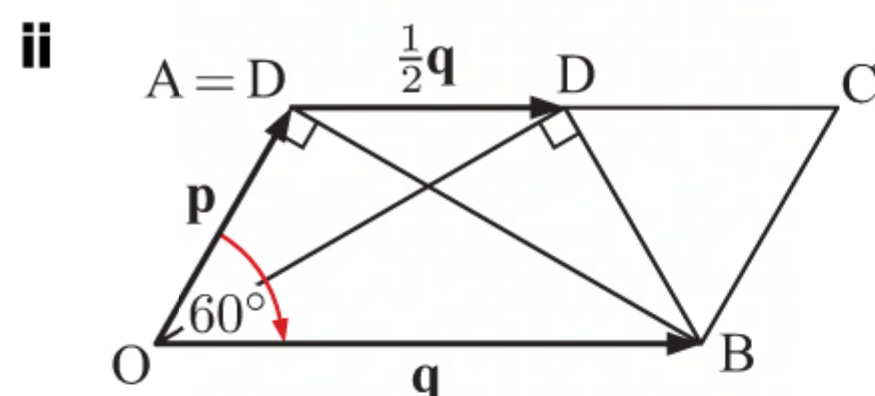
- e** When $k = \frac{1}{2}$, D is the midpoint of $[AC]$.



f i There are 2 positions for D, one when $k = \frac{1}{2}$ and the other when

$$\begin{aligned} 0 &\leq \frac{1}{2} - \cos \theta \leq 1 \quad \{\text{as } 0 \leq k \leq 1\} \\ \therefore -\frac{1}{2} &\leq -\cos \theta \leq \frac{1}{2} \\ \therefore -\frac{1}{2} &\leq \cos \theta \leq \frac{1}{2} \\ \therefore 60^\circ &\leq \theta \leq 120^\circ, \quad \theta \neq 90^\circ \\ &\{\text{if } \theta = 90^\circ, \quad \frac{1}{2} - \cos \theta = \frac{1}{2} \quad \therefore \text{only one position for D}\} \end{aligned}$$

\therefore the smallest value of θ for which there are two possible positions of D is $\theta = 60^\circ$.



9 Eddy:

Hours of sleep	5	6	7	8	9
Probability	0.4	a	0.2	0.05	0.1

Brett:

Hours of sleep	5	6	7	8	9
Probability	0.1	0.2	0.5	b	0.05

a Since these are probability distributions,

$$\begin{aligned} 0.4 + a + 0.2 + 0.05 + 0.1 &= 1 \quad \therefore a = 0.25 \\ \text{and } 0.1 + 0.2 + 0.5 + b + 0.05 &= 1 \quad \therefore b = 0.15 \end{aligned}$$

b Eddy: Expected number of hours of sleep $= 5 \times 0.4 + 6 \times 0.25 + 7 \times 0.2 + 8 \times 0.05 + 9 \times 0.1$
 $= 6.2$ hours

Brett: Expected number of hours of sleep $= 5 \times 0.1 + 6 \times 0.2 + 7 \times 0.5 + 8 \times 0.15 + 9 \times 0.05$
 $= 6.85$ hours

\therefore Eddy generally receives the least amount of sleep per night.

c Eddy: $\sigma^2 = 5^2 \times 0.4 + 6^2 \times 0.25 + 7^2 \times 0.2 + 8^2 \times 0.05 + 9^2 \times 0.1 - (6.2)^2$
 $= 1.66 \text{ hours}^2$
 $\therefore \sigma = \sqrt{1.66} \approx 1.29 \text{ hours}$

Brett: $\sigma^2 = 5^2 \times 0.1 + 6^2 \times 0.2 + 7^2 \times 0.5 + 8^2 \times 0.15 + 9^2 \times 0.05 - (6.85)^2$
 $= 0.9275 \text{ hours}^2$
 $\therefore \sigma = \sqrt{0.9275} \approx 0.963 \text{ hours}$

d From **c**, Brett has a lower standard deviation than Eddy, so Brett has the least variation in his amount of sleep.

10

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy}{x-1} \\ \therefore \frac{1}{y} \frac{dy}{dx} &= \frac{x}{x-1} \\ \therefore \frac{1}{y} \frac{dy}{dx} &= \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1} \\ \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int \left(1 + \frac{1}{x-1}\right) dx \\ \therefore \ln|y| &= x + \ln|x-1| + c \end{aligned}$$

But, when $x = 2$, $y = 2$

$$\begin{aligned} \therefore \ln 2 &= 2 + c \\ \therefore c &= \ln 2 - 2 \end{aligned}$$

$$\begin{aligned} \text{So, } \ln|y| &= x + \ln|x-1| + \ln 2 - 2 \\ \therefore \ln|y| - \ln|2(x-1)| &= x - 2 \\ \therefore \ln \left| \frac{y}{2(x-1)} \right| &= x - 2 \\ \therefore \left| \frac{y}{2(x-1)} \right| &= e^{x-2} \\ \therefore \frac{y}{2(x-1)} &= \pm e^{x-2} \end{aligned}$$

But $x = 2$, $y = 2$ does not satisfy the negative solution

$$\therefore y = 2(x-1)e^{x-2}.$$

MIXED QUESTIONS SET 17

1 $f(x) = ax^2 + bx + 7$

a We have $f(2) = 7$ and $f(4) = 23$
 $\therefore 4a + 2b + 7 = 7$ $16a + 4b + 7 = 23$
 $\therefore 4a + 2b = 0$ $16a + 4b = 16$
 $\therefore 2a + b = 0 \dots (1)$ $\therefore 4a + b = 4 \dots (2)$

b Subtracting (1) from (2), we get $2a = 4$
 $\therefore a = 2$

Substituting $a = 2$ into (1) gives $2(2) + b = 0$
 $\therefore b = -4$

c Using b, $f(x) = 2x^2 - 4x + 7$
 $\therefore f(-1) = 2(-1)^2 - 4(-1) + 7$
 $= 2 + 4 + 7$
 $= 13$

2 $f(x) = (x^2 + 1)e^{-x}$

$\therefore f(1) = (1^2 + 1)e^{-1}$
 $= \frac{2}{e}$

\therefore the point of contact is $\left(1, \frac{2}{e}\right)$.

Now $f(x) = (x^2 + 1)e^{-x}$ has derivative $f'(x) = 2xe^{-x} - (x^2 + 1)e^{-x}$ {product rule}
 $= e^{-x}(2x - (x^2 + 1))$

\therefore the tangent at $\left(1, \frac{2}{e}\right)$ has gradient $e^{-1}(2 - 2) = 0$.

\therefore the tangent is horizontal and has equation $y = \frac{2}{e}$.

- 3 a Let u_n km be the distance Hayley cycled in the n th week, and v_n km be the distance Patrick cycled in the n th week.
 Hayley cycled an additional 20 km each week.

$\therefore u_n = 60 + 20(n - 1)$

$\therefore u_5 = 60 + 20 \times 4 = 140$

So, Hayley cycled 140 km in the 5th week of training.

Patrick increased his distance by 20% each week.

$\therefore v_n = 60(1 + 0.2)^{n-1}$ {20% = 0.2}
 $= 60(1.2)^{n-1}$

$\therefore v_5 = 60(1.2)^4 \approx 124$

So, Patrick cycled about 124 km in the 5th week of training.

b $u_n = 210$ where $60 + 20(n - 1) = 210$ and $v_n = 210$ where $60(1.2)^{n-1} = 210$

$\therefore 20(n - 1) = 150$

$\therefore n - 1 = 7.5$

$\therefore n = 8.5$

$\therefore (1.2)^{n-1} = \frac{7}{2}$

$\therefore \log(1.2)^{n-1} = \log\left(\frac{7}{2}\right)$

$\therefore (n - 1) \log(1.2) = \log\left(\frac{7}{2}\right)$

$\therefore n - 1 = \frac{\log\left(\frac{7}{2}\right)}{\log(1.2)}$

$\therefore n = 1 + \frac{\log\left(\frac{7}{2}\right)}{\log(1.2)} \approx 7.87$

So, Hayley first cycled 210 km in the 9th week, and Patrick first cycled 210 km in the 8th week.

\therefore Patrick was the first to cycle 210 km in one week.

c u_n is an arithmetic sequence with $u_1 = 60$ and $d = 20$.

\therefore the total distance Hayley cycled in the first 12 weeks $= \frac{n}{2}(2u_1 + (n - 1)d)$
 $= \frac{12}{2}(2 \times 60 + 11 \times 20)$
 $= 2040$ km

v_n is a geometric sequence with $v_1 = 60$ and $r = 1.2$.

$$\begin{aligned}\therefore \text{the total distance Patrick cycled in the first 12 weeks} &= \frac{v_1(1-r^n)}{1-r} \\ &= \frac{60(1-(1.2)^{12})}{1-1.2} \\ &\approx 2375 \text{ km}\end{aligned}$$

So, Patrick cycled a greater total distance in the first 12 weeks.

4 $f(x) = \log_3(x+1) + 2$

a A translation through the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ maps $y = \log_3 x$ to $y = f(x)$.

b $\log_3(x+1)$ is defined when $x+1 > 0$, that is, when $x > -1$.

So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

c When $x = 0$, $\log_3 1 + 2 = 2$, so the y -intercept is 2.

When $y = 0$, $\log_3(x+1) + 2 = 0$

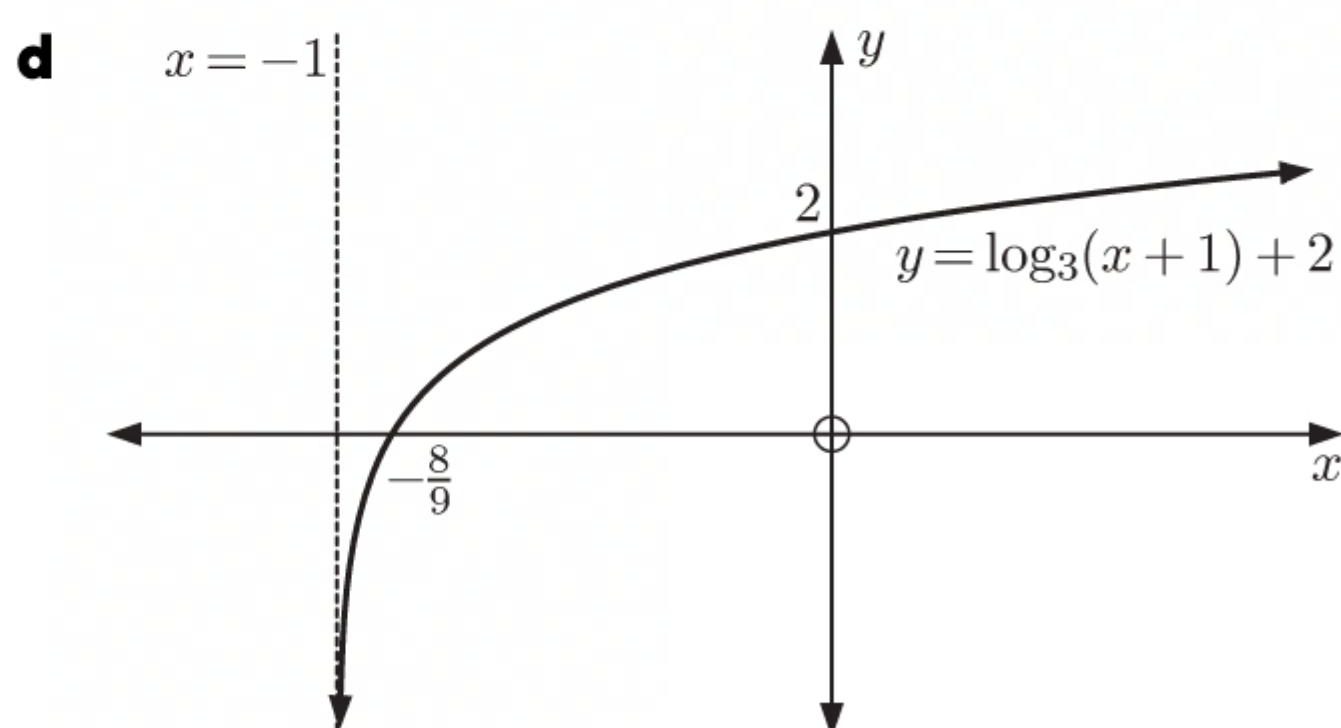
$$\therefore \log_3(x+1) = -2$$

$$\therefore x+1 = 3^{-2}$$

$$\therefore x+1 = \frac{1}{9}$$

$$\therefore x = -\frac{8}{9}$$

So, the x -intercept is $-\frac{8}{9}$.



e

$$\begin{aligned}f \text{ is } y &= \log_3(x+1) + 2 \\ \therefore f^{-1} \text{ is } x &= \log_3(y+1) + 2 \\ \therefore x-2 &= \log_3(y+1) \\ \therefore 3^{x-2} &= 3^{\log_3(y+1)} \\ \therefore 3^{x-2} &= y+1 \\ \therefore y &= 3^{x-2} - 1 \\ \therefore f^{-1}(x) &= 3^{x-2} - 1\end{aligned}$$

5 a The circle centred at X has radius [AX].

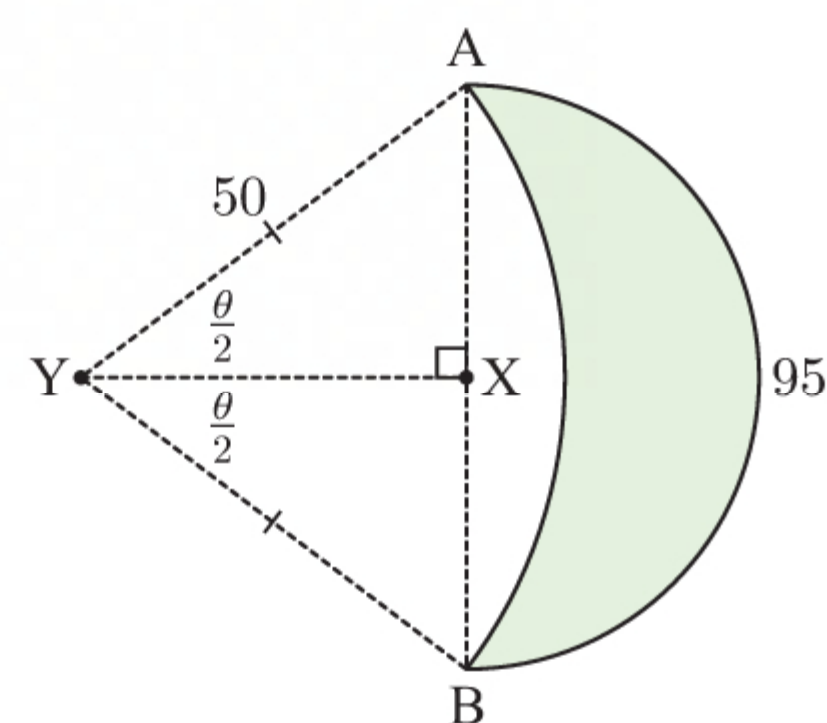
The length of the large arc [AB] is half of the circumference.

$$\therefore \frac{1}{2}(2 \times \pi \times AX) = 95$$

$$\therefore \pi \times AX = 95$$

$$\therefore AX = \frac{95}{\pi} \text{ units}$$

$$\approx 30.2 \text{ units}$$



b In $\triangle AXY$, $\sin \frac{\theta}{2} = \frac{AX}{AY}$

$$= \frac{\frac{95}{\pi}}{50}$$

$$= \frac{19}{10\pi}$$

$$\therefore \frac{\theta}{2} = \sin^{-1}\left(\frac{19}{10\pi}\right)$$

$$\therefore \theta = 2 \sin^{-1}\left(\frac{19}{10\pi}\right)$$

$$\approx 1.30$$

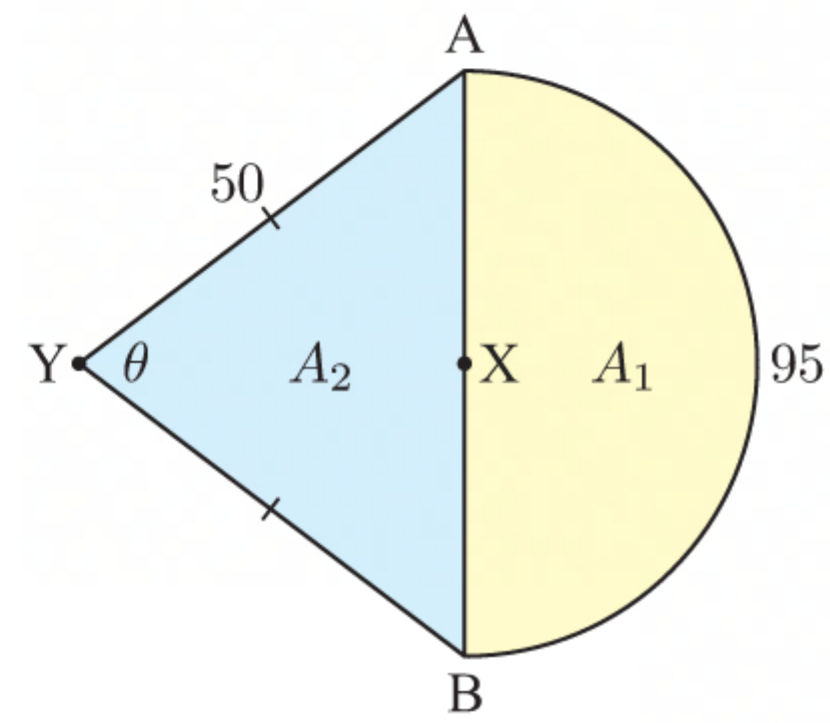
- c** We first divide the figure into areas A_1 and A_2 .

Now A_1 = area of semi-circle centred at X

$$\begin{aligned} &= \frac{1}{2} \times \pi \times \left(\frac{95}{\pi}\right)^2 \quad \{\text{from a}\} \\ &= \frac{9025}{2\pi} \text{ units}^2 \end{aligned}$$

and A_2 = area of $\triangle ABY$

$$\begin{aligned} &= \frac{1}{2} \times 50 \times 50 \times \sin \theta \\ &= 1250 \times \sin\left(2 \sin^{-1}\left(\frac{19}{10\pi}\right)\right) \quad \{\text{from b}\} \\ &= 1250 \times \sin\left(2 \sin^{-1}\left(\frac{19}{10\pi}\right)\right) \text{ units}^2 \end{aligned}$$



So, total area of figure = $A_1 + A_2$

$$= \frac{9025}{2\pi} + 1250 \times \sin\left(2 \sin^{-1}\left(\frac{19}{10\pi}\right)\right) \text{ units}^2$$

Now, shaded area = total area of figure – sector ABY

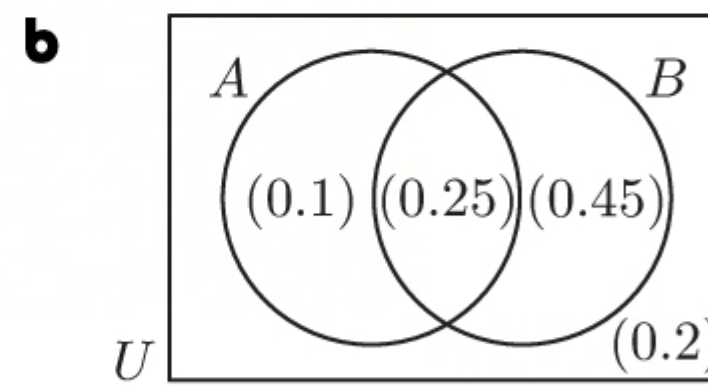
$$\begin{aligned} &= \frac{9025}{2\pi} + 1250 \times \sin\left(2 \sin^{-1}\left(\frac{19}{10\pi}\right)\right) - \frac{1}{2} \times \theta \times 50^2 \\ &= \frac{9025}{2\pi} + 1250 \times \sin\left(2 \sin^{-1}\left(\frac{19}{10\pi}\right)\right) - 1250 \times 2 \sin^{-1}\left(\frac{19}{10\pi}\right) \\ &= \frac{9025}{2\pi} + 1250 \times \sin\left(2 \sin^{-1}\left(\frac{19}{10\pi}\right)\right) - 2500 \sin^{-1}\left(\frac{19}{10\pi}\right) \\ &\approx 1020 \text{ units}^2 \end{aligned}$$

6 $P(A) = 0.35$, $P(B) = 0.7$, $P(A \cup B) = 0.8$

a $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore 0.8 = 0.35 + 0.7 - P(A \cap B)$$

$$\therefore P(A \cap B) = 1.05 - 0.8 = 0.25$$



c i $P(A' \cap B') = 0.2$

ii
$$\begin{aligned} P(A | B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.25}{0.7} \\ &= \frac{5}{14} \approx 0.357 \end{aligned}$$

d From **c ii**, $P(A | B) \neq P(A) = 0.35$.

So, events A and B are not independent.

7 a 8:30 am corresponds to time $t = 0$ hours.

$$x = 3 - 2t, \quad y = 3t + 1 \quad \therefore x(0) = 3, \quad y(0) = 1$$

So, at 8:30 am the ship is at $(3, 1)$.

b i The ship's velocity vector is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

ii The ship's speed is $\sqrt{4 + 9} = \sqrt{13} \text{ km h}^{-1}$.

c At 10:30 am, $t = 2$

So, the ship is at $(3 - 2(2), 3(2) + 1)$ or $(-1, 7)$

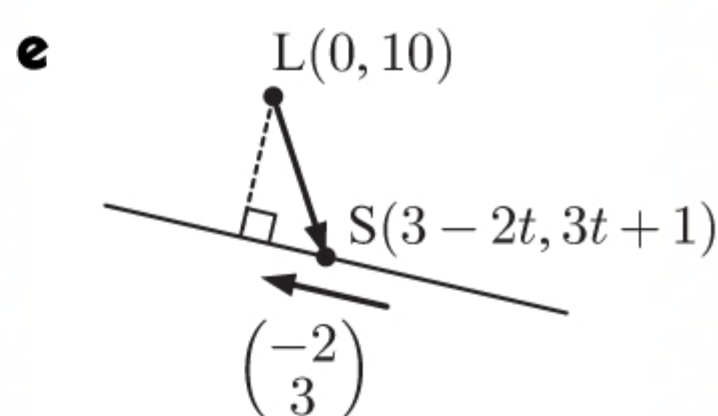
$$\begin{aligned} \text{and the distance to } (0, 10) &= \sqrt{(0 - (-1))^2 + (10 - 7)^2} \\ &= \sqrt{1 + 9} \\ &= \sqrt{10} \text{ km} \end{aligned}$$

d When the ship is directly west of the lighthouse, $3t + 1 = 10$

$$\therefore 3t = 9$$

$$\therefore t = 3$$

\therefore the time is 11:30 am.



$$\begin{aligned}\vec{LS} &= \begin{pmatrix} 3 - 2t - 0 \\ 3t + 1 - 10 \end{pmatrix} \\ &= \begin{pmatrix} 3 - 2t \\ 3t - 9 \end{pmatrix}\end{aligned}$$

The ship is closest to the lighthouse when $\vec{LS} \perp \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

$$\therefore \vec{LS} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = 0$$

$$\therefore -2(3 - 2t) + 3(3t - 9) = 0$$

$$\therefore -6 + 4t + 9t - 27 = 0$$

$$\therefore 13t = 33$$

$$\therefore t = \frac{33}{13} \approx 2.53846 \text{ h}$$

$$\therefore t \approx 2 \text{ h } 32 \text{ min}$$

So, the ship is closest to the lighthouse at about 11:02 am.

At time $t = \frac{33}{13}$,

$$\vec{LS} = \begin{pmatrix} 3 - 2(\frac{33}{13}) \\ 3(\frac{33}{13}) - 9 \end{pmatrix} = \begin{pmatrix} -\frac{27}{13} \\ -\frac{18}{13} \end{pmatrix}$$

$$\therefore |\vec{LS}| = \sqrt{\left(-\frac{27}{13}\right)^2 + \left(-\frac{18}{13}\right)^2} \approx 2.50 \text{ km}$$

So, the distance between the ship and the lighthouse is about 2.50 km.

8 a $\int_0^a \sin(0.5x) dx = 1$

$$\therefore [-2 \cos(0.5x)]_0^a = 1$$

$$\therefore -2 \cos \frac{a}{2} + 2 = 1$$

$$\therefore -2 \cos \frac{a}{2} = -1$$

$$\therefore \cos \frac{a}{2} = \frac{1}{2}$$

$$\therefore \frac{a}{2} = \frac{\pi}{3} \quad \{f(x) \geq 0 \text{ for } 0 \leq a \leq 2\pi\}$$

$$\therefore a = \frac{2\pi}{3}$$

b $E(X) = \int_0^{\frac{2}{3}} x \sin(0.5x) dx \approx 1.37$

$$\text{Var}(X) = \int_0^{\frac{2}{3}} x^2 \sin(0.5x) - [E(X)]^2 \approx 0.248$$

$$\therefore \sigma = \sqrt{\text{Var}(X)} \approx 0.498$$

c The median m is the solution of $\int_0^m f(x) dx = \frac{1}{2}$

$$\therefore \int_0^m \sin(0.5x) dx = \frac{1}{2}$$

$$\therefore [-2 \cos(0.5x)]_0^m = \frac{1}{2}$$

$$\therefore -2 \cos \frac{m}{2} + 2 = \frac{1}{2}$$

$$\therefore 2 \cos \frac{m}{2} = 1\frac{1}{2}$$

$$\therefore \cos \frac{m}{2} = \frac{3}{4}$$

$$\therefore \frac{m}{2} \approx 0.7227 \quad \{0 \leq m \leq \frac{2\pi}{3}\}$$

$$\therefore m \approx 1.45$$

$\sin(0.5x)$ on $0 \leq x \leq \frac{2\pi}{3}$ is a maximum at $x = \frac{2\pi}{3}$

\therefore the modal value of X is $\frac{2\pi}{3}$.

9 $\frac{dy}{dx} + e^x y = 5 - y$

$$\therefore \frac{dy}{dx} + (e^x + 1)y = 5$$

The integrating factor is $I(x) = e^{\int (e^x + 1) dx} = e^{e^x + x}$

Multiplying both sides of the differential equation by $e^{e^x + x}$ gives

$$e^{e^x + x} \frac{dy}{dx} + e^{e^x + x} (e^x + 1)y = 5e^{e^x + x}$$

$$\therefore \frac{d}{dx} (e^{e^x + x} y) = 5e^x e^{e^x}$$

$$\therefore e^{e^x + x} y = \int 5e^x e^{e^x} dx$$

$$\therefore e^x e^{e^x} y = \int 5e^u \frac{du}{dx} dx \quad \left\{ u = e^x, \frac{du}{dx} = e^x \right\}$$

$$\therefore e^x e^{e^x} y = 5 \int e^u du$$

$$\therefore e^x e^{e^x} y = 5e^u + c$$

$$\therefore e^x e^{e^x} y = 5e^{e^x} + c$$

$$\therefore y = 5e^{-x} + ce^{-x} e^{-e^x}$$

But $y(0) = 1$, so $1 = 5 + ce^{-1}$

$$\therefore c = -4e$$

The particular solution is $y = 5e^{-x} - 4e^{1-x-e^x}$

10 P_n is: $\frac{d^n}{dx^n} [f(x)g(x)] = \sum_{j=0}^n \binom{n}{j} f^{(n-j)}(x) g^{(j)}(x)$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

$$\begin{aligned} (1) \text{ If } n = 1, \text{ LHS} &= \frac{d}{dx} [f(x)g(x)] & \text{and RHS} &= \sum_{j=0}^1 \binom{1}{j} f^{(1-j)}(x) g^{(j)}(x) \\ &= f'(x)g(x) + f(x)g'(x) \quad \{\text{product rule}\} & &= \binom{1}{0} f^{(1)}(x) g^{(0)}(x) + \binom{1}{1} f^{(0)}(x) g^{(1)}(x) \\ & & &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

$\therefore P_1$ is true.

$$(2) \text{ If } P_k \text{ is true, then } \frac{d^k}{dx^k} [f(x)g(x)] = \sum_{j=0}^k \binom{k}{j} f^{(k-j)}(x) g^{(j)}(x).$$

$$\begin{aligned} \text{Now } \frac{d^{k+1}}{dx^{k+1}} [f(x)g(x)] &= \frac{d}{dx} \left(\frac{d^k}{dx^k} [f(x)g(x)] \right) \\ &= \frac{d}{dx} \left(\sum_{j=0}^k \binom{k}{j} f^{(k-j)}(x) g^{(j)}(x) \right) \quad \{\text{using } P_k\} \\ &= \sum_{j=0}^k \binom{k}{j} \frac{d}{dx} [f^{(k-j)}(x) g^{(j)}(x)] \\ &= \sum_{j=0}^k \binom{k}{j} \left(f^{(k-j+1)}(x) g^{(j)}(x) + f^{(k-j)}(x) g^{(j+1)}(x) \right) \quad \{\text{product rule}\} \\ &= \sum_{j=0}^k \binom{k}{j} f^{(k-j+1)}(x) g^{(j)}(x) + \sum_{j=0}^k \binom{k}{j} f^{(k-j)}(x) g^{(j+1)}(x) \\ &= \sum_{j=0}^k \binom{k}{j} f^{((k+1)-j)}(x) g^{(j)}(x) + \sum_{m=1}^{k+1} \binom{k}{m-1} f^{(k-m+1)}(x) g^{(m)}(x) \\ & \quad \{\text{letting } m = j + 1 \text{ in second summation}\} \\ &= \sum_{j=0}^k \binom{k}{j} f^{((k+1)-j)}(x) g^{(j)}(x) + \sum_{j=1}^{k+1} \binom{k}{j-1} f^{((k+1)-j)}(x) g^{(j)}(x) \quad \{\text{letting } j = m \text{ in second summation}\} \\ &= \binom{k}{0} f^{(k+1)}(x) g^{(0)}(x) + \sum_{j=1}^k \left[\binom{k}{j} + \binom{k}{j-1} \right] f^{((k+1)-j)}(x) g^{(j)}(x) + \binom{k}{k} f^{(0)}(x) g^{(k+1)}(x) \\ &= \binom{k+1}{0} f^{(k+1)}(x) g^{(0)}(x) + \sum_{j=1}^k \left[\binom{k}{j} + \binom{k}{j-1} \right] f^{((k+1)-j)}(x) g^{(j)}(x) + \binom{k+1}{k+1} f^{(0)}(x) g^{(k+1)}(x) \\ & \quad \left\{ \binom{N}{0} = 1 = \binom{N}{N} \text{ for all } N \geq 0 \right\} \end{aligned}$$

$$\begin{aligned} \text{But } \binom{k}{j} + \binom{k}{j-1} &= \frac{k!}{j!(k-j)!} + \frac{k!}{(j-1)!(k-j+1)!} \\ &= \frac{(k-j+1)k! + j \times k!}{j!(k-j+1)!} \\ &= \frac{(k+1)k!}{j!((k+1)-j)!} \\ &= \frac{(k+1)!}{j!((k+1)-j)!} \\ &= \binom{k+1}{j} \end{aligned}$$

$$\begin{aligned} \text{So, } \frac{d^{k+1}}{dx^{k+1}} [f(x)g(x)] &= \binom{k+1}{0} f^{(k+1)}(x) g^{(0)}(x) + \sum_{j=1}^k \binom{k+1}{j} f^{((k+1)-j)}(x) g^{(j)}(x) + \binom{k+1}{k+1} f^{(0)}(x) g^{(k+1)}(x) \\ &= \sum_{j=0}^{k+1} \binom{k+1}{j} f^{((k+1)-j)}(x) g^{(j)}(x) \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. $\{\text{principle of mathematical induction}\}$

MIXED QUESTIONS SET 18

$$\begin{aligned}
 1 \quad (x-1)(2-x)^9 &= -(x-1)(x-2)^9 \\
 &= -(x-1)\left[x^9 + \binom{9}{1}x^8(-2) + \binom{9}{2}x^7(-2)^2 + \binom{9}{3}x^6(-2)^3 + \dots\right] \\
 &= -(x-1)\left[x^9 + \binom{9}{1}(-2)x^8 + \binom{9}{2}(-2)^2x^7 + \binom{9}{3}(-2)^3x^6 + \dots\right]
 \end{aligned}$$

$\xrightarrow{(2)} \quad \xrightarrow{(1)}$

So, the terms containing x^7 are $-\binom{9}{3}(-2)^3x^7$ from (1)

and $\binom{9}{2}(-2)^2x^7$ from (2)

\therefore the coefficient of x^7 is $-\binom{9}{3}(-2)^3 + \binom{9}{2}(-2)^2 = 816$.

2 The ordered data set is:

132	140	149	155	159	160	161	163	164	165	(20 data values)
169	171	173	181	185	191	200	207	212	303	

a Since $n = 20$, $\frac{n+1}{2} = 10.5$ \therefore the median is the average of the 10th and 11th value.

132	140	149	155	159	160	161	163	164	165
169	171	173	181	185	191	200	207	212	303

$$\begin{aligned}
 \therefore \text{median} &= \frac{\text{10th value} + \text{11th value}}{2} \\
 &= \frac{\$165 + \$169}{2} \\
 &= \$167
 \end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set in two.

lower half									
132	140	149	155	159	160	161	163	164	165
169	171	173	181	185	191	200	207	212	303
upper half									

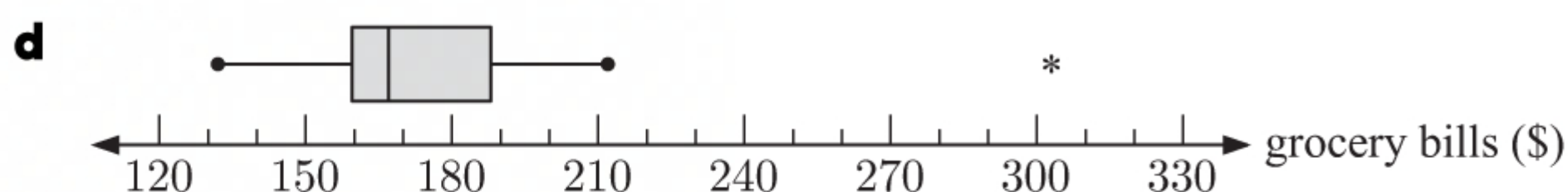
$$Q_1 = \text{median of lower half} = \frac{\$159 + \$160}{2} = \$159.50$$

$$Q_3 = \text{median of upper half} = \frac{\$185 + \$191}{2} = \$188$$

$$\begin{aligned}
 \text{b IQR} &= Q_3 - Q_1 \\
 &= \$188 - \$159.50 \\
 &= \$28.50
 \end{aligned}$$

c Test for outliers:	upper boundary	and	lower boundary
	= upper quartile + $1.5 \times \text{IQR}$		= lower quartile - $1.5 \times \text{IQR}$
	= $\$188 + 1.5 \times 28.50$		= $\$159.50 - 1.5 \times 28.50$
	= $\$230.75$		= $\$116.75$

\$303 is above the upper boundary, so it is an outlier.



$$\begin{aligned}
 3 \quad \text{a} \quad f(x) &= \frac{1}{x} - \frac{4}{x-2} \\
 &= \frac{x-2-4x}{x(x-2)} \\
 &= \frac{-3x-2}{x(x-2)}
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } f(x) = 0 \text{ when } -3x - 2 &= 0 \\
 \therefore x &= -\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= x^{-1} - 4(x-2)^{-1} \\
 \therefore f'(x) &= -x^{-2} + 4(x-2)^{-2} \\
 &= -\frac{1}{x^2} + \frac{4}{(x-2)^2} \\
 &= \frac{4x^2 - (x-2)^2}{x^2(x-2)^2} \\
 &= \frac{(2x + (x-2))(2x - (x-2))}{x^2(x-2)^2} \\
 &= \frac{(3x-2)(x+2)}{x^2(x-2)^2}
 \end{aligned}$$

$f'(x)$ has sign diagram:

$$f(-2) = \frac{1}{2} \quad \text{and} \quad f\left(\frac{2}{3}\right) = 4\frac{1}{2}$$

\therefore there is a local maximum at $(-2, \frac{1}{2})$ and a local minimum at $(\frac{2}{3}, 4\frac{1}{2})$.

$$\begin{aligned}
 \text{c} \quad f''(x) &= 2x^{-3} - 8(x-2)^{-3} \\
 &= \frac{2}{x^3} - \frac{8}{(x-2)^3}
 \end{aligned}$$

Points of inflection occur where $f''(x) = 0$

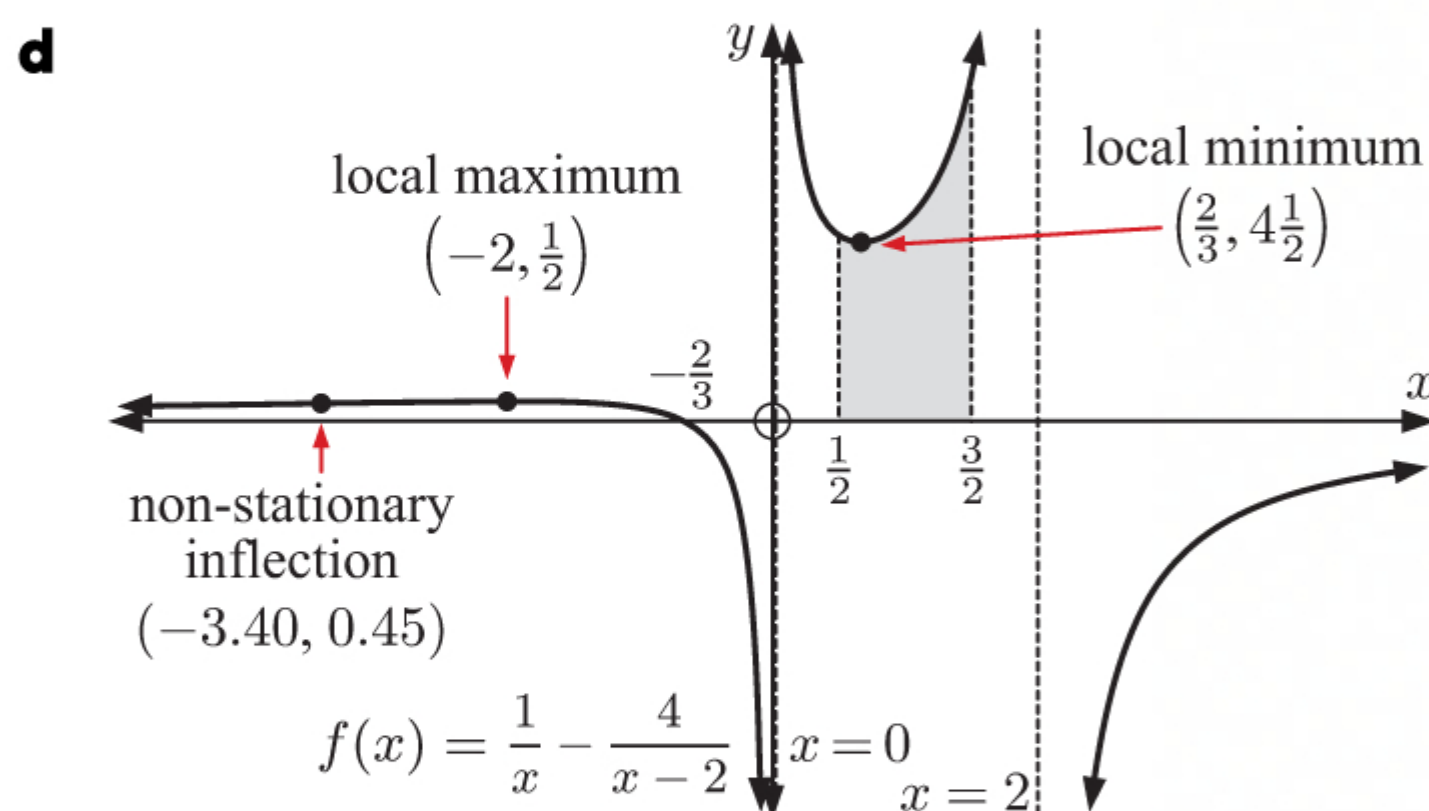
$$\therefore \frac{2}{x^3} = \frac{8}{(x-2)^3}$$

$$\therefore (x-2)^3 = 4x^3$$

$$\therefore x \approx -3.4048 \quad \{\text{using technology}\}$$

$$f(-3.4048) \approx 0.4464$$

So, $(-3.40, 0.45)$ is a non-stationary point of inflection.



$$\begin{aligned}
 \text{e} \quad \text{Area} &= \int_{\frac{1}{2}}^{\frac{3}{2}} \left(\frac{1}{x} - \frac{4}{x-2} \right) dx \\
 &= \left[\ln|x| - 4 \ln|x-2| \right]_{\frac{1}{2}}^{\frac{3}{2}} \\
 &= (\ln \frac{3}{2} - 4 \ln \frac{1}{2}) - (\ln \frac{1}{2} - 4 \ln \frac{3}{2}) \\
 &= \ln \frac{3}{2} - 4 \ln \frac{1}{2} - \ln \frac{1}{2} + 4 \ln \frac{3}{2} \\
 &= 5 \ln \frac{3}{2} - 5 \ln \frac{1}{2} \\
 &= 5(\ln \frac{3}{2} - \ln \frac{1}{2}) \\
 &= 5 \ln 3 \text{ units}^2
 \end{aligned}$$

4 a $\widehat{ADO} = \theta$ and $\widehat{OCB} = \theta$ $\{\triangle ADO \text{ and } \triangle OCB \text{ are isosceles triangles}\}$

Now $\widehat{AOD} = \pi - 2\theta$ $\{\text{angles in } \triangle ADO\}$

$\widehat{COB} = \pi - 2\theta$ $\{\text{angles in } \triangle OCB\}$

$\therefore \widehat{DOC} = \pi - 2(\pi - 2\theta)$ $\{\text{angles in a line}\}$

$$= \pi - 2\pi + 4\theta$$

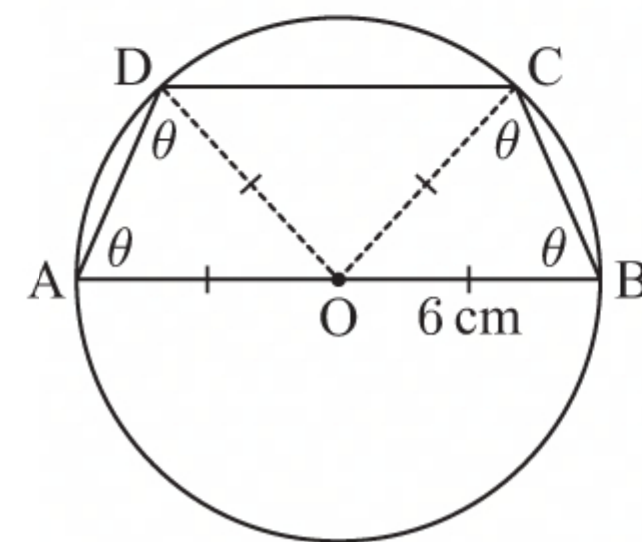
$$= 4\theta - \pi$$

$$\therefore \text{area } A = 2 \times \frac{1}{2}(6)^2 \sin(\pi - 2\theta) + \frac{1}{2}(6)^2 \sin(4\theta - \pi)$$

$$= 36 \sin(\pi - 2\theta) + 18 \sin(4\theta - \pi)$$

$$= 36 \sin 2\theta - 18 \sin 4\theta \quad \{\sin(\pi - x) = \sin x \quad \text{and} \quad \sin(x - \pi) = -\sin x\}$$

$$= 18(2 \sin 2\theta - \sin 4\theta)$$



$$\mathbf{b} \quad \frac{dA}{d\theta} = 18(4 \cos 2\theta - 4 \cos 4\theta)$$

$$\text{Now } \frac{dA}{d\theta} = 0 \quad \text{where} \quad 18(4 \cos 2\theta - 4 \cos 4\theta) = 0$$

$$\therefore 4 \cos 2\theta - 4 \cos 4\theta = 0$$

$$\therefore \cos 2\theta - \cos 4\theta = 0$$

$$\therefore \cos 2\theta - (2 \cos^2 2\theta - 1) = 0$$

$$\therefore 2 \cos^2 2\theta - \cos 2\theta - 1 = 0$$

$$\therefore 2 \cos^2 2\theta - 2 \cos 2\theta + \cos 2\theta - 1 = 0$$

$$\therefore 2 \cos 2\theta(\cos 2\theta - 1) + (\cos 2\theta - 1) = 0$$

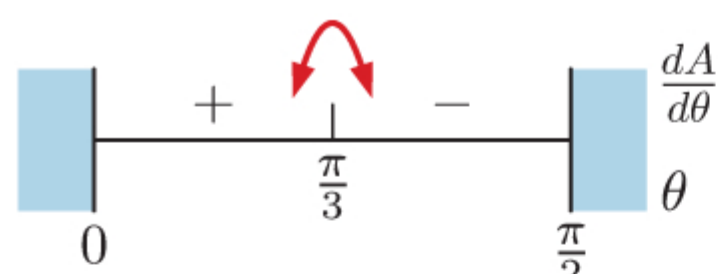
$$\therefore (\cos 2\theta - 1)(2 \cos 2\theta + 1) = 0$$

$$\therefore \cos 2\theta = 1 \quad \text{or} \quad \cos 2\theta = -\frac{1}{2}$$

$$\therefore 2\theta = 0 \quad \text{or} \quad 2\theta = \frac{2\pi}{3}$$

$$\therefore \theta = 0 \quad \text{or} \quad \theta = \frac{\pi}{3}$$

The sign diagram of $\frac{dA}{d\theta}$ is:



$\therefore A$ is maximised when $\theta = \frac{\pi}{3}$.

$\therefore \theta = \frac{\pi}{3}$ maximises the area of ABCD.

$$\mathbf{5} \quad f(x) = \tan x + \cot x, \quad 0 \leq x \leq \frac{\pi}{2}$$

$$\begin{aligned} \mathbf{a} \quad f(x) &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \\ &= \frac{1}{\sin x \cos x} \times \frac{2}{2} \\ &= \frac{2}{\sin 2x} \\ &= 2 \operatorname{cosec} 2x \end{aligned}$$

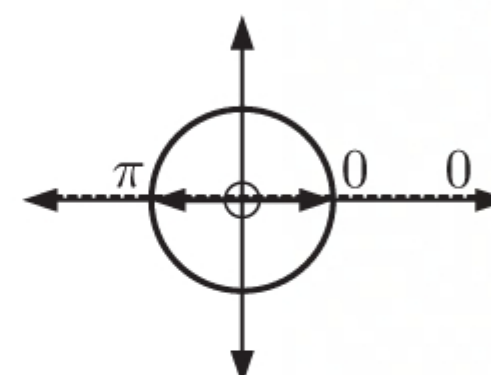
$$\mathbf{b} \quad f(x) \text{ is undefined when } \sin 2x = 0$$

$$\text{Now } 0 \leq x \leq \frac{\pi}{2}$$

$$\therefore 0 \leq 2x \leq \pi$$

$$\text{So, } 2x = 0 \text{ or } \pi$$

$$\therefore x = 0 \text{ or } \frac{\pi}{2}$$



\therefore the vertical asymptotes of $f(x)$ are $x = 0$ and $x = \frac{\pi}{2}$.

$$\mathbf{c} \quad \text{The greatest value of } \sin 2x \text{ on } 0 \leq x \leq \frac{\pi}{2} \text{ is } 1.$$

$$\therefore \text{the least value of } f(x) \text{ is } \frac{2}{1} = 2.$$

$$\text{This occurs when } \sin 2x = 1$$

$$\therefore 2x = \frac{\pi}{2} \quad \{0 \leq 2x \leq \pi\}$$

$$\therefore x = \frac{\pi}{4}$$

$$\mathbf{d} \quad \sin a = \frac{1}{3}$$

$$\therefore a = \sin^{-1}\left(\frac{1}{3}\right)$$

$$\therefore f(2a) = 2 \operatorname{cosec}(4a)$$

$$= \frac{2}{\sin\left(4 \sin^{-1}\left(\frac{1}{3}\right)\right)}$$

$$\approx 2.046$$

6	Time spent training (t hours)	2.5	1	2.5	3.5	4	2.5	2	3	3	2	1.5
	Points scored (y)	2	0	5	16	9	8	2	6	10	0	2

- a** The number of points scored can be counted exactly, whereas the time spent training is usually estimated or cannot be measured exactly.

Since the response variable y is more precisely measured than the explanatory variable t , it would be appropriate to use the regression line of t against y in this case.

b

```

LinearReg(ax+b)
a = 0.1398305
b = 1.73728813
r = 0.80201018
r^2 = 0.64322033
MSe = 0.29731638
y = ax + b
    
```

The regression line of t against y is $t \approx 0.140y + 1.74$ hours.

c i When $y = 7$, $t \approx 0.140(7) + 1.74$
 ≈ 2.72

We expect a player who scored 7 points to have spent about 2.72 hours training.

$$\begin{aligned}\text{ii} \quad & \text{When } t = 5, \quad 5 \approx 0.140y + 1.74 \\ & \therefore 3.26 \approx 0.140y \\ & \therefore y \approx 23.3\end{aligned}$$

We expect a player who spent 5 hours training to score about 23 points.

d We expect that the estimate in **c i** to be reliable because it is an interpolation.

The estimate in **c ii** is not likely to be reliable because it is an extrapolation and the linear model may not extend beyond the poles.

$$\begin{aligned}\text{7 a} \quad & \text{Since } a \text{ is a solution of the equation, } 3a^3 - 11a^2 + 8a = 12a \\ & \therefore 3a^3 - 11a^2 - 4a = 0 \\ & \therefore a(3a^2 - 11a - 4) = 0 \\ & \therefore a(3a + 1)(a - 4) = 0 \\ & \therefore a = 0, -\frac{1}{3}, \text{ or } 4\end{aligned}$$

$$\begin{aligned}\text{b} \quad & \text{If } a = 0, \quad 3x^3 - 11x^2 + 8x = 0 \\ & \therefore x(3x^2 - 11x + 8) = 0 \\ & \therefore x(3x - 8)(x - 1) = 0 \\ & \therefore x = 0, \frac{8}{3}, \text{ or } 1\end{aligned}$$

$$\begin{aligned}\text{If } a = -\frac{1}{3}, \quad & 3x^3 - 11x^2 + 8x = 12(-\frac{1}{3}) \\ & \therefore 3x^3 - 11x^2 + 8x + 4 = 0\end{aligned}$$

$x = a$ is a solution, and so $(3x + 1)$ must be a factor.

$$\therefore 3x^3 - 11x^2 + 8x + 4 = (3x + 1)(x^2 + bx + 4) \quad \text{for some } b$$

$$\begin{aligned}\text{Equating coefficients of } x^2 \text{ gives } & -11 = 1 + 3b \\ & \therefore b = -4\end{aligned}$$

$$\begin{aligned}\therefore (3x + 1)(x^2 - 4x + 4) &= 0 \\ \therefore (3x + 1)(x - 2)^2 &= 0 \\ \therefore x &= -\frac{1}{3} \text{ or } 2\end{aligned}$$

$$\begin{aligned}\text{If } a = 4, \quad & 3x^3 - 11x^2 + 8x = 12(4) \\ & \therefore 3x^3 - 11x^2 + 8x - 48 = 0\end{aligned}$$

$x = a$ is a solution, so $(x - 4)$ must be a factor.

$$\therefore 3x^3 - 11x^2 + 8x - 48 = (x - 4)(3x^2 + bx + 12) \quad \text{for some } b$$

$$\begin{aligned}\text{Equating coefficients of } x^2 \text{ gives } & -11 = b - 12 \\ & \therefore b = 1\end{aligned}$$

$$\begin{aligned}\therefore (x - 4)(3x^2 + x + 12) &= 0 \\ \therefore x = 4 \text{ or } \frac{-1 \pm \sqrt{1^2 - 4 \times 3 \times 12}}{2 \times 3} \\ \therefore x = 4 \text{ or } \frac{-1 \pm i\sqrt{143}}{6}\end{aligned}$$

$$\text{8 a} \quad \mathbf{p} = \left(1 - \frac{t}{12}\right)\mathbf{a} + \frac{t}{12}\mathbf{b}$$

$$\begin{aligned}\text{At time } t = 0, \quad & \mathbf{p} = (1 - 0)\mathbf{a} + 0\mathbf{b} \\ & \therefore \mathbf{p} = \mathbf{a}\end{aligned}$$

So, the train is at point A at time $t = 0$.

$$\begin{aligned}\text{b} \quad & \text{The train is at point B when } \mathbf{p} = 0\mathbf{a} + 1\mathbf{b} = \mathbf{b} \\ & \therefore t = 12\end{aligned}$$

It takes 12 minutes for the train to reach B.

$$\begin{aligned}\text{c i} \quad & \text{Distance from A to B} = \sqrt{(2 - 1)^2 + (2 - 3)^2 + (1 - 0)^2} \\ & = \sqrt{1 + 1 + 1} \\ & = \sqrt{3} \text{ km}\end{aligned}$$

$$\begin{aligned}\text{ii} \quad & 12 \text{ minutes} = \frac{1}{5} \text{ hour} \\ \text{speed} &= \frac{\text{distance}}{\text{time}} \\ &= \frac{\sqrt{3} \text{ km}}{\frac{1}{5} \text{ hour}} \\ &= 5\sqrt{3} \text{ km h}^{-1}\end{aligned}$$

9 $f(n) = 0.6e^{-0.6n}, \quad n \geq 0$

a $P(\text{lasts at least a year}) = P(N \geq 1)$
 $= 1 - P(0 \leq N < 1)$
 $= 1 - \int_0^1 0.6e^{-0.6n} dn$
 $= 1 + [e^{-0.6n}]_0^1$
 $= 1 + (e^{-0.6} - 1)$
 $= e^{-0.6} \approx 0.549$

b Let X be the number of operating panels after one year.

$\therefore X \sim B(8, e^{-0.6}) \quad \{\text{using a}\}$

$P(\text{calculator fails}) = P(X = 0)$
 $= \binom{8}{0} (e^{-0.6})^0 (1 - e^{-0.6})^8$
 ≈ 0.00172

10 **a** $\operatorname{arctanh} x = \frac{1}{2}[\ln(1+x) - \ln(1-x)]$

$\therefore \frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{2} \left[\frac{1}{1+x} - \frac{-1}{1-x} \right]$
 $= \frac{1}{2} \left[\frac{1}{1+x} + \frac{1}{1-x} \right]$
 $= \frac{1}{2} \left[\frac{1-x+1+x}{1-x^2} \right]$
 $= \frac{1}{1-x^2}$

b $f'(x) = x(1-x^2)^{-\frac{3}{2}}$

$\therefore f(x) = \int x(1-x^2)^{-\frac{3}{2}} dx$
 $= \int u^{-\frac{3}{2}} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \quad \left\{ u = 1-x^2, \quad \frac{du}{dx} = -2x \right\}$
 $= -\frac{1}{2} \int u^{-\frac{3}{2}} du$
 $= -\frac{1}{2} \left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \right) + c$
 $= \frac{1}{\sqrt{u}} + c$
 $= \frac{1}{\sqrt{1-x^2}} + c$

Now $V = \pi \int_0^{\frac{1}{2}} [f(x)]^2 dx$
 $= \pi \int_0^{\frac{1}{2}} \left(\frac{1}{1-x^2} + \frac{2c}{\sqrt{1-x^2}} + c^2 \right) dx$
 $= \pi \left[\operatorname{arctanh} x + 2c \arcsin x + c^2 x \right]_0^{\frac{1}{2}} \quad \{\text{using a}\}$
 $= \pi \left[\left(\operatorname{arctanh} \frac{1}{2} + 2c \arcsin \frac{1}{2} + \frac{c^2}{2} \right) - (\operatorname{arctanh} 0 + 2c \arcsin 0 + 0) \right]$
 $= \pi \left[\frac{1}{2} (\ln(\frac{3}{2}) - \ln(\frac{1}{2})) + 2c(\frac{\pi}{6}) + \frac{c^2}{2} - 0 \right] \quad \{\operatorname{arctanh} 0 = \frac{1}{2}(\ln 1 + \ln 1) = 0\}$

Thus $\pi \left[\frac{1}{2} \ln 3 + \frac{\pi}{3} c + \frac{c^2}{2} \right] \approx 14.589$

$\therefore \frac{c^2}{2} + \frac{\pi}{3} c + \frac{1}{2} \ln 3 \approx 4.6438$

$\therefore c \approx -4.09 \text{ or } 2.00$

$\therefore f(x) \approx \frac{1}{\sqrt{1-x^2}} + 2.00 \quad \text{or} \quad f(x) \approx \frac{1}{\sqrt{1-x^2}} - 4.09$

MIXED QUESTIONS SET 19

1 a $y = 3 - \frac{k}{x-1}$ has x -intercept $\frac{5}{3}$

$$\therefore 3 - \frac{k}{\frac{5}{3}-1} = 0$$

$$\therefore 3 - \frac{k}{\frac{2}{3}} = 0$$

$$\therefore \frac{3k}{2} = 3$$

$$\therefore k = 2$$

c The vertical asymptote is $x = 1$.

The horizontal asymptote is $y = 3$.

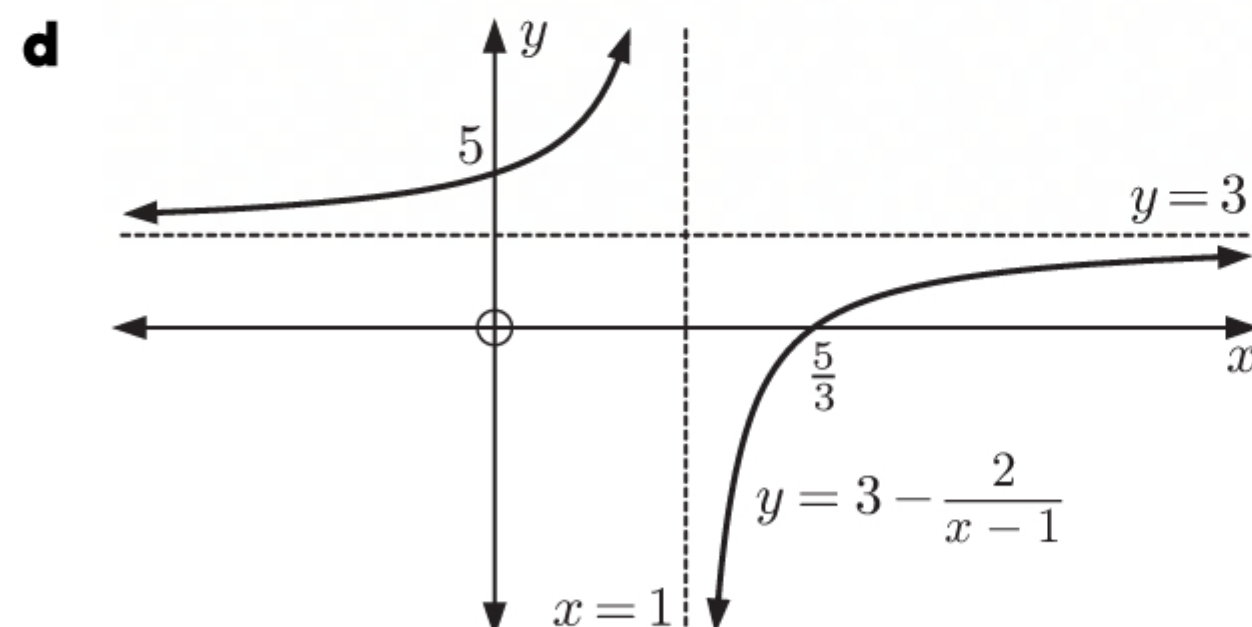
b The y -intercept occurs where $x = 0$

$$\therefore y = 3 - \frac{2}{0-1}$$

$$= 3 + 2$$

$$= 5$$

\therefore the y -intercept is 5.



2 a Total number of Year 7 students $= 30 + 27 = 57$

Let A represent a student selected from class A and

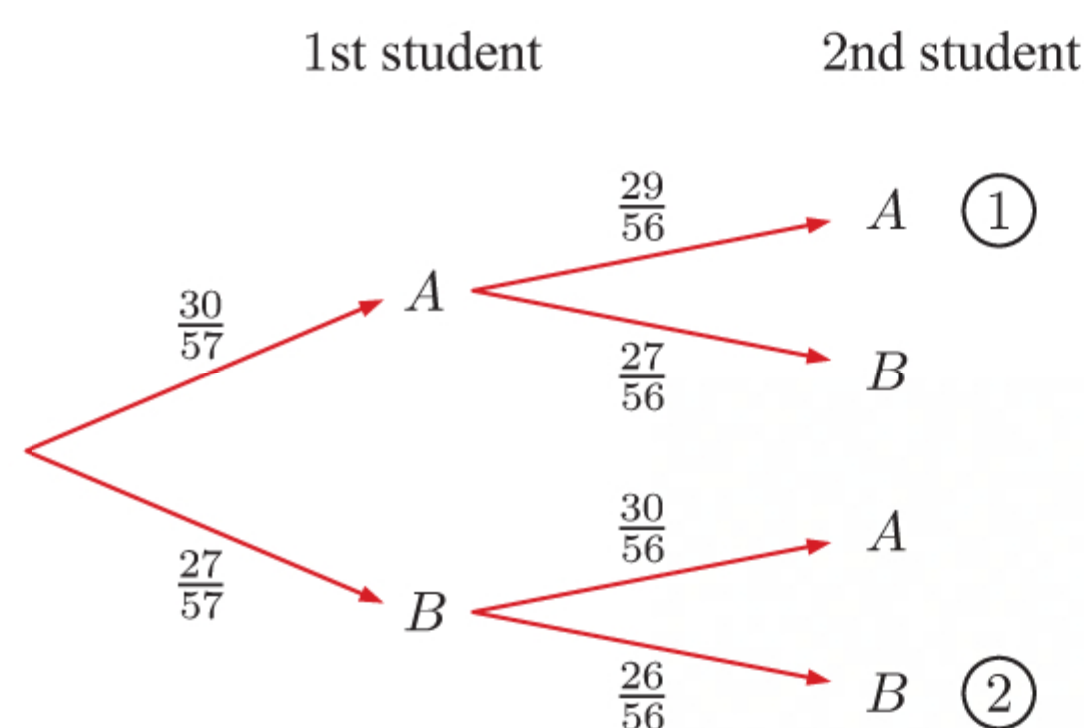
B represent a student selected from class B.

$$P(\text{same class}) = P(AA \text{ or } BB)$$

$$= \underbrace{\frac{30}{57} \times \frac{29}{56}}_{(1)} + \underbrace{\frac{27}{57} \times \frac{26}{56}}_{(2)}$$

$$= \frac{131}{266}$$

$$\approx 0.492$$



\therefore the probability that in any given week the two selected students are in the same class is $\frac{131}{266} \approx 0.492$.

b Let X be the number of weeks out of 20 that the two selected students are in the same class.

$$\therefore X \sim B\left(20, \frac{131}{266}\right) \quad \{\text{using a}\}$$

$$E(X) = np$$

$$= 20 \times \frac{131}{266}$$

$$\approx 9.85$$

\therefore we expect that the two selected students are in the same class about 9.85 times out of 20.

3 $2^a 8^b = \frac{1}{2}$ and $\frac{3^{-a}}{3^{b+1}} = 9$

$$\therefore 2^a (2^3)^b = 2^{-1}$$

$$\therefore 3^{-a} 3^{-(b+1)} = 3^2$$

$$\therefore 2^{a+3b} = 2^{-1}$$

$$\therefore 3^{-a-b-1} = 3^2$$

$$\therefore a + 3b = -1 \quad \dots (1)$$

$$\therefore -a - b - 1 = 2$$

$$\therefore a = -3 - b \quad \dots (2)$$

Substituting (2) into (1), $-3 - b + 3b = -1$

$$\therefore 2b = 2$$

$$\therefore b = 1$$

$$\therefore a = -4 \quad \{\text{using (2)}\}$$

4 a Let θ be the angle that L_1 makes with the positive x -axis.

$$\therefore \tan \theta = \frac{3}{4}$$

$$\therefore \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\approx 36.9^\circ$$

Let ϕ be the angle that L_2 makes with the positive x -axis.

$$\therefore \tan \phi = -1$$

$$\therefore \phi = 135^\circ$$

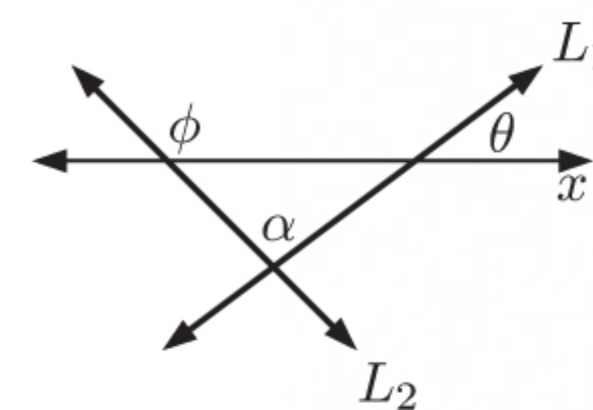
- b** Let α be the angle between L_1 and L_2 shown.

Using vertically opposite angles and the exterior angle of a triangle theorem,

$$\alpha + \theta = \phi$$

$$\begin{aligned}\therefore \alpha &\approx 135^\circ - 36.9^\circ \\ &\approx 98.1^\circ\end{aligned}$$

\therefore the acute angle between L_1 and $L_2 \approx 180^\circ - 98.1^\circ \approx 81.9^\circ$.



- 5 a** Let Maggie's eye level be at M, the car be at C, and the base of the building be at B.

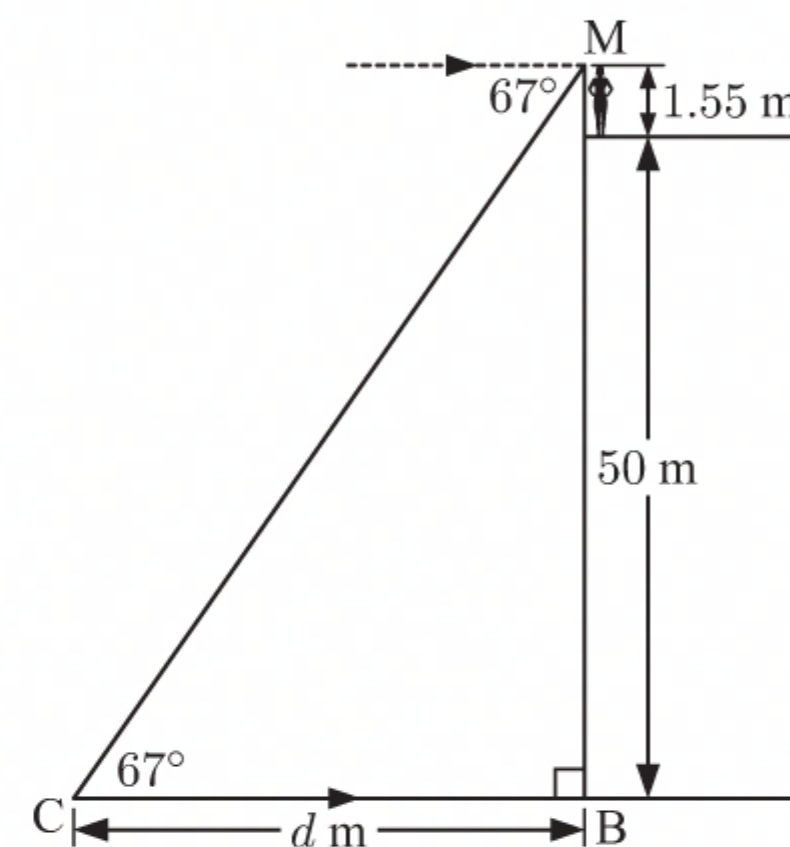
$$\begin{aligned}\text{MB} &= \text{Maggie's height} + \text{building height} \\ &= 51.55 \text{ m}\end{aligned}$$

$$\text{Now } \widehat{\text{MCB}} = 67^\circ \quad \{\text{alternate angles}\}$$

$$\therefore \tan 67^\circ = \frac{51.55}{d}$$

$$\therefore d = \frac{51.55}{\tan 67^\circ} \approx 21.9$$

So the car is about 21.9 m away from the base of the building.



- b** Let S be Sven's location.

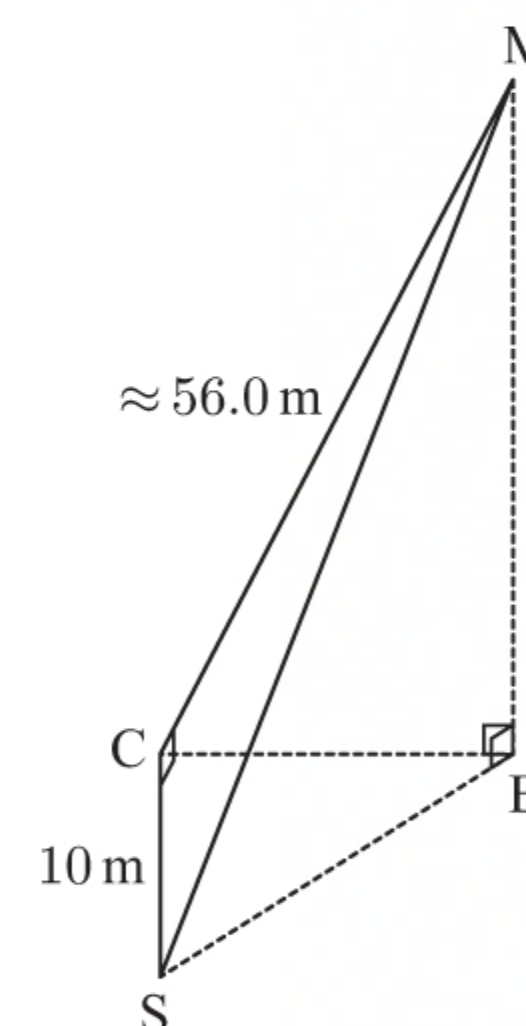
$$\begin{aligned}\text{i In } \triangle \text{MBC, } \sin 67^\circ &= \frac{51.55}{\text{MC}} \\ \therefore \text{MC} &= \frac{51.55}{\sin 67^\circ} \\ &\approx 56.0 \text{ m}\end{aligned}$$

Since the car is directly opposite to Maggie, $\triangle \text{MCS}$ is right angled at C.

$$\therefore \text{MS}^2 \approx 10^2 + 56.0^2 \quad \{\text{Pythagoras}\}$$

$$\therefore \text{MS} \approx \sqrt{3236} \approx 56.9 \text{ m}$$

The distance between Maggie and Sven is about 56.9 m.



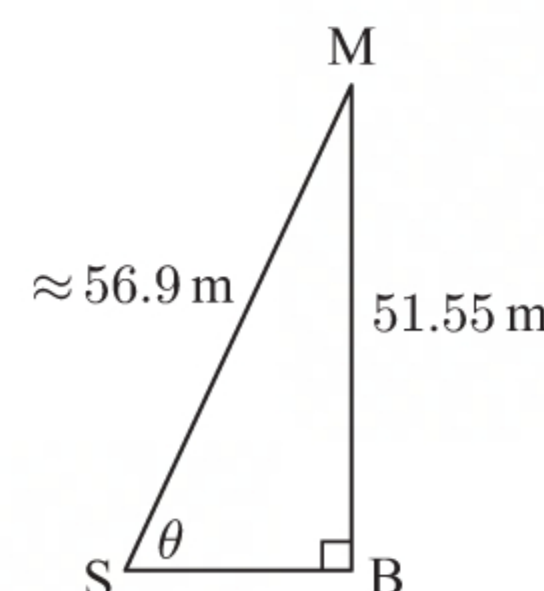
- ii** Let θ be the angle of elevation from Sven to Maggie.

Now $\triangle \text{MBS}$ is right angled at B.

$$\therefore \sin \theta \approx \frac{51.55}{56.9}$$

$$\therefore \theta \approx \sin^{-1}\left(\frac{51.55}{56.9}\right) \approx 65.0^\circ$$

Sven needs to look up at an angle of about 65.0° to see Maggie.



6 Suppose a number n is written in decimal form as

$$n = \underbrace{\text{integer part "A" with length } a}_{\text{integer part "A" with length } a} . \underbrace{\text{non-recurring decimal part "B" with length } b}_{\text{non-recurring decimal part "B" with length } b} \underbrace{\text{recurring decimal part "C" with length } c}_{\text{recurring decimal part "C" with length } c}$$

$$\therefore n = A + B \times 10^{-b} + C \times \sum_{k=1}^{\infty} 10^{-(b+kc)}$$

$$\text{Suppose } S = C \times \sum_{k=1}^{\infty} 10^{-(b+kc)}$$

$$= C \times 10^{-(b+c)} + C \times \sum_{k=2}^{\infty} 10^{-(b+kc)}$$

$$\therefore 10^c S = C \times 10^{-b} + C \times \sum_{k=2}^{\infty} 10^{-(b+(k-1)c)}$$

$$= C \times 10^{-b} + C \times \sum_{m=1}^{\infty} 10^{-(b+mc)}$$

$$= C \times 10^{-b} + S$$

$$\therefore S(10^c - 1) = C \times 10^{-b}$$

$$\therefore S = \frac{C \times 10^{-b}}{10^c - 1}$$

$$\therefore n = A + B \times 10^{-b} + \frac{C \times 10^{-b}}{10^c - 1}$$

$$= \frac{(A + B \times 10^{-b})(10^c - 1) + C \times 10^{-b}}{10^c - 1}$$

$$= \frac{(10^b A + B)(10^c - 1) + C}{10^b(10^c - 1)} \quad \text{which has the form } \frac{p}{q} \text{ where } p, q \in \mathbb{Z}.$$

Hence n is rational.

In the special case of a terminating decimal, the recurring part $C = 0$ with length $c = 1$, and $n = \frac{10^b A + B}{10^b}$.

7 $\cos 2\alpha = \sin^2 \alpha$

$$\therefore 1 - 2\sin^2 \alpha = \sin^2 \alpha$$

$$\therefore 1 = 3\sin^2 \alpha$$

$$\therefore \sin^2 \alpha = \frac{1}{3}$$

$$\therefore 1 - \cos^2 \alpha = \frac{1}{3}$$

$$\therefore \cos^2 \alpha = \frac{2}{3}$$

$$\therefore \cot^2 \alpha = \frac{\cos^2 \alpha}{\sin^2 \alpha} = 2$$

$$\therefore \cot \alpha = \pm \sqrt{2}$$

8 $f(x) = \log_2(ax + b)$, $f(6) = 4$, and $f'(1) = \frac{3}{\ln 2}$

$$\therefore f'(x) = \frac{a}{\ln 2(ax + b)}$$

$$\text{Now } f(6) = 4 \quad \therefore \log_2(6a + b) = 4$$

$$\therefore 6a + b = 2^4$$

$$\therefore 6a + b = 16 \quad \dots (1)$$

$$\text{and } f'(1) = \frac{3}{\ln 2} \quad \therefore \frac{a}{\ln 2(a + b)} = \frac{3}{\ln 2}$$

$$\therefore \frac{a}{a + b} = 3$$

$$\therefore a = 3a + 3b$$

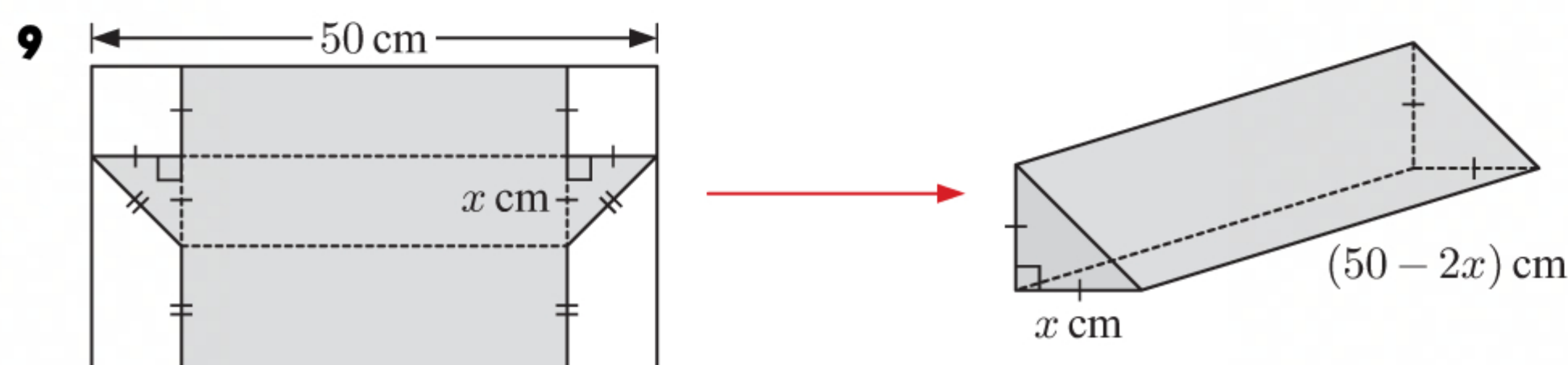
$$\therefore 2a + 3b = 0 \quad \dots (2)$$

$$\begin{array}{rcl} \text{Solving (1) and (2) simultaneously gives} & 6a + b = 16 & \{(1)\} \\ & -6a - 9b = 0 & \{(2) \times -3\} \\ \hline & \therefore -8b = 16 & \\ & \therefore b = -2 & \end{array}$$

$$\text{Substituting } b = -2 \text{ into (1) gives } 6a - 2 = 16$$

$$\therefore 6a = 18$$

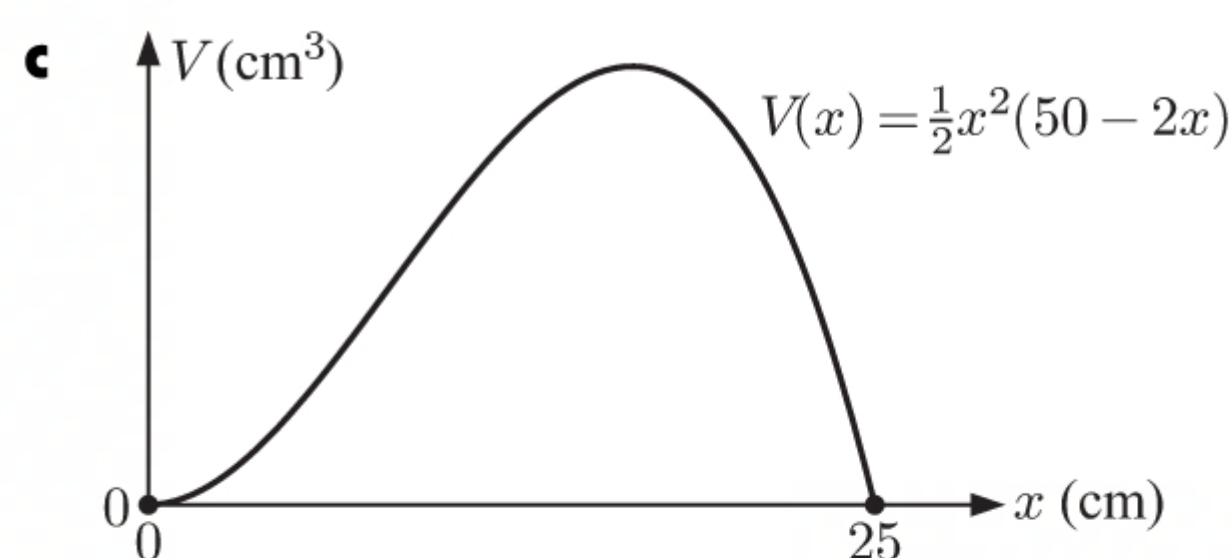
$$\therefore a = 3$$



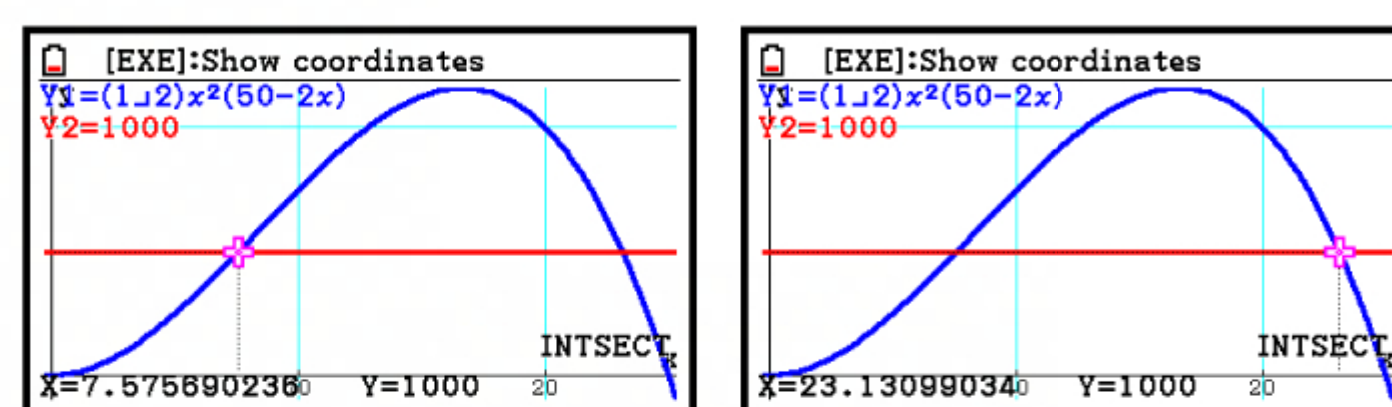
a Volume $V(x) = \text{area of cross-section} \times \text{length}$
 $= \frac{1}{2}x^2(50 - 2x) \text{ cm}^3$

b It is reasonable to use this function if $x > 0$ and $50 - 2x > 0$
 $\therefore 2x < 50$
 $\therefore x < 25$

$\therefore 0 < x < 25$

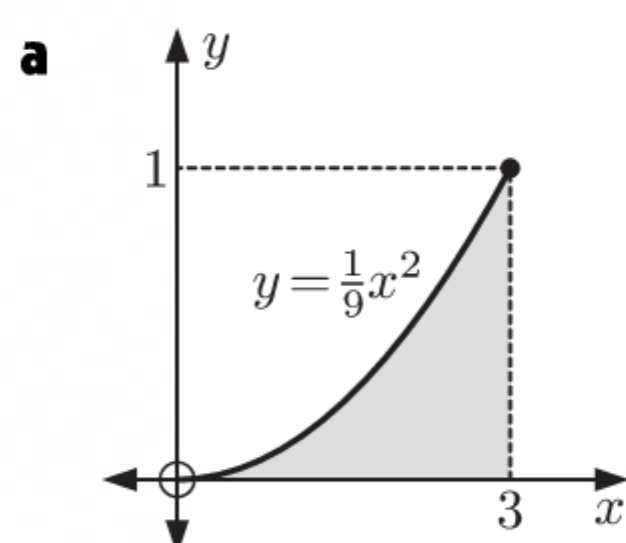


d $V(x) = 1000 \text{ cm}^3$
 $\therefore \frac{1}{2}x^2(50 - 2x) = 1000$



Using technology, $x \approx 7.58, 23.1$

10 $f(x) = \frac{1}{9}x^2, \quad 0 \leq x \leq 3$



- From the graph, $f(x) \geq 0$ for all $0 \leq x \leq 3$ ✓
- Area $= \int_0^3 \frac{1}{9}x^2 dx$
 $= \left[\frac{1}{27}x^3 \right]_0^3$
 $= \frac{27}{27}$
 $= 1$ ✓

So, $f(x)$ is a valid probability density function.

b The maximum value of $f(x)$ is when $x = 3$.

\therefore the mode $= 3$.

The median is the solution of $\int_0^m \frac{1}{9}x^2 dx = \frac{1}{2}$

$$\therefore \left[\frac{1}{27}x^3 \right]_0^m = \frac{1}{2}$$

$$\therefore \frac{m^3}{27} = \frac{1}{2}$$

$$\therefore m^3 = \frac{27}{2}$$

$$\therefore m = \sqrt[3]{\frac{27}{2}}$$

c i $E(X) = \int_0^3 x f(x) dx$
 $= \int_0^3 \frac{1}{9}x^3 dx$
 $= \left[\frac{1}{36}x^4 \right]_0^3$
 $= \frac{81}{36}$
 $= \frac{9}{4}$

ii $E(X^2) = \int_0^3 x^2 f(x) dx$
 $= \int_0^3 \frac{1}{9}x^4 dx$
 $= \left[\frac{1}{45}x^5 \right]_0^3$
 $= \frac{243}{45}$
 $= \frac{27}{5}$

iii $\sigma(X) = \sqrt{\text{Var}(X)}$
 $= \sqrt{\frac{27}{80}}$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{27}{5} - \left(\frac{9}{4}\right)^2$$

$$= \frac{27}{80}$$

d $Y = aX + b$

$$\begin{aligned}\therefore E(Y) &= E(aX + b) \\ &= aE(X) + b \\ &= a\left(\frac{9}{4}\right) + b\end{aligned}$$

Now $E(Y) = \frac{1}{2}$

$$\therefore a\left(\frac{9}{4}\right) + b = \frac{1}{2}$$

$$\therefore 9a + 4b = 2 \quad \dots (1)$$

$$\begin{aligned}\text{Var}(Y) &= \text{Var}(aX + b) \\ &= a^2 \text{Var}(X) \\ &= a^2\left(\frac{27}{80}\right)\end{aligned}$$

and $\text{Var}(Y) = \frac{27}{20}$

$$\therefore a^2\left(\frac{27}{80}\right) = \frac{27}{20}$$

$$\therefore a^2 = 40 \quad \dots (2)$$

Now from (2), $a^2 = 40$

$$\therefore a = \pm\sqrt{40} = \pm 2\sqrt{10}$$

Substituting $a = \pm 2\sqrt{10}$ into (1) gives $9(\pm 2\sqrt{10}) + 4b = 2$

$$\therefore 4b = 2 \pm 18\sqrt{10}$$

$$\therefore b = \frac{1}{2} \pm \frac{9}{2}\sqrt{10}$$

MIXED QUESTIONS SET 20

1 a f is $y = x - 2$

$$\therefore f^{-1} \text{ is } x = y - 2$$

$$\therefore x + 2 = y$$

$$\therefore f^{-1}(x) = x + 2$$

b $(g \circ f)(x) = g(f(x))$

$$= g(x - 2)$$

$$= 3 - (x - 2) - 2(x - 2)^2$$

$$= 3 - x + 2 - 2(x^2 - 4x + 4)$$

$$= -2x^2 + 7x - 3$$

c Using **b**, $(g \circ f)(-1) = -2(-1)^2 + 7(-1) - 3$
 $= -12$

2 $v = \frac{20}{\sqrt{2t+1}} \text{ m s}^{-1}, \quad 0 \leq t \leq 10$

a The brakes are applied when $t = 0$

$$\begin{aligned}\therefore v &= \frac{20}{\sqrt{0+1}} \\ &= 20 \text{ m s}^{-1}\end{aligned}$$

\therefore the speed of the truck when the brakes are applied is 20 m s^{-1} .

c The truck has acceleration $a = -2.5 \text{ m s}^{-1}$ when

$$\frac{-20}{(2t+1)^{\frac{3}{2}}} = -2.5$$

$$\therefore (2t+1)^{\frac{3}{2}} = \frac{20}{2.5} = 8$$

$$\therefore 2t+1 = 8^{\frac{2}{3}} = 4$$

$$\therefore 2t = 3$$

$$\therefore t = \frac{3}{2} \text{ seconds}$$

b $v = 20(2t+1)^{-\frac{1}{2}} \text{ m s}^{-1}$

Now $a = \frac{dv}{dt}$

$$= 20 \times \left(-\frac{1}{2}\right)(2t+1)^{-\frac{3}{2}}(2) \quad \{\text{chain rule}\}$$

$$= \frac{-20}{(2t+1)^{\frac{3}{2}}} \text{ m s}^{-2}$$

d Distance travelled $= \int_0^{10} |v| dt$

$$\begin{aligned}&= \int_0^{10} \frac{20}{\sqrt{2t+1}} dt \\ &= \int_0^{10} 20(2t+1)^{-\frac{1}{2}} dt \\ &= \left[40(2t+1)^{\frac{1}{2}}\right]_0^{10} \\ &= 40\sqrt{20+1} - 40\sqrt{1} \\ &= 40\sqrt{21} - 40 \\ &= 40(\sqrt{21} - 1) \\ &\approx 143 \text{ metres}\end{aligned}$$

- 3 a** If X kg is the mass of a sea lion, then $X \sim N(\mu, \sigma^2)$.

We start by finding z_1 and z_2 which correspond to $x_1 = 500$ and $x_2 = 900$.

$$\begin{aligned} P(X < 500) &= 0.15 \\ \therefore P\left(\frac{X - \mu}{\sigma} < \frac{500 - \mu}{\sigma}\right) &= 0.15 \\ \therefore P\left(Z < \frac{500 - \mu}{\sigma}\right) &= 0.15 \\ \therefore z_1 = \frac{500 - \mu}{\sigma} &\approx -1.0364 \\ \therefore 500 - \mu &\approx -1.0364\sigma \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{Also } P(X > 900) &= 0.1 \\ \therefore P(X \leq 900) &= 0.9 \\ \therefore P\left(\frac{X - \mu}{\sigma} \leq \frac{900 - \mu}{\sigma}\right) &= 0.9 \\ \therefore P\left(Z \leq \frac{900 - \mu}{\sigma}\right) &= 0.9 \\ \therefore z_2 = \frac{900 - \mu}{\sigma} &\approx 1.2816 \\ \therefore 900 - \mu &\approx 1.2816\sigma \quad \dots (2) \end{aligned}$$

Solving (1) and (2) simultaneously, we obtain $\mu \approx 679$ kg and $\sigma \approx 173$ kg.

$$\begin{aligned} \mathbf{b} \quad P(X < 850 \mid X > 800) &= \frac{P((X < 850) \cap (X > 800))}{P(X > 800)} \\ &= \frac{P(800 < X < 850)}{P(X > 800)} \\ &\approx \frac{0.0807}{0.242} \\ &\approx 0.333 \end{aligned}$$

So, the probability that a randomly selected sea lion weighing more than 800 kg weighs less than 850 kg is approximately 0.333.

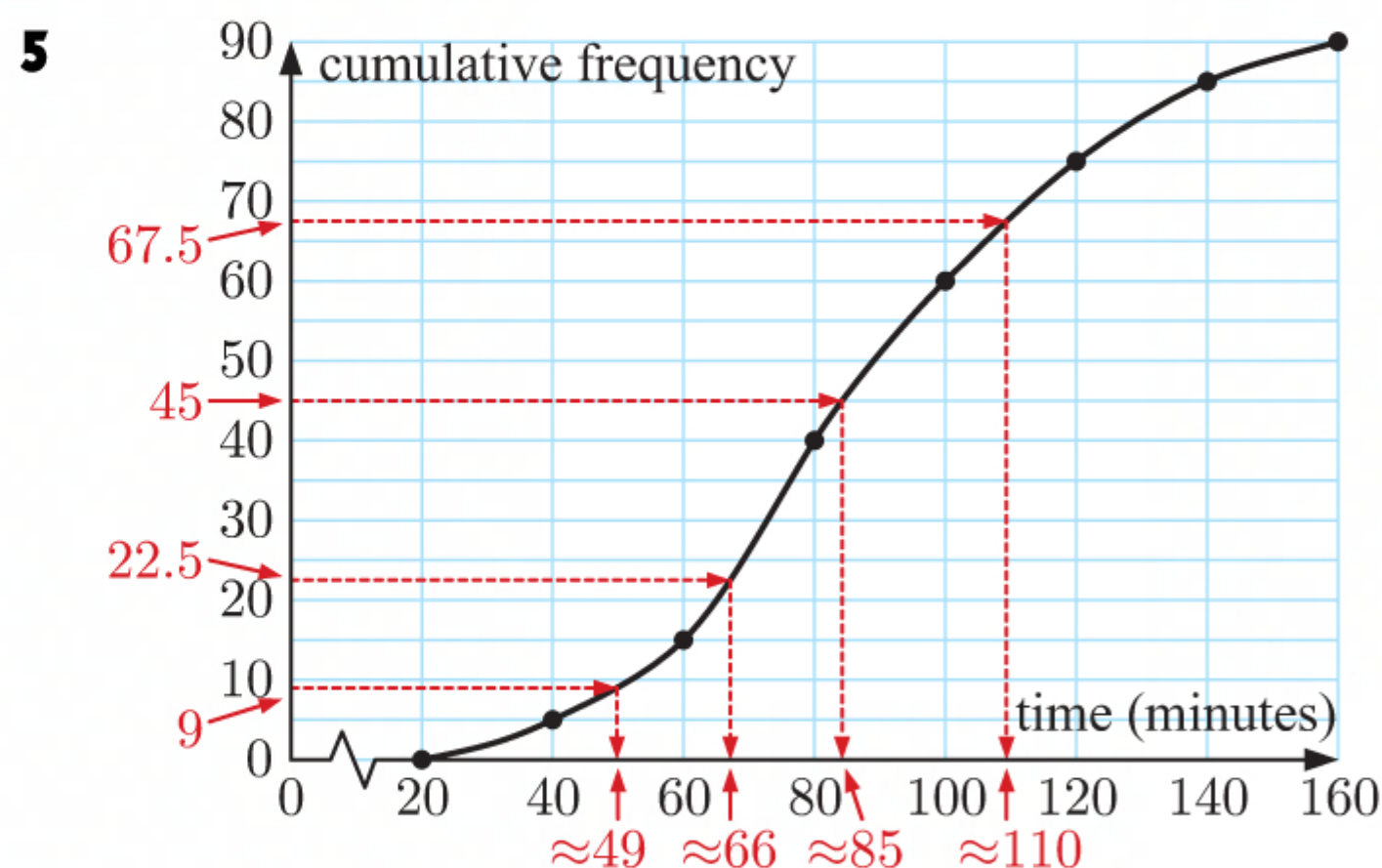
$$\begin{aligned} \mathbf{4 a} \quad f(x) &= \sin^2 x - \cos^2 x \\ \therefore f'(x) &= 2 \sin x \cos x - 2 \cos x(-\sin x) \quad \{\text{chain rule}\} \\ &= 2 \sin x \cos x + 2 \sin x \cos x \\ &= \sin 2x + \sin 2x \quad \{\text{double angle formula}\} \\ &= 2 \sin 2x \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f''(x) &= 4 \cos 2x \\ \therefore 4[f'(x)]^2 + [f''(x)]^2 &= 4(2 \sin 2x)^2 + (4 \cos 2x)^2 \\ &= 4(4 \sin^2 2x) + 16 \cos^2 2x \\ &= 16 \sin^2 2x + 16 \cos^2 2x \\ &= 16(\sin^2 2x + \cos^2 2x) \\ &= 16 \end{aligned}$$

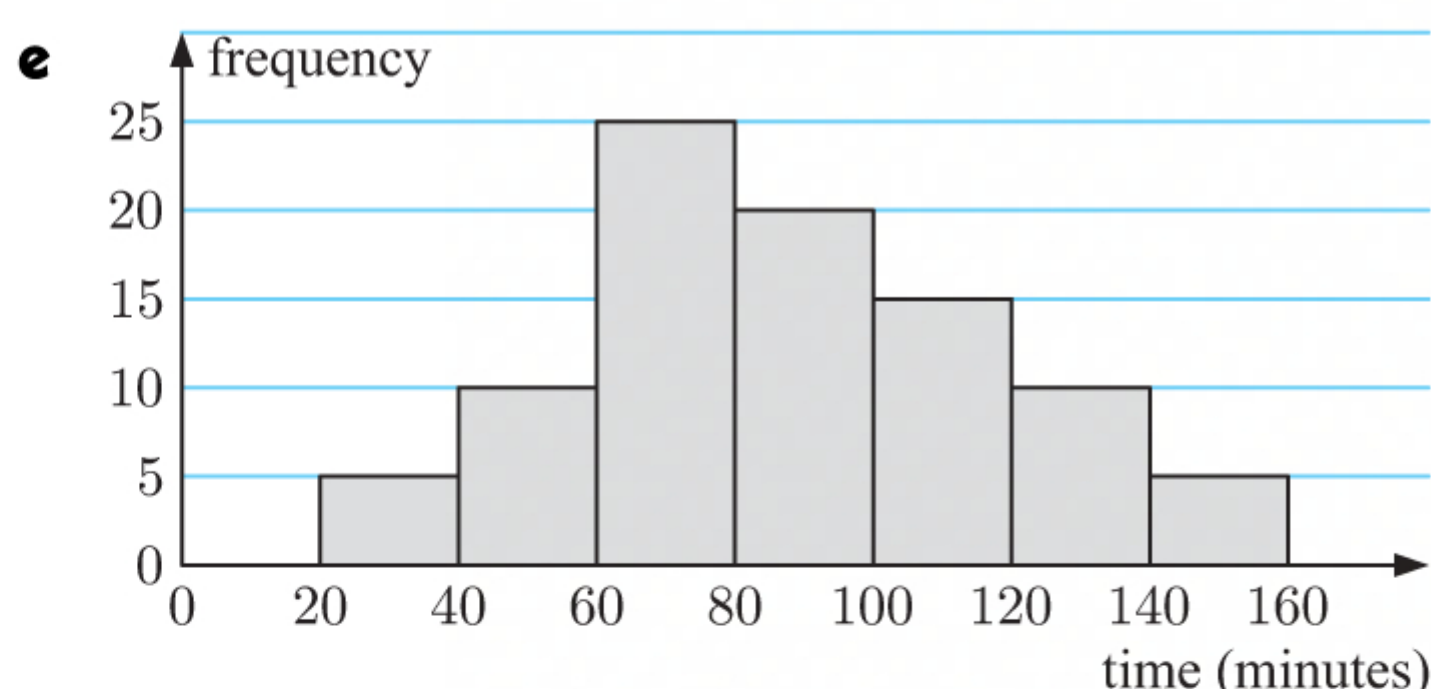
$$\begin{aligned} \mathbf{c} \quad f(\theta) &= \sin^2 \theta - \cos^2 \theta = -\cos 2\theta \quad \{\text{double angle formula}\} \\ f'(\theta) &= 2 \sin 2\theta \\ f''(\theta) &= 4 \cos 2\theta \end{aligned}$$

Now $f(\theta)$, $f'(\theta)$, and $f''(\theta)$ are consecutive terms in an arithmetic sequence.

$$\begin{aligned} \therefore f'(\theta) - f(\theta) &= f''(\theta) - f'(\theta) \quad \{\text{equating common differences}\} \\ \therefore 2 \sin 2\theta - (-\cos 2\theta) &= 4 \cos 2\theta - 2 \sin 2\theta \\ \therefore 4 \sin 2\theta &= 3 \cos 2\theta \\ \therefore \frac{\sin 2\theta}{\cos 2\theta} &= \frac{3}{4} \\ \therefore \tan 2\theta &= \frac{3}{4} \end{aligned}$$



- a** From the graph, 90 games were played.
- b** The median corresponds to cumulative frequency = 45.
Hence, the median game length is about 85 minutes.
- c** $IQR = Q_3 - Q_1$
 Q_3 corresponds to cumulative frequency = 67.5 which is ≈ 110 minutes.
 Q_1 corresponds to cumulative frequency = 22.5 which is ≈ 66 minutes.
 $\therefore IQR \approx 110 - 66 = 44$ minutes.
- d** The 10th percentile corresponds to cumulative frequency = 9 which is ≈ 49 minutes.



- 6 a** Suppose the line and the arc meet at point P where $x = k$.
 \therefore P has coordinates $(k, \sqrt{3}k)$.

$$\tan \theta = \frac{\sqrt{3}k}{k} = \sqrt{3}$$

$$\therefore \theta = \frac{\pi}{3} \text{ and } \alpha = \frac{\pi}{2} - \theta = \frac{\pi}{6}$$

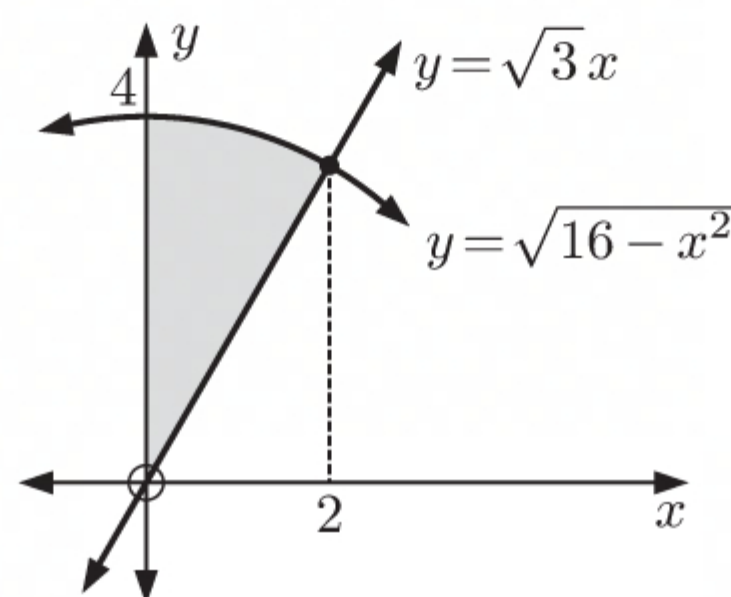
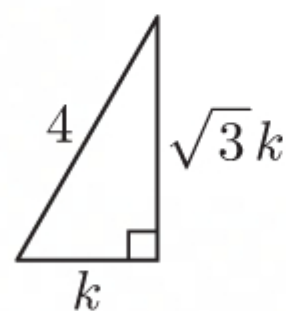
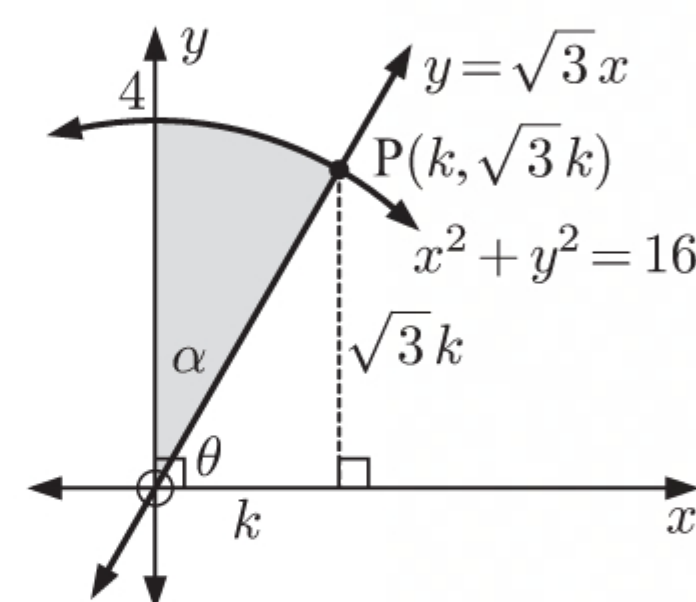
b Area $A = \frac{1}{2}\alpha r^2 = \frac{1}{2}\left(\frac{\pi}{6}\right)(4)^2 = \frac{4\pi}{3}$ units²

c Using Pythagoras, $k^2 + (\sqrt{3}k)^2 = 4^2$
 $\therefore k^2(1+3) = 16$
 $\therefore k^2 = 4$
 $\therefore k = 2 \quad \{k > 0\}$

The equation of the arc is $y^2 = 16 - x^2$

$$\therefore y = \sqrt{16 - x^2} \quad \{y > 0\}$$

Now $A = \int_0^2 (\sqrt{16 - x^2} - \sqrt{3}x) dx$
 $= \int_0^2 \sqrt{16 - x^2} dx - \sqrt{3} \int_0^2 x dx$
 $= \int_0^2 \sqrt{16 - x^2} dx - \sqrt{3} \left[\frac{1}{2}x^2 \right]_0^2$
 $= \int_0^2 \sqrt{16 - x^2} dx - \sqrt{3} \times \frac{1}{2}(2)^2$
 $= \int_0^2 \sqrt{16 - x^2} dx - 2\sqrt{3} \text{ as required}$



d Using **b** and **c**, $A = \frac{4\pi}{3} = \int_0^2 \sqrt{16-x^2} dx - 2\sqrt{3}$

$$\therefore \int_0^2 \sqrt{16-x^2} dx = \frac{4\pi}{3} + 2\sqrt{3}$$

7 Let $P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$

Since bi is a zero of the real polynomial $P(x)$, so is $-bi$.

$\therefore x^2 + b^2$ is a factor of $P(x)$

$$\begin{aligned} \therefore P(x) &= x^4 + 2x^3 + 8x^2 + 6x + 15 \\ &= (x^2 + b^2)(x^2 + cx + d) \quad \text{for some } c, d \\ &= x^4 + cx^3 + (b^2 + d)x^2 + b^2cx + b^2d \end{aligned}$$

Equating the coefficients of x^3 : $c = 2$

Equating the coefficients of x : $2b^2 = 6$

$$\therefore b = \pm\sqrt{3}$$

Equating the coefficients of x^2 : $3 + d = 8$

$$\therefore d = 5$$

$$\therefore P(x) = (x^2 + 3)(x^2 + 2x + 5)$$

Now $x^2 + 2x + 5 = 0$

$$\begin{aligned} \text{when } x &= \frac{-2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1} \\ &= -1 \pm 2i \end{aligned}$$

\therefore the zeros are $\pm\sqrt{3}i, -1 \pm 2i$.

8 Proof by contradiction:

Suppose $]a, b[$ has greatest element c .

$\therefore c \geq x$ for all $x \in]a, b[$ and $a < c < b$.

Consider $y = \frac{b+c}{2}$.

Now $c < y < b$. So $y \in]a, b[$ and $y > c$.

But c is the greatest element, so this is a contradiction.

\therefore our original supposition is false, and so $]a, b[$ has no greatest element.

9 a i Area of base plane $= \frac{1}{2} |\mathbf{b} \times \mathbf{c}|$ units²

ii Let h be the perpendicular height.

$$\therefore \sin \theta = \frac{h}{|\mathbf{a}|}$$

$$\therefore h = |\mathbf{a}| \sin \theta \text{ units}$$

b Volume

$=$ area of base \times perpendicular height

$$= \frac{1}{2} |\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \sin \theta \quad \{\text{using } \mathbf{a} \text{ i and } \mathbf{a} \text{ ii}\}$$

$$= \frac{1}{2} |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \sin \theta \text{ units}^3$$

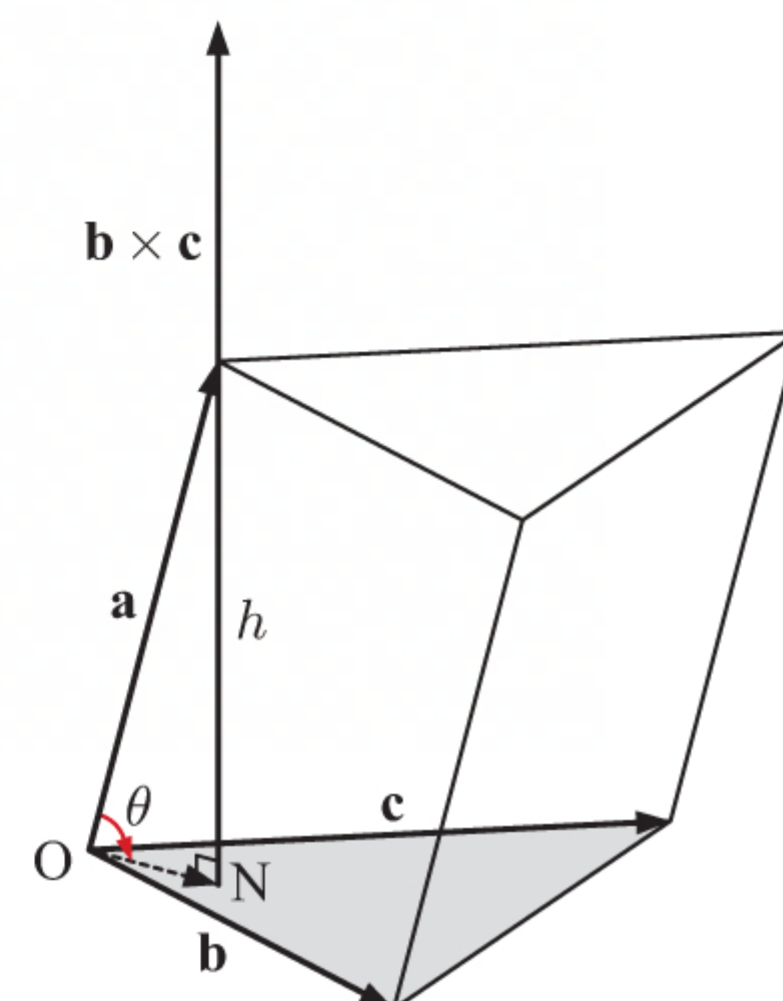
Now θ is the angle between \mathbf{a} and the base plane.

$$\therefore \theta = \sin^{-1} \left(\frac{|(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}|}{|\mathbf{b} \times \mathbf{c}| |\mathbf{a}|} \right)$$

$$\therefore \sin \theta = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a}| |\mathbf{b} \times \mathbf{c}|}$$

$$\therefore |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \sin \theta$$

$$\text{So, volume} = \frac{1}{2} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \text{ units}^3.$$



$$\bullet \quad \vec{OA} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{OB} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{OC} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \vec{OB} \times \vec{OC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k} \\ &= 4\mathbf{i} - 6\mathbf{k} \\ &= \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Volume of shape} &= \frac{1}{2} \left| \vec{OA} \bullet (\vec{OB} \times \vec{OC}) \right| \\ &= \frac{1}{2} \left| \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} \right| \\ &= \frac{1}{2} |8 - 6| \\ &= 1 \text{ unit}^3 \end{aligned}$$

10

$$x^2y + xy^3 = 6$$

$$\therefore 2xy + x^2 \frac{dy}{dx} + y^3 + 3xy^2 \frac{dy}{dx} = 0$$

$$\therefore (x^2 + 3xy^2) \frac{dy}{dx} = -2xy - y^3$$

$$\therefore \frac{dy}{dx} = \frac{-2xy - y^3}{x^2 + 3xy^2}$$

$$\begin{aligned} \text{At P, } x = -1 \text{ and } y = -2 \quad \therefore \frac{dy}{dx} &= \frac{-2(-1)(-2) - (-2)^3}{(-1)^2 + 3(-1)(-2)^2} \\ &= \frac{-4 + 8}{1 - 12} \\ &= -\frac{4}{11} \end{aligned}$$

$$\therefore \text{the tangent at P has equation } y + 2 = -\frac{4}{11}(x + 1)$$

$$\therefore y + 2 = -\frac{4}{11}x - \frac{4}{11}$$

$$\therefore y = -\frac{4}{11}x - \frac{26}{11}$$

$$\text{At Q, } x = 2 \text{ and } y = 1 \quad \therefore \frac{dy}{dx} = \frac{-2(2)(1) - (1)^3}{(2)^2 + 3(2)(1)^2}$$

$$= \frac{-4 - 1}{4 + 6}$$

$$= -\frac{5}{10}$$

$$= -\frac{1}{2}$$

$$\therefore \text{the normal at Q has equation } y - 1 = 2(x - 2)$$

$$\therefore y - 1 = 2x - 4$$

$$\therefore y = 2x - 3$$

$$\text{The tangent at P and the normal at Q intersect when } -\frac{4}{11}x - \frac{26}{11} = 2x - 3$$

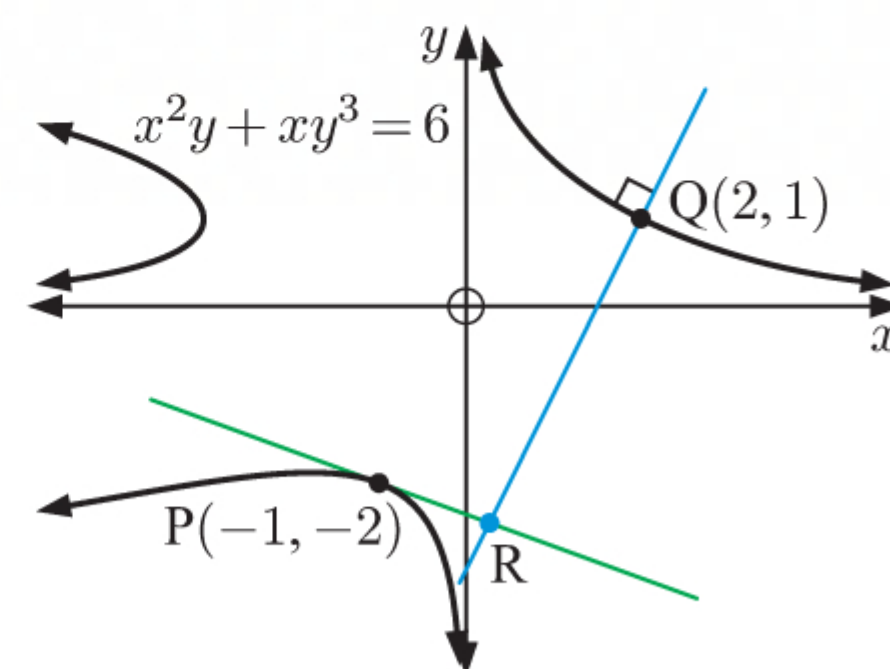
$$\therefore -4x - 26 = 22x - 33$$

$$\therefore -26x = -7$$

$$\therefore x = \frac{7}{26}$$

$$\therefore y = 2\left(\frac{7}{26}\right) - 3 = -\frac{32}{13}$$

$$\therefore \text{R has coordinates } \left(\frac{7}{26}, -\frac{32}{13}\right).$$



TRIAL EXAMINATION 1

PAPER 1

Section A

$$1 \quad \binom{n-1}{r} = \frac{(n-1)!}{r!(n-1-r)!} \quad \text{A1}$$

$$= \frac{(n-1)!}{r!(n-r-1)!} \times \frac{n-r}{n-r}$$

$$= \frac{(n-1)!(n-r)}{r!(n-r)!} \quad \text{A1}$$

$$\binom{n-1}{r-1} = \frac{(n-1)!}{(r-1)!((n-1)-(r-1))!} \quad \text{A1}$$

$$= \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= \frac{(n-1)!}{(r-1)!(n-r)!} \times \frac{r}{r}$$

$$= \frac{(n-1)! \times r}{r!(n-r)!} \quad \text{A1}$$

$$\therefore \binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!(n-r)}{r!(n-r)!} + \frac{(n-1)! \times r}{r!(n-r)!} \quad \text{A1}$$

$$= \frac{(n-1)!(n-r+r)}{r!(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

$$= \binom{n}{r} \quad \text{A1}$$

Total [6 marks]

$$2 \quad x^2 + 5kx - (5k - 11) = 0 \quad \therefore \begin{cases} \alpha + \beta = -5k \\ \alpha\beta = 11 - 5k \end{cases} \quad \text{A1A1}$$

$$\text{Now } \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (-5k)^2 - 2(11 - 5k)$$

$$= 25k^2 - 22 + 10k \quad \text{M1}$$

$$\text{So, } 25k^2 + 10k - 22 = 58 \quad \{\alpha^2 + \beta^2 = 58\} \quad \text{A1}$$

$$\therefore 25k^2 + 10k - 80 = 0$$

$$\therefore 5k^2 + 2k - 16 = 0$$

$$\therefore (5k - 8)(k + 2) = 0$$

$$\therefore k = \frac{8}{5} \text{ or } -2. \quad \text{A1A1}$$

Total [6 marks]

$$3 \quad \text{a} \quad \sin x \cos x \tan x = \sin x \cos x \left(\frac{\sin x}{\cos x} \right)$$

$$= \sin^2 x \quad \text{M1}$$

$$= 1 - \cos^2 x \quad \text{M1AG}$$

$$\text{b} \quad 4 \sin x \cos x = \frac{1}{\tan x} \Rightarrow 4 \sin x \cos x \tan x = 1 \Rightarrow \sin x \cos x \tan x = \frac{1}{4} \quad \text{M1}$$

$$\text{Using part a, we have } 1 - \cos^2 x = \frac{1}{4}$$

$$\therefore \cos^2 x = \frac{3}{4}$$

$$\therefore \cos x = \pm \frac{\sqrt{3}}{2} \quad \text{A1}$$

$$\therefore x = \pm \frac{\pi}{6}, \pm \frac{5\pi}{6} \quad \text{A1A1}$$

Total [6 marks]

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \sum P(X = x) = 1 \\
 & \therefore \frac{1}{6} + \frac{1}{3} + a + 2a + \frac{1}{6} = 1 \\
 & \therefore 3a + \frac{2}{3} = 1 \\
 & \therefore a = \frac{1}{9}
 \end{aligned}$$

A1

$$\begin{aligned}
 \mathbf{b} \quad E(X) &= \sum x P(X = x) \\
 &= 0 \times \frac{1}{6} + 1 \times \frac{1}{3} + 2 \times \frac{1}{9} + 3 \times \frac{2}{9} + 4 \times \frac{1}{6} \\
 &= \frac{17}{9}
 \end{aligned}$$

M1
A1

$$\begin{aligned}
 \mathbf{c} \quad E(X^2) &= 0^2 \times \frac{1}{6} + 1^2 \times \frac{1}{3} + 2^2 \times \frac{1}{9} + 3^2 \times \frac{2}{9} + 4^2 \times \frac{1}{6} \\
 &= \frac{1}{3} + \frac{4}{9} + \frac{18}{9} + \frac{16}{6} \\
 &= \frac{49}{9}
 \end{aligned}$$

A1

$$\begin{aligned}
 \text{Var}(X) &= E(X^2) - (E(X))^2 \\
 &= \frac{49}{9} - \left(\frac{17}{9}\right)^2 \\
 &= \frac{49}{9} - \frac{289}{81} \\
 &= \frac{152}{81}
 \end{aligned}$$

M1
A1

$$\begin{aligned}
 \mathbf{d} \quad E(Y) &= \frac{8}{3} \\
 \therefore k E(X) - k &= \frac{8}{3} \\
 \therefore \frac{17k}{9} - k &= \frac{8}{3} \\
 \therefore k\left(\frac{8}{9}\right) &= \frac{8}{3} \\
 \therefore k &= 3
 \end{aligned}$$

M1
A1

$$\begin{aligned}
 \text{So, } \text{Var}(Y) &= 3^2 \times \text{Var}(X) \\
 &= 9 \times \frac{152}{81} \\
 &= \frac{152}{9}
 \end{aligned}$$

M1
A1

Total [10 marks]

5 The centre of the circle is $(0, a)$. Substituting $y = x^2$ into the equation of the circle, we have

$$\begin{aligned}
 & x^2 + (x^2 - a)^2 = 1 \\
 \therefore x^2 + x^4 - 2ax^2 + a^2 &= 1 \\
 \therefore x^4 + (1 - 2a)x^2 + a^2 - 1 &= 0
 \end{aligned}$$

M1A1
A1

which is a quadratic in x^2 .

Since the circle and the parabola touch at their points of intersection, and are both symmetric about the y -axis, the equation has one solution for x^2 .

$$\begin{aligned}
 \therefore \text{the quadratic has discriminant zero.} \\
 \therefore (1 - 2a)^2 - 4(1)(a^2 - 1) &= 0 \\
 \therefore 1 - 4a + 4a^2 - 4a^2 + 4 &= 0 \\
 \therefore a &= \frac{5}{4}
 \end{aligned}$$

M1
A1
A1

Therefore, the coordinates of the centre of the circle are $(0, \frac{5}{4})$.

A1

Total [7 marks]

$$\mathbf{6} \quad \mathbf{a} \quad (4)^2 + (2)^3 - 3(4)(2) = 16 + 8 - 24 = 0$$

M1AG

$$\begin{aligned}
 \mathbf{b} \quad & \frac{d}{dx}(x^2 + y^3 - 3xy) = \frac{d}{dx}(0) \\
 2x + 3y^2 \frac{dy}{dx} - 3y - 3x \frac{dy}{dx} &= 0 \\
 \therefore \frac{dy}{dx}(3y^2 - 3x) &= 3y - 2x \\
 \therefore \frac{dy}{dx} &= \frac{3y - 2x}{3y^2 - 3x}
 \end{aligned}$$

M1
A1A1
M1AG

- c** At the point where $x = k$, the tangent has gradient $\frac{3y - 2k}{3y^2 - 3k}$.

This tangent is parallel to the x -axis if $\frac{3y - 2k}{3y^2 - 3k} = 0$

M1

$$\therefore 3y - 2k = 0$$

$$\therefore y = \frac{2}{3}k$$

A1

Substituting $(k, \frac{2}{3}k)$ into $x^2 + y^3 - 3xy = 0$,

$$k^2 + \left(\frac{2}{3}k\right)^3 - 3k\left(\frac{2}{3}\right) = 0$$

M1

$$\therefore k^2 + \frac{8}{27}k^3 - 2k^2 = 0$$

$$\therefore k^2\left(\frac{8}{27}k - 1\right) = 0$$

A1

$$\therefore k = \frac{27}{8} \quad \{k \neq 0\}$$

A1

Total [10 marks]

- 7 a** Using De Moivre's theorem, we may write

$$w^3 = 2i$$

$$\therefore w^3 = 2 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)$$

$$\therefore w = \sqrt[3]{2} \operatorname{cis}\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)$$

M1

$$\therefore w = \sqrt[3]{2} \operatorname{cis} \frac{\pi}{6}, \sqrt[3]{2} \operatorname{cis} \frac{5\pi}{6}, \text{ or } \sqrt[3]{2} \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

A1A1A1

- b** Note that $\sqrt[3]{2} \operatorname{cis}\left(-\frac{\pi}{2}\right) = -\sqrt[3]{2}i$, which has no real part.

(M1)

Let $w_1 = \sqrt[3]{2} \operatorname{cis} \frac{\pi}{6}$

$$\therefore w_1 = \sqrt[3]{2} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$= \frac{\sqrt{3} \sqrt[3]{2}}{2} + \frac{\sqrt[3]{2}}{2}i$$

A1

Let $w_2 = \sqrt[3]{2} \operatorname{cis} \frac{5\pi}{6}$

$$\therefore w_2 = \sqrt[3]{2} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$$

$$= -\frac{\sqrt{3} \sqrt[3]{2}}{2} + \frac{\sqrt[3]{2}}{2}i$$

A1

c $w_3 = \sqrt[3]{2} \operatorname{cis}\left(-\frac{\pi}{2}\right)$

$$\therefore w_3^{10} = \left(\sqrt[3]{2} \operatorname{cis}\left(-\frac{\pi}{2}\right)\right)^{10}$$

M1

$$= \left(2^{\frac{1}{3}}\right)^{10} \operatorname{cis}(-5\pi)$$

M1

$$= 2^{\frac{10}{3}} \times -1$$

$$= -2^3 \times \sqrt[3]{2}$$

$$= -8\sqrt[3]{2}$$

A1

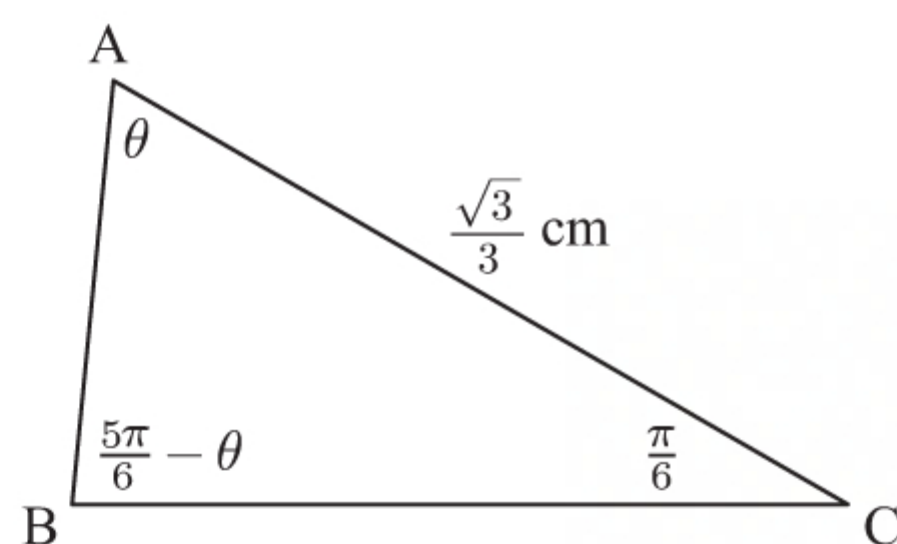
Total [14 marks]

Section B

- 8 a** $\widehat{ABC} = \pi - \frac{\pi}{6} - \theta = \frac{5\pi}{6} - \theta$

M1A1

- b** Using the sine rule, we have



$$\frac{AB}{\sin \frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{3}}{\sin\left(\frac{5\pi}{6} - \theta\right)} \quad \{\text{sine rule}\}$$

$$\therefore AB = \frac{\sqrt{3}}{6 \sin\left(\frac{5\pi}{6} - \theta\right)}$$

M1

$$= \frac{\sqrt{3}}{6 \left(\sin \frac{5\pi}{6} \cos \theta - \cos \frac{5\pi}{6} \sin \theta \right)}$$

M1

$$= \frac{\sqrt{3}}{6 \left(\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta \right)}$$

$$= \frac{\sqrt{3}}{3(\cos \theta + \sqrt{3} \sin \theta)}$$

A1AG

$$\begin{aligned}
 \text{c} \quad \frac{d(AB)}{d\theta} &= \frac{0 - (\sqrt{3})(3 \times (-\sin \theta) + 3\sqrt{3} \cos \theta)}{9(\cos \theta + \sqrt{3} \sin \theta)^2} \\
 &= \frac{3\sqrt{3} \sin \theta - 9 \cos \theta}{9(\cos \theta + \sqrt{3} \sin \theta)^2} && \text{M1} \\
 &= \frac{\sqrt{3} \sin \theta - 3 \cos \theta}{3(\cos \theta + \sqrt{3} \sin \theta)^2} && \text{A2A2}
 \end{aligned}$$

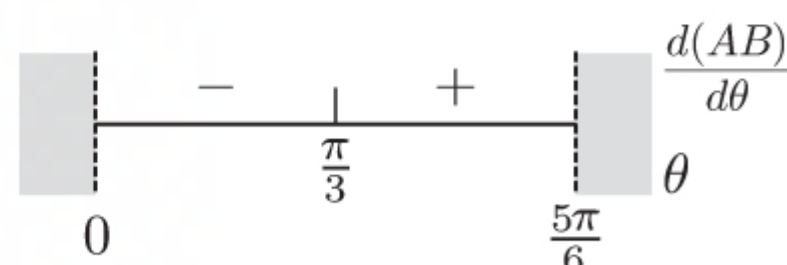
$$\text{d} \quad \frac{\sqrt{3} \sin \theta - 3 \cos \theta}{3(\cos \theta + \sqrt{3} \sin \theta)^2} = 0 \quad \text{when} \quad \sqrt{3} \sin \theta - 3 \cos \theta = 0 && \text{M1}$$

$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{3}{\sqrt{3}}$$

$$\therefore \tan \theta = \sqrt{3}$$

$$\therefore \theta = \frac{\pi}{3} \quad \{0 < \theta < \frac{5\pi}{6}\}$$

\therefore AB is minimised when $\theta = \frac{\pi}{3}$.



M1A1

$$\begin{aligned}
 \text{e} \quad \text{When } \theta = \frac{\pi}{3}, \quad AB &= \frac{\sqrt{3}}{3(\cos \frac{\pi}{3} + \sqrt{3} \sin \frac{\pi}{3})} \\
 &= \frac{\sqrt{3}}{3(\frac{1}{2} + \sqrt{3} \frac{\sqrt{3}}{2})} \\
 &= \frac{\sqrt{3}}{6} && \text{M1A1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area of triangle ABC} &= \frac{1}{2} \left(\frac{\sqrt{3}}{6} \right) \left(\frac{\sqrt{3}}{3} \right) \sin \frac{\pi}{3} \\
 &= \frac{1}{2} \times \frac{3}{18} \times \frac{\sqrt{3}}{2} \\
 &= \frac{\sqrt{3}}{24} \text{ cm}^2 && \text{M1A1}
 \end{aligned}$$

Total [17 marks]

$$\begin{aligned}
 9 \quad \text{a} \quad f'(x) &= \cos x \cos x - \sin x \sin x \\
 &= \cos^2 x - \sin^2 x && \text{M1A1} \\
 &= \cos 2x && \text{M1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore f''(x) &= -2 \sin 2x \\
 &= -2(2 \sin x \cos x) \\
 &= -4 \sin x \cos x && \text{A1AG}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f'''(x) &= (-4 \sin x)(-\sin x) + (-4 \cos x)(\cos x) \\
 &= 4 \sin^2 x - 4 \cos^2 x \\
 &= 4(\sin^2 x - \cos^2 x) && \text{M1A1} \\
 &= -4 \cos 2x && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f^{(4)}(x) &= 8 \sin 2x \\
 &= 8(2 \sin x \cos x) && \text{M1} \\
 &= 16 \sin x \cos x && \text{A1AG}
 \end{aligned}$$

$$\text{d} \quad (1) \quad \text{From a, } f^{(2)}(x) = -4 \sin x \cos x = (-4)^1 \sin x \cos x. \text{ So, } P_1 \text{ is true.} && \text{M1R1}$$

$$(2) \quad \text{If } P_k \text{ is true then let us assume } f^{(2k)}(x) = (-4)^k \sin x \cos x && \text{M1}$$

$$\begin{aligned}
 \text{Then } f^{(2k+1)}(x) &= -(-4)^k \sin^2 x + (-4)^k \cos^2 x \\
 &= (-4)^k (\cos^2 x - \sin^2 x) && \text{M1} \\
 &= (-4)^k \cos 2x && \text{M1A1}
 \end{aligned}$$

$$\therefore f^{(2k+2)}(x) = -2 \times (-4)^k \sin 2x && \text{M1}$$

$$= -2 \times (-4)^k \times 2 \sin x \cos x$$

$$= (-4)^{k+1} \sin x \cos x && \text{M1A1}$$

$$\text{Thus, } f^{(2(k+1))}(x) = (-4)^{k+1} \sin x \cos x$$

Therefore, since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for all $n \in \mathbb{Z}^+$. && R1

Total [19 marks]

$$\mathbf{10} \quad \mathbf{a} \quad \mathbf{i} \quad \overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix} \quad \mathbf{M1A1}$$

$$\overrightarrow{AC} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} \quad \mathbf{A1}$$

$$\mathbf{ii} \quad r = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} \quad \mathbf{A1A1}$$

$$\mathbf{b} \quad \begin{pmatrix} -3 \\ -6 \\ a \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -3 \\ -6 \\ a \end{pmatrix} = \begin{pmatrix} 2 + 3\lambda + \mu \\ -3 - 3\lambda - 5\mu \\ -3 - 6\lambda - 7\mu \end{pmatrix} \quad \mathbf{M1}$$

$$\therefore -3 = 2 + 3\lambda + \mu \quad \text{and} \quad -6 = -3 - 3\lambda - 5\mu$$

$$\therefore -5 = 3\lambda + \mu \quad \dots (1) \quad \therefore -3 = -3\lambda - 5\mu \quad \dots (2)$$

$$\text{Adding (1) and (2) gives} \quad -8 = -4\mu$$

$$\therefore \mu = 2 \quad \mathbf{M1A1}$$

$$\therefore \text{ in (1), } -5 = 3\lambda + 2$$

$$\therefore \lambda = -\frac{7}{3} \quad \mathbf{A1}$$

$$\text{Now } a = -3 - 6\lambda - 7\mu$$

$$\therefore a = -3 - 6\left(-\frac{7}{3}\right) - 7(2) = -3 \quad \mathbf{A1}$$

$$\mathbf{c} \quad \text{The normal vector of the plane } \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix} \times \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} \quad \mathbf{M1}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & -6 \\ 1 & -5 & -7 \end{vmatrix}$$

$$= \mathbf{i}(21 - 30) - \mathbf{j}(-21 + 6) + \mathbf{k}(-15 + 3)$$

$$= \begin{pmatrix} -9 \\ 15 \\ -12 \end{pmatrix} \quad \mathbf{A1}$$

Therefore, the Cartesian equation of plane Π is

$$-9x + 15y - 12z = -9(1) + 15(2) - 12(4)$$

$$\therefore -9x + 15y - 12z = -27$$

$$\therefore 3x - 5y + 4z = 9 \quad \mathbf{A1AG}$$

$$\mathbf{d} \quad \text{Area} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2} \left| \begin{pmatrix} -9 \\ 15 \\ -12 \end{pmatrix} \right| \quad \mathbf{M1A1}$$

$$= \frac{1}{2} \sqrt{(-9)^2 + 15^2 + (-12)^2}$$

$$= \frac{1}{2} \sqrt{450}$$

$$= \frac{15}{2} \sqrt{2} \text{ units}^2 \quad \mathbf{A1}$$

$$\begin{aligned}
 \text{e } \cos \widehat{BAC} &= \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| \times |\vec{AC}|} \\
 &= \frac{3 + 15 + 42}{\sqrt{9 + 9 + 36} \times \sqrt{1 + 25 + 49}} \\
 &= \frac{60}{\sqrt{54} \times \sqrt{75}} \\
 &= \frac{60}{3\sqrt{6} \times 5\sqrt{3}} \\
 &= \frac{60}{45\sqrt{2}} \\
 &= \frac{60\sqrt{2}}{90} \\
 &= \frac{2\sqrt{2}}{3}
 \end{aligned}$$

M1A1

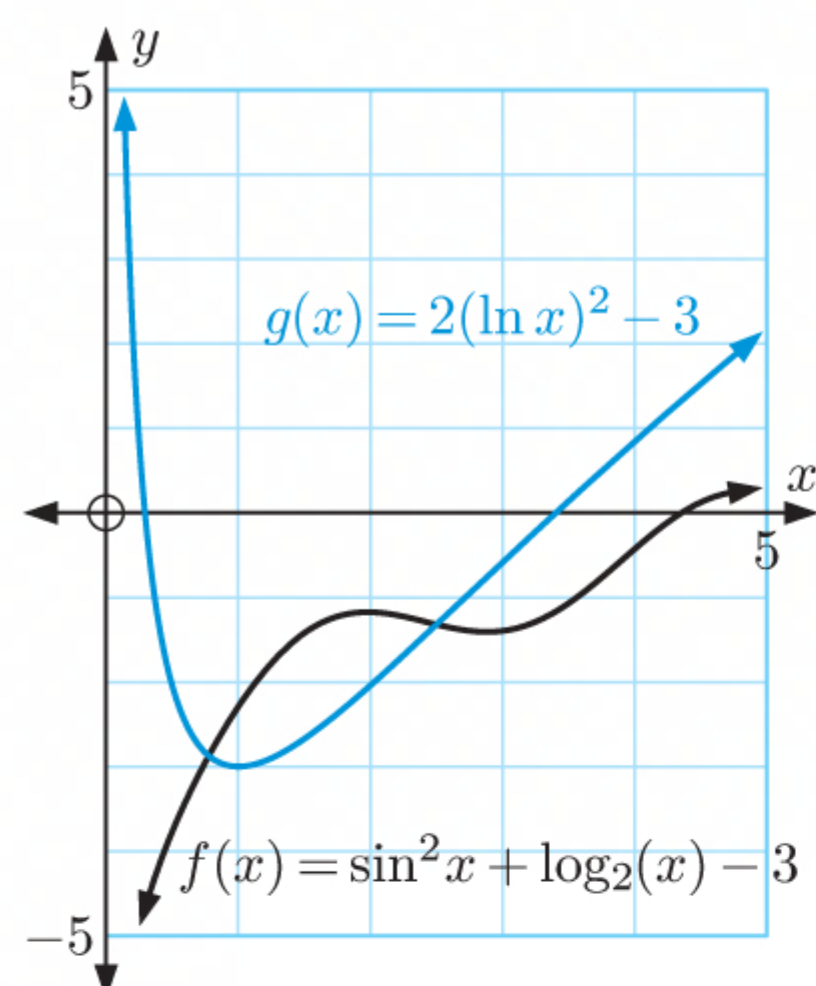
A1

Total [19 marks]

PAPER 2

Section A

1 a



A1A1A1

- b Using technology, the graphs intersect at $(0.777, -2.87)$ and $(2.50, -1.32)$.
Therefore, $f(x) \geq g(x)$ for $0.777 \leq x \leq 2.50$.

A1A1

A1

Total [6 marks]

- 2 a The coefficient of x^2 in the expansion of $(2x - 3)^5$ is $\binom{5}{3}(2)^2(-3)^3$
 $= 10 \times 4 \times (-27)$
 $= -1080$

M1A1

The coefficient of x^3 in the expansion of $(-3ax + 4)^4$ is $\binom{4}{1}(-3a)^3(4)^1$
 $= 4 \times (-27a^3) \times 4$
 $= -432a^3$

M1A1

Equating the coefficients gives $-1080 = -432a^3$

$$\begin{aligned}
 \therefore a^3 &= \frac{5}{2} \\
 \therefore a &= \sqrt[3]{\frac{5}{2}}
 \end{aligned}$$

A1

- b The coefficient of x^4 in the expansion of $(2x)(-3ax + 4)^4$ is $(2) \times \binom{4}{1}(-3a)^3(4)^1$
 $= (2) \times (-432(\frac{5}{2}))$
 $= -2160$

M1

A1

Total [7 marks]

- 3 a $y = 2 + \ln[(3x + 6)^2]$
 $= 2 + 2\ln(3x + 6) \quad \{x > -2 \Rightarrow 3x + 6 > 0\}$
 $= 2 + 2\ln(3(x + 2))$
 $= 2 + 2(\ln 3 + \ln(x + 2)) \quad \{x + 2 > 0\}$
 $= 2 + 2\ln 3 + 2\ln(x + 2)$

M1

M1

Therefore, $a = -2$, $b = 2$, and $c = 2 + 2\ln 3$.

A1A1A1

$$\begin{aligned} \mathbf{b} \quad V &= \pi \int_e^{e^2} (2 + \ln[(3x + 6)^2])^2 dx \\ &\approx 957.057\,466 \dots \\ &\approx 957 \text{ units}^3 \end{aligned}$$

M1A1

Total [7 marks]

- 4 a** The equation of the vertical asymptote is $2x + 5 = 0$

$$\therefore x = -\frac{5}{2}$$

A1

- b i** Given the oblique asymptote, $x^2 + bx + c$ divided by $2x + 5$ results in $\frac{1}{2}x + \frac{9}{4}$, plus some constant remainder, R .

$$\therefore x^2 + bx + c = (2x + 5)\left(\frac{1}{2}x + \frac{9}{4}\right) + R$$

M1

$$\therefore x^2 + bx + c = x^2 + 7x + \frac{45}{4} + R$$

As $R \in \mathbb{R}$, we have $b = 7$.

A1

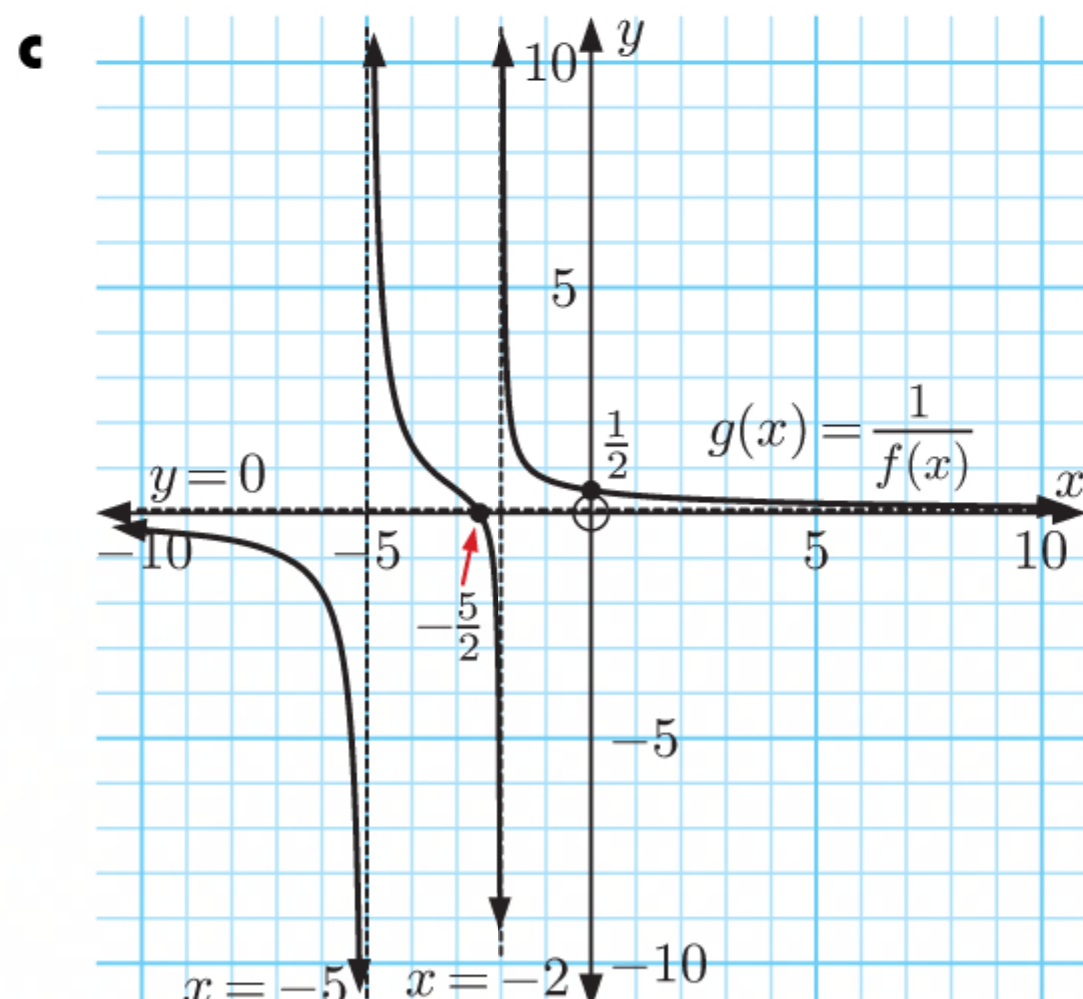
- ii** $(-3, 2)$ lies on the graph of $f(x)$, so $\frac{(-3)^2 + 7(-3) + c}{2(-3) + 5} = 2$

M1

$$\therefore \frac{c - 12}{-1} = 2$$

$$\therefore c = 10$$

A1



A1A1A1

Total [8 marks]

- 5 a** Using technology, $y = 145.5349 \dots x - 127.5259 \dots$

$$\therefore y \approx 146x - 128$$

A1A1

- b** Using technology, $r = 0.987\,26 \dots \approx 0.987$

A1

- c** Since $0.95 \leq r < 1$, there is a very strong positive correlation between the variables.

A1

- d i** The gradient $m \approx 146$ means that for every additional custom-made bike sold, the bike shop's profit will increase by approximately 146 dollars.

A1

- ii** The y -intercept $c \approx -128$ means that when no custom-made bikes are sold, the average profit is about -128 dollars (or the average loss is about 128 dollars).

A1

- e** We substitute $x = 30$ in the equation of the regression line:

M1

$$\begin{aligned} y &= 145.5349 \dots (30) - 127.5259 \dots \\ &\approx \$4238.52 \end{aligned}$$

A1

- f** Solving the inequality $146x - 128 \geq 8000$ gives

M1

$$x \geq 55.671$$

Thus, the bike shop needs to sell at least 56 custom-made bikes to make a profit of at least 8000 dollars.

A1

Total [10 marks]

- 6 a i** Lines L_1 and L_2 are not parallel, since for any $k \in \mathbb{R}$

$$\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} \neq k \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

R1AG

- ii** If the two lines intersect, $2 - 2\lambda = 2 + 4\mu$ and $3 + 3\lambda = -3 + 2\mu$.

M1

Solving the equations simultaneously, we find $\lambda = -\frac{3}{2}$ and $\mu = \frac{3}{4}$.

A1

However, in the Z -coordinate, $5 + 4(-\frac{3}{2}) = -1$ and $5 - 3(\frac{3}{4}) = \frac{11}{4}$.

So, the lines do not intersect.

R1AG

b $\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = (-2)(4) + (3)(2) + (4)(-3) = -14$

A1

$$\left| \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} \right| = \sqrt{(-2)^2 + (3)^2 + (4)^2} = \sqrt{29}$$

A1

$$\left| \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \right| = \sqrt{(4)^2 + (2)^2 + (-3)^2} = \sqrt{29}$$

A1

$$\therefore \cos \theta = \frac{-14}{29}$$

$$\therefore \theta \approx 119^\circ$$

M1A1

Total [9 marks]

7 a $\int_0^2 \frac{1}{k}x \, dx + \int_2^4 \frac{1}{2k}x \, dx = 1$

M1

$$\therefore \left[\frac{1}{2k}x^2 \right]_0^2 + \left[\frac{1}{4k}x^2 \right]_2^4 = 1$$

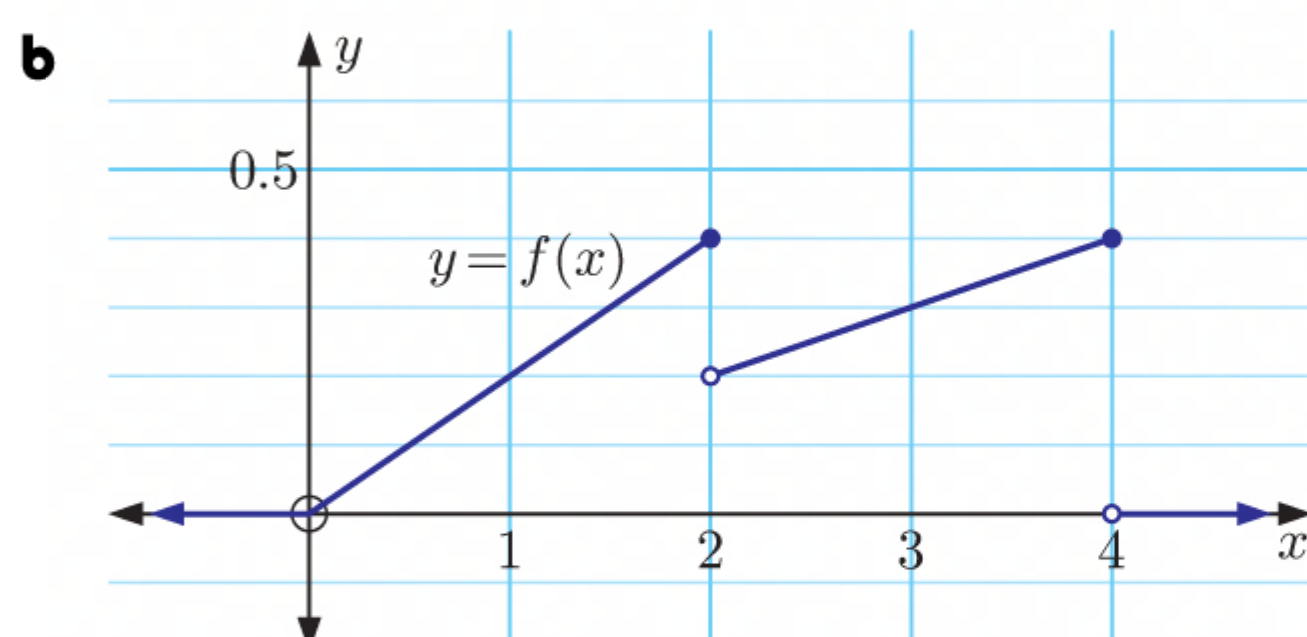
A1

$$\therefore \frac{1}{2k} \times 4 + \frac{1}{4k} \times 16 - \frac{1}{4k} \times 4 = 1$$

$$\therefore \frac{5}{k} = 1$$

$$\therefore k = 5$$

A1



A1A1

c $E(X) = \int_0^2 \frac{1}{5}x^2 \, dx + \int_2^4 \frac{1}{10}x^2 \, dx$

M1

$$= \left[\frac{1}{15}x^3 \right]_0^2 + \left[\frac{1}{30}x^3 \right]_2^4$$

A1

$$= \frac{1}{15} \times 8 + \frac{1}{30} \times 64 - \frac{1}{30} \times 8$$

$$= 2.4$$

A1

Total [8 marks]

Section B

8 a $f(x) = \frac{5x}{x^2 - x - 6} = \frac{A}{(x+2)} + \frac{B}{(x-3)}$

$$\therefore 5x = A(x-3) + B(x+2)$$

M1

This gives $5 = A + B$ and $0 = -3A + 2B$.

Solving simultaneously gives $A = 2$, $B = 3$

M1

$$\therefore f(x) = \frac{2}{(x+2)} + \frac{3}{(x-3)}$$

A1A1

$$\begin{aligned} \mathbf{b} \quad f(x) &= \frac{2}{(x+2)} + \frac{3}{(x-3)} \\ \therefore f'(x) &= \frac{-2}{(x+2)^2} + \frac{-3}{(x-3)^2} \end{aligned} \quad \mathbf{M1A1}$$

Since $(x+2)^2 > 0$ and $(x-3)^2 > 0$ for all $x \neq -2$ or 3 , $f'(x) < 0$ for all $x \neq -2$ or 3 , which means that $f(x)$ is decreasing for all $x \neq -2$ or 3 . **R1**

$$\begin{aligned} \mathbf{c} \quad \int_{-1}^0 f(x) dx &= \int_{-1}^0 \left(\frac{2}{(x+2)} + \frac{3}{(x-3)} \right) dx & \mathbf{M1} \\ &= [2 \ln|x+2| + 3 \ln|x-3|]_{-1}^0 \\ &= (2 \ln|2| + 3 \ln|-3|) - (2 \ln|1| + 3 \ln|-4|) & \mathbf{M1} \\ &= 2 \ln 2 + 3 \ln 3 - 2 \ln 1 - 3 \ln 4 \\ &= \ln \frac{2^2 \times 3^3}{1^2 \times 4^3} \\ &= \ln \frac{27}{16} & \mathbf{A1AG} \end{aligned}$$

Total [10 marks]

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad \frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(6xy) & \mathbf{M1} \\ \therefore 3x^2 + 3y^2 \frac{dy}{dx} &= 6y + 6x \frac{dy}{dx} & \mathbf{M1} \\ \therefore 3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} &= 6y - 3x^2 \\ \therefore \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x} & \mathbf{A1A1} \end{aligned}$$

$$\mathbf{b} \quad \text{At } (3, 3), \frac{dy}{dx} = \frac{2(3) - (3)^2}{(3)^2 - 2(3)} = \frac{6 - 9}{9 - 6} = -1 \quad \mathbf{M1A1}$$

Therefore, the equation of the tangent at $(3, 3)$ is $y - 3 = (-1)(x - 3)$
which is $y = -x + 6$ **M1A1**

$$\begin{aligned} \mathbf{c} \quad \mathbf{i} \quad \text{The tangent is horizontal when } \frac{dy}{dx} &= 0 & \mathbf{M1} \\ \therefore \frac{2y - x^2}{y^2 - 2x} &= 0 \\ \therefore 2y - x^2 &= 0 \\ \therefore y &= \frac{1}{2}x^2 & \mathbf{M1A1} \end{aligned}$$

$$\begin{aligned} \text{Substituting this into the original equation gives } x^3 + \left(\frac{1}{2}x^2\right)^3 &= 6x\left(\frac{1}{2}x^2\right) & \mathbf{M1} \\ \therefore x^3 + \frac{1}{8}x^6 &= 3x^3 \\ \therefore x^6 - 16x^3 &= 0 \\ \therefore x^3(x^3 - 16) &= 0 \end{aligned}$$

We reject $x^3 = 0$ since $x > 0$ **R1**

$$\therefore x^3 = 16, \text{ so } x = \sqrt[3]{16} = 2\sqrt[3]{2}. \quad \mathbf{A1}$$

$$\text{When } x = 2\sqrt[3]{2}, y = \frac{1}{2}(2\sqrt[3]{2})^2 = 2\sqrt[3]{4}$$

So, the tangent is horizontal at $(2\sqrt[3]{2}, 2\sqrt[3]{4})$. **A1**

$$\begin{aligned} \mathbf{ii} \quad \text{The tangent is vertical when } y^2 - 2x &= 0 \\ \therefore x &= \frac{1}{2}y^2 & \mathbf{M1A1} \end{aligned}$$

$$\begin{aligned} \text{Substituting this into the original equation gives } \left(\frac{1}{2}y^2\right)^3 + y^3 &= 6\left(\frac{1}{2}y^2\right)y & \mathbf{M1} \\ \therefore \frac{1}{8}y^6 + y^3 &= 3y^3 \\ \therefore y^3 &= 0 \text{ or } 16 \quad \{\text{as in } \mathbf{ci}\} \end{aligned}$$

We reject $y^3 = 0$ since $y > 0$ **R1**

$$\therefore y^3 = 16, \text{ so } y = \sqrt[3]{16} = 2\sqrt[3]{2}$$

$$\text{When } y = 2\sqrt[3]{2}, x = \frac{1}{2}(2\sqrt[3]{2})^2 = 2\sqrt[3]{4}$$

So, the tangent is vertical at $(2\sqrt[3]{4}, 2\sqrt[3]{2})$. **A1**

Total [20 marks]

- 10 a i** $f(x) = \tan x$
 $\therefore f'(x) = \sec^2 x$ **A1**
- ii** Using the chain rule, $f''(x) = 2 \sec x \times \sec x \tan x = 2 \sec^2 x \tan x$ **A1A1**
- b** Using the product rule,
- $$f'''(x) = 2 \sec^2 x \times \sec^2 x + 4 \sec x \times \sec x \tan x \times \tan x$$
- $$= 2 \sec^4 x + 4 \tan^2 x \sec^2 x$$
- M1A1**
AG
- c** $f(0) = \tan 0 = 0$
 $f'(0) = \sec^2 0 = 1$
 $f''(0) = 2 \sec^2 0 \tan 0 = 0$
 $f'''(0) = 2 \sec^4 0 + 4 \tan^2 0 \sec^2 0 = 2$ **M1**
- $$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$
- $$= 0 + x + 0 + \frac{1}{3}x^3 + \dots$$
- $$= x + \frac{1}{3}x^3 + \dots$$
- M1**
A1A1
- d** $g(x) = e^{2x} \quad \therefore g(0) = 1$
 $g'(x) = 2e^{2x} \quad \therefore g'(0) = 2$
 $g''(x) = 4e^{2x} \quad \therefore g''(0) = 4$
 $g'''(x) = 8e^{2x} \quad \therefore g'''(0) = 8$ **A1A1**
- $$\therefore g(x) = g(0) + xg'(0) + \frac{x^2}{2!}g''(0) + \frac{x^3}{3!}g'''(0) + \dots$$
- $$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$
- M1**
A1A1A1A1
- e** $h(x) = f(x) \times g(x) = (x + \frac{1}{3}x^3 + \dots) \times (1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots)$ **M1**

$$= x + 2x^2 + \frac{7}{3}x^3 + \dots$$
A1A1A1
- f** $h(x) = f(x) \times g(x) = e^{2x} \tan x$
 $\therefore h'(x) = 2e^{2x} \tan x + e^{2x} \sec^2 x$ **M1A1**
- Therefore, $2e^{2x} \tan x + e^{2x} \sec^2 x = \frac{d}{dx}(x + 2x^2 + \frac{7}{3}x^3 + \dots)$

$$= 1 + 4x + 7x^2 + \dots$$
 A1A1A1

Total [25 marks]**PAPER 3**

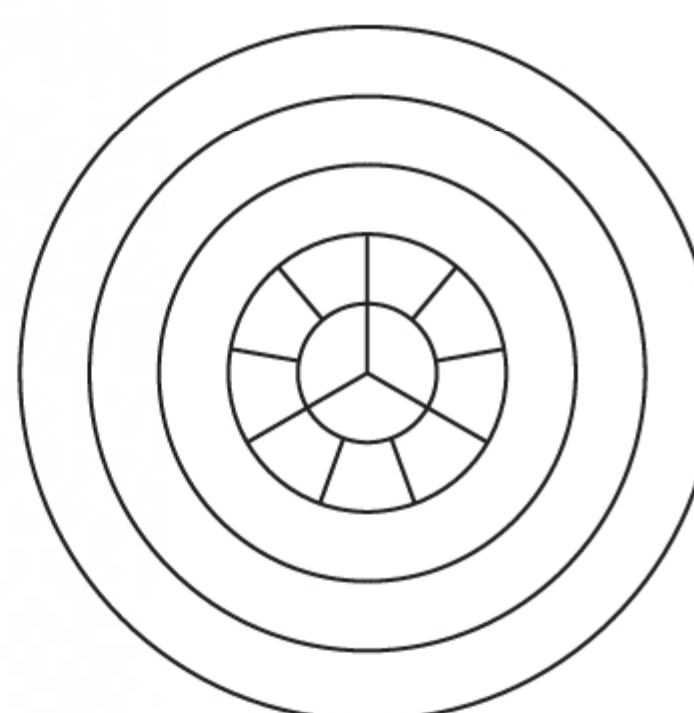
- 1 a i** radial height $= \frac{1}{5}$ **A1**
- ii** $\text{Area}_{\text{cell}} = \frac{1}{3} \times \pi \times \left(\frac{1}{5}\right)^2$ **M1**

$$= \frac{1}{75}\pi$$
 A1
- b i** $\text{Area}_{\text{ring}} = \pi\left(\frac{2}{5}\right)^2 - \pi\left(\frac{1}{5}\right)^2$ **M1**

$$= \frac{4}{25}\pi - \frac{1}{25}\pi$$

$$= \frac{3}{25}\pi$$
 A1
- ii** Dividing the area of the second smallest ring by the area of a cell in the smallest ring gives the required number of cells:

$$\begin{aligned} \text{Number of cells} &= \frac{\text{Area}_{\text{ring}}}{\text{Area}_{\text{cell}}} \\ &= \frac{\frac{3}{25}\pi}{\frac{1}{75}\pi} \\ &= \frac{3}{25} \times \frac{75}{1} \\ &= 9 \end{aligned}$$

**M1****A1**

c The third smallest ring has area $= \pi\left(\frac{3}{5}\right)^2 - \pi\left(\frac{2}{5}\right)^2$ **M1**

$$= \frac{9}{25}\pi - \frac{4}{25}\pi$$

$$= \frac{1}{5}\pi$$
 A1

So, if the ring is divided into 15 cells, each cell has area $\frac{\frac{1}{5}\pi}{15} = \frac{1}{75}\pi$ which is the same area as the cells in the smallest and second smallest rings. **AG**

d i The fourth smallest (second largest) ring has area $= \pi\left(\frac{4}{5}\right)^2 - \pi\left(\frac{3}{5}\right)^2 = \frac{7}{25}\pi$ **M1**

$$\therefore a = \frac{\frac{7}{25}\pi}{\frac{1}{75}\pi} = 21$$
 M1A1

ii The largest ring has area $= \pi(1)^2 - \pi\left(\frac{4}{5}\right)^2 = \frac{9}{25}\pi$ **M1**

$$\therefore b = \frac{\frac{9}{25}\pi}{\frac{1}{75}\pi} = 27$$
 A1

e radial height $= \frac{1}{N}$ **A1**

f The area of the n th ring is $A_n = \pi\left(\frac{n}{N}\right)^2 - \pi\left(\frac{n-1}{N}\right)^2$ **M1**

$$= \pi \frac{n^2 - n^2 + 2n - 1}{N^2}$$

$$= \pi \frac{2n - 1}{N^2}$$
 A1

g i area $= \frac{A_1}{k}$

$$= \frac{\pi\left(\frac{1}{N}\right)^2}{k}$$
 M1

$$= \frac{\pi}{kN^2}$$
 A1

ii Number of cells $= \frac{A_n}{\frac{\pi}{kN^2}}$

$$= \frac{\pi \frac{2n-1}{N^2}}{\frac{\pi}{kN^2}}$$
 M1

$$= k(2n - 1)$$
 A1

iii Total number of cells $= \sum_{n=1}^N k(2n - 1)$ **M1**

$$= k \sum_{n=1}^N (2n - 1)$$

$$= k(1 + 3 + 5 + \dots + (2N - 1))$$

$$= kN^2$$
 A1

h $P = \frac{1}{N} + \frac{1}{N} + \frac{2\pi\left(\frac{n}{N}\right)}{k(2n-1)} + \frac{2\pi\left(\frac{n-1}{N}\right)}{k(2n-1)}$ **M1**

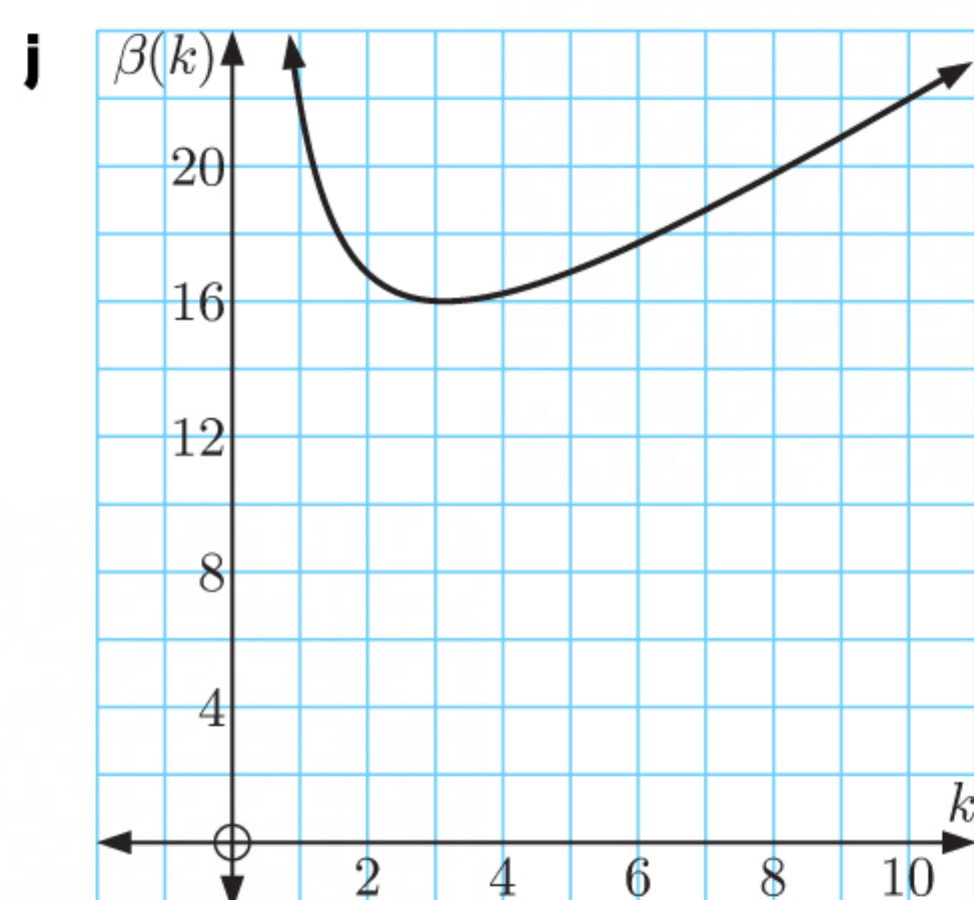
$$= \frac{2}{N} + \frac{2\pi n + 2\pi n - 2\pi}{k(2n-1)N} = \frac{2}{N} + \frac{2\pi(2n-1)}{k(2n-1)N} = \frac{2}{N} + \frac{2\pi}{kN} \text{ or } \frac{2}{N}\left(1 + \frac{\pi}{k}\right)$$
 A1

i $\beta = \frac{\left(\frac{2}{N}\left(1 + \frac{\pi}{k}\right)\right)^2}{\frac{\pi}{kN^2}}$

$$= \frac{4}{N^2}\left(1 + \frac{\pi}{k}\right)^2 \times \frac{kN^2}{\pi}$$

$$= \frac{4k}{\pi} \times \frac{(k + \pi)^2}{k^2}$$

$$= \frac{4(k + \pi)^2}{k\pi}$$
 M1AG



A1

k i

$$\beta = \frac{4(k + \pi)^2}{k\pi}$$

$$\therefore \frac{d\beta}{dk} = \frac{8(k + \pi)k\pi - 4\pi(k + \pi)^2}{k^2\pi^2}$$

$$= \frac{4\pi(k + \pi)(2k - (k + \pi))}{k^2\pi^2}$$

$$= \frac{4(k + \pi)(k - \pi)}{k^2\pi}$$

M1

A1

ii $\frac{d\beta}{dk} = 0$ when $4(k + \pi)(k - \pi) = 0$

$$\therefore k = -\pi \text{ or } \pi$$

M1A1

However, given that $k \in \mathbb{Z}^+$, we test the integer values of k that are closest to π .

When $k = 3$, $\beta \approx 16.009$

When $k = 4$, $\beta \approx 16.235$

$\therefore \beta$ is minimised when $k = 3$.

R1

Total [32 marks]

2 a i $f_1(-x) = \frac{3}{(-x)^2 + 2} = \frac{3}{x^2 + 2}$

M1

Since $f_1(-x) = f_1(x)$, the function $f_1(x)$ is an even function.

A1

ii $f_2(-x) = 2(-x)^3 - 4(-x) = -2x^3 + 4x = -(2x^3 - 4x)$

M1

Since $f_2(-x) = -f_2(x)$, the function $f_2(x)$ is an odd function.

A1

b i $g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$

$$= \frac{2f(x)}{2}$$

$$= f(x)$$

M1

AG

ii $g(-x) = \frac{f(-x) + f(x)}{2} = g(x)$

$\therefore g(x)$ is an even function.

A1

$$h(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -h(x)$$

$\therefore h(x)$ is an odd function.

A1

Since $f(x) = g(x) + h(x)$, any function $f(x)$ can be written as the sum of an even function and an odd function.

R1

c $f(-x) = (-x)^3 + 2(-x)^2 - (-x) = -x^3 + 2x^2 + x$

M1

The even part of $f(x)$ is

$$g(x) = \frac{(x^3 + 2x^2 - x) + (-x^3 + 2x^2 + x)}{2} = \frac{4x^2}{2} = 2x^2$$

A1

The odd part of $f(x)$ is

$$h(x) = \frac{(x^3 + 2x^2 - x) - (-x^3 + 2x^2 + x)}{2} = \frac{2x^3 - 2x}{2} = x^3 - x$$

A1

- d i** If $f(x)$ is an even function then $f(-x) = f(x)$. Differentiating both sides of this equation gives

$$\begin{aligned} -f'(-x) &= f'(x) & \text{M1} \\ \therefore f'(-x) &= -f'(x) \end{aligned}$$

Thus, $f'(x)$ is an odd function.

AG

- ii** If $f(x)$ is an odd function then $f(-x) = -f(x)$. Differentiating both sides of this equation gives

$$\begin{aligned} -f'(-x) &= -f'(x) & \text{M1} \\ \therefore f'(-x) &= f'(x) \end{aligned}$$

Thus, $f'(x)$ is an even function.

AG

- e i** If $f(x)$ is an odd function then $f(-x) = -f(x)$.

$$\therefore f(-0) = -f(0) \quad \text{M1}$$

$$\therefore f(0) = -f(0)$$

$$\therefore f(0) = 0 \quad \text{AG}$$

ii $f^{(0)}(0) = f(0) = 0$ {from **e i**} A1

Now $f(x)$ is odd, so $f'(x)$ is even {from **d ii**}

$$\therefore f''(x) \text{ is odd } \quad \text{{from **d i**}}$$

$$\therefore f''(0) = 0 \quad \text{{from **e i**}} \quad \text{A1}$$

Continuing this pattern, $f^{(n)}(0) = 0$ for all even values of n .

A1

- iii** The Maclaurin series expansion of the odd function $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(x) + \dots$$

From part **e ii**, $f^{(n)}(0) = 0$ for all even n , which means the only non-zero terms remaining in the Maclaurin series expansion are those with odd powers of x .

R1AG

- f** From **d i**, since $f(x)$ is an even function, $f'(x)$ must be an odd function.

M1

$$\therefore f'(-0) = -f'(0)$$

$$\therefore f'(0) = -f'(0)$$

$$\therefore f'(0) = 0 \quad \text{A1}$$

Now $f'(x)$ is odd, so $f''(x)$ is even

$$\therefore f^{(3)}(x) \text{ is odd}$$

$$\therefore f^{(3)}(0) = 0 \quad \text{M1}$$

Continuing this pattern,

$$f^{(n)}(0) = 0 \text{ for all odd values of } n. \quad \text{M1}$$

The Maclaurin series expansion of the even function $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(0) + \dots$$

$f^{(n)}(0) = 0$ for all odd n , which means the only non-zero terms remaining in the Maclaurin series expansion are those with even powers of x , including the constant term, which is a term in x^0 .

R1AG

Total [23 marks]

TRIAL EXAMINATION 2

PAPER 1

Section A

1 $f(x) = x^4 + bx^2 - 3ax + (4 - 2b)$

If $(x - 1)$ and $(x - 2)$ are linear factors, then $f(1) = 0$ and $f(2) = 0$.

$f(1) = 0$, so $1 + b - 3a + 4 - 2b = 0$

$\therefore -3a - b = -5 \quad \dots (1)$

M1A1

$f(2) = 0$, so $16 + 4b - 6a + 4 - 2b = 0$

$\therefore -6a + 2b + 20 = 0$

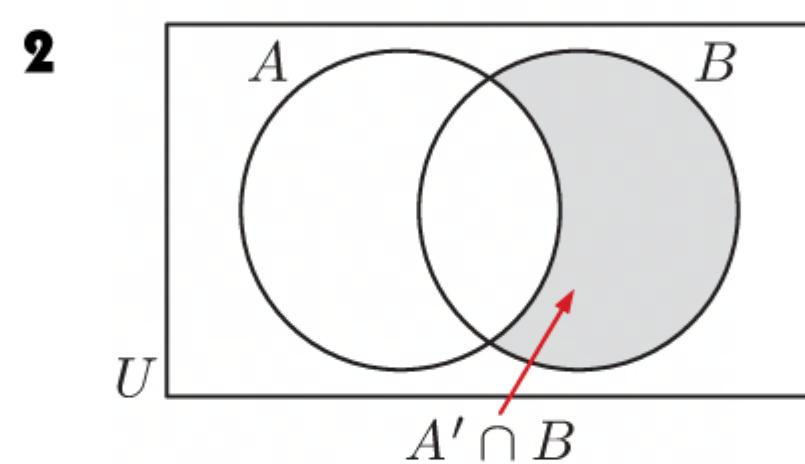
$\therefore 3a - b = 10 \quad \dots (2)$

M1A1

Solving (1) and (2) simultaneously gives $a = \frac{5}{2}$, $b = \frac{-5}{2}$.

A1A1

Total [6 marks]



$P(A) + P(A' \cap B) = P(A \cup B)$

M1

$\therefore P(A) + 0.4 = 0.9$

$\therefore P(A) = 0.5$

A1

Now $P(A \cup B) = P(A) + P(B) - P(A)P(B)$ {A and B are independent}

M1

$\therefore 0.9 = 0.5 + 0.5P(B)$

$\therefore P(B) = 0.8$

A1

Total [4 marks]

3 $S_3 = 24$

$\therefore \frac{3}{2}(2u_1 + 2d) = 24$

M1A1

$\therefore 3u_1 + 3u_1 = 24 \quad \{u_1 = d\}$

$\therefore u_1 = 4, d = 4$

A1A1

Now $u_n = 124$ when $u_1 + (n - 1)d = 124$

M1

$\therefore 4 + 4(n - 1) = 124$

$\therefore 4n = 124$

$\therefore n = 31$

A1

Total [6 marks]

4 $49^{x^3-2} = \left(\frac{1}{\sqrt[3]{7}}\right)^{7-x}$

$\therefore (7^2)^{x^3-2} = \left(7^{-\frac{1}{3}}\right)^{7-x}$

$\therefore 7^{2x^3-4} = 7^{-\frac{7}{3} + \frac{1}{3}x}$

M1

$\therefore 2x^3 - 4 = -\frac{7}{3} + \frac{1}{3}x$

A1

$\therefore 6x^3 - 12 = -7 + x$

$\therefore 6x^3 - x - 5 = 0$

By inspection, $x = 1$ is a solution.

$$1 \left| \begin{array}{ccc|c} 6 & 0 & -1 & -5 \\ 0 & 6 & 6 & 5 \\ \hline 6 & 6 & 5 & 0 \end{array} \right.$$

$\therefore (x - 1)(6x^2 + 6x + 5) = 0$

(M1)

But $6x^2 + 6x + 5$ is irreducible as $\Delta < 0$

M1

$\therefore x = 1$ is the only real solution.

A1

Total [5 marks]

$$\begin{aligned}
 \mathbf{5} \quad \sec \theta &= \frac{5}{2}, \quad \frac{3\pi}{2} < \theta < 2\pi \\
 \therefore \sec^2 \theta &= \frac{25}{4} \\
 \therefore 1 + \tan^2 \theta &= \frac{25}{4} \\
 \therefore \tan^2 \theta &= \frac{21}{4} \\
 \therefore \tan \theta &= -\frac{\sqrt{21}}{2} \quad \text{as } \tan \theta < 0 \\
 \therefore \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\
 &= \frac{-\sqrt{21}}{1 - \frac{21}{4}} \\
 &= \frac{-\sqrt{21}}{-\frac{17}{4}} \\
 &= \frac{4\sqrt{21}}{17}
 \end{aligned}$$

M1

M1A1

M1

A1

Total [5 marks]

$$\mathbf{6} \quad 2x^2 + 4x + k = 0 \text{ has roots } \alpha \text{ and } \beta.$$

$$\begin{aligned}
 \text{Sum of roots: } \alpha + \beta &= \frac{-b}{a} \\
 \therefore \alpha + \beta &= -2 \quad \dots (1)
 \end{aligned}$$

A1

$$\begin{aligned}
 \text{Product of roots: } \alpha\beta &= \frac{c}{a} \\
 \therefore \alpha\beta &= \frac{k}{2} \quad \dots (2)
 \end{aligned}$$

A1

Suppose the roots of a quadratic equation are α^2 and β^2 .

$$\begin{aligned}
 \text{Sum: } \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\
 &= (-2)^2 - k \quad \{\text{using (1) and (2)}\} \\
 &= 4 - k
 \end{aligned}$$

M1

A1

$$\begin{aligned}
 \text{Product: } \alpha^2\beta^2 &= (\alpha\beta)^2 \\
 &= \left(\frac{k}{2}\right)^2 \quad \{\text{using (2)}\} \\
 &= \frac{k^2}{4}
 \end{aligned}$$

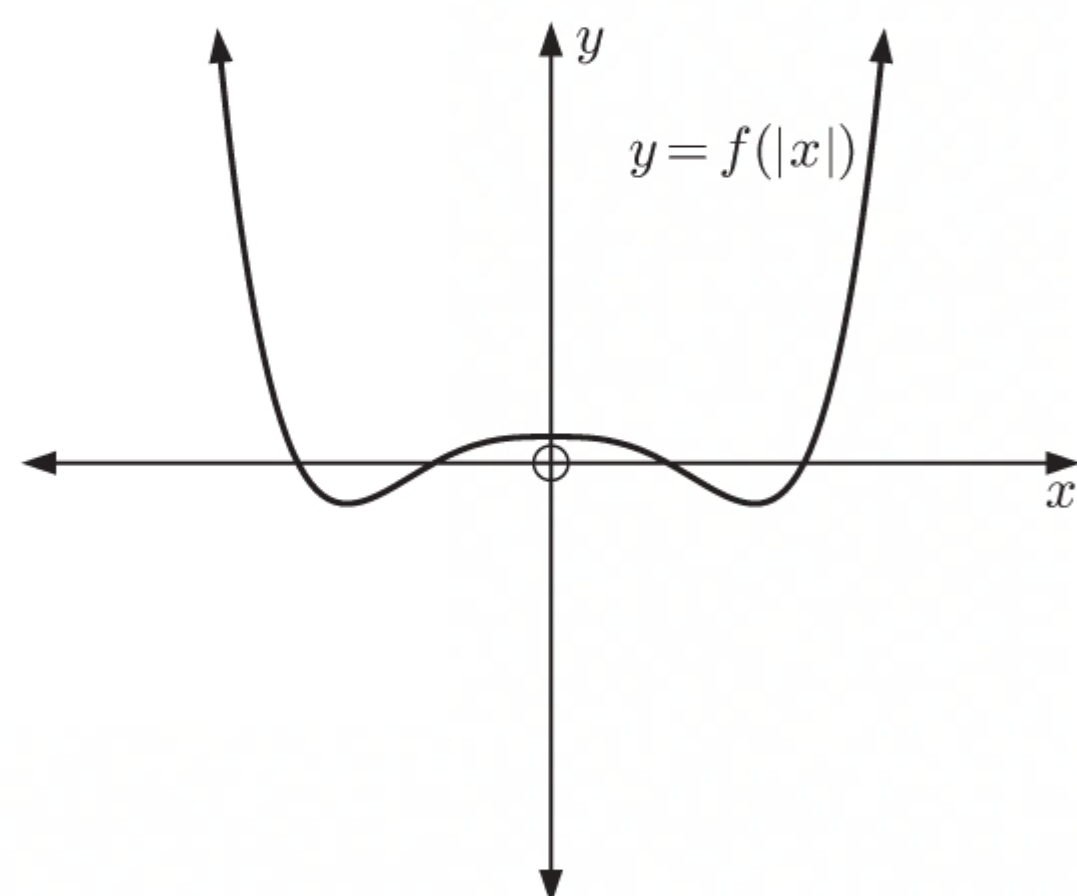
M1

A1

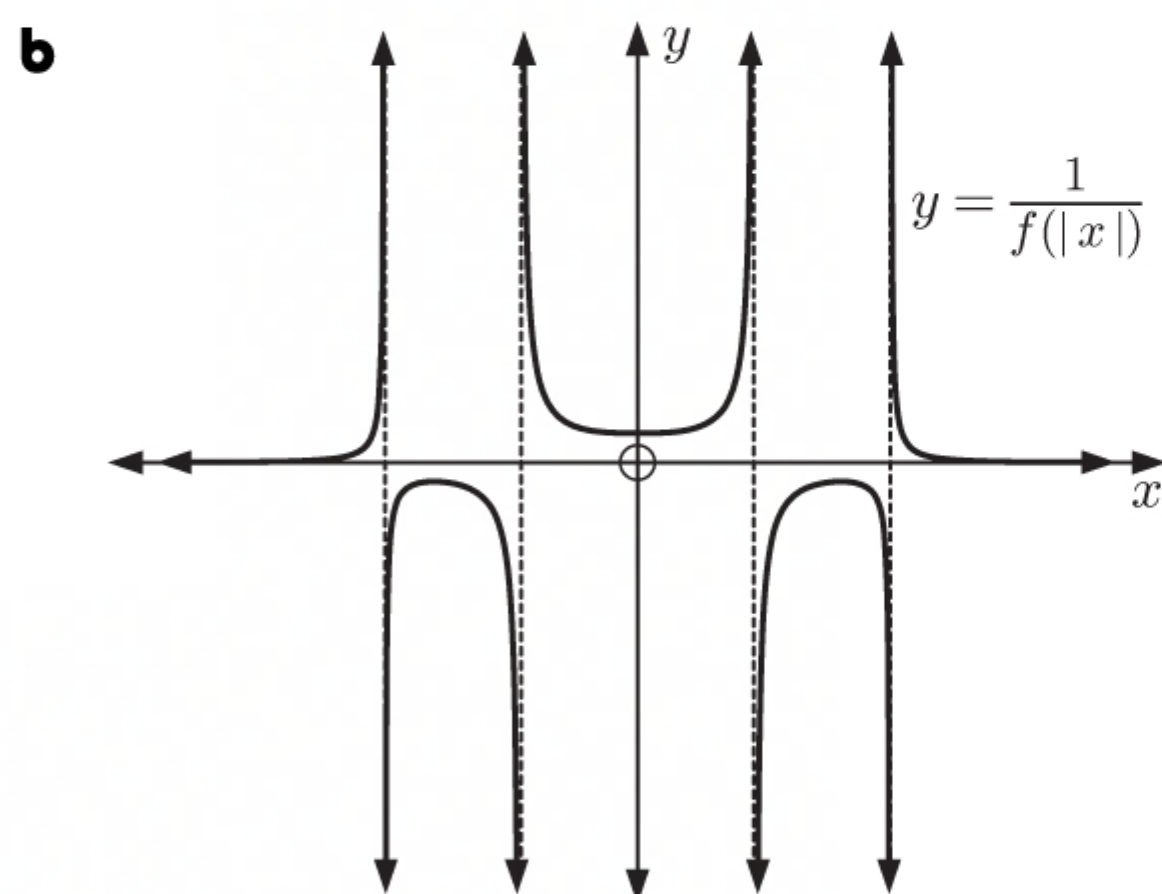
$$\text{A quadratic equation is } x^2 - (4 - k)x + \frac{k^2}{4} = 0.$$

A1

Total [7 marks]

 $\mathbf{7 \quad a}$


A1 : correct for $x > 0$
 A1 : reflect in y -axis for $x < 0$



A1 : 4 vertical asymptotes

A1 : maxima for $y < 0$ A1 : minimum for $y > 0$

A1 : correct curves

Total [6 marks]

8 $f(x) = x \ln(2x + 1)$

a $f'(x) = \ln(2x + 1) + x \left(\frac{2}{2x + 1} \right)$ (M1)

$$= \ln(2x + 1) + \frac{2x}{2x + 1}$$
 A1

b Using **a**, $\int \left(\ln(2x + 1) + \frac{2x}{2x + 1} \right) dx = x \ln(2x + 1) + c$ M1

$$\therefore \int \ln(2x + 1) dx + \int \frac{2x}{2x + 1} dx = x \ln(2x + 1) + c$$

$$\therefore \int \ln(2x + 1) dx + \int \left(1 - \frac{1}{2x + 1} \right) dx = x \ln(2x + 1) + c$$
 M1

$$\therefore \int \ln(2x + 1) dx + x - \frac{1}{2} \ln|2x + 1| = x \ln(2x + 1) + c$$
 M1A1

$$\therefore \int \ln(2x + 1) dx = x \ln(2x + 1) - x + \frac{1}{2} \ln(2x + 1) + c$$
 A1
 {the integrand has domain $2x + 1 > 0$ }

Total [7 marks]

9 $\lim_{\theta \rightarrow 0} \left(\frac{2 \tan \theta - 2 \sin \theta}{\sin 2\theta - 2 \sin \theta} \right)$

$$= \lim_{\theta \rightarrow 0} \left(\frac{2 \sec^2 \theta - 2 \cos \theta}{2 \cos 2\theta - 2 \cos \theta} \right) \quad \{\text{l'Hôpital's rule}\} \quad \text{M1A1}$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{4 \sec \theta \sec \theta \tan \theta + 2 \sin \theta}{-4 \sin 2\theta + 2 \sin \theta} \right) \quad \{\text{l'Hôpital's rule}\} \quad \text{M1A1}$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{2 \sec^2 \theta \tan \theta + \sin \theta}{\sin \theta - 2 \sin 2\theta} \right)$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{2 \sec^2 \theta \sec^2 \theta + 4 \tan^2 \theta \sec^2 \theta + \cos \theta}{\cos \theta - 4 \cos 2\theta} \right) \quad \{\text{l'Hôpital's rule}\} \quad \text{M1A1}$$

$$= \frac{3}{-3} \quad \{\text{direct substitution}\} \quad \text{(M1)}$$

$$= -1 \quad \text{A1}$$

Total [8 marks]

Section B

10 a $(\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{k}) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} \quad \text{(M1)}$$

$$= \mathbf{i} \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \quad \text{(M1)}$$

$$= -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \quad \text{A1}$$

$$\mathbf{b} \quad \mathbf{i} \quad \mathbf{n}_1 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \quad \{\text{using } \mathbf{a}\} \quad \text{A1}$$

$$\mathbf{n}_2 = \begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix} \quad \text{A1}$$

$$\mathbf{ii} \quad \cos \theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \quad \text{M1}$$

$$= \frac{|6 + 4 + 6|}{\sqrt{9}\sqrt{49}} \quad \text{A1A1}$$

$$= \frac{16}{3 \times 7} = \frac{16}{21} \quad \text{as required.} \quad \text{AG}$$

$$\mathbf{c} \quad L_2 \text{ has equation } \frac{x+1}{2} = \frac{3(11-2y)}{4} = z-1$$

$$\therefore \frac{x+1}{2} = \frac{-2y+11}{\frac{4}{3}} = \frac{z-1}{1}$$

$$\therefore \frac{x+1}{2} = \frac{y-\frac{11}{2}}{-\frac{2}{3}} = \frac{z-1}{1}$$

$$\therefore \text{ the direction vector of } L_2 \text{ is } \begin{pmatrix} 2 \\ -\frac{2}{3} \\ 1 \end{pmatrix} \quad \text{M1A1}$$

$$\therefore \text{ since } \begin{pmatrix} 2 \\ -\frac{2}{3} \\ 1 \end{pmatrix} = -\frac{1}{3}\mathbf{n}_2, L_2 \text{ is parallel to the normal vector of } P_2. \quad \text{R1A1}$$

$$\therefore L_2 \text{ is normal to } P_2. \quad \text{AG}$$

$\mathbf{d} \quad \mathbf{i}$ The parametric equations of L_2 are:

$$x = 2t - 1, \quad y = \frac{11}{2} - \frac{2t}{3}, \quad z = 1 + t, \quad t \in \mathbb{R}.$$

$$\therefore L_2 \text{ intersects } P_2 \text{ where } -6(2t-1) + 2\left(\frac{11}{2} - \frac{2t}{3}\right) - 3(1+t) = 63 \quad \text{M1A1}$$

$$\therefore -12t + 6 + 11 - \frac{4}{3}t - 3 - 3t = 63$$

$$\therefore \frac{-49t}{3} = 49$$

$$\therefore t = -3 \quad \text{A1}$$

$$\therefore Q\left(-7, \frac{15}{2}, -2\right) \quad \text{A3}$$

Total [18 marks]

11 $z = \cos \theta + i \sin \theta$

$$\mathbf{a} \quad \mathbf{i} \quad z^4 = (\cos \theta + i \sin \theta)^4$$

$$= \cos^4 \theta + \binom{4}{1}(\cos \theta)^3(i \sin \theta) + \binom{4}{2}(\cos \theta)^2(i \sin \theta)^2 + \binom{4}{3}(\cos \theta)(i \sin \theta)^3 + (i \sin \theta)^4 \quad \text{M1}$$

$$= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta \quad \text{A1A1}$$

$$\mathbf{ii} \quad z = \text{cis } \theta$$

$$\therefore z^4 = (\text{cis } \theta)^4$$

$$= \text{cis } 4\theta \quad \{\text{De Moivre}\} \quad \text{M1}$$

$$= \cos 4\theta + i \sin 4\theta$$

$$\therefore \cos 4\theta + i \sin 4\theta = \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta \quad \text{M1A1}$$

$$\text{Equating real parts, } \cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta \quad \text{M1A1}$$

$$= \cos^4 \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2$$

$$= \cos^4 \theta - 6 \cos^2 \theta + 6 \cos^4 \theta + 1 - 2 \cos^2 \theta + \cos^4 \theta \quad \text{A1}$$

$$= 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \quad \text{as required.} \quad \text{AG}$$

$$\begin{aligned}
\text{b LHS} &= \frac{\sin 4\theta - 2 \sin 2\theta}{\cos 4\theta + 4 \cos 2\theta + 3} \\
&= \frac{2 \sin 2\theta \cos 2\theta - 2 \sin 2\theta}{8 \cos^4 \theta - 8 \cos^2 \theta + 1 + 4 \cos 2\theta + 3} && \text{M1A1} \\
&= \frac{2 \sin 2\theta (\cos 2\theta - 1)}{8 \cos^4 \theta - 8 \cos^2 \theta + 4 + 4(2 \cos^2 \theta - 1)} && \text{A1} \\
&= \frac{2 \sin 2\theta (\cos 2\theta - 1)}{8 \cos^4 \theta - 8 \cos^2 \theta + 4 + 8 \cos^2 \theta - 4} \\
&= \frac{2 \sin 2\theta (\cos 2\theta - 1)}{8 \cos^4 \theta} && \text{A1} \\
&= \frac{\sin 2\theta (1 - 2 \sin^2 \theta - 1)}{4 \cos^4 \theta} && \text{M1} \\
&= \frac{2 \sin \theta \cos \theta (-2 \sin^2 \theta)}{4 \cos^4 \theta} && \text{M1} \\
&= \frac{-4 \sin^3 \theta \cos \theta}{4 \cos^4 \theta} \\
&= \frac{-\sin^3 \theta}{\cos^3 \theta} \\
&= -\tan^3 \theta \\
&= \text{RHS} && \text{AG}
\end{aligned}$$

$$\begin{aligned}
\text{c From part b, } -\tan^3 \theta &= \frac{\sin 4\theta - 2 \sin 2\theta}{\cos 4\theta + 4 \cos 2\theta + 3} \\
\therefore \tan^3 \theta &= \frac{2 \sin 2\theta - \sin 4\theta}{\cos 4\theta + 4 \cos 2\theta + 3} \\
\therefore \tan^3(15^\circ) &= \frac{2 \sin 30^\circ - \sin 60^\circ}{\cos 60^\circ + 4 \cos 30^\circ + 3} && \text{M1A1} \\
&= \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2} + \frac{4\sqrt{3}}{2} + 3} && \text{A1} \\
&= \frac{\frac{2-\sqrt{3}}{2}}{\frac{7+4\sqrt{3}}{2}} \\
&= \frac{2-\sqrt{3}}{7+4\sqrt{3}} \times \frac{7-4\sqrt{3}}{7-4\sqrt{3}} && \text{M1} \\
&= \frac{14-8\sqrt{3}-7\sqrt{3}+12}{49-48} \\
&= 26-15\sqrt{3} && \text{A1A1}
\end{aligned}$$

Total [21 marks]

$$12 \quad g(x) = \tan x$$

$$\text{a Domain of } g: \quad x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \quad \text{A1}$$

$$\text{Range of } g: \quad y \in \mathbb{R} \quad \text{A1}$$

$$\text{b } g(x) = \tan x \quad \therefore g(0) = 0$$

$$g'(x) = \sec^2 x \quad \therefore g'(0) = 1 \quad \text{A1}$$

$$g''(x) = 2 \sec^2 x \tan x \quad \therefore g''(0) = 0 \quad \text{A1}$$

$$g'''(x) = 2 \sec^4 x + 4 \sec^2 x \tan^2 x \quad \therefore g'''(0) = 2 \quad \text{A2}$$

$$\therefore g(x) \approx x + \frac{2}{3!}x^3 \quad \text{M1}$$

$$\approx x + \frac{1}{3}x^3 \quad \text{A1}$$

$$\text{c If } e^z \approx 1 + z + \frac{1}{2}z^2$$

$$\text{then } e^{\tan x} \approx 1 + \tan x + \frac{1}{2}(\tan x)^2 \quad \text{M1}$$

$$\approx 1 + \left(x + \frac{1}{3}x^3\right) + \frac{1}{2}\left(x + \frac{1}{3}x^3\right)^2 \quad \text{M1A1}$$

$$\approx 1 + x + \frac{1}{3}x^3 + \frac{1}{2}\left(x^2 + \frac{2}{3}x^4 + \frac{1}{9}x^6\right) \quad \text{A1}$$

$$\approx 1 + x + \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x^4 + \frac{1}{18}x^6$$

$$\approx 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{18}x^6 \quad \text{A1}$$

$$\begin{aligned}
 \mathbf{d} \quad & \lim_{x \rightarrow 0} \left(\frac{e^{\tan x} - 1}{\tan x} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{18}x^6 - 1}{x + \frac{1}{3}x^3} \right) & \mathbf{M1} \\
 &= \lim_{x \rightarrow 0} \left(\frac{x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{18}x^6}{x + \frac{1}{3}x^3} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{x \left(1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^5 \right)}{x \left(1 + \frac{1}{3}x^2 \right)} \right) & \mathbf{M1} \\
 &= \lim_{x \rightarrow 0} \left(\frac{1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^5}{1 + \frac{1}{3}x^2} \right) \quad \{x \neq 0\} & \mathbf{A1} \\
 &= 1 & \mathbf{A1}
 \end{aligned}$$

Total [17 marks]

PAPER 2

Section A

- 1 a** There are $12 \times 3 = 36$ time periods.

$$\text{The rate per period} = \frac{2.5\%}{12} = 0.208\bar{3}\% \quad \mathbf{(M1)}$$

$$\begin{aligned}
 \therefore u_{36} &= u_0(1+i)^{36} \\
 &= 28\,000(1+0.002\,08\bar{3})^{36} & \mathbf{(M1)} \\
 &= 30\,178 \text{ pounds} & \mathbf{A1}
 \end{aligned}$$

- b** If inflation is 1.8% per annum during this time, the real value of the investment can be calculated as:

$$\text{Real value} \times (1.018)^3 = 30\,178 \quad \mathbf{M1}$$

$$\begin{aligned}
 \therefore \text{real value} &= \frac{30\,178}{(1.018)^3} \\
 &= 28\,605 \text{ pounds} & \mathbf{A1}
 \end{aligned}$$

Total [5 marks]

$$\mathbf{2} \quad \sum P(X = x) = 1$$

$$\therefore \frac{1}{10} + p + \frac{1}{4}p + 4p^2 + \frac{27}{80} = 1 \quad \mathbf{M1A1}$$

$$\therefore 4p^2 + \frac{5}{4}p - \frac{9}{16} = 0$$

$$\therefore p = 0.25 \quad \{0 < p < 1\} \quad \mathbf{A1}$$

Hence the probability distribution is:

X	1	3	6	8	x
$P(X = x)$	0.1	0.25	0.0625	0.25	0.3375

$$\text{Now } E(X) = 7.95$$

$$\therefore 0.1 + 0.75 + 0.375 + 2 + 0.3375x = 7.95 \quad \mathbf{M1A1}$$

$$0.3375x = 4.725$$

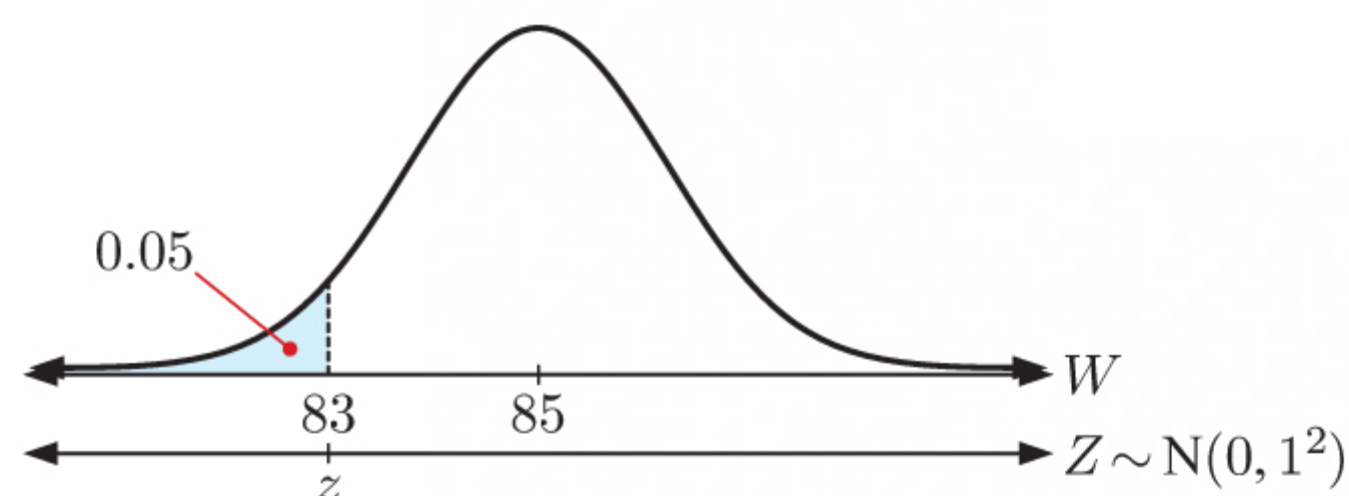
$$x = 14 \quad \mathbf{A1}$$

Total [6 marks]

3 $W \sim N(85, \sigma^2)$

$P(W < 83) = 0.05$

a i



$z \approx -1.64485$

Now $z = \frac{x - \mu}{\sigma}$

$$\therefore \sigma = \frac{x - \mu}{z}$$

$$\approx \frac{83 - 85}{-1.64485}$$

$\sigma \approx 1.22$

ii $W \sim N(85, 1.22^2)$

$\therefore P(W > 86.5) \approx 0.109$

b Let X represent the number of packets that weigh more than 86.5 g.

$X \sim B(60, 0.109)$

$P(X \geq 9) \approx 0.203$

(M1)

A1

(M1)

A1

M1A1

A1

Total [8 marks]

4 Let any odd integer be represented by $2n + 1$, $n \in \mathbb{Z}^+$.

\therefore three consecutive odd integers are $2n + 1$, $2n + 3$, $2n + 5$.

$$(2n + 1) + (2n + 3) + (2n + 5) = 6n + 9$$

$$= 2(3n + 4) + 1 \quad \text{which is odd as } 3n + 4 \in \mathbb{Z}^+.$$

M1A1

M1R1

Total [4 marks]

5 Normal vector of plane $\mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

A1

Direction vector of line is $\mathbf{d} = \begin{pmatrix} 2 \\ 2 \\ \frac{1}{4} \end{pmatrix}$.

A1

\therefore the angle θ between the line and the plane is given by $\sin \theta = \frac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|}$

M1

$$= \frac{|6 + 0 - \frac{1}{4}|}{\sqrt{10} \sqrt{8.0625}}$$

A1A1

$$\approx 0.640$$

$$\therefore \theta \approx 39.8^\circ$$

A1

Total [6 marks]

6 Let P_n be “ $7^n - 3^n$ is divisible by 4” for all $n \in \mathbb{Z}^+$.

When $n = 1$, $7^1 - 3^1 = 4$ which is divisible by 4

$\therefore P_1$ is true.

A1

Assume $7^k - 3^k = 4A$ where $A \in \mathbb{Z}^+$ (*)

M1

$$\therefore 7^{k+1} - 3^{k+1} = 7 \times 7^k - 3 \times 3^k$$

M1A1

$$= 7 \times (4A + 3^k) - 3 \times 3^k \quad \{\text{using (*)}\}$$

M1

$$= 28A + 7 \times 3^k - 3 \times 3^k$$

$$= 28A + 4 \times 3^k$$

$$= 4(7A + 3^k) \quad \text{which is divisible by 4}$$

A1R1

$\therefore P_{k+1}$ is true whenever P_k is true, and since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$.

Total [7 marks]

7 a $2^t - \frac{t^2}{4} > 0$ for all $t \geq 0$

Now $\int_0^k \left(2^t - \frac{t^2}{4}\right) dt = 1$ M1A1

$\therefore \left[\frac{1}{\ln 2} 2^t - \frac{1}{12} t^3\right]_0^k = 1$ A1

$\frac{1}{\ln 2} 2^k - \frac{1}{12} k^3 - \frac{1}{\ln 2} = 1$

\therefore using technology, $k \approx 0.783\,16$ {5 significant figures} M1A1

b $P\left(\frac{1}{10} \leq T \leq \frac{1}{2}\right) = \int_{0.1}^{0.5} \left(2^t - \frac{t^2}{4}\right) dt$ (M1)
 ≈ 0.484 A1

Total [7 marks]

8 $\arg(z + i) = \frac{\pi}{6}$, $\arg\left(z - \frac{\sqrt{3}}{2}\right) = -\frac{\pi}{2}$

Let $z = a + bi$

$\arg(a + bi + i) = \frac{\pi}{6}$

$\therefore \arg(a + i(b + 1)) = \frac{\pi}{6}$

$\therefore \frac{b+1}{a} = \tan \frac{\pi}{6}$ M1

$\therefore \frac{b+1}{a} = \frac{1}{\sqrt{3}}$

$\therefore a = \sqrt{3}(b + 1) \dots (*)$ A1

Also, $\arg\left(a + bi - \frac{\sqrt{3}}{2}\right) = -\frac{\pi}{2}$

$\therefore \arg\left(a - \frac{\sqrt{3}}{2} + bi\right) = -\frac{\pi}{2}$

$\therefore a - \frac{\sqrt{3}}{2} + bi$ is purely imaginary (M1)

$\therefore a = \frac{\sqrt{3}}{2}$ A1

Substituting $a = \frac{\sqrt{3}}{2}$ into (*) gives

$\frac{\sqrt{3}}{2} = \sqrt{3}(b + 1)$ M1

$\therefore \frac{1}{2} = b + 1$

$\therefore b = -\frac{1}{2}$ A1

$\therefore z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ A1

Total [7 marks]

9 a Let A and B be random variables which represent the number of calls Anne and Billie fail to answer respectively.

$A \sim B(30, 0.03)$ and $B \sim B(50, 0.05)$

$\therefore E(A) = 30 \times 0.03 = 0.9$ and $E(B) = 50 \times 0.05 = 2.5$ A1A1

$\therefore E(A + B) = E(A) + E(B)$

$= 0.9 + 2.5$

$= 3.4$ calls A1

b $P(A + B \geq 2) = 1 - P(A + B = 1) - P(A + B = 0)$ (M1)

$= 1 - P(A = 0, B = 1 \text{ or } A = 1, B = 0) - P(A = 0, B = 0)$

$\approx 1 - (0.401 \times 0.202 + 0.372 \times 0.0769) - 0.401 \times 0.0769$ (M1)

≈ 0.859 A1

Total [6 marks]

Section B

10 8 girls, 2 boys

$$\mathbf{a} \quad \mathbf{i} \quad 9! \times 2! = 725\,760 \quad (\text{M1})\text{A1}$$

$$\mathbf{ii} \quad 10! - 725\,760 = 2\,903\,040 \quad (\text{M1})\text{A1}$$

\mathbf{iii} There are 8 possible positions for the boys, and $2!$ ways in which they can be arranged. (M1)

$$8 \times 2! \times 8! = 645\,120 \quad (\text{M1})\text{A1}$$

$$\mathbf{b} \quad \mathbf{i} \quad {}^8C_2 \times {}^2C_2 = 28 \quad (\text{M1})\text{A1}$$

$$\mathbf{ii} \quad {}^8C_4 {}^2C_0 = 70 \quad (\text{M1})\text{A1}$$

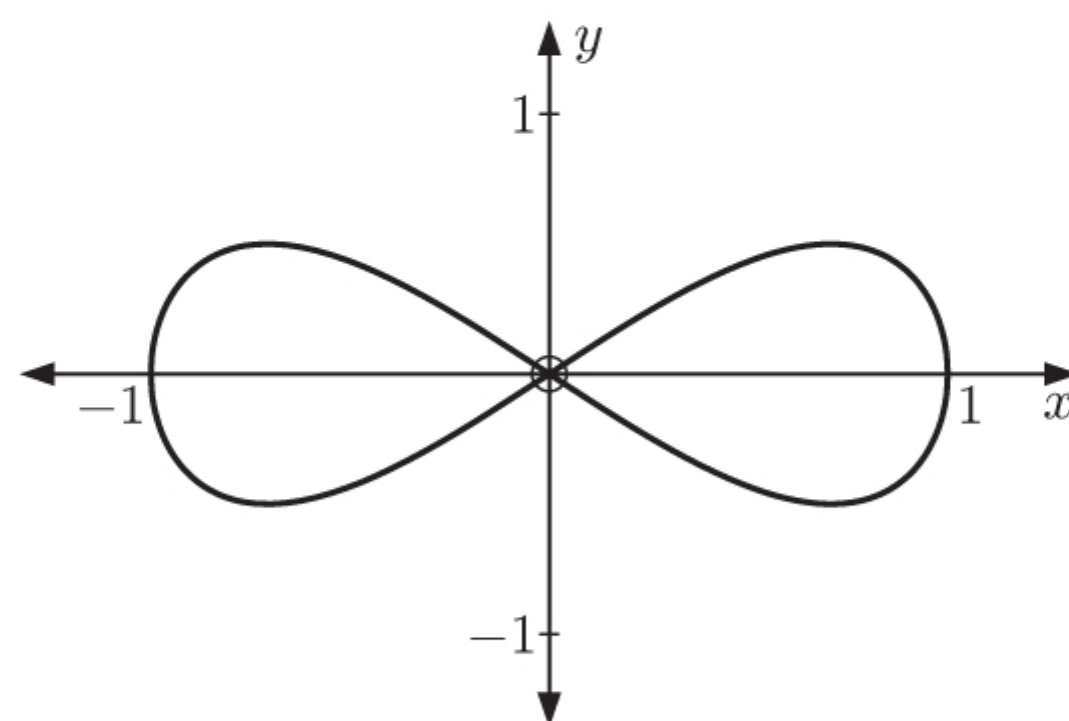
$$\mathbf{c} \quad \text{P(a team contains all 3 sisters)} = \frac{{}^3C_3 {}^7C_1}{{}^{10}C_4} \quad \text{M1A1}$$

$$= \frac{1}{30} \quad \text{A1}$$

Total [14 marks]

$$\mathbf{11} \quad y^2 = x^2 - x^4, \quad -1 \leq x \leq 1$$

$$\mathbf{a} \quad y = \pm \sqrt{x^2 - x^4}$$

A1 : correct for $y > 0$ A1 : correct for $y < 0$

A1 : correct end points

$$\mathbf{b} \quad \text{Consider } y = (x^2 - x^4)^{\frac{1}{2}}, \quad y > 0$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(x^2 - x^4)^{-\frac{1}{2}}(2x - 4x^3) \quad \text{M1A1}$$

$$= \frac{x(1 - 2x^2)}{\sqrt{x^2 - x^4}}, \quad x \neq 0$$

$$\therefore \frac{dy}{dx} = 0 \quad \text{when } 1 - 2x^2 = 0 \quad \text{M1}$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}} \quad \text{A1}$$

$$\therefore \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \text{ and } \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \text{ are the stationary points for } y > 0. \quad \text{A1A1}$$

$$\mathbf{c} \quad \text{The gradient is undefined when } x^2 - x^4 = 0 \quad \text{A1}$$

$$x^2(1 - x^2) = 0$$

$$x = 0, \pm 1 \quad \text{A1A1}$$

\mathbf{d} Consider the region under the curve for $x \geq 0$ and $y \geq 0$.

$$A = \int_0^1 \sqrt{x^2 - x^4} dx \quad (\text{M1})$$

$$= \frac{1}{3} \quad \{\text{technology}\} \quad (\text{A1})$$

$$\therefore \text{total area} = 4 \times \frac{1}{3} \quad \text{A1}$$

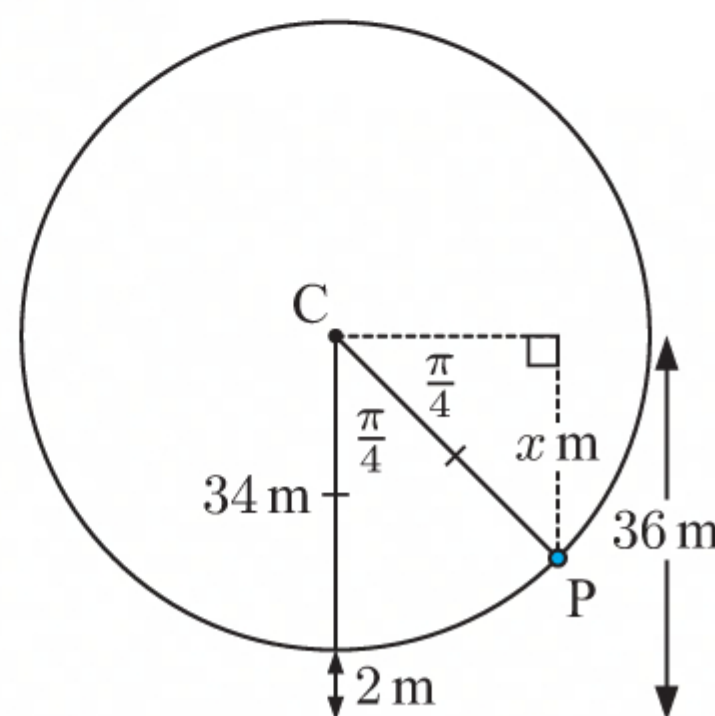
$$= \frac{4}{3}$$

Total [15 marks]

- 12 a** After 5 minutes, the wheel has rotated 225° or $\frac{5\pi}{4}$.

A1

(M1)



$$\sin \frac{\pi}{4} = \frac{x}{34}$$

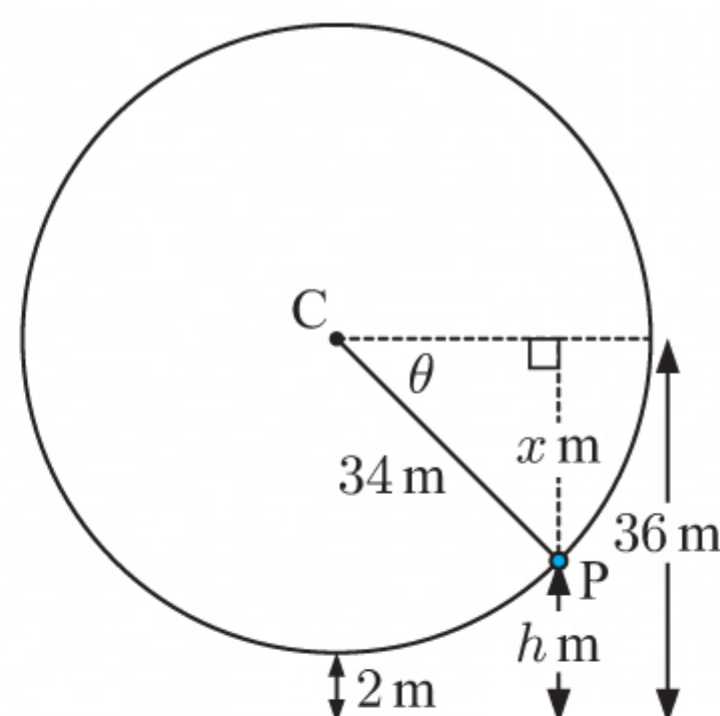
$$\therefore x = 34 \left(\frac{1}{\sqrt{2}} \right)$$

(A1)

$$\therefore P \text{ is } 36 - \frac{34}{\sqrt{2}} \approx 11.96 \text{ m above ground level.}$$

M1A1

b



Let h be the vertical height of point P above ground level.

$$\text{Now } \sin \theta = \frac{x}{34}$$

$$h = 36 - 34 \sin \theta$$

M1A1

$$\therefore \frac{dh}{dt} = -34 \cos \theta \frac{d\theta}{dt}$$

A1

$$\text{After 5 minutes, } \theta = \frac{\pi}{4}, \quad \frac{d\theta}{dt} = -\frac{2\pi}{8} = -\frac{\pi}{4}$$

A1

$$\therefore \frac{dh}{dt} = -34 \times \cos \frac{\pi}{4} \times -\frac{\pi}{4}$$

M1

$$= \frac{17\pi}{2\sqrt{2}}$$

$$\approx 18.9 \text{ m per min}$$

A1

Total [11 marks]

- 13 a** $\frac{dy}{dt} = ky(2 - ky), \quad 0 < y < \frac{k}{2}, \quad k \in \mathbb{R}^+$

$$\therefore \frac{1}{y(2 - ky)} \frac{dy}{dt} = k$$

$$\therefore \int \frac{1}{y(2 - ky)} dy = \int k dt$$

M1A1

$$\text{Now let } \frac{1}{y(2 - ky)} = \frac{A}{y} + \frac{B}{2 - ky}$$

M1

$$\therefore \frac{1}{y(2 - ky)} = \frac{A(2 - ky) + By}{y(2 - ky)}$$

$$\therefore \frac{1}{y(2 - ky)} = \frac{2A - Ak y + By}{y(2 - ky)}$$

$$\therefore 2A = 1 \quad \text{and} \quad B - Ak = 0$$

$$\therefore A = \frac{1}{2} \quad B = \frac{1}{2}k$$

A1A1

$$\therefore \frac{1}{y(2 - ky)} = \frac{1}{2y} + \frac{k}{2(2 - ky)}$$

A1

$$\text{So, } \int \left(\frac{1}{2y} + \frac{k}{2(2-ky)} \right) dy = \int k dt$$

$$\therefore \frac{1}{2} \int \left(\frac{1}{y} + \frac{k}{2-ky} \right) dy = kt + c, \quad c \in \mathbb{R}$$

$$\therefore \int \left(\frac{1}{y} + \frac{k}{2-ky} \right) dy = 2(kt + c)$$

$$\therefore \ln|y| + \frac{k}{(-k)} \ln|2-ky| = 2(kt + c)$$

A1

$$\therefore \ln y - \ln(2-ky) = 2(kt + c) \quad \left\{ 0 < y < \frac{2}{k} \right\}$$

$$\therefore \ln \left(\frac{y}{2-ky} \right) = 2(kt + c)$$

A1

$$\therefore \frac{y}{2-ky} = e^{2(kt+c)}$$

$$\therefore y = (2-ky)e^{2(kt+c)}$$

A1

$$\therefore y = 2e^{2(kt+c)} - kye^{2(kt+c)}$$

$$\therefore y + kye^{2(kt+c)} = 2e^{2(kt+c)}$$

$$\therefore y(1 + ke^{2(kt+c)}) = 2e^{2(kt+c)}$$

A1

$$\therefore y = \frac{2e^{2(kt+c)}}{1 + ke^{2(kt+c)}}$$

AG

b $y(0) = e$

$$\therefore \frac{2e^{2c}}{1 + \frac{1}{e}(e^{2c})} = e$$

M1

$$\therefore 2e^{2c} = e + e^{2c}$$

$$\therefore e^{2c} = e$$

$$\therefore c = \frac{1}{2}$$

A1

$$\therefore y(e) = \frac{2e^{2(1+\frac{1}{2})}}{1 + \frac{1}{e}e^{2(1+\frac{1}{2})}}$$

M1

$$= \frac{2e^3}{1 + e^2}$$

A1

Total [14 marks]

PAPER 3

1 a i $p(2) = 2^3 - 3(2)^2 - 10(2) + 24$
 $= 8 - 12 - 20 + 24$
 $= 0$

M1

$$\therefore x = 2 \text{ is a root of } p(x) = 0.$$

$$\text{Let } p(x) = (x-2)(x^2 + \alpha x + \beta)$$

$$= x^3 + (\alpha-2)x^2 + (\beta-2\alpha)x - 2\beta$$

$$\text{Equating coefficients, } \alpha - 2 = -3$$

$$\beta - 2\alpha = -10$$

$$-2\beta = 24$$

$$\therefore \alpha = -1 \text{ and } \beta = -12$$

(M1)

$$\therefore p(x) = (x-2)(x^2 - x - 12)$$

$$= (x-2)(x+3)(x-4)$$

A1A1

ii $p'(x) = 3x^2 - 6x - 10$

$$\therefore p'(3) = 3 \times 9 - 6 \times 3 - 10 = -1$$

A1

$$\text{Also, } p(3) = 3^3 - 3(3)^2 - 10(3) + 24 = -6$$

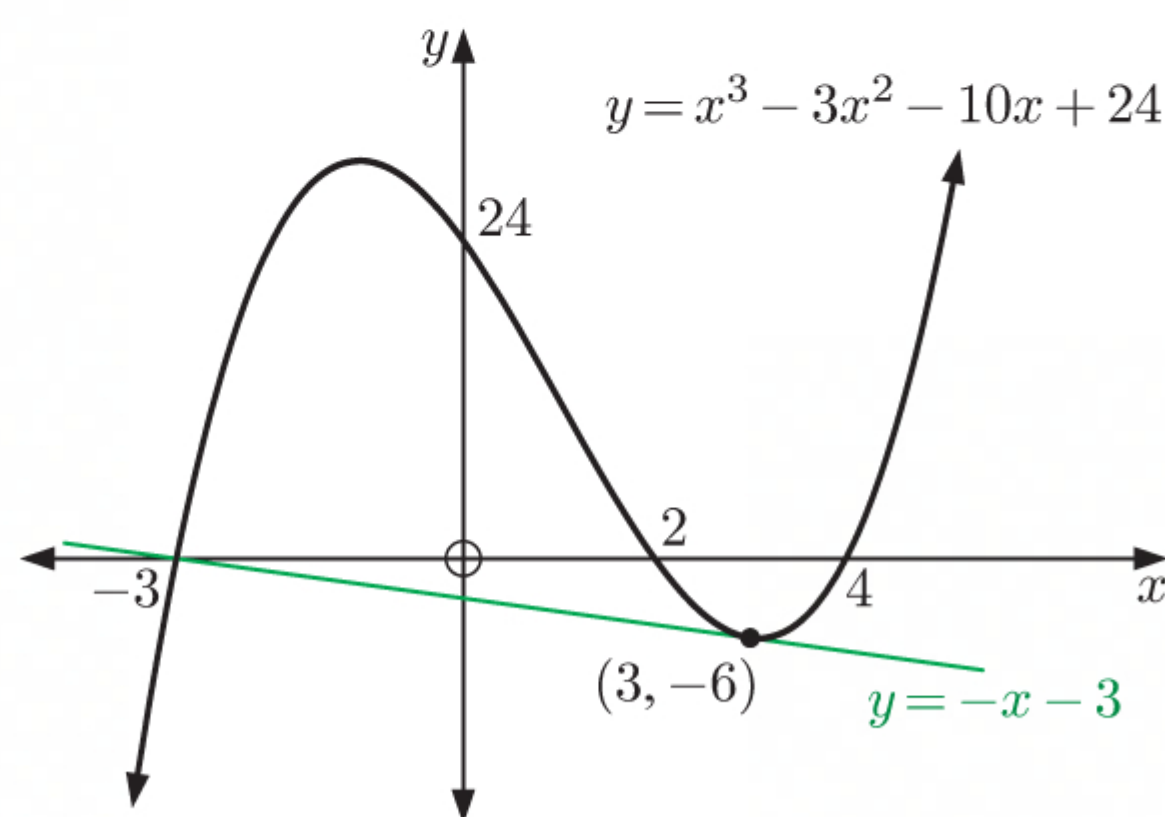
A1

$$\therefore \text{the equation of the tangent is } y - (-6) = -(x - 3)$$

$$\text{or } y = -x - 3$$

A1

iii



A1A1

 b i $f(x) = L(x)$ has roots x_1, x_2, x_3

$$\therefore f(x) - L(x) = a(x - x_1)(x - x_2)(x - x_3)$$

$$\therefore ax^3 + bx^2 + cx + d - mx - 5 = a(x - x_1)(x^2 - (x_2 + x_3)x + x_2x_3)$$

$$\begin{aligned} \therefore ax^3 + bx^2 + (c - m)x + (d - 5) &= ax^3 - a(x_1 + x_2 + x_3)x^2 \\ &\quad + a(x_1x_2 + x_2x_3 + x_1x_3)x - ax_1x_2x_3 \end{aligned}$$

 Equating coefficients of x^2 , $-a(x_1 + x_2 + x_3) = b$

$$\therefore x_1 + x_2 + x_3 = -\frac{b}{a} \quad \text{which is independent of the choice of } L.$$

A1R1

ii $f'(x) = 3ax^2 + 2bx + c$

$$\therefore f''(x) = 6ax + 2b$$

A1

 At the inflection point, $f''(x^*) = 0$

M1

$$\therefore 6ax^* + 2b = 0$$

$$\therefore x^* = -\frac{2b}{6a}$$

$$= \frac{1}{3} \left(-\frac{b}{a} \right)$$

A1

$$= \frac{x_1 + x_2 + x_3}{3} \quad \{\text{using i}\}$$

AG

 iii (1) If the line is a tangent to the cubic at $x = x_1$, then $f(x_1) = L(x_1)$
and $f'(x_1) = L'(x_1)$

$$\therefore f(x_1) - L(x_1) = 0 \quad \dots (1)$$

$$\text{and } f'(x_1) - L'(x_1) = 0 \quad \dots (2)$$

M1

 From (2), $f'(x) - L'(x)$ has a factor $(x - x_1)$ (3)

M1

$$\text{Now } f(x) - L(x) = ax^3 + bx^2 + (c - m)x + (d - 5)$$

$$\therefore f'(x) - L'(x) = 3ax^2 + 2bx + (c - m)$$

A1

 \therefore using (3), $f'(x) - L'(x) = 3a(x - x_1)(x - \beta)$ for some β .

AG

 (2) Integrating both sides with respect to x ,

M1

$$f(x) - L(x)$$

$$= \int 3a(x - x_1)(x - \beta) dx$$

$$= 3a \int (x - x_1)(x - \beta) dx$$

$$= 3a \left[\frac{1}{2}(x - x_1)^2(x - \beta) - \int \frac{1}{2}(x - x_1)^2 dx \right] \quad \begin{cases} u = x - \beta & v' = x - x_1 \\ u' = 1 & v = \frac{1}{2}(x - x_1)^2 \end{cases} \quad \text{(M1)}$$

$$= \frac{3a}{2}(x - x_1)^2(x - \beta) - \frac{a}{2}(x - x_1)^3 + c$$

A1

 But using (1), $c = 0$

$$\therefore f(x) - L(x) = \frac{3a}{2}(x - x_1)^2(x - \beta) - \frac{a}{2}(x - x_1)^3$$

M1A1

$$= \frac{a}{2}(x - x_1)^2(3x - 3\beta - x + x_1)$$

$$= \frac{a}{2}(x - x_1)^2(2x + x_1 - 3\beta)$$

AG

(3) Let the other intersection point occur at $x = \hat{x}$.

$$\therefore f(x) - L(x) = a(x - x_1)^2(x - \hat{x})$$

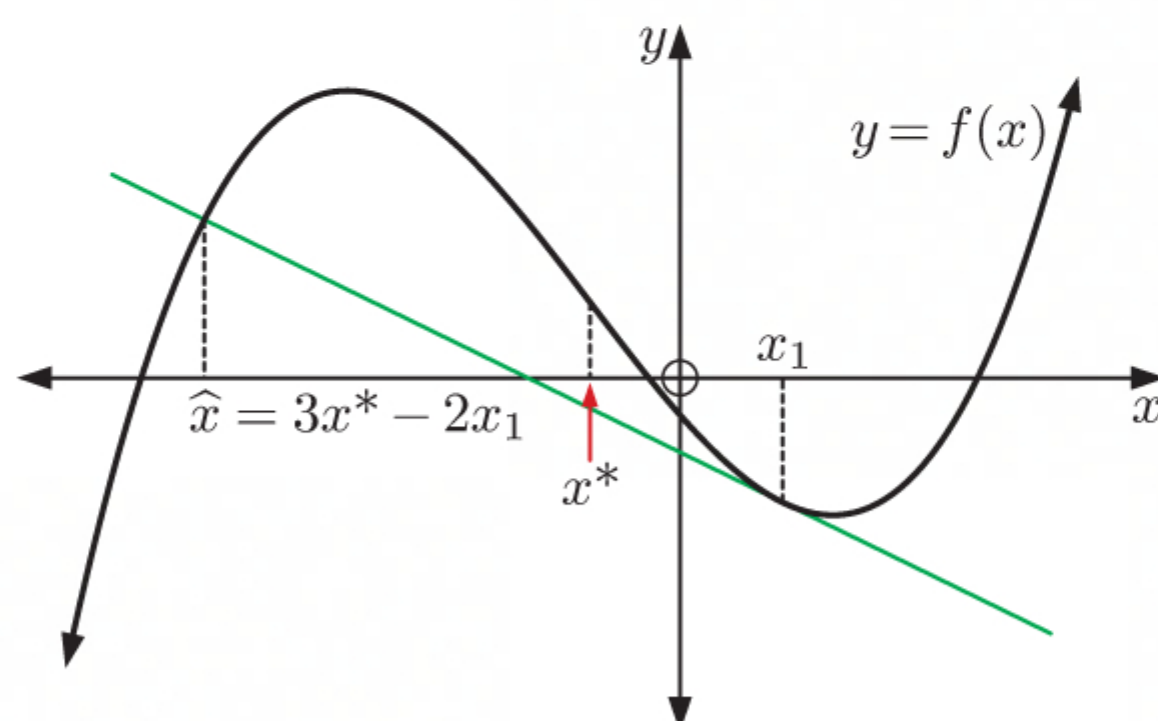
A1

Now $x_1 + x_1 + \hat{x} = 3x^*$ {from **b ii**}

$$\therefore \hat{x} = 3x^* - 2x_1$$

A1

A1



(4) If a cubic function with inflection point at $x = x^*$ touches the x -axis at $x = x_1$, the cubic cuts the x -axis again at $x = 3x^* - 2x_1$.

A1A1

Total [27 marks]

2 a $R \sin(x - \alpha) = R \sin x \cos \alpha - R \cos x \sin \alpha$

A1

b i $\sqrt{3} \sin x - \cos x = R \sin(x - \alpha)$

$$\therefore \sqrt{3} \sin x - \cos x = R \sin x \cos \alpha - R \cos x \sin \alpha$$

Equating coefficients of $\sin x$ and $\cos x$ gives

M1

$$\sqrt{3} = R \cos \alpha \quad \text{and} \quad -R \sin \alpha = -1$$

$$\therefore R \sin \alpha = 1$$

AG

ii $\cos \alpha = \frac{\sqrt{3}}{R} \quad \text{and} \quad \sin \alpha = \frac{1}{R}$

Now $\cos^2 \alpha + \sin^2 \alpha = 1$

$$\therefore \left(\frac{\sqrt{3}}{R}\right)^2 + \left(\frac{1}{R}\right)^2 = 1$$

M1A1

$$\therefore \frac{4}{R^2} = 1$$

$$\therefore R = 2 \quad \{R > 0\}$$

A1

So, $\cos \alpha = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin \alpha = \frac{1}{2}$

$$\therefore \alpha = \frac{\pi}{6} \quad \{0 < \alpha < \frac{\pi}{2}\}$$

A1

c $f(x) - 1 = 0$

$$\therefore 2 \sin(x - \frac{\pi}{6}) - 1 = 0$$

M1

$$\therefore \sin(x - \frac{\pi}{6}) = \frac{1}{2}$$

$$\therefore x - \frac{\pi}{6} = \frac{\pi}{6} \quad \{x \in [0, \frac{\pi}{2}], \text{ so } x - \frac{\pi}{6} \in [-\frac{\pi}{6}, \frac{\pi}{3}]\}$$

A1

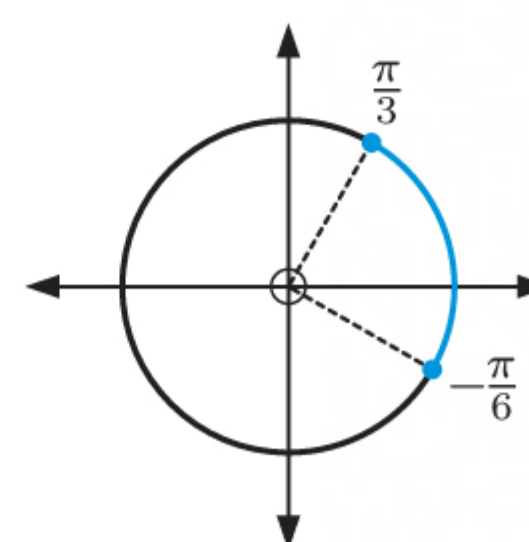
$$\therefore x = \frac{\pi}{3}$$

A1

d i $f(x) = 2 \sin(x - \frac{\pi}{6}), \quad x - \frac{\pi}{6} \in [-\frac{\pi}{6}, \frac{\pi}{3}]$

On the interval $[-\frac{\pi}{6}, \frac{\pi}{3}]$, $\sin \theta$ is one-to-one.

$\therefore f(x)$ is one-to-one and hence invertible.



R1AG

ii $f^{-1}(x)$ is given by $x = 2 \sin(y - \frac{\pi}{6})$

M1

$$\therefore \sin(y - \frac{\pi}{6}) = \frac{x}{2}$$

$$\therefore y - \frac{\pi}{6} = \arcsin \frac{x}{2}$$

A1

$$\therefore y = \frac{\pi}{6} + \arcsin \frac{x}{2}$$

$$\therefore f^{-1}(x) = \frac{\pi}{6} + \arcsin \frac{x}{2}$$

AG

$$\begin{aligned}
 \mathbf{e} \quad \int_0^1 f^{-1}(x) dx &= \int_0^1 \left(\frac{\pi}{6} + \arcsin \frac{x}{2} \right) dx \\
 &= \left[\frac{\pi}{6} x \right]_0^1 + \int_0^1 1 \times \arcsin \frac{x}{2} dx \quad \begin{cases} u = \arcsin \frac{x}{2} & v' = 1 \\ u' = \frac{1}{2\sqrt{1 - (\frac{x}{2})^2}} & v = x \end{cases} & \mathbf{M1A1} \\
 &= \frac{\pi}{6} + \left[x \arcsin \frac{x}{2} \right]_0^1 - \int_0^1 \frac{x}{\sqrt{4 - x^2}} dx & \mathbf{A1} \\
 &= \frac{\pi}{6} + \arcsin \frac{1}{2} + \frac{1}{2} \int_0^1 \frac{-2x}{\sqrt{4 - x^2}} dx & \mathbf{M1} \\
 &= \frac{\pi}{6} + \arcsin \frac{1}{2} + \frac{1}{2} \int_4^3 \frac{1}{\sqrt{z}} dz \quad \begin{cases} z = 4 - x^2 \\ \frac{dz}{dx} = -2x \\ x = 0 \Rightarrow z = 4 \\ x = 1 \Rightarrow z = 3 \end{cases} & \mathbf{M1A1} \\
 &= \frac{\pi}{6} + \frac{\pi}{6} + \frac{1}{2} \left[2z^{\frac{1}{2}} \right]_4^3 & \mathbf{M1} \\
 &= \frac{\pi}{3} + \frac{1}{2} (2\sqrt{3} - 4) & \mathbf{A1} \\
 &= \frac{\pi}{3} + \sqrt{3} - 2 & \mathbf{AG}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad \int_0^1 f^{-1}(x) dx &= \int_{\frac{\pi}{3}}^a \frac{4}{f(x)} dx, \quad a > \frac{\pi}{3} \\
 \therefore \int_{\frac{\pi}{3}}^a \frac{4}{f(x)} dx &= \frac{\pi}{3} + \sqrt{3} - 2 & \mathbf{M1} \\
 \therefore 4 \int_{\frac{\pi}{3}}^a \frac{1}{2 \sin(x - \frac{\pi}{6})} dx &= \frac{\pi}{3} + \sqrt{3} - 2 \\
 \therefore 2 \int_{\frac{\pi}{3}}^a \frac{1}{\sin(x - \frac{\pi}{6})} dx &= \frac{\pi}{3} + \sqrt{3} - 2 \\
 \therefore \int_{\frac{\pi}{3}}^a \operatorname{cosec}(x - \frac{\pi}{6}) dx &= \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 & \mathbf{A1} \\
 \therefore \left[-\ln \left| \operatorname{cosec}(x - \frac{\pi}{6}) + \cot(x - \frac{\pi}{6}) \right| \right]_{\frac{\pi}{3}}^a &= \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 & \mathbf{M1A1} \\
 \therefore -\ln \left| \operatorname{cosec}(a - \frac{\pi}{6}) + \cot(a - \frac{\pi}{6}) \right| + \ln \left| \operatorname{cosec} \frac{\pi}{6} + \cot \frac{\pi}{6} \right| &= \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 \\
 \therefore \ln \left(\frac{\left| \operatorname{cosec} \frac{\pi}{6} + \cot \frac{\pi}{6} \right|}{\left| \operatorname{cosec}(a - \frac{\pi}{6}) + \cot(a - \frac{\pi}{6}) \right|} \right) &= \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 & \mathbf{A1} \\
 \therefore \frac{2 + \sqrt{3}}{\left| \operatorname{cosec}(a - \frac{\pi}{6}) + \cot(a - \frac{\pi}{6}) \right|} &= e^{\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1} & \mathbf{A1} \\
 \therefore \left| \operatorname{cosec}(a - \frac{\pi}{6}) + \cot(a - \frac{\pi}{6}) \right| &= \frac{2 + \sqrt{3}}{e^{\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1}} & \mathbf{AG}
 \end{aligned}$$

$$\mathbf{g} \quad \text{Solving using technology gives } a \approx 1.2770 \quad \{5 \text{ s.f.}\} \quad \mathbf{(M1)A1}$$

Total [28 marks]

TRIAL EXAMINATION 3

PAPER 1

Section A

$$\begin{aligned}
 1 \quad a \quad \binom{2n-1}{2n-3} &= \frac{(2n-1)!}{(2n-3)!((2n-1)-(2n-3))!} && \text{M1} \\
 &= \frac{(2n-1)!}{(2n-3)! \times 2} \\
 &= \frac{(2n-1)(2n-2)}{2} && \text{A1} \\
 &= (2n-1)(n-1) && \text{A1A1}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \binom{2n-1}{2n-3} &\leq 10 \\
 \therefore (2n-1)(n-1) &\leq 10 \\
 \therefore 2n^2 - 3n + 1 &\leq 10 && \text{M1} \\
 \therefore 2n^2 - 3n - 9 &\leq 0 \\
 \therefore (2n+3)(n-3) &\leq 0 \\
 \therefore -\frac{3}{2} &\leq n \leq 3
 \end{aligned}$$

However, $2n-3 \geq 0$, so $n \geq \frac{3}{2}$.

Therefore, the possible values of n , where $n \in \mathbb{Z}^+$, are $n = 3$ and $n = 2$.

A1A1

Total [7 marks]

- 2 Adding the first equation to twice the second equation (to eliminate z) results in

$$\begin{aligned}
 2x + 3y - 4z &= -7 \\
 + 2x - 2y + 4z &= 12 \\
 \hline
 4x + y &= 5 \quad \dots (1)
 \end{aligned}$$

Adding the last two equations (to eliminate z) results in

$$\begin{aligned}
 x - y + 2z &= 6 \\
 + 3x - 2y - 2z &= 11 \\
 \hline
 4x - 3y &= 17 \quad \dots (2) && \text{M1A1}
 \end{aligned}$$

$$\begin{aligned}
 (1) - (2) \text{ gives } & \begin{aligned} &4x + y = 5 \\ &- (4x - 3y = 17) \\ \hline &4y = -12 \Rightarrow y = -3 \end{aligned} && \text{M1A1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ in (1), } 4x + (-3) &= 5 \\
 \therefore 4x &= 8 \\
 \therefore x &= 2 && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ in the first equation, } 2(2) + 3(-3) - 4z &= -7 \\
 \therefore 4 - 9 - 4z &= -7 \\
 \therefore z &= \frac{1}{2}
 \end{aligned}$$

So, the point of intersection is $(2, -3, \frac{1}{2})$.

A1

Total [6 marks]

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad S_8 = 6, \text{ so } \frac{8}{2}(2u_1 + (8-1)d) &= 6 \\ \therefore 8u_1 + 28d &= 6 \quad \dots (1) \end{aligned} \quad \mathbf{M1}$$

$$\begin{aligned} S_{12} - S_8 = 39, \text{ so } \left(\frac{12}{2}(2u_1 + (12-1)d)\right) - (8u_1 + 28d) &= 39 \\ \therefore 4u_1 + 38d &= 39 \quad \dots (2) \end{aligned} \quad \mathbf{M1}$$

$$\begin{aligned} (1) - 2 \times (2) \text{ gives } -48d &= -72 \\ \therefore d &= 1.5 \end{aligned} \quad \begin{array}{l} \mathbf{M1} \\ \mathbf{A1} \end{array}$$

$$\begin{aligned} \therefore \text{ in (1), } 8u_1 + 28(1.5) &= 6 \\ \therefore u_1 &= -4.5 \end{aligned} \quad \mathbf{A1}$$

$$\begin{aligned} \mathbf{b} \quad u_n &= u_1 + (n-1)d \\ &= -4.5 + (n-1)(1.5) \\ &= 1.5n - 6 \end{aligned} \quad \begin{array}{l} \mathbf{M1} \\ \mathbf{A1} \end{array}$$

$$\begin{aligned} \mathbf{c} \quad S_n &> 45 \\ \therefore \frac{n}{2}(-4.5 + (1.5n - 6)) &> 45 \quad \left\{ S_n = \frac{n}{2}(u_1 + u_n) \right\} \\ \therefore 0.75n^2 - 5.25n &> 45 \\ \therefore 3n^2 - 21n &> 180 \\ \therefore n^2 - 7n - 60 &> 0 \\ \therefore (n-12)(n+5) &> 0 \\ \therefore n > 12 \text{ or } n < -5 \end{aligned} \quad \begin{array}{l} \mathbf{M1} \\ \mathbf{M1} \end{array}$$



However, as $n \in \mathbb{Z}^+$, the smallest value of n for which $S_n > 45$ is $n = 13$. **A1**

Total [10 marks]

$$\mathbf{4} \quad \text{Suppose } f(x) \text{ intersects } g_k(x) \text{ only at } x = a. \text{ Then, } f(a) = g_k(a) \text{ and } f'(a) = g'_k(a). \quad \mathbf{M1}$$

$$\text{So, } e^{2a} = k\sqrt{a} \quad \dots (1) \quad \mathbf{A1}$$

$$\text{and } 2e^{2a} = \frac{k}{2\sqrt{a}} \Rightarrow e^{2a} = \frac{k}{4\sqrt{a}} \quad \dots (2) \quad \mathbf{M1A1}$$

$$\begin{aligned} \text{Equating (1) and (2) gives } k\sqrt{a} &= \frac{k}{4\sqrt{a}} \\ \therefore (\sqrt{a})^2 &= \frac{1}{4} \\ \therefore a &= \frac{1}{4} \end{aligned} \quad \mathbf{A1}$$

$$\begin{aligned} \therefore \text{ in (1), } e^{\frac{1}{2}} &= \frac{1}{2}k \\ \therefore k &= 2\sqrt{e} \end{aligned} \quad \mathbf{A1}$$

Total [6 marks]

$$\mathbf{5} \quad \mathbf{a} \quad \text{A normal to the plane is } \mathbf{n} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}. \quad \mathbf{A1}$$

$$\mathbf{b} \quad \mathbf{i} \quad \text{As the plane and the line are perpendicular, the normal } \mathbf{n} \text{ and the direction vector of the line are parallel.} \quad \mathbf{R1}$$

$$\therefore \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = k \begin{pmatrix} t+1 \\ -3 \\ 2-t \end{pmatrix} \quad \mathbf{M1}$$

$$\therefore k = -\frac{1}{3} \text{ and } t = 8 \quad \mathbf{A1}$$

$$\begin{aligned} \mathbf{ii} \quad -3(2+9\lambda) + (3-3\lambda) + 2(-1-6\lambda) &= 2 \\ \therefore -5-42\lambda &= 2 \\ \therefore -42\lambda &= 7 \\ \therefore \lambda &= -\frac{1}{6} \end{aligned} \quad \begin{array}{l} \mathbf{M1A1} \\ \mathbf{A1} \end{array}$$

Therefore, the coordinates of the point of intersection are

$$\left(2+9\left(-\frac{1}{6}\right), 3-3\left(-\frac{1}{6}\right), -1-6\left(-\frac{1}{6}\right)\right) = \left(\frac{1}{2}, \frac{7}{2}, 0\right) \quad \mathbf{A1}$$

Total [8 marks]

$$\mathbf{6} \quad \mathbf{a} \quad \text{Differentiating the equation implicitly with respect to } x \text{ gives } 4x^3 + 4y^3 \frac{dy}{dx} = 0 \quad \mathbf{M1}$$

$$\therefore \frac{dy}{dx} = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3} \quad \mathbf{AG}$$

$$\mathbf{b} \quad \frac{dy}{dx} = -\frac{x^3}{y^3} \quad \mathbf{M1}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{(3x^2)y^3 - x^3\left(3y^2\frac{dy}{dx}\right)}{(y^3)^2} \quad \mathbf{M1A1}$$

$$= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} \quad \mathbf{M1A1}$$

$$= -\frac{3x^2y^3 + 3x^6y^{-1}}{y^6}$$

$$= -\frac{3x^2(y^4 + x^4)}{y^7}$$

$$= -\frac{3x^2}{y^7} \{x^4 + y^4 = 1\} \quad \mathbf{M1AG}$$

$$\mathbf{c} \quad \frac{d^2y}{dx^2} = 0$$

$$\therefore 3x^2 = 0$$

$$\therefore x = 0 \quad \mathbf{A1}$$

$$\text{When } x = 0, \quad y^4 = 1$$

$$\therefore y = \pm 1$$

$$\therefore \text{the points are } (0, 1) \text{ and } (0, -1). \quad \mathbf{A1}$$

Total [9 marks]

$$\mathbf{7} \quad \mathbf{a} \quad \int_0^{\frac{\pi}{6}} k \cos x \, dx = 1 \quad \mathbf{M1}$$

$$\therefore k \int_0^{\frac{\pi}{6}} \cos x \, dx = 1$$

$$\therefore k \left[\sin x \right]_0^{\frac{\pi}{6}} = 1 \quad \mathbf{A1}$$

$$\therefore k \left(\sin\left(\frac{\pi}{6}\right) + \sin 0 \right) = 1 \quad \mathbf{A1}$$

$$\therefore k \times \frac{1}{2} = 1$$

$$\therefore k = 2 \quad \mathbf{A1}$$

$$\mathbf{b} \quad E(X) = \int_0^{\frac{\pi}{6}} 2x \cos x \, dx = 2 \int_0^{\frac{\pi}{6}} x \cos x \, dx \quad \mathbf{M1}$$

$$\text{Using integration by parts where } u = x \quad v = \sin x \quad \mathbf{M1}$$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \cos x$$

$$\text{we have } \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + c \quad \mathbf{A1}$$

$$\therefore E(X) = 2 \left[x \sin x + \cos x \right]_0^{\frac{\pi}{6}} \quad \mathbf{A1}$$

$$= 2 \left(\frac{\pi}{6} \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) - \cos 0 \right)$$

$$= 2 \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \right)$$

$$= \frac{\pi}{6} + \sqrt{3} - 2 \quad \mathbf{A1}$$

- c** Let Q_3 represent the upper quartile.

$$\therefore \int_0^{Q_3} 2 \cos x \, dx = \frac{3}{4} \quad \text{M1}$$

$$\therefore 2[\sin x]_0^{Q_3} = \frac{3}{4} \quad \text{A1}$$

$$\therefore 2(\sin Q_3 - \sin 0) = \frac{3}{4}$$

$$\therefore \sin Q_3 = \frac{3}{8}$$

$$\therefore Q_3 = \arcsin\left(\frac{3}{8}\right) \quad \text{A1}$$

Total [12 marks]

Section B

- 8 a i** L_1 has equation $y = \tan \theta_1 x$. M1

So, $m_1 = \tan \theta_1$. A1

ii Similarly, $m_2 = \tan \theta_2$. A1

b $\theta_2 = \theta_1 + \beta$ {exterior angle of triangle} M1

$\therefore \beta = \theta_2 - \theta_1$ A1

$$\begin{aligned} \therefore \tan \beta &= \tan(\theta_2 - \theta_1) \\ &= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1} \end{aligned} \quad \text{M1}$$

$$= \frac{m_2 - m_1}{1 + m_2 m_1} \quad \text{AG}$$

- c** If β is acute then $\tan \beta = \frac{m_2 - m_1}{1 + m_2 m_1}$ is positive.

In this case $\alpha = \beta$, so $\tan \alpha = \tan \beta = \frac{m_2 - m_1}{1 + m_2 m_1}$. A1

If β is obtuse then $\tan \beta = \frac{m_2 - m_1}{1 + m_2 m_1}$ is negative.

In this case $\alpha = \pi - \beta$, so $\tan \alpha = \tan(\pi - \beta) = -\tan \beta = -\frac{m_2 - m_1}{1 + m_2 m_1}$. M1A1

So, $\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|$. AG

- d** $3x + y = 4$ has gradient $m_1 = -3$ and $y = -\frac{1}{2}x + 2$ has gradient $m_2 = -\frac{1}{2}$ M1A1A1

Thus, $\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|$ M1

$$= \left| \frac{\left(-\frac{1}{2}\right) - (-3)}{1 + \left(-\frac{1}{2}\right)(-3)} \right|$$

$$= \left| \frac{2.5}{2.5} \right|$$

$$= 1 \quad \text{A1}$$

$$\therefore \alpha = \frac{\pi}{4} \quad \text{A1}$$

- e** The curves intersect when $x^2 = (x - 3)^2 + 1$ M1

$$\therefore x^2 = x^2 - 6x + 9 + 1$$

$$\therefore 6x = 10$$

$$\therefore x = \frac{5}{3} \quad \text{A1}$$

Now $f'(x) = 2x$, so $f'\left(\frac{5}{3}\right) = 2\left(\frac{5}{3}\right) = \frac{10}{3}$

$$g'(x) = 2(x - 3), \text{ so } g'\left(\frac{5}{3}\right) = 2\left(\frac{5}{3} - 3\right) = -\frac{8}{3} \quad \text{M1A1A1}$$

Thus, the tangent of the angle between the given curves is

$$\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right| = \left| \frac{\left(-\frac{8}{3}\right) - \frac{10}{3}}{1 + \left(-\frac{8}{3}\right)\left(\frac{10}{3}\right)} \right| = \left| \frac{-6}{-\frac{71}{9}} \right| = \frac{54}{71} \quad \text{A1}$$

Total [21 marks]

- 9 a We rearrange the equation of the circle to obtain

$$x^2 + (y - k)^2 = 1$$

$$\therefore y = k \pm \sqrt{1 - x^2}$$

M1A1

So, the equation of the lower semicircle is

$$y = k - \sqrt{1 - x^2} \quad \{y \leq k\}$$

A1

- b $y = |2x|$ for $x > 0$, and the lower semicircle intersect where

$$k - \sqrt{1 - x^2} = 2x$$

M1

$$\therefore \sqrt{1 - x^2} = k - 2x$$

$$\therefore 1 - x^2 = k^2 - 4kx + 4x^2$$

$$\therefore 5x^2 - 4kx + k^2 - 1 = 0$$

A1

As the two curves intersect at exactly one point, the discriminant must equal zero:

M1

$$\therefore (-4k)^2 - 4(5)(k^2 - 1) = 0$$

A1

$$\therefore 16k^2 - 20k^2 + 20 = 0$$

$$\therefore 4k^2 = 20$$

$$\therefore k = \sqrt{5} \quad \{k > 0\}$$

A1

- c From b, the intersection point for $x > 0$ occurs where

$$5x^2 - 4\sqrt{5}x + 4 = 0$$

M1A1

$$\therefore (\sqrt{5}x - 2)^2 = 0$$

$$\therefore x = \frac{2}{\sqrt{5}}$$

A1

When $x = \frac{2}{\sqrt{5}}$, $y = \left| 2\left(\frac{2}{\sqrt{5}}\right) \right| = \frac{4}{\sqrt{5}}$

A1

\therefore the intersection points are $\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ and by symmetry $\left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$.

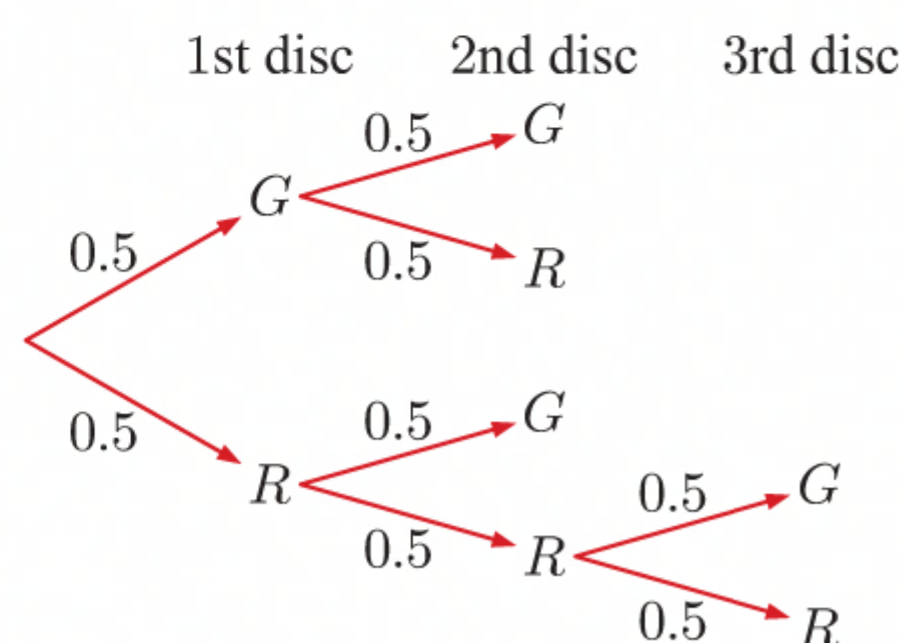
A1

d Area = $\int_{-\frac{2}{\sqrt{5}}}^{\frac{2}{\sqrt{5}}} \left[(\sqrt{5} - \sqrt{1 - x^2}) - |2x| \right] dx$ or $2 \int_0^{\frac{2}{\sqrt{5}}} \left[(\sqrt{5} - \sqrt{1 - x^2}) - (2x) \right] dx$

A1A1A1

Total [16 marks]

- 10 a



A1A1A1

- b Earning 6 coins is only possible if both discs show a red face the first time, and then a red face shows again when one of the discs is thrown again.

M1

$$\therefore P(X = 6) = 0.5 \times 0.5 \times 0.5 = \frac{1}{8}$$

AG

- c Earning 1 coin is only possible when one green and one red face are shown after the initial two disc flips.

M1

$$\therefore P(X = 1) = P(GR \text{ or } RG) = 0.5 \times 0.5 + 0.5 \times 0.5 = \frac{1}{2}$$

A1

d

1st flip	G	G	R	R	R
2nd flip	G	R	G	R	R
3rd flip	-	-	-	G	R
x	$-1 - 1 = -2$	$-1 + 2 = 1$	$2 - 1 = 1$	$2 + 2 - 1 = 3$	$2 + 2 + 2 = 6$
$P(X = x)$	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

A1A1A1

$$\begin{aligned} \mathbf{e} \quad E(X) &= (-2) \times \frac{1}{4} + 1 \times \frac{1}{4} + 1 \times \frac{1}{4} + 3 \times \frac{1}{8} + 6 \times \frac{1}{8} \\ &= -\frac{2}{4} + \frac{1}{4} + \frac{1}{4} + \frac{3}{8} + \frac{6}{8} \\ &= \frac{9}{8} \end{aligned}$$

M1
A1

- f** As playing each round costs 1 coin, playing 40 rounds costs 40 coins. Given that the expected number of coins won per round is $\frac{9}{8}$, the expected number of coins won after playing 40 rounds is

$$40 \times \frac{9}{8} - 40 = 45 - 40 = 5$$

M1A1

- g** Let n represent the number of rounds Siobhan needs to play in order to earn 15 coins.

$$\therefore \frac{9}{8} \times n - n = 15$$

M1

$$\therefore \frac{1}{8}n = 15$$

$$\therefore n = 120$$

Therefore, Siobhan needs to play 120 rounds in order to expect to earn 15 coins.

A1

Total [15 marks]

PAPER 2

Section A

$$\mathbf{1} \quad \ln x^3 + \ln y^5 = 3 \Rightarrow 3 \ln x + 5 \ln y = 3 \quad \dots (1)$$

M1

$$3 \ln \frac{y}{x} = \frac{1}{5} \Rightarrow 3 \ln y - 3 \ln x = \frac{1}{5} \quad \dots (2)$$

M1

$$(1) + (2) \text{ gives } 8 \ln y = \frac{16}{5}$$

$$\therefore \ln y = \frac{2}{5}$$

$$\therefore y = e^{\frac{2}{5}}$$

M1A1

$$\therefore \text{in (1), } 3 \ln x + 5\left(\frac{2}{5}\right) = 3$$

$$\therefore 3 \ln x = 1$$

$$\therefore \ln x = \frac{1}{3}$$

$$\therefore x = e^{\frac{1}{3}}$$

M1A1

Total [6 marks]

$$\mathbf{2} \quad \mathbf{a} \quad (1 + 2x)^{-2} = 1 + (-2)(2x) + \frac{(-2)(-3)}{2!}(2x)^2 + \frac{(-2)(-3)(-4)}{3!}(2x)^3 + \dots$$

M1

$$= 1 - 4x + 12x^2 - 32x^3 + \dots$$

A1A1A1

- b** Using **a** with $x = 0.03$ gives

$$(1.06)^{-2} \approx 1 - 4(0.03) + 12(0.03)^2 - 32(0.03)^3$$

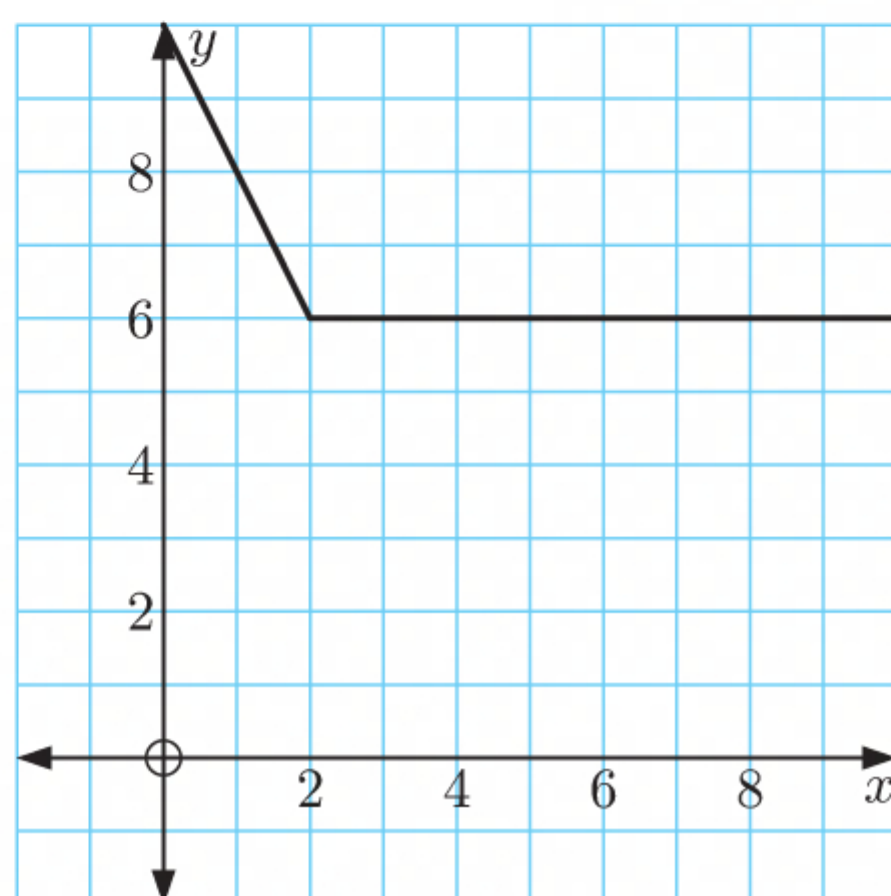
$$\approx 0.8899 \quad \{4 \text{ d.p.}\}$$

A1

Total [6 marks]

$$\mathbf{3} \quad \mathbf{a}$$

A1A1

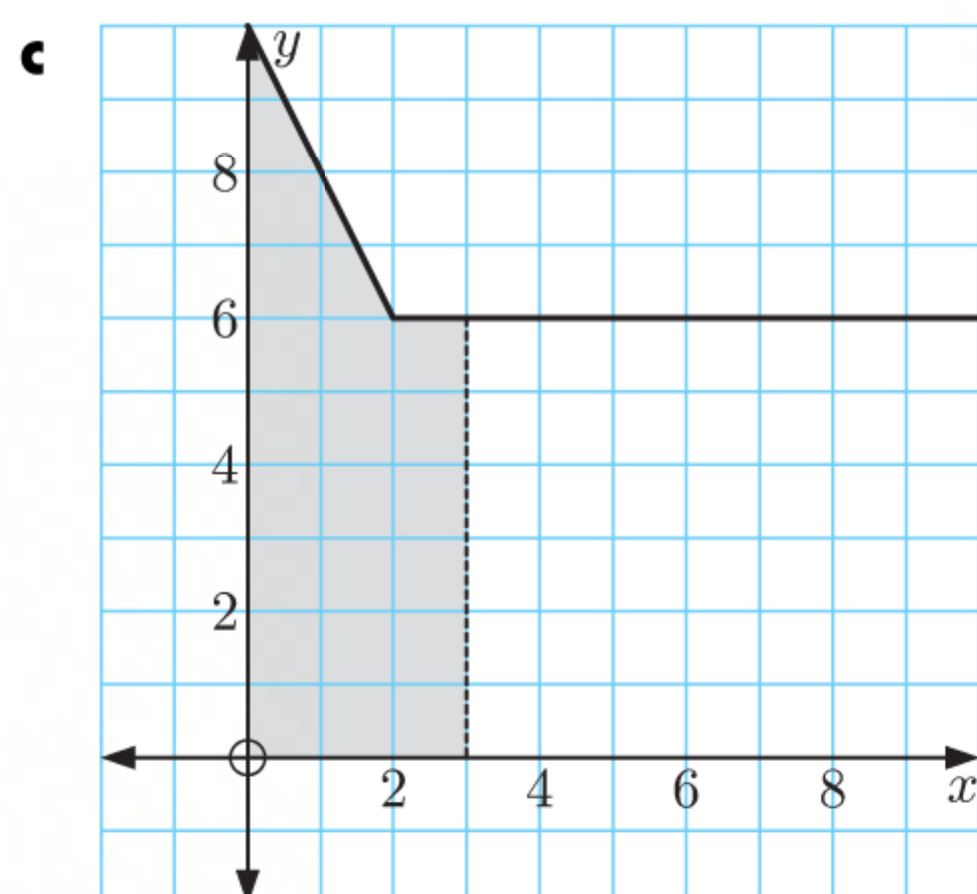


b i $h'(1) = -2$

A1

ii $h'(3) = 0$

A1



A1

d $\int_0^3 h(x) dx = 22$

A1

Total [6 marks]

- 4** Given that $2 + 3i$ is a root, $2 - 3i$ must also be a root.

A1

Similarly, given that $-1 - \sqrt{2}i$ is a root, $-1 + \sqrt{2}i$ must also be a root.

A1

The roots $2 \pm 3i$ have sum $= 4$ and product $= 2^2 + 3^2 = 13$.

\therefore quadratics with roots $2 \pm 3i$ have the form $a_1(x^2 - 4x + 13)$, $a_1 \neq 0$.

A1

The roots $-1 \pm \sqrt{2}i$ have sum $= -2$ and product $= (-1)^2 + (\sqrt{2})^2 = 3$.

\therefore quadratics with roots $-1 \pm \sqrt{2}i$ have the form $a_2(x^2 + 2x + 3)$, $a_2 \neq 0$.

A1

Since the coefficient of x^4 in $p(x)$ is 1, we let $a_1 = a_2 = 1$.

Thus,
$$p(x) = (x^2 - 4x + 13)(x^2 + 2x + 3)$$
$$= x^4 - 2x^3 + 8x^2 + 14x + 39$$

A1

$\therefore a = -2, b = 8, c = 14, d = 39$

A1

Total [6 marks]

5 a i $p = 1 - 0.25 = 0.75$

A1

ii $q = 1 - 0.45 = 0.55$

A1

b
$$P(B) = P(AB \cup A'B)$$
$$= P(AB) + P(A'B)$$
$$= 0.75 \times 0.3 + 0.25 \times 0.35$$
$$= 0.225 + 0.0875$$
$$= 0.3125$$

M1

M1

A1

c Using $P(A | B) = \frac{P(A \cap B)}{P(B)}$, we find

M1

$$P(A' \cap B' | C) = \frac{P(A' \cap B' \cap C)}{P(C)}$$
$$= \frac{0.25 \times 0.65 \times 0.55}{0.25 \times 0.65 \times 0.55 + 0.75 \times 0.7 \times 0.15}$$
$$= \frac{0.089375}{0.089375 + 0.07875}$$
$$= 0.5315985 \dots$$
$$\approx 0.532$$

A1A1

Total [8 marks]

6 a
$$\vec{AB} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

A1

b
$$\vec{AC} = \begin{pmatrix} 3 \\ -4 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

A1

$$\begin{aligned}
 \text{c } \vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} \\
 &= (4 - 6)\mathbf{i} - (12 + 2)\mathbf{j} + (-9 - 1)\mathbf{k} \\
 &= -2\mathbf{i} - 14\mathbf{j} - 10\mathbf{k} \quad \text{M1A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } |\vec{AB} \times \vec{AC}| &= \sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \sqrt{300} = 10\sqrt{3} \quad \text{A1} \\
 \therefore \text{ area} &= \frac{1}{2} |\vec{AB} \times \vec{AC}| \quad \text{M1} \\
 &= 5\sqrt{3} \\
 &\approx 8.66 \quad \text{A1}
 \end{aligned}$$

Total [7 marks]

$$\begin{aligned}
 \text{7 a } 1 + \cot^2 \theta &= 1 + \frac{\cos^2 \theta}{\sin^2 \theta} \quad \text{A1} \\
 &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \\
 &= \frac{1}{\sin^2 \theta} \quad \text{A1} \\
 &= \operatorname{cosec}^2 \theta \quad \text{AG}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } 8 \operatorname{cosec}^2 \theta + 14 \cot \theta - 23 &= 0 \\
 \therefore 8(1 + \cot^2 \theta) + 14 \cot \theta - 23 &= 0 \quad \text{M1} \\
 \therefore 8 \cot^2 \theta + 14 \cot \theta - 15 &= 0 \\
 \therefore (4 \cot \theta - 3)(2 \cot \theta + 5) &= 0 \quad \text{A1} \\
 \therefore \cot \theta &= \frac{3}{4} \text{ or } -\frac{5}{2} \\
 \therefore \tan \theta &= \frac{4}{3} \text{ or } -\frac{2}{5} \quad \text{A1}
 \end{aligned}$$

$$\text{Now } \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927 \text{ and } \tan^{-1}\left(-\frac{2}{5}\right) \approx -0.381$$

$$\therefore \text{ on } -\frac{\pi}{2} < x < \frac{\pi}{2}, \text{ the solutions are } \theta \approx -0.381 \text{ or } 0.927. \quad \text{A1}$$

Total [6 marks]

8 a Let X represent the amount of liquid in the large-size DEIT bottle of drink.

$$X \sim N(2.08, 0.05^2) \quad \text{A1}$$

$$\therefore P(X > 2) \approx 0.945 \quad \text{A1}$$

b Let Y be the number of bottles which contain more than 2 litres of liquid.

$$\text{Then } Y \sim B(12, 0.945) \quad \text{M1A1}$$

$$P(Y \geq 10) = 1 - P(Y \leq 9) \quad \text{M1}$$

$$\approx 1 - 0.0249$$

$$\approx 0.975 \quad \text{A1}$$

c i Let W represent the amount of liquid in the small-size DEIT bottle of drink.

$$W \sim N(324, \sigma^2)$$

$$\text{Now } P(W > 340) = 0.1$$

$$\therefore \frac{340 - 324}{\sigma} \approx 1.28155 \quad \text{M1}$$

$$\therefore \sigma \approx \frac{16}{1.28155} \approx 12.5 \quad \text{A1}$$

$$\text{ii } W \sim N(324, 12.5^2)$$

$$P(W < 320) \approx 0.374 \quad \text{A1}$$

$$\therefore \text{ from a batch of 50 small-size cans, we would expect } \approx 50 \times 0.374 \approx 19 \text{ cans to contain less than 320 millilitres.} \quad \text{A1}$$

Total [10 marks]

Section B

$$9 \quad \mathbf{a} \quad \left(\frac{12}{13}\right)^2 + \sin^2 \widehat{ABC} = 1 \quad \mathbf{M1}$$

$$\therefore \sin^2 \widehat{ABC} = \frac{25}{169}$$

$$\therefore \sin \widehat{ABC} = \pm \frac{5}{13} \quad \mathbf{A1}$$

Recognise that $\sin \widehat{ABC} > 0$, so $\sin \widehat{ABC} = \frac{5}{13}$. $\mathbf{R1}$

$$\mathbf{b} \quad (AC)^2 = 8^2 + 12^2 - 2 \times 8 \times 12 \times \frac{12}{13} \quad \mathbf{M1}$$

$$\therefore AC \approx 5.55 \quad \mathbf{A1}$$

$$\mathbf{c} \quad \mathbf{i} \quad (AD_1)^2 = 3^2 + 12^2 - 2 \times 3 \times 12 \times \frac{12}{13} \quad \mathbf{A1}$$

$$\therefore AD_1 \approx 9.30 \quad \mathbf{A1}$$

$$\mathbf{ii} \quad (AD_2)^2 = 13^2 + 12^2 - 2 \times 13 \times 12 \times \frac{12}{13} \quad \mathbf{A1}$$

$$\therefore AD_2 = 5 \quad \mathbf{A1}$$

or notice that triangle ABD_2 is a right angled triangle and 5-12-13 is a Pythagorean triple.

$$\mathbf{d} \quad \mathbf{i} \quad \text{Using the sine rule, } \frac{\sin D_1 \widehat{AB}}{3} \approx \frac{\frac{5}{13}}{9.30} \quad \mathbf{M1}$$

$$\therefore \sin D_1 \widehat{AB} \approx \frac{3 \times \frac{5}{13}}{9.30}$$

$$\therefore D_1 \widehat{AB} \approx 7.13^\circ \quad \left\{ D_1 \widehat{AB} \text{ is acute} \right\} \quad \mathbf{A1}$$

$$\mathbf{ii} \quad \text{Using the sine rule, } \frac{\sin D_2 \widehat{AB}}{13} = \frac{\frac{5}{13}}{5} \quad \mathbf{M1}$$

$$\therefore \sin D_2 \widehat{AB} = 1$$

$$\therefore D_2 \widehat{AB} = 90^\circ \quad \mathbf{A1}$$

or notice that triangle ABD_2 is a right angled triangle so angle $D_2 \widehat{AB}$ is a right angle.

\mathbf{e} Consider CD_1 and CD_2 the bases of triangles ACD_1 and ACD_2 respectively. As $CD_1 = CD_2 (= 5)$ and the heights of the triangle drawn to these bases is equal for both triangles, the areas must also be equal. $\mathbf{R1}$

$$\mathbf{f} \quad \text{Area} = \frac{1}{2}(D_2 A)(D_2 C) \sin(\widehat{AD_2 C}) \quad \mathbf{M1}$$

$$= \frac{1}{2}(5)(5)\left(\frac{12}{13}\right)$$

$$= \frac{150}{13} \quad \mathbf{A1}$$

$$\mathbf{g} \quad \frac{1}{2}(CD_2)(CA) \sin(\widehat{ACD_2}) = \frac{150}{13} \quad \mathbf{M1}$$

$$\therefore \frac{1}{2}(5)(5.55) \sin(\widehat{ACD_2}) \approx \frac{150}{13} \quad \mathbf{A1}$$

$$\therefore \sin(\widehat{ACD_2}) \approx 0.832 \quad \mathbf{A1}$$

Total [19 marks]

$$10 \quad \mathbf{a} \quad f(x) = \frac{3x+1}{(x-3)(x+2)} \quad \mathbf{M1}$$

Thus, the two vertical asymptotes are $x = 3$ and $x = -2$. $\mathbf{A1A1}$

$$\mathbf{b} \quad \text{Let } f(x) = \frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} \quad \mathbf{M1}$$

$$\therefore 3x+11 = A(x+2) + B(x-3) \quad \mathbf{M1}$$

$$\therefore 3x+11 = (A+B)x + (2A-3B) \Rightarrow \begin{cases} A+B=3 \\ 2A-3B=11 \end{cases} \quad \mathbf{A1A1}$$

Solving the system of equations gives $A = 4$ and $B = -1$.

$$\therefore f(x) = \frac{3x+11}{x^2-x-6} = \frac{4}{x-3} + \frac{-1}{x+2} \quad \mathbf{A1A1}$$

$$\begin{aligned}
 \text{c} \quad \int_4^5 f(x) dx &= \int_4^5 \left(\frac{4}{x-3} + \frac{-1}{x+2} \right) dx && \text{M1} \\
 &= [4 \ln |x-3| - \ln |x+2|]_4^5 && \text{M1} \\
 &= (4 \ln 2 - \ln 7) - (4 \ln 1 - \ln 6) \\
 &= 4 \ln 2 - \ln 7 + \ln 6 && \text{A1} \\
 &= \ln \frac{2^4 \times 6}{7} = \ln \frac{96}{7} && \text{M1AG}
 \end{aligned}$$

$$\begin{aligned}
 \text{d i} \quad \int_4^a f(x) dx &= [4 \ln |x-3| - \ln |x+2|]_4^a && \text{M1} \\
 &= (4 \ln(a-3) - \ln(a+2)) - (4 \ln(1) - \ln(6)) && \text{M1} \\
 &= \ln \frac{6(a-3)^4}{(a+2)} && \text{A1}
 \end{aligned}$$

$$\text{So, we have } \ln \frac{6(a-3)^4}{(a+2)} = 2(\ln \frac{96}{7}) \quad \text{M1}$$

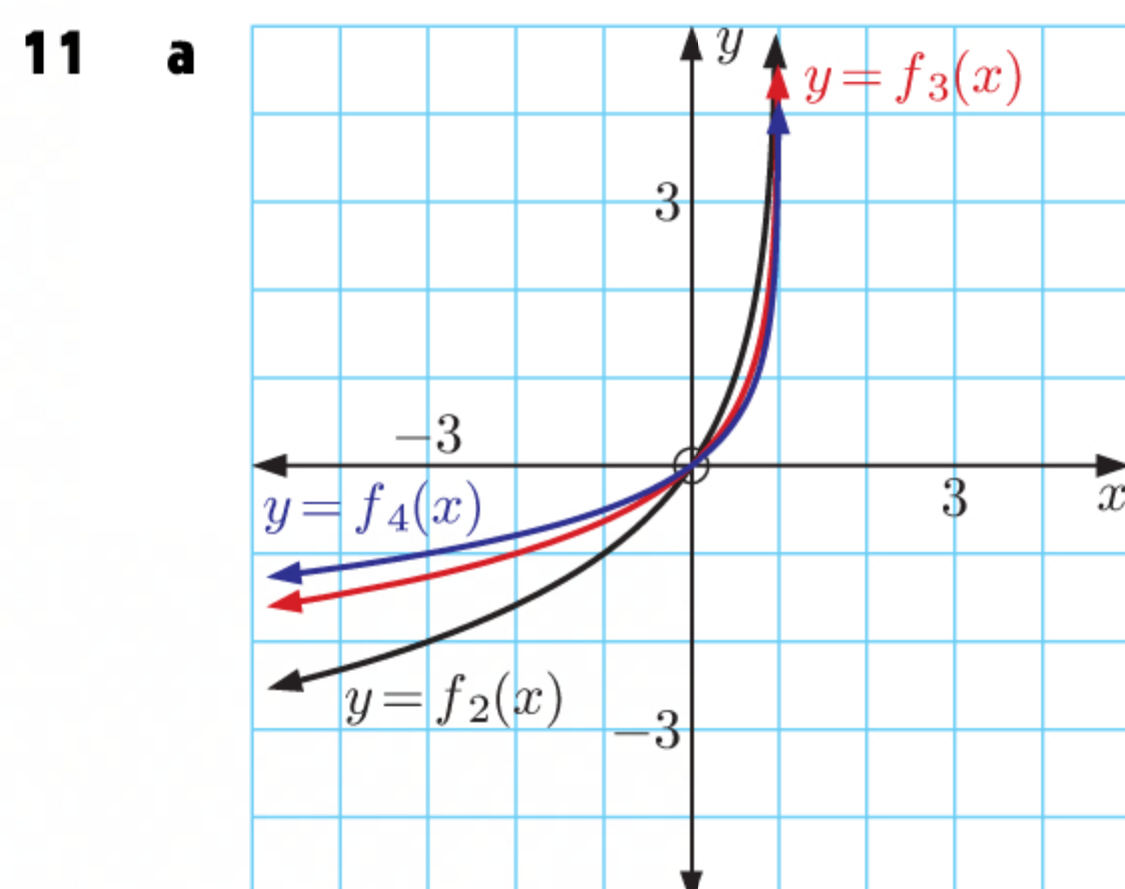
$$\therefore \frac{6(a-3)^4}{(a+2)} = \left(\frac{96}{7} \right)^2 \quad \text{M1}$$

$$\therefore \frac{(a-3)^4}{a+2} = \frac{1536}{49} \quad \text{AG}$$

$$\text{ii Using technology to solve for } a, \text{ we find } a \approx 7.1109234 \dots \approx 7.11 \quad \text{A1}$$

Total [19 marks]

A1A1A1



$$\begin{aligned}
 \text{b} \quad f_2(x) &= \log_2 \left(\frac{1}{1-x} \right) \\
 &= \log_2 1 - \log_2(1-x) && \text{M1}
 \end{aligned}$$

$$\therefore f_2'(x) = \frac{1}{\ln 2(1-x)} \quad \text{A1}$$

$$\therefore f_2''(x) = \frac{1}{\ln 2(1-x)^2} \quad \text{A1}$$

$$\begin{aligned}
 \text{c} \quad f_2''(x) &= \frac{1}{\ln 2}(1-x)^{-2} \\
 \therefore f_2'''(x) &= \frac{2}{(\ln 2)(1-x)^3} && \text{M1AG}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f_2(x) &= 0 + \frac{\frac{1}{\ln 2}}{1!}x + \frac{\frac{1}{\ln 2}}{2!}x^2 + \frac{\frac{2}{\ln 2}}{3!}x^3 + \dots && \text{M1} \\
 &= \frac{1}{\ln 2}x + \frac{1}{2\ln 2}x^2 + \frac{1}{3\ln 2}x^3 + \dots && \text{A1A1A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad f_3(x) &= \log_3 \left(\frac{1}{1-x} \right) \\
 &= \log_3 1 - \log_3(1-x) \\
 \therefore f_3'(x) &= \frac{1}{\ln 3(1-x)} && \text{A1} \\
 \therefore f_3''(x) &= \frac{1}{\ln 3(1-x)^2} = \frac{1}{\ln 3}(1-x)^{-2} && \text{A1} \\
 \therefore f_3'''(x) &= \frac{2}{\ln 3(1-x)^3} && \text{A1}
 \end{aligned}$$

Therefore, the Maclaurin series for $f_3(x)$ is

$$f_3(x) = \frac{1}{\ln 3}x + \frac{1}{2\ln 3}x^2 + \frac{1}{3\ln 3}x^3 + \dots \quad \text{M1AG}$$

$$\mathbf{f} \quad \mathbf{i} \quad f_n(x) = \frac{1}{\ln n}x + \frac{1}{2\ln n}x^2 + \frac{1}{3\ln n}x^3 + \dots \quad \mathbf{A1}$$

\mathbf{ii} Using the Maclaurin series for $f_n(x) = \log_n\left(\frac{1}{1-x}\right)$, we have

$$f_e(x) = \frac{1}{\ln e}x + \frac{1}{2\ln e}x^2 + \frac{1}{3\ln e}x^3 + \dots \quad \mathbf{M1}$$

Given that $\ln e = 1$, this simplifies to

$$f_e(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots \quad \mathbf{AG}$$

Total [17 marks]

PAPER 3

$\mathbf{1} \quad \mathbf{a} \quad \mathbf{i}$ F_1 is created by removing the middle third of each side of F_0 , and replacing it with two line segments that form an equilateral triangle with sides whose length is one-third of the previous side's length. $\mathbf{M1}$

$$\therefore N_1 = 4 \times 3 = 12 \quad \mathbf{A1}$$

$$\text{and } l_1 = \frac{1}{3} \times 1 = \frac{1}{3} \quad \mathbf{A1}$$

$$\mathbf{ii} \quad N_2 = 4 \times 12 = 48 \quad \mathbf{A1}$$

$$\text{and } l_2 = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \quad \mathbf{A1}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{i} \quad P_1 &= N_1 \times l_1 \\ &= 12 \times \frac{1}{3} \\ &= 4 \end{aligned} \quad \mathbf{A1}$$

$$\begin{aligned} \mathbf{ii} \quad P_2 &= N_2 \times l_2 \\ &= 48 \times \frac{1}{9} \\ &= \frac{16}{3} \end{aligned} \quad \mathbf{A1}$$

$$\mathbf{c} \quad \mathbf{i} \quad N_n = 4^n \times 3 \quad \mathbf{A1}$$

$$\mathbf{ii} \quad l_n = \left(\frac{1}{3}\right)^n \quad \mathbf{A1}$$

$$\begin{aligned} \mathbf{iii} \quad P_n &= N_n \times l_n \\ &= 4^n \times 3 \times \left(\frac{1}{3}\right)^n \\ &= 3 \times \left(\frac{4}{3}\right)^n \end{aligned} \quad \mathbf{M1} \quad \mathbf{A1}$$

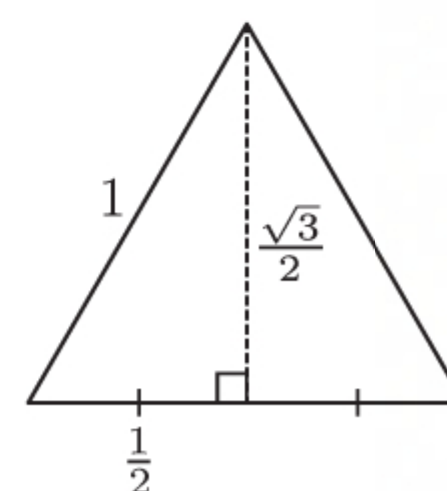
$$\begin{aligned} \mathbf{d} \quad \text{The common ratio } r &= \frac{P_n}{P_{n-1}} \\ &= \frac{3 \times \left(\frac{4}{3}\right)^n}{3 \times \left(\frac{4}{3}\right)^{n-1}} \\ &= \frac{4}{3} \end{aligned} \quad \mathbf{M1} \quad \mathbf{A1}$$

\mathbf{e} Since $r = \frac{4}{3} > 1$, the sequence $\{P_n\}$ is divergent.

The perimeter of Koch's snowflake $= \lim_{n \rightarrow \infty} \left(3 \times \left(\frac{4}{3}\right)^n\right)$ which is infinite. $\mathbf{A1}$

\mathbf{f} Since F_0 is an equilateral triangle with sides of 1 unit in length,

$$T_0 = \frac{\sqrt{3}}{4} \quad \text{and} \quad A_0 = \frac{\sqrt{3}}{4}$$



$\mathbf{A1A1}$

$\mathbf{g} \quad \mathbf{i}$ For $n \geq 1$, the lengths of the sides of the new triangle are $\frac{1}{3}$ of the length of the sides of F_{n-1} , we have

$$T_n = \left(\frac{1}{3}\right)^2 \times T_{n-1} \quad \mathbf{M1}$$

$$= \frac{1}{9}T_{n-1} \quad \mathbf{A1}$$

ii Given $T_n = \frac{1}{9}T_{n-1}$,

$$T_1 = \frac{1}{9}T_0 = \frac{1}{9} \times \frac{\sqrt{3}}{4}$$

$$T_2 = \frac{1}{9}T_1 = \left(\frac{1}{9}\right)^2 \times \frac{\sqrt{3}}{4}$$

$$\therefore T_n = \left(\frac{1}{9}\right)^n \times \frac{\sqrt{3}}{4}$$

M1A1

h Given that a new triangle is formed on each side of F_{n-1} to form F_n ,

$$A_n = A_{n-1} + N_{n-1} \times T_n$$

M1

$$= A_{n-1} + (4^{n-1} \times 3) \times \left(\left(\frac{1}{9}\right)^n \times \frac{\sqrt{3}}{4}\right)$$

A1

$$= A_{n-1} + \frac{3}{4} \times \frac{\sqrt{3}}{4} \times \left(\frac{4}{9}\right)^n$$

AG

i Writing out the first few terms A_0 , A_1 , and A_2 , we have

$$A_0 = \frac{\sqrt{3}}{4}$$

$$A_1 = \frac{\sqrt{3}}{4} + \frac{3}{4} \times \frac{\sqrt{3}}{4} \times \left(\frac{4}{9}\right)^1 = \frac{\sqrt{3}}{4} \left(1 + \frac{3}{4} \times \frac{4}{9}\right)$$

$$A_2 = \frac{\sqrt{3}}{4} \left(1 + \frac{3}{4} \times \frac{4}{9}\right) + \frac{3}{4} \times \frac{\sqrt{3}}{4} \times \left(\frac{4}{9}\right)^2 = \frac{\sqrt{3}}{4} \left(1 + \frac{3}{4} \times \frac{4}{9} + \frac{3}{4} \times \left(\frac{4}{9}\right)^2\right) = \frac{\sqrt{3}}{4} \left[1 + \frac{3}{4} \left(\frac{4}{9} + \left(\frac{4}{9}\right)^2\right)\right]$$

M1

$$\text{Thus, more generally, } A_n = \frac{\sqrt{3}}{4} \left[1 + \frac{3}{4} \left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \dots + \left(\frac{4}{9}\right)^n\right)\right]$$

A1

$$= \frac{\sqrt{3}}{4} \left[1 + \frac{1}{3} \left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \dots + \left(\frac{4}{9}\right)^{n-1}\right)\right]$$

$$= \frac{\sqrt{3}}{4} \left[1 + \frac{1}{3} \left(\frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}}\right)\right]$$

A1

$$= \frac{\sqrt{3}}{4} \left[1 + \frac{1}{3} \left(\frac{9}{5} \left(1 - \left(\frac{4}{9}\right)^n\right)\right)\right]$$

$$= \frac{\sqrt{3}}{4} \left[1 + \frac{3}{5} \left(1 - \left(\frac{4}{9}\right)^n\right)\right]$$

A1

$$= \frac{\sqrt{3}}{4} \left(\frac{8}{5} - \frac{3}{5} \left(\frac{4}{9}\right)^n\right)$$

AG

j The area of Koch's snowflake $= \lim_{n \rightarrow \infty} A_n$

M1

$$= \lim_{n \rightarrow \infty} \left(\frac{\sqrt{3}}{4} \left[\frac{8}{5} - \frac{3}{5} \left(\frac{4}{9}\right)^n\right]\right)$$

$$= \frac{\sqrt{3}}{4} \times \frac{8}{5}$$

$$= \frac{2\sqrt{3}}{5}$$

A1

Total [28 marks]

2 a $\int e^{\frac{x}{\alpha}} \sin \beta x \, dx = \alpha e^{\frac{x}{\alpha}} \sin \beta x - \int \alpha e^{\frac{x}{\alpha}} \beta \cos \beta x \, dx \quad \begin{cases} u' = e^{\frac{x}{\alpha}} & v = \sin \beta x \\ u = \alpha e^{\frac{x}{\alpha}} & v' = \beta \cos \beta x \end{cases}$

M1

$$= \alpha e^{\frac{x}{\alpha}} \sin \beta x - \alpha \beta \int e^{\frac{x}{\alpha}} \cos \beta x \, dx$$

A1

$$= \alpha e^{\frac{x}{\alpha}} \sin \beta x - \alpha \beta \left(\alpha e^{\frac{x}{\alpha}} \cos \beta x - \int \alpha e^{\frac{x}{\alpha}} (-\beta \sin \beta x) \, dx \right)$$

M1

$$\begin{cases} u' = e^{\frac{x}{\alpha}} & v = \cos \beta x \\ u = \alpha e^{\frac{x}{\alpha}} & v' = -\beta \sin \beta x \end{cases}$$

$$= \alpha e^{\frac{x}{\alpha}} \sin \beta x - \alpha^2 \beta e^{\frac{x}{\alpha}} \cos \beta x - \alpha^2 \beta^2 \int e^{\frac{x}{\alpha}} \sin \beta x \, dx$$

A1

$$\therefore (1 + \alpha^2 \beta^2) \int e^{\frac{x}{\alpha}} \sin \beta x \, dx = \alpha e^{\frac{x}{\alpha}} \sin \beta x - \alpha^2 \beta e^{\frac{x}{\alpha}} \cos \beta x + c$$

M1

$$\therefore \int e^{\frac{x}{\alpha}} \sin \beta x \, dx = \frac{e^{\frac{x}{\alpha}}}{1 + \alpha^2 \beta^2} (\alpha \sin \beta x - \alpha^2 \beta \cos \beta x) + c$$

AG

b i $a(t) = v'(t)$

$$= \frac{4}{3} e^{-\frac{t}{24}} \sin 2t - 64 e^{-\frac{t}{24}} \cos 2t$$

M1A1

$$\therefore a(0) = -64 \text{ cm s}^{-2}$$

A1

The pendulum is accelerating at 64 cm s^{-2} towards its mean position.

$$\begin{aligned}
\text{ii } s(t) &= \int v(t) dt \\
&= -32 \int e^{-\frac{t}{24}} \sin 2t dt && \text{M1} \\
&= -\frac{32}{1 + (-24)^2 \times 2^2} e^{-\frac{t}{24}} (-24 \sin 2t - (-24)^2 \times 2 \cos 2t) + c && \text{M1A1} \\
&= e^{-\frac{t}{24}} \left(\frac{768}{2305} \sin 2t + \frac{36864}{2305} \cos 2t \right) + c \\
&\approx e^{-\frac{t}{24}} (0.33319 \sin 2t + 15.993 \cos 2t) + c \\
&\approx e^{-\frac{t}{24}} \left(\frac{1}{3} \sin 2t + 16 \cos 2t \right) + c && \text{A1}
\end{aligned}$$

$$\text{As } t \rightarrow \infty, s(t) \rightarrow c \quad \text{R1}$$

$$\therefore c = 0$$

$$\therefore s(t) \approx e^{-\frac{t}{24}} \left(\frac{1}{3} \sin 2t + 16 \cos 2t \right) \quad \text{AG}$$

$$\begin{aligned}
\text{iii } \text{The first two times the pendulum changes direction are when } 2t \approx \pi \quad \text{and} \quad 2t \approx 2\pi \\
\therefore t \approx \frac{\pi}{2} \quad \text{and} \quad t \approx \pi && \text{A1A1}
\end{aligned}$$

iv (1) Using the approximation in **ii**,

$$s(t) \approx e^{-\frac{t}{24}} \left(\frac{1}{3} \sin 2t + 16 \cos 2t \right)$$

$$\therefore s(0) \approx 16 \text{ cm} \quad \text{A1}$$

$$\text{and } s\left(\frac{\pi}{2}\right) \approx e^{-\frac{\pi}{48}} \left(\frac{1}{3} \sin \pi + 16 \cos \pi \right) \approx -14.99 \text{ cm} \quad \text{A1}$$

$$\therefore \text{the first swing has length } \approx 16 + 14.99 \approx 30.99 \text{ cm.} \quad \text{A1}$$

$$\text{(2) } s(\pi) \approx e^{-\frac{\pi}{24}} \left(\frac{1}{3} \sin 2\pi + 16 \cos 2\pi \right) \approx 14.04 \text{ cm} \quad \text{A1}$$

$$\text{The swing back towards the initial position has length } \approx 14.99 + 14.04 \approx 29.03 \text{ cm.} \quad \text{A1}$$

$$\text{v (1) The ratio of lengths for the first two swings } \approx \frac{29.03}{30.99} \approx 0.9368 \quad \text{M1}$$

$$\therefore \text{we estimate we will lose } \approx 6.32\% \text{ distance with each swing.} \quad \text{A1}$$

(2) Assuming this loss continues, which is not unreasonable considering the exponential decay term in $s(t)$ and $v(t)$, we model the distance travelled after n swings with the geometric series

$$S_n = \sum_{i=1}^n 30.99(0.9368)^{n-1} \quad \text{M1}$$

As $n \rightarrow \infty$, this series converges to the sum

$$S = \frac{30.99}{1 - 0.9368} \approx 490 \text{ cm} \quad \text{M1A1}$$

We expect the pendulum to travel a total distance of about 4.90 m.

$$\text{vi Total distance travelled} = \int_0^\infty |v(t)| dt \quad \text{M1}$$

$$= \int_0^\infty 32e^{-\frac{t}{24}} |\sin 2t| dt \quad \text{A1}$$

Total [27 marks]

TRIAL EXAMINATION 4

PAPER 1

Section A

1 a $f(x) = e^{\sin kx} + c$

$$\begin{aligned}\therefore f(0) &= e^0 + c \\ &= 1 + c\end{aligned}$$

A1

The tangent has y -intercept 3, so $1 + c = 3$

$$\therefore c = 2$$

M1A1

b $f'(x) = e^{\sin kx}(k \cos kx)$

$$\therefore f'(0) = e^{\sin 0}(k \cos 0) = k$$

M1A1

The tangent has gradient -1 , so $k = -1$.

A1

Total [6 marks]

2 $P(Y) = P(X \cap Y) + P(X' \cap Y) = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$

A1

Since X and Y are independent, $P(X \cap Y) = P(X)P(Y)$

$$\therefore \frac{1}{5} = P(X) \times \frac{7}{10}$$

$$\therefore P(X) = \frac{1}{5} \times \frac{10}{7} = \frac{2}{7}$$

M1A1

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

M1

$$= \frac{2}{7} + \frac{7}{10} - \frac{1}{5}$$

$$= \frac{11}{14}$$

A1

Total [5 marks]

3 Since $a + b + c$ is divisible by 3, $a + b + c = 3n$ for some $n \in \mathbb{Z}$

M1

$$\therefore 100a + 10b + c = 3n + 99a + 9b$$

M1

$$\therefore "abc" = 3(n + 33a + 3b)$$

A1

Since $n + 33a + 3b \in \mathbb{Z}$, " abc " is divisible by 3.

R1

Total [4 marks]

4 a g is $y = \frac{2x+1}{3}$

$$\therefore g^{-1} \text{ is } x = \frac{2y+1}{3}$$

M1

$$\therefore 3x = 2y + 1$$

$$\therefore 2y = 3x - 1$$

A1

$$\therefore y = \frac{3x-1}{2}$$

$$\therefore g^{-1}(x) = \frac{3x-1}{2}$$

AG

b $(f \circ g^{-1})(x) = 4$

$$\therefore f\left(\frac{3x-1}{2}\right) = 4$$

M1

$$\therefore 3 - \frac{3x-1}{2} = 4$$

A1

$$\therefore 3 - \frac{3}{2}x + \frac{1}{2} = 4$$

$$\therefore -\frac{3}{2}x = \frac{1}{2}$$

$$\therefore x = -\frac{1}{3}$$

A1

Total [5 marks]

$$\begin{aligned}
 5 \quad & 9^x + 18 = 3^{x+2} \\
 & \therefore 9^x - 3^{x+2} + 18 = 0 \\
 & \therefore (3^x)^2 - 9 \times 3^x + 18 = 0 \quad \text{M1A1} \\
 & \therefore (3^x - 3)(3^x - 6) = 0 \quad \text{A1} \\
 & \therefore 3^x = 3 \quad \text{or} \quad 3^x = 6 \\
 & \therefore x = 1 \quad \text{or} \quad x = \log_3 6 \quad \text{A1A1}
 \end{aligned}$$

Total [5 marks]

$$\begin{aligned}
 6 \quad \text{a} \quad & \text{Let } f(x) = \frac{1}{1-x} = (1-x)^{-1} \\
 & \therefore f'(x) = (1-x)^{-2} \\
 & \therefore f''(x) = 2(1-x)^{-3} \\
 & \therefore f'''(x) = 6(1-x)^{-4} \\
 & \quad \vdots \\
 & \therefore f^{(k)}(x) = k!(1-x)^{-k-1} \quad \text{(M1)} \\
 & \therefore f^{(k)}(0) = k! \quad \text{A1} \\
 & \therefore \text{the Maclaurin series is } \sum_{k=0}^{\infty} \frac{k!}{k!} x^k = \sum_{k=0}^{\infty} x^k. \quad \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{1}{\sin^2 x} = \frac{1}{1 - \cos^2 x} \quad \text{M1} \\
 & = \sum_{k=0}^{\infty} (\cos^2 x)^k \quad \text{whenever } \cos^2 x \neq 1 \quad \text{A1} \\
 & = 1 + \cos^2 x + \cos^4 x + \cos^6 x + \dots \quad \text{provided } x \neq k\pi, k \in \mathbb{Z}. \quad \text{AG}
 \end{aligned}$$

Total [5 marks]

$$\begin{aligned}
 7 \quad \text{a} \quad & \text{For } f(x) \text{ to be a valid probability density function, } \int_0^1 f(x) dx = 1 \quad \text{M1} \\
 & \therefore k \int_0^1 \frac{1}{x^2 + 1} dx = 1 \\
 & \therefore k [\arctan x]_0^1 = 1 \quad \text{A1} \\
 & \therefore k(\arctan 1 - \arctan 0) = 1 \\
 & \therefore \frac{\pi}{4} k = 1 \\
 & \therefore k = \frac{4}{\pi} \quad \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & E(X) = \int_0^1 x f(x) dx \quad \text{M1} \\
 & = \int_0^1 \frac{4}{\pi} \frac{x}{x^2 + 1} dx \quad \text{A1} \\
 & = \frac{2}{\pi} \int_0^1 \frac{2x}{x^2 + 1} dx \\
 & = \frac{2}{\pi} [\ln |x^2 + 1|]_0^1 \quad \text{A1} \\
 & = \frac{2}{\pi} (\ln 2 - \ln 1) \\
 & = \frac{2}{\pi} \ln 2 \quad \text{A1}
 \end{aligned}$$

Total [7 marks]

$$\begin{aligned}
 8 \quad \text{a} \quad & y = f(x) \text{ has vertical asymptote } x = 0 \quad \text{M1} \\
 & \text{and oblique asymptote } y = x \\
 & \therefore f(x) = x + \frac{c}{x} \quad \text{for some } c \quad \text{A1} \\
 & \therefore f'(x) = 1 - \frac{c}{x^2} \\
 & \text{But } f'(x) = 0 \text{ when } x = 2, \text{ so } 1 - \frac{c}{4} = 0 \\
 & \therefore c = 4 \quad \text{A1} \\
 & \therefore f(x) = x + \frac{4}{x} = \frac{x^2 + 4}{x} \quad \text{AG}
 \end{aligned}$$

b $f'(x) = 1 - \frac{4}{x^2}$

$\therefore f'(x) = 0$ when $x = \pm 2$

Now $f(2) = 2 + \frac{4}{2} = 4$

and $f(-2) = -2 - \frac{4}{2} = -4$

\therefore the local maximum is $(-2, -4)$

and the local minimum is $(2, 4)$.

A1

A1

c $f(x) = 0$ when $x^2 + 4 = 0$

$\therefore x = \pm 2i$

\therefore the complex zeros of $f(x)$ are $\pm 2i$.

A1A1

d $[f(x)]^2 = \left(x + \frac{4}{x}\right)^2$

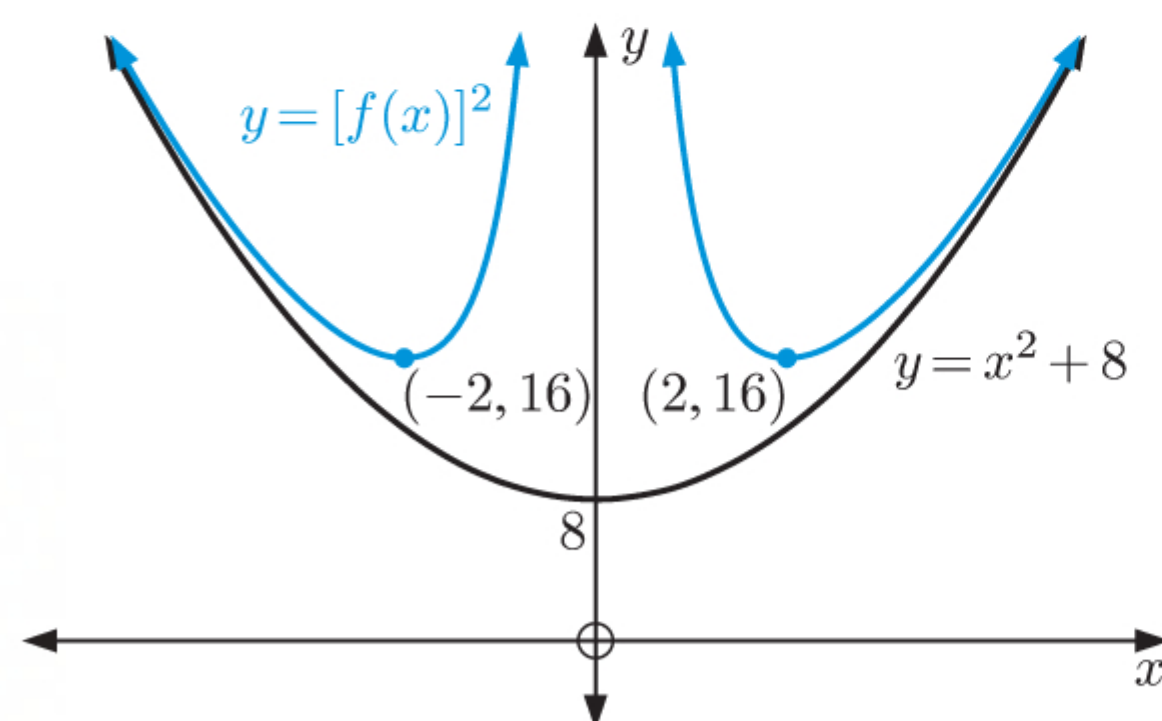
$= x^2 + 8 + \frac{16}{x^2}$

$> x^2 + 8$ for all x . $\left\{\frac{16}{x^2} > 0 \text{ for all } x\right\}$

A1

R1AG

e



A1A1A1

Total [12 marks]

9 a In augmented matrix form, the system is

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 2 & 2 & -3 & 3 \\ 4 & 0 & 2 & k \end{array}\right)$$

A1

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & k-4 \end{array}\right) \quad \begin{array}{l} R_2 - R_1 \rightarrow R_1 \\ R_3 - 2R_1 \rightarrow R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & k-2 \end{array}\right) \quad R_3 + 2R_2 \rightarrow R_3$$

A1A1

b There are no solutions if $k \neq 2$, as the system would be inconsistent.

A1

In this case, the intersection of any two planes is parallel to the third plane.

A1

c There are infinitely many solutions if $k = 2$.

A1

Letting $z = t$ in row 2 gives $y - 2t = 1$

$\therefore y = 1 + 2t$

Using row 1, $2x + (1 + 2t) - t = 2$

$\therefore 2x = 1 - t$

$\therefore x = \frac{1-t}{2}$

\therefore the solutions have the form $x = \frac{1-t}{2}$, $y = 1 + 2t$, $z = t$, $t \in \mathbb{R}$.

A1

In this case, the three planes meet in a line.

A1

Total [8 marks]

Section B

10 a $f(x) = x^3 - 6x^2 + 9x - 2$

$$\therefore f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3)$$

A1

M1A1



A1

b $f(1) = (1)^3 - 6(1)^2 + 9(1) - 2 = 2$

$$f(3) = (3)^3 - 6(3)^2 + 9(3) - 2 = -2$$

Using the sign diagram from **a**, there is a local maximum at $(1, 2)$ and a local minimum at $(3, -2)$.

A1A1

c $f''(x) = 6x - 12$

$$= 6(x - 2)$$

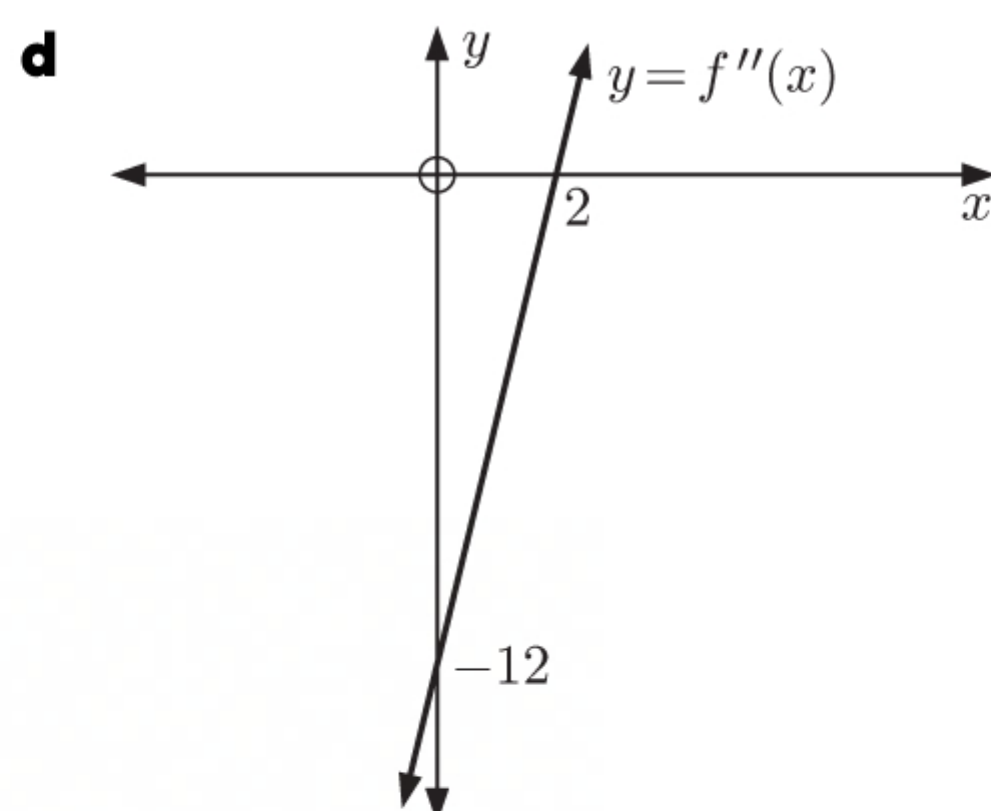
$$f(2) = 8 - 24 + 18 - 2 = 0$$

M1

A1

\therefore the inflection point is at $(2, 0)$.

A1



A1

$f''(x) \leq 0$ for $x \leq 2$, so $y = f(x)$ is concave down for $x \leq 2$.

$f''(x) \geq 0$ for $x \geq 2$, so $y = f(x)$ is concave up for $x \geq 2$.

A1

e i $f'(2) = 3(2)^2 - 12(2) + 9 = -3$

M1

\therefore the gradient of the normal at the inflection point is $\frac{1}{3}$, and it passes through $(2, 0)$

\therefore its equation is $y = \frac{1}{3}(x - 2)$.

M1A1

ii The normal meets the curve again when

$$x^3 - 6x^2 + 9x - 2 = \frac{1}{3}(x - 2)$$

M1

We know $x - 2$ is a factor of the LHS, so we write

$$x^3 - 6x^2 + 9x - 2 = (x - 2)(x^2 + ax + 1)$$

M1

Equating coefficients of x^2 , $-6 = a - 2$

Equating coefficients of x , $9 = 1 - 2a$

M1

Both equations give $a = -4$

A1

\therefore the normal meets the curve again when $(x - 2)(x^2 - 4x + 1) = \frac{1}{3}(x - 2)$

$$\therefore x^2 - 4x + 1 = \frac{1}{3}$$

$$\therefore x^2 - 4x + \frac{2}{3} = 0$$

A1

$$\therefore 3x^2 - 12x + 2 = 0$$

$$\therefore x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(2)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{144 - 24}}{6}$$

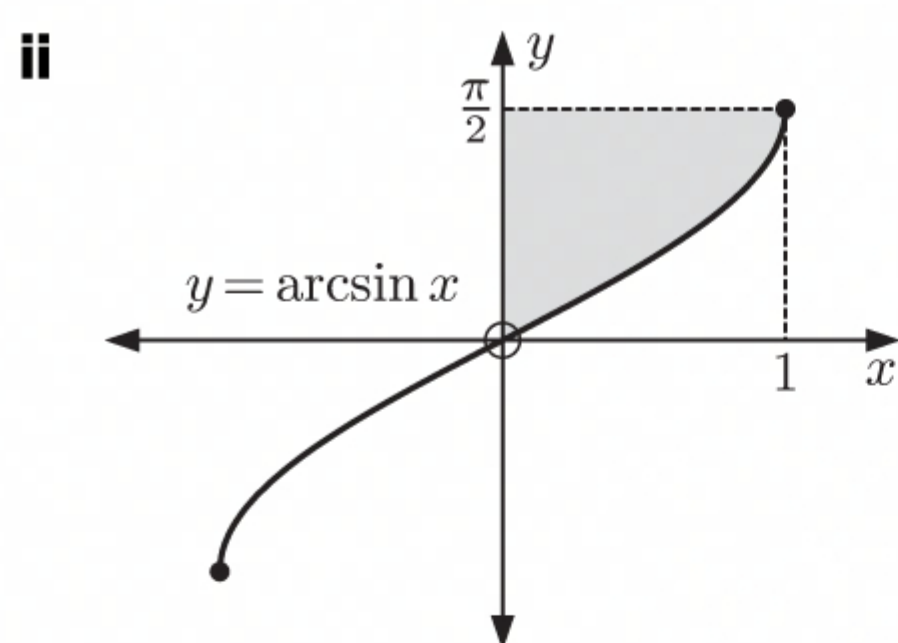
$$= \frac{12 \pm 2\sqrt{30}}{6}$$

$$= 2 \pm \frac{1}{3}\sqrt{30}$$

A1A1

Total [21 marks]

$$\begin{aligned}
 \mathbf{11} \quad \mathbf{a} \quad \mathbf{i} \quad & \int_0^1 \arcsin x \, dx \\
 &= [x \arcsin x]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx \quad \begin{cases} u = \arcsin x & v' = 1 \\ u' = \frac{1}{\sqrt{1-x^2}} & v = x \end{cases} & \mathbf{A1} \\
 &= \arcsin 1 - 0 - \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx \\
 &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{\sin \theta \cos \theta}{\sqrt{1-\sin^2 \theta}} \, d\theta \quad \begin{cases} x = \sin \theta, \frac{dx}{d\theta} = \cos \theta \\ \text{When } x = 0, \theta = 0 \\ \text{When } x = 1, \theta = \frac{\pi}{2} \end{cases} & \mathbf{(M1)} \\
 &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin \theta \, d\theta & \mathbf{(M1)} \\
 &= \frac{\pi}{2} + [\cos \theta]_0^{\frac{\pi}{2}} & \mathbf{(M1)} \\
 &= \frac{\pi}{2} + \cos \frac{\pi}{2} - \cos 0 \\
 &= \frac{\pi}{2} - 1 & \mathbf{A1}
 \end{aligned}$$



$$\begin{aligned}
 \text{Shaded area} &= \int_0^{\frac{\pi}{2}} \sin y \, dy & \mathbf{M1} \\
 &= [-\cos y]_0^{\frac{\pi}{2}} \\
 &= -\cos \frac{\pi}{2} + \cos 0 \\
 &= 1 \text{ unit}^2 & \mathbf{A1} \\
 \therefore \int_0^1 \arcsin x \, dx &= \text{area of rectangle} - \text{shaded area} & \mathbf{R1} \\
 &= \frac{\pi}{2} - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} x^2 \, dy \\
 &= \pi \int_0^{\frac{\pi}{2}} \sin^2 y \, dy & \mathbf{M1} \\
 &= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos 2y \right) \, dy & \mathbf{M1} \\
 &= \pi \left[\frac{1}{2} y - \frac{1}{4} \sin 2y \right]_0^{\frac{\pi}{2}} & \mathbf{A1} \\
 &= \pi \left(\frac{\pi}{4} - 0 \right) \\
 &= \frac{\pi^2}{4} \text{ units}^3 & \mathbf{A1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad p &= \lim_{x \rightarrow 0} \frac{\arcsin x}{x} & \mathbf{M1} \\
 &= \frac{\lim_{x \rightarrow 0} \frac{1}{\sqrt{1-x^2}}}{\lim_{x \rightarrow 0} 1} \quad \{\text{l'Hôpital}\} & \mathbf{A1} \\
 &= \frac{1}{1} \\
 &= 1 & \mathbf{A1}
 \end{aligned}$$

Total [15 marks]

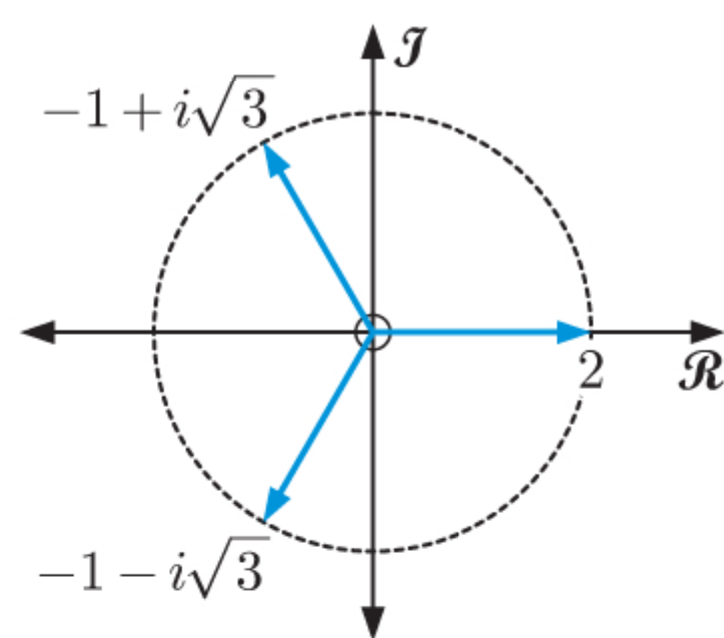
$$\begin{aligned}
 \mathbf{12} \quad \mathbf{a} \quad \text{Let } z^3 &= 8 = 8 \operatorname{cis}(k2\pi), \quad k \in \mathbb{Z} \\
 \therefore z &= \sqrt[3]{8} \operatorname{cis} \frac{k2\pi}{3} & \mathbf{M1} & \quad \{\text{De Moivre}\} \\
 &= 2 \operatorname{cis} \frac{k2\pi}{3} \\
 &= 2 \operatorname{cis}\left(-\frac{2\pi}{3}\right), 2 \operatorname{cis} 0, 2 \operatorname{cis} \frac{2\pi}{3} & \mathbf{M1} & \quad \{\text{letting } k = -1, 0, 1\} \\
 &= 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right), 2, 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
 &= -1 - i\sqrt{3}, 2, -1 + i\sqrt{3} & \mathbf{A1A1A1}
 \end{aligned}$$

b The complex conjugates $-1 \pm i\sqrt{3}$ have sum -2 and product $1 - 3i^2 = 1 + 3 = 4$.

M1

$$\therefore z^3 - 8 = (z - 2)(z^2 + 2z + 4)$$

A1A1

c

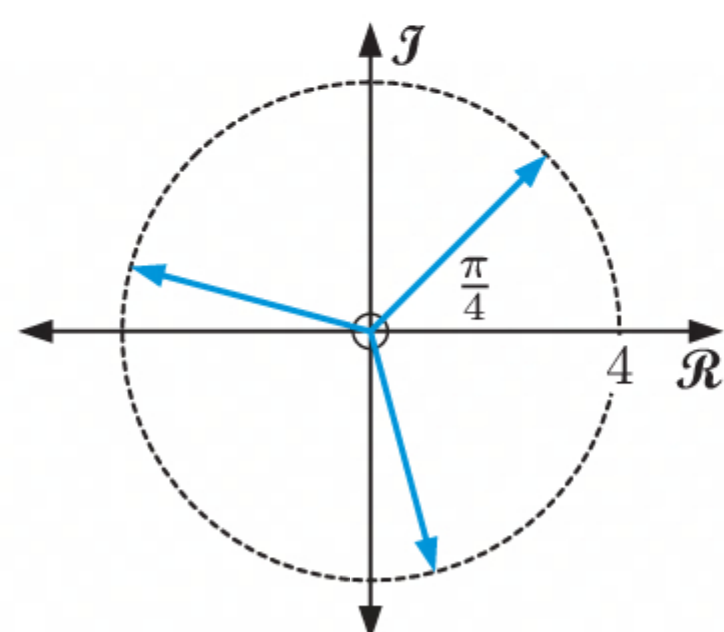
A1A1A1

d i Multiplication by 2 is an enlargement with scale factor 2.

A1

Multiplication by $e^{i\frac{\pi}{4}}$ is an anticlockwise rotation about O through $\frac{\pi}{4}$.

A1



A1

ii The cube root 2 is transformed to $4 \operatorname{cis} \frac{\pi}{4}$, whose cube is $4^3 \operatorname{cis} \frac{3\pi}{4}$. {De Moivre}

M1A1

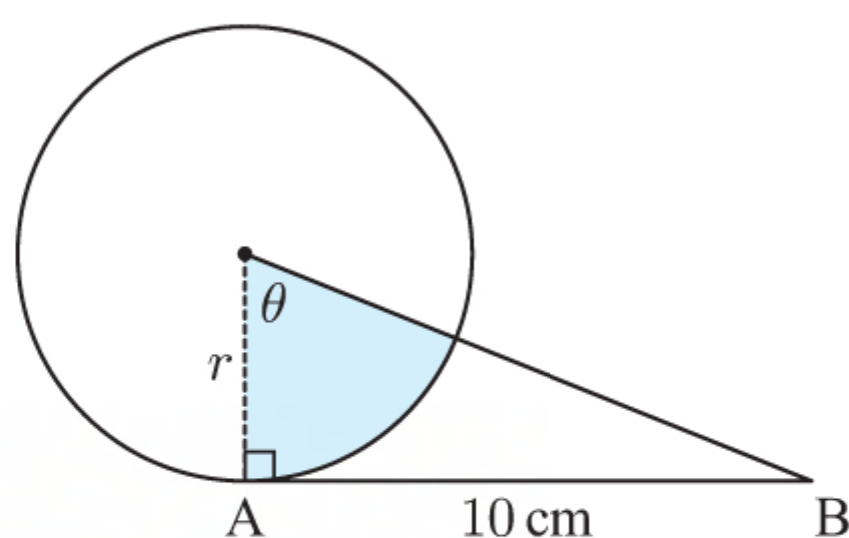
Noting that each of the values has the same modulus, and their arguments are $\frac{2\pi}{3}$ apart, they are all cube roots of $64 \operatorname{cis} \frac{3\pi}{4}$ or $32\sqrt{2}(-1 + i)$.

A1

Total [17 marks]

PAPER 2

Section A

1

$$\begin{aligned} \text{Since the shaded area is } 20 \text{ cm}^2, \quad \frac{\theta}{2\pi} \times \pi r^2 &= 20 \\ \therefore \theta r^2 &= 40 \end{aligned}$$

M1A1

$$\begin{aligned} \text{Since AB is a tangent, } \tan \theta &= \frac{10}{r} \\ \therefore r &= \frac{10}{\tan \theta} \\ \therefore \theta \times \frac{100}{\tan^2 \theta} &= 40 \\ \therefore \frac{\theta}{\tan^2 \theta} &= \frac{2}{5} \end{aligned}$$

M1

A1

M1

Using technology, $\theta \approx 1.0088 \dots \approx 1.01$ {since $0 < \theta < \frac{\pi}{2}$ }

A1

$$\text{and } r = \frac{10}{\tan \theta} \approx 6.30 \text{ cm}$$

A1

Total [7 marks]

2 a Each quarter, the interest paid is $\text{€}10\,000 \times \frac{0.05}{4}$

A1

$$\begin{aligned} \therefore \text{after } n \text{ quarters, the investment is worth } &\text{€}10\,000 + n \times \text{€}10\,000 \times \frac{0.05}{4} \\ &= \text{€}10\,000(1 + 0.0125n) \end{aligned}$$

A1

b i Each quarter, the value of the investment is multiplied by $1 + \frac{0.044}{4} = 1.011$

A1

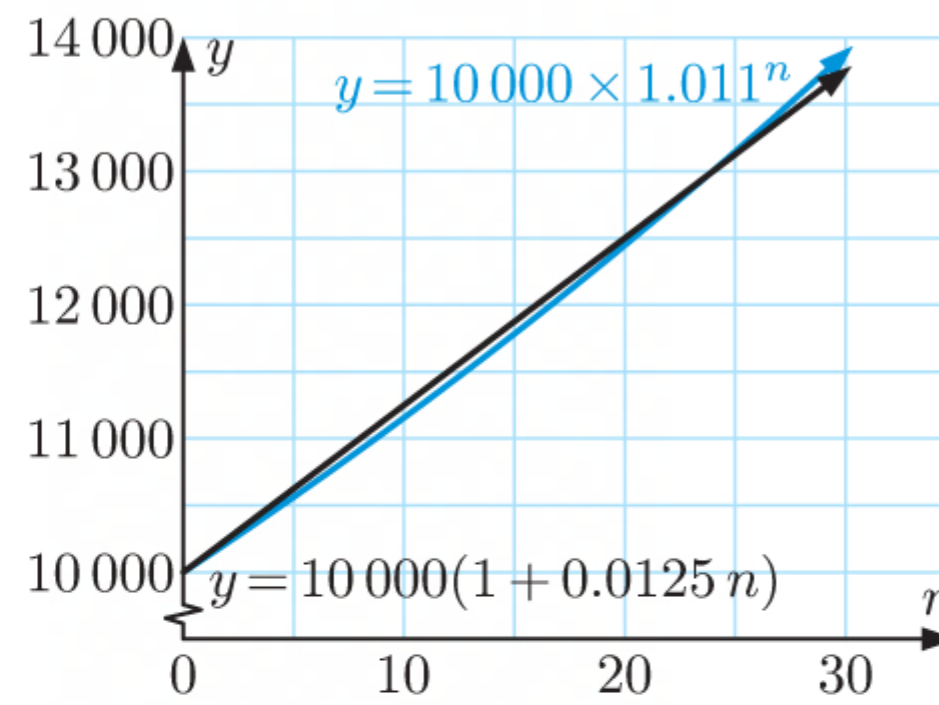
\therefore after n quarters, the investment is worth $\text{€}10\,000 \times 1.011^n$

A1

ii After 7 quarters, the compound interest investment would be worth $\text{€}10\,000 \times 1.011^7 = \text{€}10\,795.88$

A1

- c** We use technology to compare the graphs of $y = 10\,000(1 + 0.0125n)$ and $y = 10\,000 \times 1.011^n$. **M1**



The graphs intersect when $n \approx 23.85$ **A1**

\therefore it will take 24 quarters, which is 6 years, for the compound interest investment to be the better option. **A1**

Total [8 marks]

- 3 a** There are $6 \times 6 = 36$ possible outcomes when rolling the dice. **A1**

Of these, the outcomes with sum 5 are: $(1, 4), (2, 3), (3, 2), (4, 1)$. **A1**

So, the probability that the sum is 5 is $\frac{4}{36} = \frac{1}{9}$. **A1**

- b** When the pair of dice is rolled 10 times, let X be the number of times that the sum of the dice is 5.

$\therefore X \sim B(10, \frac{1}{9})$ **M1A1**

Using technology, $P(X \geq 2) \approx 0.307$ **A1**

Total [5 marks]

- 4 a** The time taken to get to work *depends* on the distance the worker needs to travel, so the time taken is the dependent variable, and the distance is the independent variable. **R1**

- b i** Since the workers have measured their distance to work by the path they move along rather than the straight-line distance, the values for x are less accurately measured than the values for y . It is therefore appropriate to use the x against y regression line. **R1AG**

ii Using technology, $x \approx 0.3489y - 2.4926$ **A1A1**

iii Letting $y = 50$, $x \approx 15.0$

Jody would expect the worker to live about 15.0 km from the office. **A1**

Total [5 marks]

- 5 a** The lines have direction vectors $\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. **A1**

If θ is the acute angle between the lines, then

$$\cos \theta = \frac{\left| \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right|} = \frac{|-2 + 0 - 1|}{\sqrt{1+9+1}\sqrt{4+0+1}} = \frac{3}{\sqrt{11}\sqrt{5}}$$

$\therefore \theta \approx 66.1^\circ$ **A1**

- b** A vector normal to the plane is $\mathbf{n} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ **M1**

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -1 \\ 2 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} \mathbf{k} \\ &= 3\mathbf{i} - \mathbf{j} - 6\mathbf{k} \end{aligned}$$

A1

From L_1 , the point $(1, 5, 0)$ lies on the plane.

\therefore the plane has equation $3x - y - 6z = 3(1) - (5) - 6(0)$ **M1**

which is $3x - y - 6z = -2$. **A1**

Total [8 marks]

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad \frac{z^* - i}{z} &= \frac{a - bi - i}{a + bi} \\
 &= \frac{a - (b+1)i}{a + bi} \times \frac{a - bi}{a - bi} \\
 &= \frac{a^2 - abi - a(b+1)i + b(b+1)i^2}{a^2 - b^2i^2} \\
 &= \frac{a^2 - b(b+1) - (2ab+a)i}{a^2 + b^2}
 \end{aligned}$$

M1

$\frac{z^* - i}{z}$ is undefined if $a^2 + b^2 = 0$, which requires both $a = 0$ and $b = 0$. **A1A1**

$$\begin{aligned}
 \mathbf{b} \quad \frac{z^* - i}{z} \text{ is real if } 2ab + a &= 0 & \text{M1} \\
 \therefore a(2b + 1) &= 0 \\
 \therefore b = -\frac{1}{2} \text{ or } a = 0, b &\neq 0 & \text{A1}
 \end{aligned}$$

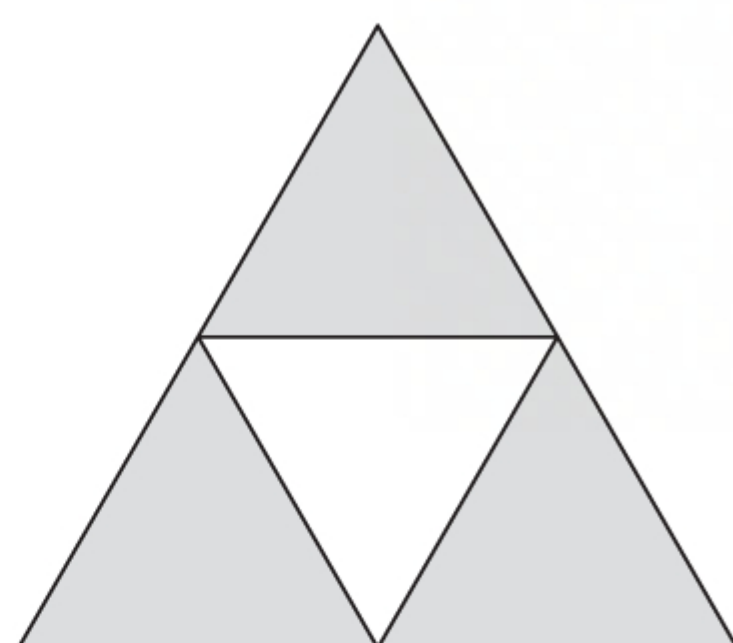
$$\begin{aligned}
 \mathbf{c} \quad \frac{z^* - i}{z} \text{ is purely imaginary if } a^2 - b(b+1) &= 0 & \text{M1} \\
 \therefore a^2 = b(b+1), a, b &\neq 0 & \text{A1}
 \end{aligned}$$

Total [7 marks]

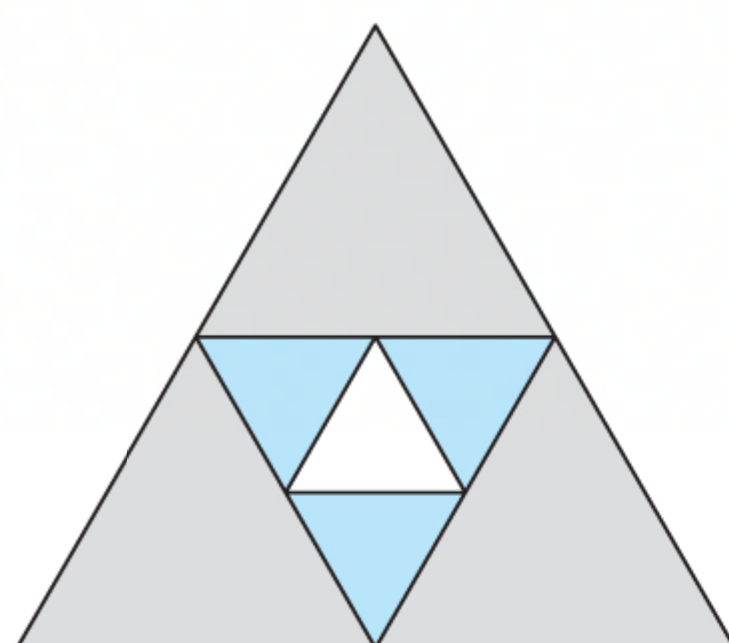
$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad S_n &= \frac{3}{4} \left(\frac{1 - (\frac{1}{4})^n}{1 - \frac{1}{4}} \right) \quad \{\text{sum of a geometric series}\} & \text{M1} \\
 &= 1 - \left(\frac{1}{4} \right)^n & \text{A1}
 \end{aligned}$$

$$\mathbf{b} \quad \lim_{n \rightarrow \infty} S_n = 1 \quad \text{A1}$$

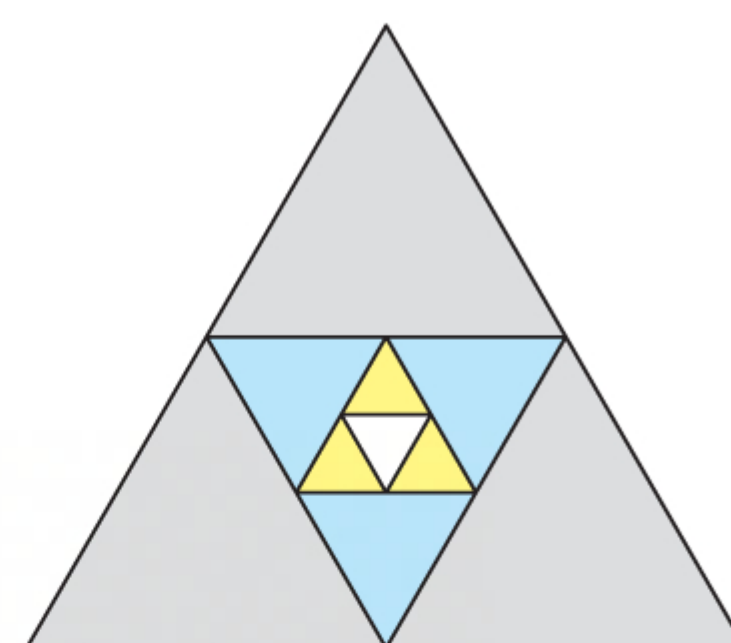
c Suppose the whole triangle has area = 1.



$$S_1 = \frac{3}{4}$$



$$\begin{aligned}
 S_2 &= \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} \\
 &= \frac{3}{4} + \frac{3}{16}
 \end{aligned}$$



$$\begin{aligned}
 S_3 &= \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} \\
 &= \frac{3}{4} + \frac{3}{16} + \frac{3}{64}
 \end{aligned}$$

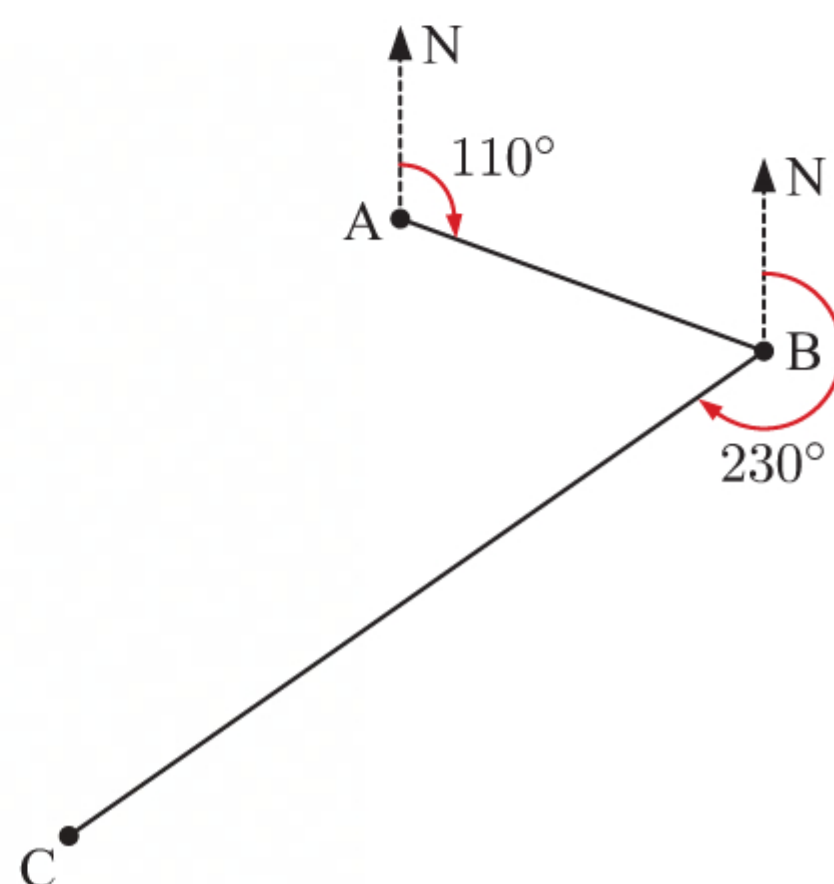
M1

If we continued this indefinitely, we would obtain

$$\lim_{n \rightarrow \infty} S_n = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \dots = 1 \quad \text{M1A1}$$

Total [6 marks]

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad \widehat{ABC} &= 360^\circ - 230^\circ - (180 - 110)^\circ & \text{M1} \\
 &= 60^\circ & \text{A1}
 \end{aligned}$$



b After t seconds, Alan has run $3t$ m.

\therefore he is $(600 - 3t)$ m from B.

After t seconds, Belinda has cycled $8t$ m.

\therefore she is $8t$ m from B.

Let the distance between Alan and Belinda after t seconds be D m.

Using the cosine rule,

$$\begin{aligned} D^2 &= (600 - 3t)^2 + (8t)^2 - 2(600 - 3t)(8t) \cos 60^\circ \\ &= 360\,000 - 3600t + 73t^2 - 8t(600 - 3t) \\ &= 97t^2 - 8400t + 360\,000 \end{aligned}$$

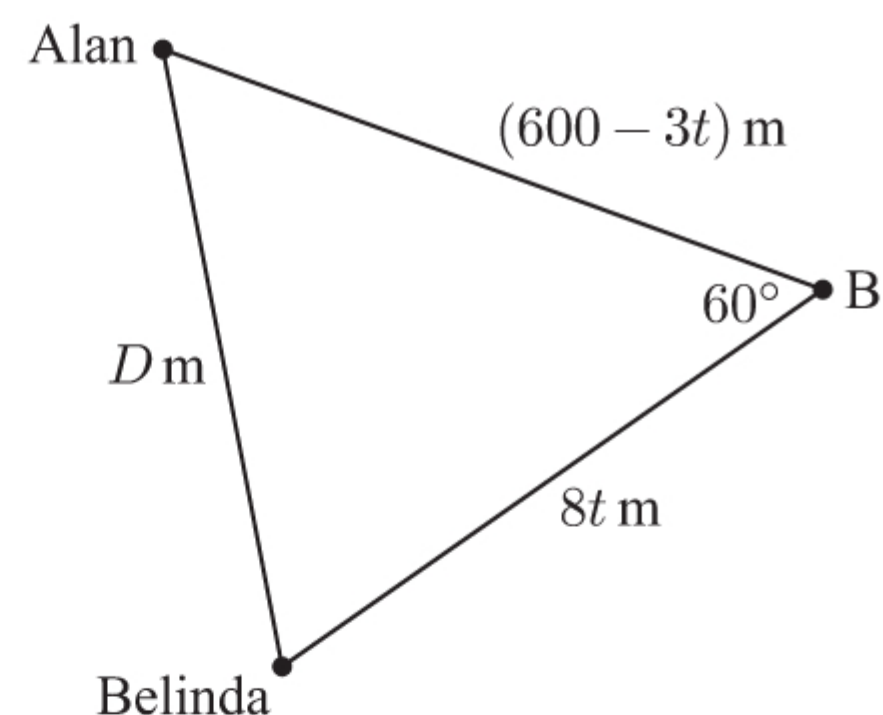
$$\therefore D = \sqrt{97t^2 - 8400t + 360\,000} \text{ m}$$

c D is minimised when D^2 is minimised.

D^2 is a quadratic, so is minimised when $t = \frac{8400}{2 \times 97} \approx 43.3$ seconds.

At this time, $D \approx 422$ m.

The minimum distance between Alan and Belinda is about 422 m. This occurs after about 43.3 seconds.



Total [9 marks]

9 a $\frac{\sin 3x}{\sin x} = \frac{\sin(2x + x)}{\sin x}$

$$= \frac{\sin 2x \cos x + \cos 2x \sin x}{\sin x}$$

$$= \frac{2 \sin x \cos^2 x + \cos 2x \sin x}{\sin x}$$

$$= 2 \cos^2 x + \cos 2x \quad \text{when } \sin x \neq 0 \text{ which is when } x \neq k\pi, k \in \mathbb{Z}$$

$$= \cos 2x + 1 + \cos 2x$$

$$= 1 + 2 \cos 2x$$

b Proof by mathematical induction:

Let P_n be that $\frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx = \frac{\sin((n + \frac{1}{2})x)}{2 \sin \frac{x}{2}}$ for all $n \in \mathbb{Z}^+$.

Consider P_1 : LHS = $\frac{1}{2} + \cos x$

$$\text{RHS} = \frac{\sin \frac{3x}{2}}{2 \sin \frac{x}{2}}$$

$$= \frac{1}{2}(1 + 2 \cos x) \quad \{\text{using a}\}$$

$$= \frac{1}{2} + \cos x$$

$\therefore P_1$ is true.

Suppose P_k is true.

$$\therefore \frac{1}{2} + \cos x + \cos 2x + \dots + \cos kx + \cos((k+1)x)$$

$$= \frac{\sin((k + \frac{1}{2})x)}{2 \sin \frac{x}{2}} + \cos((k+1)x)$$

$$= \frac{\sin((k+1)x - \frac{1}{2}x)}{2 \sin \frac{x}{2}} + \cos((k+1)x)$$

$$= \frac{\sin((k+1)x) \cos \frac{x}{2} - \cos((k+1)x) \sin \frac{x}{2}}{2 \sin \frac{x}{2}} + \cos((k+1)x)$$

$$= \frac{\sin((k+1)x) \cos \frac{x}{2} + \cos((k+1)x) \sin \frac{x}{2}}{2 \sin \frac{x}{2}}$$

$$= \frac{\sin((k+1)x + \frac{x}{2})}{2 \sin \frac{x}{2}}$$

$$= \frac{\sin((k+1) + \frac{1}{2})x)}{2 \sin \frac{x}{2}}$$

$\therefore P_{k+1}$ is true if P_k is true, and since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$.

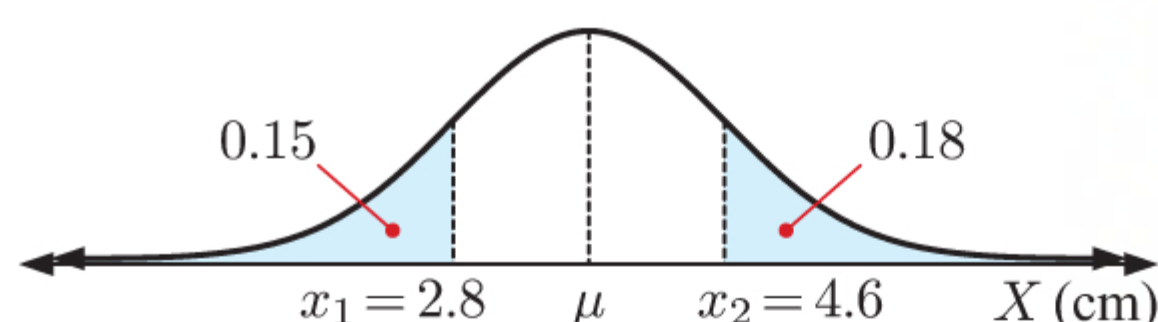
Total [9 marks]

Section B

- 10 a i** Let X cm be the length of an acorn.

$$\therefore X \sim N(\mu, \sigma^2)$$

Let z_1 and z_2 be the z -scores corresponding to $x_1 = 2.8$ cm and $x_2 = 4.6$ cm.



$$\text{Now } P(X \leq x_1) = 0.15$$

$$\therefore P\left(Z \leq \frac{2.8 - \mu}{\sigma}\right) = 0.15$$

$$\therefore z_1 = \frac{2.8 - \mu}{\sigma} \approx -1.0364 \quad \{\text{technology}\}$$

M1A1

$$\therefore 2.8 - \mu \approx -1.0364\sigma \quad \dots (1)$$

$$\text{and } P(X \geq x_2) = 0.18$$

$$\therefore P(X \leq x_2) = 0.82$$

$$\therefore P\left(Z \leq \frac{4.6 - \mu}{\sigma}\right) = 0.82$$

$$\therefore z_2 = \frac{4.6 - \mu}{\sigma} \approx 0.9154 \quad \{\text{technology}\}$$

M1A1

$$\therefore 4.6 - \mu \approx 0.9154\sigma \quad \dots (2)$$

- ii** (2) - (1) gives $1.8 \approx 1.9518\sigma$

$$\therefore \sigma \approx 0.9222$$

A1

Substituting into (1), $2.8 - \mu \approx -1.0364 \times 0.9222$

$$\therefore \mu \approx 3.756$$

A1

So, the population mean $\mu \approx 3.76$ cm and standard deviation $\sigma \approx 0.922$ cm.

- b i** $X \sim N(3.756, 0.9222)$

M1

$$\therefore P(X > 4) \approx 0.396$$

A1

$$\therefore Y \sim B(12, 0.396)$$

M1A1

$$\therefore E(Y) \approx 12 \times 0.396$$

$$\approx 4.75 \text{ acorns}$$

A1

- ii** $P(Y = 6) \approx \binom{12}{6} 0.396^6 (1 - 0.396)^6$

M1

$$\approx 0.173$$

A1

Total [13 marks]

- 11 a i** $f_2(0) = 0$

$$\therefore 2 \ln(\cos 0 + 1) + a = 0$$

M1

$$\therefore a = -2 \ln 2$$

A1

- b i** $f_2(k) = -4$

$$\therefore 2 \ln\left(\cos \frac{k\pi}{12} + 1\right) - 2 \ln 2 = -4$$

M1A1

$$\therefore \ln\left(\cos \frac{k\pi}{12} + 1\right) = \ln 2 - 2$$

$$= \ln 2 - \ln(e^2)$$

M1

$$= \ln\left(\frac{2}{e^2}\right)$$

A1

$$\therefore \cos \frac{k\pi}{12} = \frac{2}{e^2} - 1$$

AG

- ii** $\frac{k\pi}{12} \approx 2.388$

$$\therefore k \approx 9.12$$

A1

$$\begin{aligned}
 \text{c} \quad f_1(x) &= \ln\left(\cos \frac{\pi x}{12} + 2\right) \\
 \therefore f_1'(x) &= \frac{-\frac{\pi}{12} \sin \frac{\pi x}{12}}{\cos \frac{\pi x}{12} + 2} && \text{M1A1} \\
 \therefore f_1''(x) &= \frac{-\frac{\pi^2}{144} \cos \frac{\pi x}{12} \left(\cos \frac{\pi x}{12} + 2\right) + \frac{\pi}{12} \sin \frac{\pi x}{12} \left(-\frac{\pi}{12} \sin \frac{\pi x}{12}\right)}{\left(\cos \frac{\pi x}{12} + 2\right)^2} && \text{M1A1} \\
 \therefore f_1''(x) = 0 \quad \text{when} \quad &-\frac{\pi^2}{144} \left(\cos^2\left(\frac{\pi x}{12}\right) + \sin^2\left(\frac{\pi x}{12}\right) + 2 \cos \frac{\pi x}{12}\right) = 0 && \text{M1} \\
 &\therefore -\frac{\pi^2}{144} (1 + 2 \cos \frac{\pi x}{12}) = 0 \\
 &\therefore \cos \frac{\pi x}{12} = -\frac{1}{2} && \text{A1} \\
 &\therefore \frac{\pi x}{12} = \pm \frac{2\pi}{3} \quad \{\text{for } -12 \leq x \leq 12\} \\
 &\therefore x = \pm 8 && \text{A1}
 \end{aligned}$$

$$\text{Now } f'(-8) = \frac{\frac{\pi}{12} \sin\left(-\frac{2\pi}{3}\right)}{\cos\left(-\frac{2\pi}{3}\right) + 2} = \frac{-\frac{\pi}{12} \left(-\frac{\sqrt{3}}{2}\right)}{\frac{3}{2}} = \frac{\pi}{12\sqrt{3}} \quad \text{A1}$$

\therefore the maximum gradient of the road is $\frac{\pi}{12\sqrt{3}}$ at distance 8 m from the centre of the bridge, whichever side you are approaching from.

d Since the defining functions are both even,

$$\begin{aligned}
 &\text{shaded area} \\
 &= 2 \left(\int_0^k (f_1(x) - f_2(x)) dx + \int_k^{12} (f_1(x) - (-4)) dx \right) && \text{M1A1A1} \\
 &\approx 2 \left(\int_0^{9.12} \left(\ln\left(\cos \frac{\pi x}{12} + 2\right) - 2 \ln\left(\cos \frac{\pi x}{12} + 1\right) + 2 \ln 2 \right) dx + \int_{9.12}^{12} \left(\ln\left(\cos \frac{\pi x}{12} + 2\right) + 4 \right) dx \right) \\
 &\approx 2(17.66 + 11.77) \\
 &\approx 58.9 \text{ m}^2 && \text{A1}
 \end{aligned}$$

Total [19 marks]

$$\begin{aligned}
 \text{12 a} \quad \text{When } t = 0, \quad \frac{dP}{dt} &= \frac{1}{10} \times 500 \times \left(1 - \frac{500}{10000}\right)(1 - 2) && \text{M1} \\
 &= -47.5
 \end{aligned}$$

\therefore the population will immediately start decreasing at a significant rate relative to the population size. A1

$$\begin{aligned}
 \text{b} \quad \frac{dP}{dt} &= \frac{1}{10} P \left(1 - \frac{P}{10000}\right) (1 - 2 \cos 2\pi t) \\
 \therefore \frac{1}{P \left(1 - \frac{P}{10000}\right)} \frac{dP}{dt} &= \frac{1}{10} (1 - 2 \cos 2\pi t) \\
 \therefore \frac{10000}{P(10000 - P)} \frac{dP}{dt} &= \frac{1}{10} (1 - 2 \cos 2\pi t) \\
 \therefore \left(\frac{1}{P} + \frac{1}{10000 - P} \right) \frac{dP}{dt} &= \frac{1}{10} (1 - 2 \cos 2\pi t) \quad \{\text{partial fractions}\} && \text{M1} \\
 \therefore \int \left(\frac{1}{P} + \frac{1}{10000 - P} \right) \frac{dP}{dt} dt &= \int \frac{1}{10} (1 - 2 \cos 2\pi t) dt \\
 \therefore \int \left(\frac{1}{P} + \frac{1}{10000 - P} \right) dP &= \frac{1}{10} \int (1 - 2 \cos 2\pi t) dt && \text{A1} \\
 \therefore \ln|P| - \ln|10000 - P| &= \frac{1}{10} \left(t - \frac{1}{\pi} \sin 2\pi t\right) + c && \text{M1} \\
 \therefore \ln \left| \frac{P}{10000 - P} \right| &= \frac{1}{10} \left(t - \frac{1}{\pi} \sin 2\pi t\right) + c \\
 \therefore \frac{P}{10000 - P} &= \pm e^{\frac{1}{10} \left(t - \frac{1}{\pi} \sin 2\pi t\right) + c} \\
 \therefore \frac{10000 - P}{P} &= b e^{-\frac{1}{10} \left(t - \frac{1}{\pi} \sin 2\pi t\right)} \quad \{\text{letting } b = \pm e^{-c}\}
 \end{aligned}$$

When $t = 0$, $P = 500$

M1

$$\therefore \frac{9500}{500} = be^0$$

$$\therefore b = 19$$

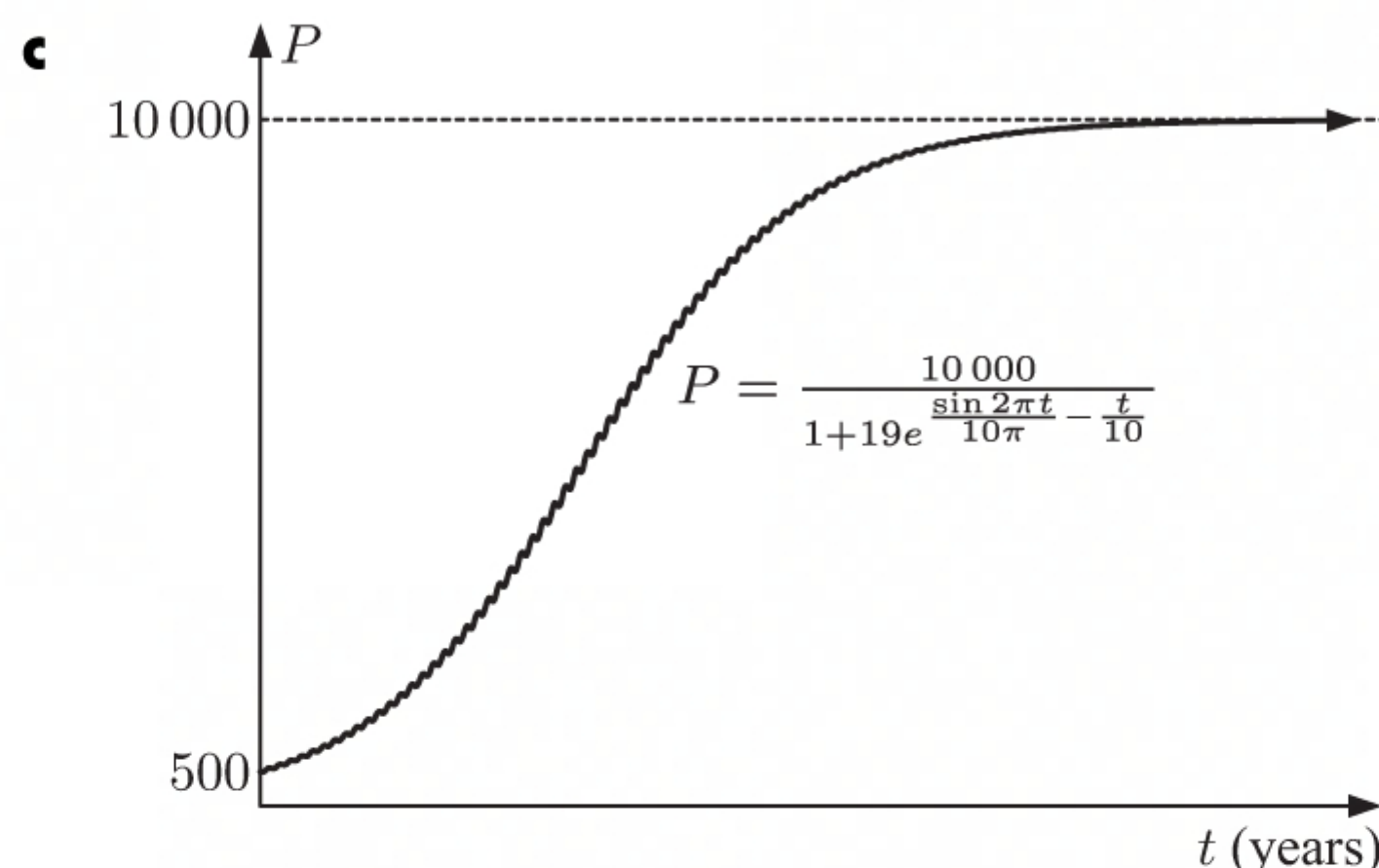
A1

So, $\frac{10\,000}{P} - 1 = 19e^{\frac{\sin 2\pi t}{10\pi} - \frac{t}{10}}$

$$\therefore \frac{10\,000}{P} = 1 + 19e^{\frac{\sin 2\pi t}{10\pi} - \frac{t}{10}}$$

$$\therefore P = \frac{10\,000}{1 + 19e^{\frac{\sin 2\pi t}{10\pi} - \frac{t}{10}}}$$

A1



A1A1

d The population growth is approximately logistic, but there is annual oscillation as the lizards struggle through winter. In the long term, the population stabilises at 10 000, and the lizards adapt to the climate.

A1A1

e Using technology, the population reaches 5000 in the 30th year.

M1A1

Total [14 marks]

PAPER 3

1 a $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \text{cis } \frac{2\pi}{3}$

i $\omega^3 = (\text{cis } \frac{2\pi}{3})^3$
 $= \text{cis } 2\pi$ {De Moivre}
 $= 1$

M1

A1

ii $\omega^2 = (\text{cis } \frac{2\pi}{3})^2$
 $= \text{cis } \frac{4\pi}{3}$ {De Moivre}
 $= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

M1

A1

$$\therefore \omega + \omega^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -1$$

M1

AG

b i $(x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b)$
 $= (x + a + b)(x^2 + \omega^2 ax + \omega bx + \omega^3 a^2 + \omega^2 ab$
 $\quad + \omega^2 bx + \omega ax$
 $\quad \quad + \omega^3 b^2 + \omega^4 ab)$
 $= (x + a + b)(x^2 + (a + b)(\omega + \omega^2)x + \omega^3(a^2 + b^2) + (\omega^2 + \omega)ab)$ {using **a i**}
 $= (x + a + b)(x^2 - (a + b)x + a^2 + b^2 - ab)$ {using **a i, ii**}
 $= x^3 - (a + b)x^2 + (a^2 + b^2 - ab)x$
 $\quad + (a + b)x^2 - (a + b)^2 x + (a + b)(a^2 + b^2 - ab)$
 $= x^3 + (a^2 + b^2 - ab - a^2 - 2ab - b^2)x + a^3 + ab^2 - a^2 b + a^2 b + b^3 - ab^2$
 $= x^3 - 3abx + a^3 + b^3$

(M1)

M1

A1

Equating coefficients, $ab = p$ and $a^3 + b^3 = 2q$.

M1AG

$$\text{ii} \quad ab = p$$

$$\therefore a^3 b^3 = p^3$$

$$\therefore b^3 = \frac{p^3}{a^3}$$

$$\text{Substituting in } a^3 + b^3 = 2q,$$

$$a^3 + \frac{p^3}{a^3} = 2q$$

$$\therefore a^6 + p^3 = 2qa^3$$

$$\therefore a^6 - 2qa^3 + p^3 = 0$$

$$\therefore (a^3 - q)^2 = q^2 - p^3 \quad \{\text{completing the square}\}$$

$$\therefore a^3 = q \pm \sqrt{q^2 - p^3}$$

$$\therefore a = \sqrt[3]{q \pm \sqrt{q^2 - p^3}}$$

$$\text{Also, } b^3 = 2q - a^3$$

$$= 2q - \left(q \pm \sqrt{q^2 - p^3} \right)$$

$$= q \mp \sqrt{q^2 - p^3}$$

$$\therefore b = \sqrt[3]{q \mp \sqrt{q^2 - p^3}}$$

$$\text{iii} \quad \text{Since } x^3 - 3px + 2q = (x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b), \text{ a solution to } x^3 - 3px + 2q = 0 \text{ is } x = -a - b$$

$$= -\sqrt[3]{q + \sqrt{q^2 - p^3}} - \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

$$\text{iv} \quad \text{For } x^3 - 6x + 6 = 0, \text{ we have } p = 2 \text{ and } q = 3.$$

$$\therefore \text{a solution to the equation is } x = -\sqrt[3]{3 + \sqrt{3^2 - 2^3}} - \sqrt[3]{3 - \sqrt{3^2 - 2^3}}$$

$$= -\sqrt[3]{3 + \sqrt{1}} - \sqrt[3]{3 - \sqrt{1}}$$

$$= -\sqrt[3]{4} - \sqrt[3]{2}$$

$$\text{c} \quad \text{i} \quad p(x - \alpha) = a(x - \alpha)^3 + b(x - \alpha)^2 + c(x - \alpha) + d$$

$$= ax^3 - 3a\alpha x^2 + 3a\alpha^2 x - a\alpha^3$$

$$+ bx^2 - 2b\alpha x + b\alpha^2$$

$$+ cx - c\alpha + d$$

$$= ax^3 + (b - 3a\alpha)x^2 + (3a\alpha^2 - 2b\alpha + c)x + (-a\alpha^3 + b\alpha^2 - c\alpha + d)$$

$$\therefore \text{the coefficient of } x^2 \text{ is zero if } b - 3a\alpha = 0$$

$$\therefore \alpha = \frac{b}{3a}$$

$$\text{ii} \quad \text{To apply Cardano's formula, we need to use } \alpha = \frac{b}{3a} = \frac{3}{3(1)} = 1$$

$$\therefore \text{we let } z = x - 1$$

$$\therefore z^3 + 3z^2 - 4 = (x - 1)^3 + 3(x - 1)^2 - 4$$

$$= x^3 - 3x^2 + 3x - 1$$

$$+ 3x^2 - 6x + 3$$

$$- 4$$

$$= x^3 - 3x - 2$$

$$\therefore x^3 - 3x - 2 = 0 \text{ which is of the form needed for Cardano's formula with } p = 1, q = -1$$

$$\therefore \text{a solution to the equation is } x = -\sqrt[3]{-1 + \sqrt{(-1)^2 - 1^3}} - \sqrt[3]{-1 - \sqrt{(-1)^2 - 1^3}}$$

$$= -\sqrt[3]{-1 + 0} - \sqrt[3]{-1 - 0}$$

$$= 2$$

$$\therefore z = 1 \text{ is a solution.}$$

Total [29 marks]

- 2 a** $G(1) = \sum_i P(X = x_i)$ **M1**
 $= 1$ since X is a discrete random variable. **A1**
- b i** $G'(z) = \sum_i x_i P(X = x_i) z^{x_i-1}$ **M1**
 $\therefore G'(1) = \sum_i x_i P(X = x_i)$ **A1**
 $= E(X)$ **AG**
- ii** $G''(z) = \sum_i x_i(x_i - 1) P(X = x_i) z^{x_i-2}$ **M1**
 $\therefore G''(1) = \sum_i x_i(x_i - 1) P(X = x_i)$ **A1**
 $= E(X(X - 1))$
 $= E(X^2 - X)$ **AG**
- iii** $G''(1) + G'(1) - [G'(1)]^2 = E(X^2 - X) + E(X) - [E(X)]^2$ **A1**
 $= E(X^2) - E(X) + E(X) - [E(X)]^2$ **A1**
 $= E(X^2) - [E(X)]^2$
 $= \text{Var}(X)$ **AG**
- c i** For $X \sim B(n, p)$, $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$, $k = 0, 1, 2, \dots, n$. **A1**
 $\therefore G(z) = \sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k} z^k$ **A1**
 $= \sum_{k=0}^n \binom{n}{k} (pz)^k (1 - p)^{n-k}$ **A1**
 $= (pz + 1 - p)^n \quad \{\text{binomial theorem}\}$ **AG**
- ii** $G'(z) = np(pz + 1 - p)^{n-1}$ **A1**
 $\therefore G''(z) = n(n - 1)p^2(pz + 1 - p)^{n-2}$ **A1**
Using **b i**, $E(X) = G'(1)$ **M1**
 $= np(p + 1 - p)^{n-1}$
 $= np$ **A1**
- Using **b iii**, $\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$ **M1**
 $= n(n - 1)p^2(p + 1 - p)^{n-2} + np - (np)^2$ **A1**
 $= n(n - 1)p^2 + np - (np)^2$
 $= n^2p^2 - np^2 + np - n^2p^2$
 $= np(1 - p)$ **A1**
- d i** For this uniform discrete random variable, $P(X = k) = \frac{1}{n}$ for $k = 1, 2, 3, \dots, n$. **M1**
 $\therefore G(z) = \sum_{k=1}^n \frac{1}{n} z^k$ **A1**
 $= \frac{1}{n} \sum_{k=1}^n z^k$
 $= \frac{1}{n} (z + z^2 + z^3 + \dots + z^n)$ **AG**
- ii** $G'(z) = \frac{1}{n} (1 + 2z + 3z^2 + \dots + nz^{n-1})$ **A1**
 $\therefore G''(z) = \frac{1}{n} (1 \times 2 + 2 \times 3z + 3 \times 4z^2 + \dots + (n - 1) \times nz^{n-2})$ **A1**
Using **b i**, $E(X) = G'(1)$
 $= \frac{1}{n} (1 + 2 + 3 + \dots + n)$
 $= \frac{1}{n} \times \frac{n(n + 1)}{2}$ **A1**
 $= \frac{n + 1}{2}$ **A1**

Using **b ii**, $\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$

$$= \frac{1}{n}(1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + (n-1)n) + \frac{n+1}{2} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n} \times \frac{(n-1)n(n+1)}{3} + \frac{n+1}{2} - \frac{(n+1)^2}{4} \quad \{\text{using given summation}\} \quad \mathbf{A1}$$

$$= (n+1) \left(\frac{n-1}{3} + \frac{1}{2} - \frac{n+1}{4} \right)$$

$$= (n+1) \frac{4(n-1) + 6 - 3(n+1)}{12}$$

$$= \frac{(n+1)(n-1)}{12}$$

$$= \frac{n^2 - 1}{12} \quad \mathbf{A1}$$

Total [26 marks]

TRIAL EXAMINATION 5

PAPER 1

Section A

1	x	A	5	7	9
	$P(X = x)$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{4}{9}$	p

$$\begin{aligned} \mathbf{a} \quad p &= 1 - \left(\frac{1}{3} + \frac{1}{9} + \frac{4}{9}\right) \\ &= 1 - \frac{8}{9} \\ &= \frac{1}{9} \end{aligned}$$

M1

A1

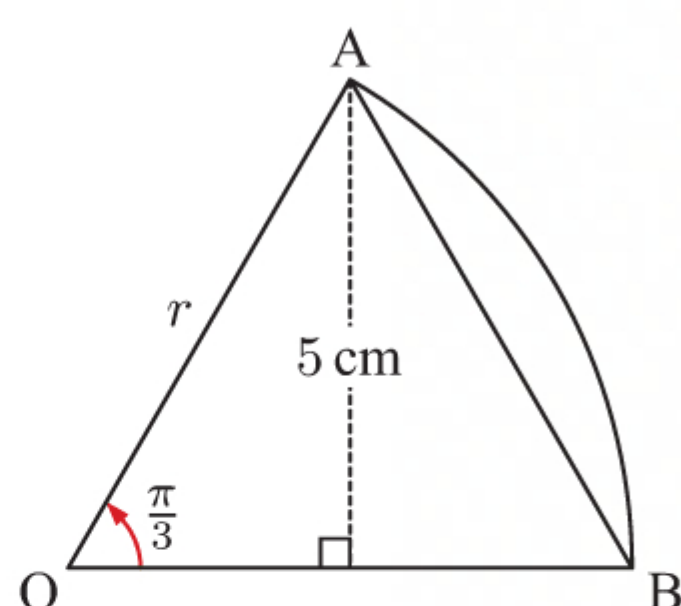
$$\begin{aligned} \mathbf{b} \quad E(X) &= 6 \\ \therefore \frac{A}{3} + \frac{5}{9} + \frac{28}{9} + 1 &= 6 \\ \therefore \frac{33 + 3A}{9} &= 5 \\ \therefore 33 + 3A &= 45 \\ \therefore 3A &= 12 \\ \therefore A &= 4 \end{aligned}$$

M1A1

A1

Total [5 marks]

2 a



$$\begin{aligned} \sin \frac{\pi}{3} &= \frac{5}{r} \\ \therefore \frac{\sqrt{3}}{2} &= \frac{5}{r} \\ \therefore r &= \frac{10}{\sqrt{3}} \\ \therefore r &= \frac{10\sqrt{3}}{3} \text{ cm} \end{aligned}$$

M1

A1

AG

$$\begin{aligned} \mathbf{b} \quad P &= 2r + r\theta \\ &= 2\left(\frac{10\sqrt{3}}{3}\right) + \frac{10\sqrt{3}}{3} \times \frac{\pi}{3} \\ &= \left(\frac{20\sqrt{3}}{3} + \frac{10\sqrt{3}\pi}{9}\right) \text{ cm} \end{aligned}$$

M1

A1

A1

Total [5 marks]

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad P(A | B) &= \frac{P(A \cap B)}{P(B)} \\ \therefore P(A \cap B) &= \frac{1}{4} \times \frac{2}{5} \\ &= \frac{1}{10} \end{aligned}$$

M1

A1

$$\begin{aligned} \mathbf{b} \quad P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{3}{10} + \frac{2}{5} - \frac{1}{10} \\ &= \frac{3 + 4 - 1}{10} \\ &= \frac{3}{5} \end{aligned}$$

M1

A1

$$\begin{aligned} \mathbf{c} \quad P(A) &\neq P(A | B) \quad \therefore \text{not independent} \\ \text{or } P(A) \times P(B) &\neq P(A \cap B) \text{ as } \frac{3}{10} \times \frac{2}{5} \neq \frac{1}{10} \\ \therefore &\text{not independent} \end{aligned}$$

R1

R1

Total [5 marks]

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \frac{4}{a} &= 2 \\ \therefore a &= 2 \\ 2\left(\frac{1}{2}\right) + b &= 0 \\ \therefore b &= -1 \end{aligned}$$

A1

A1

$$\mathbf{b} \quad y = \frac{4x-1}{2x-1}$$

$$\therefore \text{the inverse function is } x = \frac{4y-1}{2y-1}$$

M1

$$\therefore x(2y-1) = 4y-1$$

$$\therefore 2xy - x = 4y - 1$$

$$\therefore 2xy - 4y = x - 1$$

$$\therefore y(2x-4) = x-1$$

M1

$$\therefore y = \frac{x-1}{2x-4}$$

$$\therefore f^{-1}(x) = \frac{x-1}{2x-4}$$

A1

$$\mathbf{c} \quad f^{-1}(x) \text{ has asymptotes } x = 2 \text{ and } y = \frac{1}{2}.$$

$$\therefore \text{domain} = \{x \mid x \neq 2\}$$

A1

$$\text{range} = \{y \mid y \neq \frac{1}{2}\}$$

A1

Total [7 marks]

$$\mathbf{5} \quad \cos \theta \cos 30^\circ + \sin \theta \sin 30^\circ = 2 \sin \theta$$

M1

$$\therefore \frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta = 2 \sin \theta$$

A1

$$\therefore \frac{\sqrt{3}}{2} \cos \theta = \frac{3}{2} \sin \theta$$

M1

$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{3}}{3}$$

$$\therefore \tan \theta = \frac{\sqrt{3}}{3}$$

A1

Total [4 marks]

$$\mathbf{6} \quad \mathbf{a} \quad \overrightarrow{\text{OA}} \bullet \overrightarrow{\text{OB}} = 0$$

$$\therefore k(k-1) + 3k - 35 = 0$$

M1

$$\therefore k^2 - k + 3k - 35 = 0$$

$$\therefore k^2 + 2k - 35 = 0$$

A1

$$\therefore (k+7)(k-5) = 0$$

$$\therefore k = -7 \text{ or } k = 5$$

A1

$$\mathbf{b} \quad \text{If } k = 5, \overrightarrow{\text{OA}} = \begin{pmatrix} 5 \\ 3 \\ -7 \end{pmatrix}, \overrightarrow{\text{OB}} = \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix}$$

(A1)

$$\overrightarrow{\text{AB}} = \overrightarrow{\text{OB}} - \overrightarrow{\text{OA}} = \begin{pmatrix} 4-5 \\ 5-3 \\ 5-(-7) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 12 \end{pmatrix}$$

A1

$$|\overrightarrow{\text{AB}}| = \sqrt{(-1)^2 + (2)^2 + (12)^2}$$

M1

$$= \sqrt{1 + 4 + 144}$$

$$= \sqrt{149}$$

$$\therefore \text{unit vector in opposite direction to } \overrightarrow{\text{AB}} = \frac{-1}{\sqrt{149}} \begin{pmatrix} -1 \\ 2 \\ 12 \end{pmatrix}.$$

M1A1

Total [8 marks]

$$\mathbf{7} \quad \mathbf{a} \quad \text{The graphs meet where } (x+3)^2 = -x^2 + bx + c$$

M1

$$\therefore x^2 + 6x + 9 = -x^2 + bx + c$$

$$\therefore 2x^2 + 6x - bx + 9 - c = 0$$

$$\therefore 2x^2 + (6-b)x + (9-c) = 0$$

A1

$$\text{Now } \Delta = (6-b)^2 - 4(2)(9-c) = 0$$

M1

$$\therefore 36 - 12b + b^2 - 72 + 8c = 0$$

A1

$$\therefore b^2 - 12b + 8c = 36 \quad \dots (*)$$

AG

b $y = -x^2 + bx + c$ passes through $(-4, 1)$, so $1 = -(-4)^2 - 4b + c$ **A1**

$$\therefore 1 = -16 - 4b + c$$

$$\therefore 4b - c + 17 = 0$$
 AG

c Substituting $c = 4b + 17$ into (*) gives $b^2 - 12b + 8(4b + 17) - 36 = 0$ **M1**

$$\therefore b^2 - 12b + 32b + 136 - 36 = 0$$

$$\therefore b^2 + 20b + 100 = 0$$

$$\therefore (b + 10)^2 = 0$$

$$\therefore b = -10$$
 A1

$$\therefore c = 4(-10) + 17 = -23$$
 A1

Total [8 marks]

8 $y^3 - 5xy = 7 + e^{\sin x}$

$$\therefore 3y^2 \frac{dy}{dx} - 5x \frac{dy}{dx} - 5y = \cos x e^{\sin x}$$
 M1A1

$$\therefore \frac{dy}{dx}(3y^2 - 5x) = 5y + \cos x e^{\sin x}$$
 M1

$$\therefore \frac{dy}{dx} = \frac{5y + \cos x e^{\sin x}}{3y^2 - 5x}$$
 A1A1

Total [5 marks]

9 a $\int e^{3x} \cos x \, dx = e^{3x} \sin x - 3 \int e^{3x} \sin x \, dx$ **M1A1**

$$\therefore \int e^{3x} \cos x \, dx = e^{3x} \sin x - 3 \left[-e^{3x} \cos x - 3 \int -e^{3x} \cos x \, dx \right]$$
 M1

$$\therefore \int e^{3x} \cos x \, dx = e^{3x} \sin x + 3e^{3x} \cos x - 9 \int e^{3x} \cos x \, dx$$

$$\therefore 10 \int e^{3x} \cos x \, dx = e^{3x} \sin x + 3e^{3x} \cos x + c$$
 M1

$$\therefore \int e^{3x} \cos x \, dx = \frac{1}{10} e^{3x} \sin x + \frac{3}{10} e^{3x} \cos x + c$$
 A1A1

b From **a**, $y = \frac{1}{10} e^{3x} \sin x + \frac{3}{10} e^{3x} \cos x + c$

$$\therefore 3 = \frac{1}{10} e^0 \sin(0) + \frac{3}{10} e^{(0)} \cos(0) + c$$
 M1

$$\therefore 3 = \frac{3}{10} + c$$

$$\therefore c = \frac{27}{10}$$
 A1

$$\therefore y = \frac{1}{10} e^{3x} \sin x + \frac{3}{10} e^{3x} \cos x + \frac{27}{10}$$
 A1

Total [9 marks]

Section B

10 a If $n = 1$, $u_1 = 5 \times 2^0 + 3$

$$= 5 \times 1 + 3$$

$$= 8 \quad \checkmark \quad \therefore \text{true for } n = 1.$$
 M1A1

Assume true for $n = k$: $u_k = 5 \times 2^{k-1} + 3$ **A1**

$$u_{k+1} = 2(5 \times 2^{k-1} + 3) - 3$$
 M1

$$= 5 \times 2^k + 6 - 3$$

$$= 5 \times 2^{(k+1)-1} + 3 \quad \therefore \text{true for } n = k + 1$$
 A1

The result is true for $n = k \Rightarrow$ it is true for $n = k + 1$.

It is true for $n = 1$, so the result is proved by mathematical induction. **R1**

$$\mathbf{b} \quad u_k = \underbrace{5 \times 2^{k-1}}_{\substack{\text{geometric with} \\ u_1=5, \quad r=2}} + 3 \quad (\text{M1})$$

$$\begin{aligned} \therefore S_8 &= \frac{5(2^8 - 1)}{2 - 1} + 8(3) \\ &= 5(256 - 1) + 24 \\ &= 5 \times 255 + 24 \\ &= 1299 \end{aligned} \quad \begin{array}{l} \text{M1A1} \\ \\ \text{A1} \end{array}$$

$$\begin{aligned} \mathbf{c} \quad u_1 &= \frac{1}{\log_3 2x}, \quad d = \frac{1}{\log_{27} 2x} - \frac{1}{\log_3 2x} \\ &= \frac{1}{\left(\frac{\log_3 2x}{\log_3 27}\right)} - \frac{1}{\log_3 2x} \\ &= \frac{3}{\log_3 2x} - \frac{1}{\log_3 2x} \\ &= \frac{2}{\log_3 2x} \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{M1} \\ \\ \text{A1} \end{array}$$

$$\text{Now } S_{30} = 450$$

$$\begin{aligned} \therefore \frac{30}{2} \left(\frac{2}{\log_3 2x} + \frac{58}{\log_3 2x} \right) &= 450 \\ \therefore 15 \times \frac{60}{\log_3 2x} &= 450 \\ \therefore \frac{900}{\log_3 2x} &= 450 \\ \therefore \log_3 2x &= 2 \\ \therefore 2x &= 9 \\ \therefore x &= 4.5 \end{aligned} \quad \begin{array}{l} \text{M1} \\ \\ \\ \text{A1} \end{array}$$

Total [16 marks]

$$\begin{aligned} \mathbf{11} \quad \mathbf{a} \quad y &= kx(x+1)^2 \\ \therefore \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{k(x+h)[(x+h)+1]^2 - kx(x+1)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{k(x+h)[(x+h)^2 + 2(x+h) + 1] - kx(x^2 + 2x + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{k(x+h)^3 + 2k(x+h)^2 + k(x+h) - kx^3 - 2kx^2 - kx}{h} \\ &= \lim_{h \rightarrow 0} \frac{k(x^3 + 3x^2h + 3xh^2 + h^3) + 2k(x^2 + 2hx + h^2) + \cancel{kx} + kh - kx^3 - 2kx^2 - \cancel{kx}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{kx^3} + 3kx^2h + 3kxh^2 + kh^3 + \cancel{2kx^2} + 4kxh + 2kh^2 + kh - \cancel{kx^3} - \cancel{2kx^2}}{h} \\ &= \lim_{h \rightarrow 0} (3kx^2 + 3kxh + kh^2 + 4kx + 2kh + k) \quad \{h \neq 0\} \\ &= 3kx^2 + 4kx + k \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{M1} \\ \\ \text{M1} \\ \text{A1} \\ \text{A1} \\ \text{AG} \end{array}$$

$$\begin{aligned} \mathbf{b} \quad \text{Turning points occur when } 3kx^2 + 4kx + k &= 0 \\ \therefore k(3x^2 + 4x + 1) &= 0 \\ \therefore k(3x + 1)(x + 1) &= 0 \\ \therefore x = -\frac{1}{3} \text{ or } -1 \quad \{k > 0\} \end{aligned} \quad \begin{array}{l} \text{A1} \\ \\ \text{A1} \\ \text{AG} \end{array}$$

Turning points are at $x = -\frac{1}{3}$ and $x = -1$.

$$\begin{aligned} \text{When } x = -\frac{1}{3}, \quad y &= -\frac{k}{3} \left(\frac{2}{3} \right)^2 \\ &= -\frac{4k}{27} \end{aligned}$$

When $x = -1$, $y = 0$.

The turning points are $\left(-\frac{1}{3}, -\frac{4k}{27}\right)$ and $(-1, 0)$. A1A1

$$\begin{aligned} \mathbf{c} \quad \frac{dy}{dx} &= 3kx^2 + 4kx + k \\ \therefore \frac{d^2y}{dx^2} &= 6kx + 4k \end{aligned} \quad \mathbf{A1}$$

$$\begin{aligned} \text{When } x = -\frac{1}{3}, \quad \frac{d^2y}{dx^2} &= -2k + 4k \\ &= 2k > 0 \quad \{k > 0\} \quad \therefore \text{minimum at } \left(-\frac{1}{3}, -\frac{4k}{27}\right) \end{aligned} \quad \mathbf{A1}$$

$$\begin{aligned} \text{When } x = -1, \quad \frac{d^2y}{dx^2} &= -6k + 4k \\ &= -2k < 0 \quad \{k > 0\} \quad \therefore \text{maximum at } (-1, 0) \end{aligned} \quad \mathbf{A1}$$

$$\mathbf{d} \quad y \text{ is a strictly increasing function when } \frac{dy}{dx} > 0. \quad \mathbf{(M1)}$$

$$\frac{dy}{dx} \text{ is a quadratic with } a > 0.$$

$$\frac{dy}{dx} > 0 \text{ when } x < -1 \text{ and } x > -\frac{1}{3}. \quad \mathbf{A1A1}$$

$$\begin{aligned} \mathbf{e} \quad v &= 2y \\ \therefore \frac{dv}{dx} &= \frac{dv}{dy} \times \frac{dy}{dx} \end{aligned} \quad \mathbf{(M1)}$$

$$\begin{aligned} &= 2(3kx^2 + 4kx + k) \\ &= 6kx^2 + 8kx + 2k \end{aligned} \quad \mathbf{A1}$$

$$\text{Now } \frac{dv}{dx} = 20 \text{ at } x = -2$$

$$\therefore 6k(-2)^2 + 8k(-2) + 2k = 20 \quad \mathbf{M1}$$

$$\therefore 24k - 16k + 2k = 20$$

$$\therefore 10k = 20$$

$$\therefore k = 2 \quad \mathbf{A1}$$

Total [19 marks]

$$\mathbf{12} \quad \mathbf{a} \quad f(x) = \frac{-6x^2 + 12x + 4}{(2-3x)(1+2x)} = A + \frac{B}{2-3x} + \frac{C}{1+2x}$$

$$f(x) = \frac{-6x^2 + 12x + 4}{-6x^2 + x + 2} = \frac{-6x^2 + x + 2 + 11x + 2}{-6x^2 + x + 2} \quad \mathbf{M1}$$

$$= \frac{-6x^2 + x + 2}{-6x^2 + x + 2} + \frac{11x + 2}{-6x^2 + x + 2}$$

$$= 1 + \frac{11x + 2}{(2-3x)(1+2x)}$$

$$\therefore A = 1 \quad \mathbf{A1}$$

$$\text{Now } \frac{11x + 2}{(2-3x)(1+2x)} = \frac{B}{2-3x} + \frac{C}{1+2x}$$

$$\therefore B(1+2x) + C(2-3x) = 11x + 2 \quad \mathbf{M1}$$

$$\text{Equating coefficients:} \quad \mathbf{M1}$$

$$B + 2C = 2 \quad \dots (1)$$

$$2B - 3C = 11 \quad \dots (2)$$

$$\therefore 2B + 4C = 4 \quad \{2 \times (1)\}$$

$$\text{Subtracting,} \quad \frac{-7C = 7}{-7C = 7}$$

$$\therefore C = -1 \quad \mathbf{A1}$$

$$\therefore B = 4 \quad \mathbf{A1}$$

$$\therefore f(x) = \frac{-6x^2 + 12x + 4}{(2-3x)(1+2x)} = 1 + \frac{4}{2-3x} - \frac{1}{1+2x}$$

$$\mathbf{b} \quad \int f(x) dx = \int \left(1 + \frac{4}{2-3x} - \frac{1}{1+2x}\right) dx \quad \mathbf{M1}$$

$$= x - \frac{4}{3} \ln|2-3x| - \frac{1}{2} \ln|1+2x| + c \quad \mathbf{A1A1}$$

$$\begin{aligned}
 \text{c } f(x) &= 1 + 4(2 - 3x)^{-1} - (1 + 2x)^{-1} \\
 &= 1 + 4(2)^{-1}(1 - \frac{3}{2}x)^{-1} - (1 + 2x)^{-1} && \text{M1} \\
 &= 1 + 2(1 - \frac{3}{2}x)^{-1} - (1 + 2x)^{-1} && \text{A1}
 \end{aligned}$$

$$\text{Now } (1 - \frac{3}{2}x)^{-1} = 1 - 1\left(-\frac{3x}{2}\right) + \frac{(-1)(-2)(-\frac{3}{2}x)^2}{2} + \frac{(-1)(-2)(-3)(-\frac{3}{2}x)^3}{6} + \dots \quad \text{M1A1}$$

$$\text{and } (1 + 2x)^{-1} = 1 - 1(2x) + \frac{(-1)(-2)(2x)^2}{2} + \frac{(-1)(-2)(-3)(2x)^3}{6} + \dots \quad \text{M1A1}$$

$$\begin{aligned}
 \text{So, } f(x) &= 1 + 2(1 - \frac{3}{2}x)^{-1} - (1 + 2x)^{-1} \\
 &= 1 + 2\left(1 + \frac{3x}{2} + \frac{9x^2}{4} + \frac{27}{8}x^3 + \dots\right) - (1 - 2x + 4x^2 - 8x^3 + \dots) && \text{A1} \\
 &= 1 + 2 + 3x + \frac{9x^2}{2} + \frac{27}{4}x^3 - 1 + 2x - 4x^2 + 8x^3 + \dots \\
 &= 2 + 5x + \frac{1}{2}x^2 + \frac{59}{4}x^3 + \dots && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \text{The expansion is valid when } \left| -\frac{3x}{2} \right| < 1 \quad \text{and} \quad |2x| < 1 && \text{M1} \\
 -\frac{2}{3} < x < \frac{2}{3} \quad \text{and} \quad -\frac{1}{2} < x < \frac{1}{2} \\
 \therefore -\frac{1}{2} < x < \frac{1}{2} && \text{A1}
 \end{aligned}$$

Total [19 marks]

PAPER 2

Section A

$$1 \quad \text{a } X \sim B(15, 0.35) \quad \text{(M1)}$$

$$P(X \geq 8) = 1 - P(X \leq 7) \approx 0.113 \quad \text{M1A1}$$

$$\text{b } \mu = 15 \times 0.35 = 5.25$$

$$\text{and } \sigma = \sqrt{15 \times 0.35(1 - 0.35)} = 1.8472 \dots \quad \text{M1A1}$$

$$\begin{aligned}
 \text{Now } P(3.4027 \dots < X < 7.0972 \dots) &= P(X \leq 7) - P(X \leq 3) && \text{M1} \\
 &\approx 0.714 && \text{A1}
 \end{aligned}$$

Total [7 marks]

$$\begin{aligned}
 2 \quad \text{a } s(t) &= \int v(t) dt \\
 &= \int (2t + 3)(5t^{\frac{3}{2}} + 8) dt \\
 &= \int (10t^{\frac{5}{2}} + 16t + 15t^{\frac{3}{2}} + 24) dt && \text{M1} \\
 &= \frac{20}{7}t^{\frac{7}{2}} + 8t^2 + 6t^{\frac{5}{2}} + 24t + c && \text{A1}
 \end{aligned}$$

$$\text{When } t = 0, \quad s = 0 \Rightarrow c = 0 \quad \text{M1}$$

$$\therefore s(t) = \frac{20}{7}t^{\frac{7}{2}} + 8t^2 + 6t^{\frac{5}{2}} + 24t \quad \text{A1}$$

$$\text{b } a = \frac{dv}{dt} = 25t^{\frac{3}{2}} + 16 + \frac{45}{2}t^{\frac{1}{2}} \quad \text{M1}$$

$$\text{Hence, when } t = 0, \quad a = 16 \text{ m/s}^2 \quad \text{A1}$$

Total [6 marks]

$$3 \quad P(X < 80) = 0.24 \quad \text{and} \quad P(X > 120) = \frac{1}{3} \quad \text{A1}$$

$$\therefore P\left(Z < \frac{80 - \mu}{\sigma}\right) = 0.24 \quad \text{and} \quad P\left(Z > \frac{120 - \mu}{\sigma}\right) = \frac{1}{3} \quad \text{M1}$$

$$\text{Hence, using the GDC, we have that } \frac{80 - \mu}{\sigma} = -0.70630 \dots \quad \text{and} \quad \frac{120 - \mu}{\sigma} = 0.43072 \dots \quad \text{M1}$$

$$\text{Solving simultaneously gives } \mu \approx 105 \text{ minutes} \quad \text{and} \quad \sigma \approx 35.2 \text{ minutes.} \quad \text{M1A1}$$

Total [5 marks]

$$\begin{aligned}
 4 \quad a \quad 2z^*w &= 2(1 - \sqrt{2}i)(3 - 2\sqrt{2}i) && \text{M1} \\
 &= 2(3 - 2\sqrt{2}i - 3\sqrt{2}i - 4) \\
 &= -2 - 10\sqrt{2}i && \text{A1} \\
 \therefore \operatorname{Re}(2z^*w) &= -2 && \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \frac{z}{w^*} &= \left(\frac{1 + \sqrt{2}i}{3 + 2\sqrt{2}i} \right) \times \left(\frac{3 - 2\sqrt{2}i}{3 - 2\sqrt{2}i} \right) && \text{M1} \\
 &= \frac{3 - 2\sqrt{2}i + 3\sqrt{2}i + 4}{9 - \cancel{6\sqrt{2}i} + \cancel{6\sqrt{2}i} + 8} && \text{A1} \\
 &= \frac{7}{17} + \frac{\sqrt{2}}{17}i \\
 \therefore \operatorname{Im}\left(\frac{z}{w^*} + i\right) &= \frac{\sqrt{2} + 17}{17} && \text{A1}
 \end{aligned}$$

Total [6 marks]

$$\begin{aligned}
 5 \quad a \quad V &= \pi r^2 h = 240 && \text{M1} \\
 \therefore h &= \frac{240}{\pi r^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } A &= 2\pi r h + \pi r^2 && \text{M1} \\
 &= 2\pi r \left(\frac{240}{\pi r^2} \right) + \pi r^2 && \text{A1} \\
 &= \frac{480}{r} + \pi r^2 && \text{AG}
 \end{aligned}$$

$$b \quad \frac{dA}{dr} = -\frac{480}{r^2} + 2\pi r \quad \text{M1}$$

$$A \text{ is minimised when } \frac{dA}{dr} = 0 \quad \text{M1}$$

$$\therefore -\frac{480}{r^2} + 2\pi r = 0$$

$$\therefore 2\pi r = \frac{480}{r^2}$$

$$\therefore r = \sqrt[3]{\frac{240}{\pi}} \quad \text{A1}$$

$$\therefore h = \frac{240}{\pi \left(\frac{240}{\pi} \right)^{\frac{2}{3}}} = \sqrt[3]{\frac{240}{\pi}} = r \quad \text{M1AG}$$

$$A \text{ is minimised when } r = h.$$

Total [7 marks]

$$6 \quad a \quad \int_0^2 kxe^{-2x} dx = 1$$

$$\text{Integrating by parts gives } k \left(\left[-\frac{1}{2}xe^{-2x} \right]_0^2 + \frac{1}{2} \int_0^2 e^{-2x} dx \right) = 1 \quad \text{M1A1}$$

$$\therefore k \left(-\frac{1}{2}(2)e^{-4} - \frac{1}{4}[e^{-2x}]_0^2 \right) = 1 \quad \text{A1}$$

$$\therefore k \left(-e^{-4} - \frac{1}{4}(e^{-4} - 1) \right) = 1 \quad \text{A1}$$

$$\therefore k \left(-\frac{5}{4}e^{-4} + \frac{1}{4} \right) = 1$$

$$\therefore \frac{k}{4}(1 - 5e^{-4}) = 1$$

$$\therefore k = \frac{4}{1 - 5e^{-4}} \quad \text{A1}$$

$$b \quad \operatorname{Var}(X) = E(X^2) - (E(X))^2$$

$$= \int_0^2 x^2 f(x) dx - \left(\int_0^2 xf(x) dx \right)^2 \quad \text{(M1)}$$

$$= \int_0^2 kx^3 e^{-2x} dx - \left(\int_0^2 kx^2 e^{-2x} dx \right)^2$$

$$= \int_0^2 \frac{4x^3 e^{-2x}}{1 - 5e^{-4}} dx - \left(\int_0^2 \frac{4x^2 e^{-2x}}{1 - 5e^{-4}} dx \right)^2 \quad \text{(M1)}$$

$$\approx 0.232 \quad \{\text{GDC}\} \quad \text{A1}$$

Total [8 marks]

$$\begin{aligned}
 7 \quad & 3(4^x) - 10(2^x) = -3 \\
 & \therefore 3(2^2)^x - 10(2^x) + 3 = 0 \quad \text{M1} \\
 & \therefore 3(2^x)^2 - 10(2^x) + 3 = 0 \quad \text{A1} \\
 & \text{Let } y = 2^x: \\
 & \therefore 3y^2 - 10y + 3 = 0 \quad \text{M1} \\
 & \therefore (3y - 1)(y - 3) = 0 \quad \text{A1} \\
 & \therefore y = 2^x = \frac{1}{3} \text{ or } 3 \\
 & \therefore x = \log_2\left(\frac{1}{3}\right) \text{ or } \log_2 3 \quad \text{A1} \\
 & \left(\text{accept } x = -\frac{\log 3}{\log 2} \text{ or } \frac{\log 3}{\log 2}\right)
 \end{aligned}$$

Total [5 marks]

$$\begin{aligned}
 8 \quad \lim_{x \rightarrow 0} \frac{e^{\sin x} - x - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{(\cos x)e^{\sin x} - 1}{2x} \quad \text{M1A1} \\
 &= \lim_{x \rightarrow 0} \frac{(e^{\sin x})(-\sin x) + (\cos^2 x)e^{\sin x}}{2} \quad \text{M1A1} \\
 &= \frac{1}{2} \quad \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 9 \quad \frac{dy}{dx} &= \frac{(x+y)^2 - xy}{x^2} \\
 &= 1 + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \quad \text{M1}
 \end{aligned}$$

which is homogeneous.

$$\text{Let } y = vx \Rightarrow \frac{dy}{dx} = x \frac{dv}{dx} + v$$

$$\text{Substituting into the differential equation gives } x \frac{dv}{dx} + v = 1 + v^2 + v \quad \text{M1A1}$$

$$\therefore \frac{1}{1+v^2} \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int \frac{1}{1+v^2} dv = \int \frac{1}{x} dx \quad \text{M1}$$

$$\therefore \arctan v = \ln|x| + c$$

$$\text{When } x = 1, y = v = \sqrt{3}, \text{ so } c = \frac{\pi}{3} \quad \text{M1}$$

$$\therefore \arctan\left(\frac{y}{x}\right) = \ln|x| + \frac{\pi}{3} \quad \text{A1}$$

$$\therefore \frac{y}{x} = \tan\left(\ln|x| + \frac{\pi}{3}\right)$$

$$\therefore y = x \tan\left(\ln|x| + \frac{\pi}{3}\right) \quad \text{A1}$$

Total [7 marks]

Section B

$$\begin{aligned}
 10 \quad \mathbf{a} \quad h(x) &= 3g(x-1) - 2 \quad \text{(M1)(A1)} \\
 &= 3e^{2(x-1)} - 2 \quad \text{A1}
 \end{aligned}$$

$$\mathbf{b} \quad \text{The graph crosses the } x\text{-axis when } 3e^{2(x-1)} - 2 = 0 \quad \text{M1}$$

$$\therefore e^{2(x-1)} = \frac{2}{3}$$

$$\therefore 2(x-1) = \ln\left(\frac{2}{3}\right) \quad \text{M1}$$

$$\therefore x = 1 + \frac{1}{2} \ln\left(\frac{2}{3}\right) \quad \text{A1}$$

$$h(x) \text{ crosses the } x\text{-axis at } \left(1 + \frac{1}{2} \ln\left(\frac{2}{3}\right), 0\right). \quad \text{A1}$$

$$\mathbf{c} \quad f'(x) = 3x^2 - 1, \text{ so } f'(2) = 11 \quad \text{M1}$$

$$\therefore \text{gradient of the normal at } x = 2 \text{ is } -\frac{1}{11}. \quad \text{A1}$$

$$f(2) = 6$$

$$\therefore \text{equation of the normal is } y - 6 = -\frac{1}{11}(x - 2) \quad \text{M1}$$

$$\therefore y = -\frac{1}{11}x + \frac{68}{11} \quad \text{A1}$$

$$\mathbf{d} \quad gf(x) = g(x^3 - x) = e^{2(x^3 - x)} \quad \mathbf{A1}$$

Using technology, the graphs of $y = x^3 - x$ and $y = e^{2(x^3 - x)} - 1$ intersect at $x \approx -1.27$, $x = -1$, $x = 0$, and $x = 1$. **(M1)**

$$\begin{aligned} \therefore \text{the total area of the enclosed regions} &\approx \int_{-1.27}^1 \left| (x^3 - x) - (e^{2(x^3 - x)} - 1) \right| dx & \mathbf{M1} \\ &\approx 0.596 \text{ units}^2 & \mathbf{A1} \end{aligned}$$

Total [15 marks]

$$\mathbf{11} \quad \mathbf{a} \quad \mathbf{r}_P = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} \quad \mathbf{A1}$$

$$\mathbf{b} \quad \mathbf{v} \times \mathbf{B} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -b - 5 \\ 3 + a \\ 5a - 3b \end{pmatrix} \quad \mathbf{M1A1}$$

$$\therefore \begin{pmatrix} -3 \\ 10 \\ 41 \end{pmatrix} = \begin{pmatrix} -b - 5 \\ 3 + a \\ 5a - 3b \end{pmatrix} \quad \mathbf{M1}$$

$$\therefore a = 7 \text{ and } b = -2 \quad \mathbf{A1}$$

$$\text{Hence, the position vector of Q is } \mathbf{r}_Q = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 7 \\ -2 \\ 1 \end{pmatrix} \quad \mathbf{A1}$$

$$\mathbf{c} \quad \cos \theta = \frac{\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{1^2 + 3^2 + (-4)^2} \sqrt{7^2 + (-2)^2 + 1^2}} \quad \mathbf{M1A1}$$

$$\therefore \cos \theta = \frac{-3}{\sqrt{26}\sqrt{54}} \quad \mathbf{A1}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-3}{\sqrt{26}\sqrt{54}}\right) \approx 94.6^\circ \quad \mathbf{A1}$$

$$\mathbf{d} \quad \mathbf{r}_P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 + t \\ 5 + 3t \\ 1 - 4t \end{pmatrix}$$

$$\text{The particle P meets the plane when } 2(-1 + t) + 5(5 + 3t) + (1 - 4t) = 50 \quad \mathbf{M1A1}$$

$$\therefore -2 + 2t + 25 + 15t + 1 - 4t = 50$$

$$\therefore t = 2 \quad \mathbf{A1}$$

$$\text{When } t = 2, \quad \mathbf{r}_P = \begin{pmatrix} -1 + 2 \\ 5 + 3(2) \\ 1 - 4(2) \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \\ -7 \end{pmatrix}$$

$$\therefore \text{particle P meets the plane at } (1, 11, -7). \quad \mathbf{A1}$$

Total [14 marks]

$$\mathbf{12} \quad \mathbf{a} \quad \sec^2 x - \sec x \tan x = \frac{1}{\cos^2 x} - \frac{1}{\cos x} \frac{\sin x}{\cos x} \quad \mathbf{M1}$$

$$= \frac{1 - \sin x}{\cos^2 x} \quad \mathbf{A1}$$

$$= \frac{1 - \sin x}{1 - \sin^2 x} \quad \mathbf{A1}$$

$$= \frac{1 - \sin x}{(1 - \sin x)(1 + \sin x)} \quad \mathbf{A1}$$

$$= \frac{1}{1 + \sin x} \quad \mathbf{AG}$$

b i The Maclaurin series for y is $y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \dots$

We are given $y(0) = 1$.

From the differential equation, $y'(0) = \frac{3(1)}{1 + \sin 0} = 3$

M1A1

$$\text{Now } \frac{d^2y}{dx^2} = \frac{(1 + \sin x)\left(3 \frac{dy}{dx}\right) - (3y)(\cos x)}{(1 + \sin x)^2}$$

M1A1

$$\therefore y''(0) = \frac{(1)(9) - (3)(1)}{(1)^2} = 6$$

A1

$$\begin{aligned} \therefore \text{the first 3 non-zero terms are } y &= 1 + x(3) + \frac{x^2}{2!}(6) \\ &= 1 + 3x + 3x^2 \end{aligned}$$

A1

AG

$$\begin{aligned} \text{ii Using b i, } y(1.5) &\approx 1 + 3(1.5) + 3(1.5)^2 \\ &\approx 12.25 \end{aligned}$$

A1

c Euler's method with step size 0.5 is $x_{n+1} = x_n + 0.5$

$$y_{n+1} = y_n + 0.5\left(\frac{3y_n}{1 + \sin x_n}\right)$$

M1A1

Starting with $(x_0, y_0) = (0, 1)$, we obtain $(x_1, y_1) = (0.5, 2.5)$

$$(x_2, y_2) \approx (1, 5.0348)$$

M1

$$(x_3, y_3) \approx (1.5, 9.1359)$$

Hence, $y(1.5) \approx 9.136$ (3 d.p.)

A1

d Separating the variables gives $\int \frac{1}{y} dy = 3 \int \frac{1}{1 + \sin x} dx$

M1

$$\therefore \int \frac{1}{y} dy = 3 \int (\sec^2 x - \sec x \tan x) dx \quad (\text{using a})$$

M1

$$\therefore \ln|y| = 3(\tan x - \sec x) + c$$

A1

Substituting $x = 0, y = 1$ gives $c = 3$

M1

$$\therefore y = e^{3(\tan x - \sec x + 1)} \quad (y > 0)$$

A1

e From **d**, $y(1.5) = e^{3(\tan 1.5 - \sec 1.5 + 1)}$

$$\approx 18.06111$$

A1

$$\therefore \% \text{ error from b} = \left| \frac{18.06111 \dots - 12.25}{18.06111 \dots} \right| \times 100\%$$

M1

$$\approx 32.2\%$$

A1

$$\% \text{ error from c} = \left| \frac{18.06111 \dots - 9.136}{18.06111 \dots} \right| \times 100\%$$

$$\approx 49.4\%$$

A1

The estimate in **b** could be improved by including more higher order terms in the Maclaurin series.

A1

The estimate in **c** could be improved by reducing the step size.

A1

Total [25 marks]

PAPER 3

1 a When $n = 0$, (1) becomes $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \cos x$,

(M1)

which can be solved using an integrating factor $e^{\int \frac{1}{x} dx} = e^{\ln x} \quad \{x \geq \frac{\pi}{2}\}$

A1

$$= x$$

AG

- b** Multiplying (1) through by the integrating factor gives $x \frac{dy}{dx} + y = x \cos x$ **M1**
- $$\therefore \frac{d}{dx}(xy) = x \cos x$$
- A1**
- $$\therefore xy = \int x \cos x \, dx$$
- Integrating the RHS by parts gives $xy = x \sin x - \int \sin x \, dx$ **M1A1**
- $$\therefore xy = x \sin x + \cos x + A \quad (A \text{ is a constant})$$
- A1**
- $$\therefore y = \sin x + \frac{A + \cos x}{x}$$
- AG**
- c** When $n = 1$, (1) becomes $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = y \cos x$
- $$\therefore \frac{dy}{dx} = y \left(\cos x - \frac{1}{x} \right)$$
- A1**
- which is separable. **AG**
- Separating the variables gives $\int \frac{1}{y} \, dy = \int \left(\cos x - \frac{1}{x} \right) \, dx$ **M1**
- $$\therefore \ln|y| = \sin x - \ln x + c \quad \{x \geq \frac{\pi}{2}\}$$
- A1**
- But $y = 1$ when $x = \frac{\pi}{2}$
- $$\therefore 0 = 1 - \ln \frac{\pi}{2} + c$$
- M1**
- $$\therefore c = \ln \frac{\pi}{2} - 1$$
- A1**
- Hence the particular solution is $\ln y = \sin x - \ln x + \ln \frac{\pi}{2} - 1 \quad \{y \geq 0\}$
- $$\therefore y = e^{\sin x - \ln x + \ln \frac{\pi}{2} - 1}$$
- $$\therefore y = \frac{\pi}{2x} e^{\sin x - 1}$$
- A1(oe)**
- d** Differentiating both sides of $u = y^{1-n}$ with respect to x gives $\frac{du}{dx} = (1-n)y^{-n} \frac{dy}{dx}$ **M1A1**
- $$\therefore \frac{dy}{dx} = \frac{y^n}{1-n} \frac{du}{dx}$$
- M1**
- $$\therefore \frac{dy}{dx} = \frac{u^{\frac{n}{1-n}}}{1-n} \frac{du}{dx}$$
- AG**
- e** Substituting $y = u^{\frac{1}{1-n}}$ and the result from part **d** into (1) gives $\frac{u^{\frac{n}{1-n}}}{1-n} \frac{du}{dx} + p(x)u^{\frac{1}{1-n}} = q(x)u^{\frac{n}{1-n}}$ **M1A1**
- Multiplying through by $u^{-\frac{1}{1-n}}$:
- $$\frac{1}{(1-n)u} \frac{du}{dx} + p(x) = \frac{q(x)}{u}$$
- M1A1**
- $$\therefore \frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$$
- AG**
- f** Using **e** and the substitution $u = y^{-1}$, (1) becomes $\frac{du}{dx} - xu = -x$. **A1**
- The integrating factor $= e^{\int -x \, dx} = e^{-\frac{1}{2}x^2}$ **(A1)**
- Multiplying both sides by $e^{-\frac{1}{2}x^2}$ gives $e^{-\frac{1}{2}x^2} \frac{du}{dx} - xe^{-\frac{1}{2}x^2}u = -xe^{-\frac{1}{2}x^2}$ **M1**
- $$\therefore \frac{d}{dx} \left(ue^{-\frac{1}{2}x^2} \right) = -xe^{-\frac{1}{2}x^2}$$
- A1**
- $$\therefore ue^{-\frac{1}{2}x^2} = \int -xe^{-\frac{1}{2}x^2} \, dx$$
- By inspection or using the substitution $t = -\frac{1}{2}x^2$, $ue^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}x^2} + c$ (c is a constant) **A1**
- $$\therefore u = 1 + ce^{\frac{1}{2}x^2}$$
- Since $u = y^{-1}$, $y = \frac{1}{1 + ce^{\frac{1}{2}x^2}}$ **A1**
- (Also accept solution by separation of variables.)

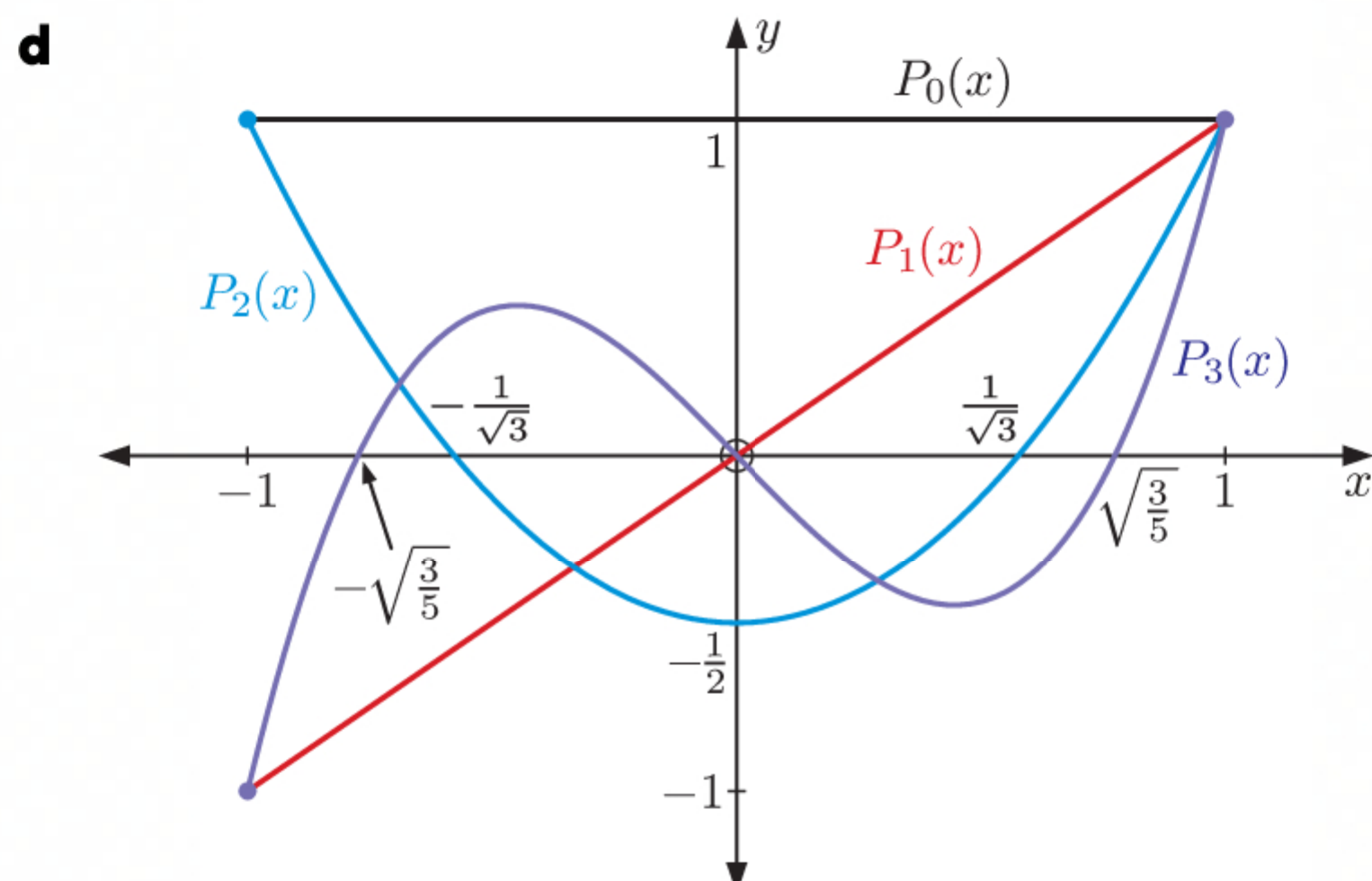
Total [26 marks]

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad P_2(x) &= \frac{1}{2!2^2} \frac{d^2}{dx^2} [(x^2 - 1)^2] & \mathbf{M1} \\
 &= \frac{1}{8} \frac{d}{dx} [2(x^2 - 1)(2x)] & \mathbf{M1} \\
 &= \frac{1}{8} \frac{d}{dx} (4x^3 - 4x) \\
 &= \frac{1}{8} (12x^2 - 4) \\
 &= \frac{1}{2} (3x^2 - 1) & \mathbf{A1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad P_2(x) &= \frac{1}{\pi} \int_0^\pi \left(x + \sqrt{x^2 - 1} \cos t \right)^2 dt & \mathbf{A1} \\
 &= \frac{1}{\pi} \int_0^\pi \left(x^2 + 2x\sqrt{x^2 - 1} \cos t + (x^2 - 1) \cos^2 t \right) dt \\
 &= \frac{1}{\pi} \left[x^2 t + 2x\sqrt{x^2 - 1} \sin t + (x^2 - 1) \left(\frac{1}{4} \sin 2t + \frac{1}{2} t \right) \right]_0^\pi & \mathbf{M1A1} \\
 &= \frac{1}{\pi} \left(\pi x^2 + (x^2 - 1) \left(\frac{\pi}{2} \right) \right) & \mathbf{A1} \\
 &= \frac{1}{2} (3x^2 - 1)
 \end{aligned}$$

c Substitute $n = 2$ into the recursion relation:

$$\begin{aligned}
 3P_3(x) &= 5xP_2(x) - 2P_1(x) & \mathbf{M1} \\
 \therefore P_3(x) &= \frac{1}{3} \left(\frac{5}{2} x(3x^2 - 1) - 2x \right) & \mathbf{A1} \\
 &= \frac{15}{6} x^3 - \frac{5}{6} x - \frac{2}{3} x \\
 &= \frac{1}{2} x(5x^2 - 3) & \mathbf{A1}
 \end{aligned}$$



A1 : shape of $P_0(x)$ and $P_1(x)$
A1 : x and y -intercepts of $P_0(x)$ and $P_1(x)$
A1 : shape of $P_2(x)$ and $P_3(x)$
A1 : x and y -intercepts of $P_2(x)$ and $P_3(x)$

$$\begin{aligned}
 \mathbf{e} \quad P_2(x) \text{ and } P_3(x) \text{ intersect when } \frac{1}{2} (3x^2 - 1) &= \frac{1}{2} x(5x^2 - 3) \\
 \therefore 3x^2 - 1 &= 5x^3 - 3x \\
 \therefore 5x^3 - 3x^2 - 3x + 1 &= 0 & \mathbf{A1}
 \end{aligned}$$

Either by inspection of this equation or from the graphs in part **d**, one intersection point is at $x = 1$. **A1**

By the factor theorem, $(x - 1)$ is a factor of $5x^3 - 3x^2 - 3x + 1$

$$\therefore 5x^3 - 3x^2 - 3x + 1 = (x - 1)(Ax^2 + Bx + C)$$

Equating coefficients gives $A = 5$, $B = 2$, and $C = -1$. **(A1)**

Hence, $P_2(x)$ and $P_3(x)$ also intersect when $5x^2 + 2x - 1 = 0$

$$\therefore x = \frac{-2 \pm \sqrt{24}}{10}$$

$$\therefore x = \frac{1}{5} (-1 \pm \sqrt{6}) \quad \mathbf{A1A1}$$

$$\begin{aligned}
 \mathbf{f} \quad 4P_2(\cos \theta) - P_0(\cos \theta) &= 0 \\
 \therefore 2(3 \cos^2 \theta - 1) - 1 &= 0 & \mathbf{M1} \\
 \therefore 6 \cos^2 \theta - 3 &= 0 \\
 \therefore \cos^2 \theta &= \frac{1}{2} \\
 \therefore \cos \theta &= \pm \frac{1}{\sqrt{2}} & \mathbf{A1}
 \end{aligned}$$

For $0 \leq \theta \leq 2\pi$, $\cos \theta = \pm \frac{1}{\sqrt{2}}$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **A1**

$$\mathbf{g} \quad \text{LHS} = \frac{\sin 2\theta}{\sin \theta} = \frac{2 \sin \theta \cos \theta}{\sin \theta} = 2 \cos \theta \quad \mathbf{A1}$$

$$\text{RHS} = P_0(\cos \theta)P_1(\cos \theta) + P_1(\cos \theta)P_0(\cos \theta) \quad \mathbf{A1}$$

$$= 2P_0(\cos \theta)P_1(\cos \theta)$$

$$= 2(1)(\cos \theta) \quad \mathbf{A1}$$

$$= \text{LHS}$$

$$\mathbf{h} \quad \text{Letting } n = 2, \quad \frac{\sin 3\theta}{\sin \theta} = P_0(\cos \theta)P_2(\cos \theta) + P_1(\cos \theta)P_1(\cos \theta) + P_2(\cos \theta)P_0(\cos \theta) \quad \mathbf{M1}$$

$$\therefore \frac{\sin 3\theta}{\sin \theta} = 1\left(\frac{1}{2}(3 \cos^2 \theta - 1)\right) + \cos \theta(\cos \theta) + \frac{1}{2}(3 \cos^2 \theta - 1)(1) \quad \mathbf{A1}$$

$$\therefore \frac{\sin 3\theta}{\sin \theta} = 3 \cos^2 \theta - 1 + \cos^2 \theta$$

$$\therefore \frac{\sin 3\theta}{\sin \theta} = 4 \cos^2 \theta - 1$$

$$\therefore \frac{\sin 3\theta}{\sin \theta} = 4(1 - \sin^2 \theta) - 1 \quad \mathbf{M1}$$

$$\therefore \frac{\sin 3\theta}{\sin \theta} = 3 - 4 \sin^2 \theta$$

$$\therefore \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \quad \mathbf{A1}$$

$$\therefore \sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta \quad \mathbf{AG}$$

Total [29 marks]

PAPER 3 PRACTICE

- 1 a $1 + 2 + 3 + \dots + n$ is an arithmetic series of n terms, with $u_1 = 1$ and $u_n = n$.

$$\begin{aligned} \therefore \text{the sum of the series } S_n &= \frac{n}{2}(1 + n) \\ &= \frac{n(n+1)}{2} \end{aligned}$$

- b P_n is: $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $1 \times 2 = 2$ and RHS = $\frac{1 \times 2 \times 3}{3} = 2$

$\therefore P_1$ is true.

(2) If P_k is true then $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}$.

$$\begin{aligned} \text{Thus } 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \{\text{using } P_k\} \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

c i $P_3(k+1) - P_3(k) = \frac{(k+1)(k+2)(k+3)(k+4)}{4} - \frac{k(k+1)(k+2)(k+3)}{4}$

$$\begin{aligned} &= \frac{(k+1)(k+2)(k+3)}{4} (k+4-k) \\ &= (k+1)(k+2)(k+3) \end{aligned}$$

ii P_n is: $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $1 \times 2 \times 3 = 6$ and RHS = $\frac{1 \times 2 \times 3 \times 4}{4} = 6$

$\therefore P_1$ is true.

(2) If P_k is true then $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$.

$$\begin{aligned} \text{Thus } 1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\ &= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \quad \{\text{using } P_k\} \\ &= \frac{(k+1)(k+2)(k+3)(k+4)}{4} \quad \{\text{using c i}\} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

d i $P_l(k+1) - P_l(k) = \frac{(k+1)(k+2) \dots (k+l)(k+l+1)}{l+1} - \frac{k(k+1)(k+2) \dots (k+l)}{l+1}$

$$\begin{aligned} &= \frac{(k+1)(k+2) \dots (k+l)}{l+1} [k+l+1-k] \\ &= (k+1)(k+2) \dots (k+l) \end{aligned}$$

ii P_n is: $(1 \times 2 \times 3 \times \dots \times l) + (2 \times 3 \times 4 \times \dots \times (l+1)) + \dots + (n(n+1)(n+2) \dots (n+l-1))$

$$= \frac{n(n+1)(n+2) \dots (n+l)}{l+1} \text{ for } n \in \mathbb{Z}^+.$$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $1 \times 2 \times 3 \times \dots \times l = l!$ and RHS = $\frac{1 \times 2 \times 3 \times \dots \times l \times (l+1)}{l+1} = l!$

$\therefore P_1$ is true.

$$(2) \text{ If } P_k \text{ is true then } (1 \times 2 \times \dots \times l) + (2 \times 3 \times \dots \times (l+1)) + \dots + (k(k+1) \dots (k+l-1)) \\ = \frac{k(k+1)(k+2) \dots (k+l)}{l+1}.$$

$$\text{Thus } (1 \times 2 \times \dots \times l) + (2 \times 3 \times \dots \times (l+1)) \\ + (k(k+1) \dots (k+l-1)) + (k+1)(k+2) \dots (k+l) \\ = \frac{k(k+1)(k+2) \dots (k+l)}{l+1} + (k+1)(k+2) \dots (k+l) \quad \{\text{using } P_k\} \\ = \frac{(k+1)(k+2) \dots (k+l)(k+l+1)}{l+1} \quad \{\text{using d i}\}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
 P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

e From **d ii**, $S_n = \frac{n(n+1)(n+2) \dots (n+10)}{11}$

i The numerator of S_n is the product of 11 consecutive integers, at least one of which must be a multiple of 9.

The denominator does not contain a multiple of 9, so S_n must be divisible by 9 for any $n \in \mathbb{Z}^+$.

ii The numerator of S_n contains exactly one multiple of 11.

Since the denominator is 11, S_n will only be divisible by 11 if the multiple of 11 in the numerator is also a multiple of 11^2 .

The smallest value of n for which this occurs is when $n+10 = 11^2$
 $\therefore n = 111$

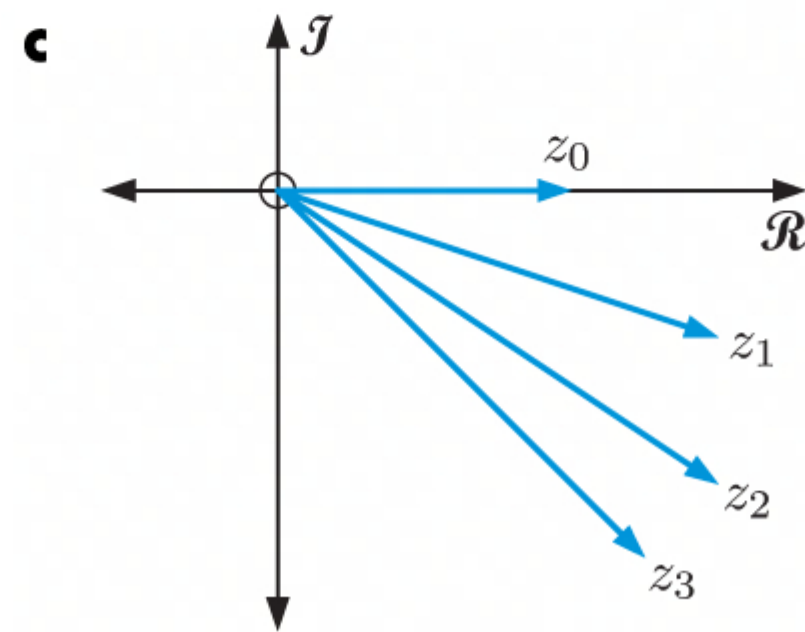
2 a $\frac{1}{1+i} = \frac{1}{1+i} \times \left(\frac{1-i}{1-i}\right) = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$

b $z_0 = \frac{1}{(1+i)^0} = 1$

$$z_1 = 1 + \frac{1}{2} - \frac{1}{2}i = \frac{3}{2} - \frac{1}{2}i$$

$$z_2 = \frac{3}{2} - \frac{1}{2}i + \left(\frac{1}{2} - \frac{1}{2}i\right)^2 \\ = \frac{3}{2} - \frac{1}{2}i + \frac{1}{4} - \frac{1}{2}i + \frac{1}{4}i^2 \\ = \frac{3}{2} - i$$

$$z_3 = \frac{3}{2} - i + \left(\frac{1}{2} - \frac{1}{2}i\right)^3 \\ = \frac{3}{2} - i + \frac{1}{8} - \frac{3}{8}i + \frac{3}{8}i^2 - \frac{1}{8}i^3 \\ = \frac{5}{4} - \frac{5}{4}i$$



d z_n is a geometric series with $u_1 = 1$ and $r = \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i$

$$\therefore z_n = \sum_{k=0}^n \left(\frac{1}{1+i}\right)^k = \frac{1 - \left(\frac{1}{2} - \frac{1}{2}i\right)^{n+1}}{1 - \left(\frac{1}{2} - \frac{1}{2}i\right)} \quad \{\text{including the } k=0 \text{ term}\} \\ = \frac{1 - \left(\frac{1}{2} - \frac{1}{2}i\right)^{n+1}}{\frac{1}{2} + \frac{1}{2}i} \times \left(\frac{\frac{1}{2} - \frac{1}{2}i}{\frac{1}{2} - \frac{1}{2}i}\right) \\ = \frac{\frac{1}{2} - \frac{1}{2}i - \left(\frac{1}{2} - \frac{1}{2}i\right)^{n+2}}{\frac{1}{2}} \\ = 1 - i - 2\left(\frac{1}{\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^{n+2} \\ = 1 - i - \frac{\operatorname{cis}\left(-\frac{\pi}{4}(n+2)\right)}{\sqrt{2}^n} \quad \{\text{De Moivre}\} \\ = 1 - i + \frac{\operatorname{cis} \pi \operatorname{cis}\left(-\frac{\pi}{2} - \frac{\pi}{4}n\right)}{\sqrt{2}^n} \\ = 1 - i + \frac{\operatorname{cis}\left(\frac{\pi}{2} - \frac{\pi}{4}n\right)}{\sqrt{2}^n}$$

e $z_{10} = 1 - i + \frac{\operatorname{cis}(-2\pi)}{\sqrt{2}^{10}} = 1 - i + \frac{1}{32} = \frac{33}{32} - i$

$$\begin{aligned}
 \text{f } z &= \sum_{k=0}^{\infty} \frac{1}{(1+i)^k} \\
 &= \lim_{n \rightarrow \infty} z_n \\
 &= \lim_{n \rightarrow \infty} \left(1 - i + \frac{\operatorname{cis}\left(\frac{\pi}{2} - \frac{\pi}{4}n\right)}{\sqrt{2}^n} \right) \\
 &= 1 - i
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } \sum_{n=1}^4 n \times 3^n &= 1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + 4 \times 3^4 \\
 &= 426
 \end{aligned}$$

$$\text{b } \frac{d}{dr} \left(\sum_{n=1}^5 r^n \right) = \sum_{n=1}^5 \frac{d}{dr} (r^n) = \sum_{n=1}^5 n r^{n-1}$$

$$\begin{aligned}
 \text{c i } \sum_{n=1}^N n r^n - r \sum_{n=1}^N n r^n &= r + 2r^2 + 3r^3 + 4r^4 + \dots + N r^N \\
 &\quad - r^2 - 2r^3 - 3r^4 - \dots - (N-1)r^N - N r^{N+1} \\
 &= r + r^2 + r^3 + r^4 + \dots + r^N - N r^{N+1} \\
 &= \sum_{n=1}^N r^n - N r^{N+1}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii From i, } (1-r) \sum_{n=1}^N n r^n &= r \sum_{n=1}^N r^{n-1} - N r^{N+1} \\
 &= r \frac{1-r^N}{1-r} - N r^{N+1} \quad \{\text{geometric series}\} \\
 \therefore \sum_{n=1}^N n r^n &= \frac{r(1-r^N) - (1-r)N r^{N+1}}{(1-r)^2} \\
 &= \frac{r - r^{N+1} - N r^{N+1} + N r^{N+2}}{(1-r)^2} \\
 &= \frac{r - (N+1)r^{N+1} + N r^{N+2}}{(1-r)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{iii Substituting } r=3, N=4, \text{ we have } \sum_{n=1}^4 n \times 3^n &= \frac{3 - 5 \times 3^5 + 4 \times 3^6}{(-2)^2} \\
 &= 426
 \end{aligned}$$

$$\text{d i } \frac{d}{dr} \left(\sum_{n=1}^N r^n \right) = \sum_{n=1}^N \frac{d}{dr} (r^n) = \sum_{n=1}^N n r^{n-1}$$

$$\begin{aligned}
 \text{ii } \sum_{n=1}^N n r^n &= r \sum_{n=1}^N n r^{n-1} \\
 &= r \frac{d}{dr} \left(\sum_{n=1}^N r^n \right) \quad \{\text{using i}\} \\
 &= r \frac{d}{dr} \left(r \sum_{n=1}^N r^{n-1} \right) \\
 &= r \frac{d}{dr} \left(r \frac{1-r^N}{1-r} \right) \quad \{\text{geometric series}\} \\
 &= r \frac{d}{dr} \left(\frac{r - r^{N+1}}{1-r} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \therefore \sum_{n=1}^N n r^n &= r \frac{(1 - (N+1)r^N)(1-r) - (r - r^{N+1})(-1)}{(1-r)^2} \quad \{\text{quotient rule}\} \\
 &= r \frac{1-r - (N+1)r^N + (N+1)r^{N+1} + r - r^{N+1}}{(1-r)^2} \\
 &= r \frac{1 - (N+1)r^N + N r^{N+1}}{(1-r)^2} \\
 &= \frac{r - (N+1)r^{N+1} + N r^{N+2}}{(1-r)^2}
 \end{aligned}$$

e P_N is: $\sum_{n=1}^N nr^n = \frac{r - (N+1)r^{N+1} + Nr^{N+2}}{(1-r)^2}$ for $N \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

$$\begin{aligned} (1) \text{ If } N=1, \text{ LHS} &= \sum_{n=1}^1 nr^n = r \quad \text{and} \quad \text{RHS} = \frac{r - 2r^2 + r^3}{(1-r)^2} \\ &= \frac{r(1 - 2r + r^2)}{(1-r)^2} \\ &= \frac{r(1-r)^2}{(1-r)^2} \\ &= r \quad \text{provided } r \neq 1 \end{aligned}$$

$\therefore P_1$ is true.

$$(2) \text{ If } P_k \text{ is true then } \sum_{n=1}^k nr^n = \frac{r - (k+1)r^{k+1} + kr^{k+2}}{(1-r)^2}.$$

$$\begin{aligned} \text{Thus } \sum_{n=1}^{k+1} nr^n &= \sum_{n=1}^k nr^n + (k+1)r^{k+1} \\ &= \frac{r - (k+1)r^{k+1} + kr^{k+2}}{(1-r)^2} + (k+1)r^{k+1} \quad \{\text{using } P_k\} \\ &= \frac{r - (k+1)r^{k+1} + kr^{k+2} + (k+1)r^{k+1}(1-r)^2}{(1-r)^2} \\ &= \frac{\cancel{r - (k+1)r^{k+1}} + kr^{k+2} + \cancel{(k+1)r^{k+1}} - 2(k+1)r^{k+2} + (k+1)r^{k+3}}{(1-r)^2} \\ &= \frac{r - (k+2)r^{k+2} + (k+1)r^{k+3}}{(1-r)^2} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true if P_k is true,

P_N is true for all $N \in \mathbb{Z}^+$. {principle of mathematical induction}

f Let $L = \lim_{r \rightarrow 1} \left(\frac{r - (N+1)r^{N+1} + Nr^{N+2}}{(1-r)^2} \right)$ which has the form $\frac{0}{0}$, so we apply l'Hôpital's rule.

$$\begin{aligned} \therefore L &= \frac{\lim_{r \rightarrow 1} \left[\frac{d}{dr} (r - (N+1)r^{N+1} + Nr^{N+2}) \right]}{\lim_{r \rightarrow 1} \left[\frac{d}{dr} (1-r)^2 \right]} \\ &= \frac{\lim_{r \rightarrow 1} [1 - (N+1)^2 r^N + N(N+2)r^{N+1}]}{\lim_{r \rightarrow 1} [-2(1-r)]} \end{aligned}$$

which again has the form $\frac{0}{0}$, so we apply l'Hôpital's rule again.

$$\begin{aligned} \therefore L &= \frac{\lim_{r \rightarrow 1} \left[\frac{d}{dr} (1 - (N+1)^2 r^N + N(N+2)r^{N+1}) \right]}{\lim_{r \rightarrow 1} [-2(1-r)]} \\ &= \frac{\lim_{r \rightarrow 1} [-N(N+1)^2 r^{N-1} + N(N+1)(N+2)r^N]}{\lim_{r \rightarrow 1} (2)} \\ &= \frac{-N(N+1)^2 + N(N+1)(N+2)}{2} \\ &= \frac{N(N+1)(N+2 - N - 1)}{2} \\ &= \frac{N(N+1)}{2} \quad \text{which is consistent with the value of } \sum_{n=1}^N n. \end{aligned}$$

g i $\sum_{n=1}^{\infty} nr^n = r^1 + 2r^2 + 3r^3 + 4r^4 + 5r^5 + \dots$

$$\begin{aligned} &= (r + r^2 + r^3 + r^4 + r^5 + \dots) \\ &\quad + (r^2 + r^3 + r^4 + r^5 + \dots) \\ &\quad + (r^3 + r^4 + r^5 + \dots) \\ &\quad + (r^4 + r^5 + \dots) \\ &\quad + \dots \end{aligned}$$

$$\begin{aligned}
 \text{ii From g i, } \sum_{n=1}^{\infty} nr^n &= r \sum_{n=1}^{\infty} r^{n-1} + r^2 \sum_{n=1}^{\infty} r^{n-1} + r^3 \sum_{n=1}^{\infty} r^{n-1} + \dots \\
 &= (r + r^2 + r^3 + \dots) \sum_{n=1}^{\infty} r^{n-1} \\
 &= r \left(\sum_{n=1}^{\infty} r^{n-1} \right)^2 \\
 &= \frac{r}{(1-r)^2} \quad \text{when } |r| < 1
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \frac{1}{\sqrt{2}} + \frac{2}{2} + \frac{3}{2\sqrt{2}} + \frac{4}{4} + \frac{5}{4\sqrt{2}} + \dots &= \sum_{n=1}^{\infty} n \left(\frac{1}{\sqrt{2}} \right)^n \\
 &= \frac{\frac{1}{\sqrt{2}}}{\left(1 - \frac{1}{\sqrt{2}} \right)^2} \quad \{\text{using g ii}\} \\
 &= \frac{\frac{1}{\sqrt{2}}}{1 - \frac{2}{\sqrt{2}} + \frac{1}{2}} \\
 &= \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{2}-4}{2\sqrt{2}}} \\
 &= \frac{2}{3\sqrt{2}-4} \times \left(\frac{3\sqrt{2}+4}{3\sqrt{2}+4} \right) \\
 &= \frac{6\sqrt{2}+8}{18-16} \\
 &= 4 + 3\sqrt{2}
 \end{aligned}$$

iv The third term can only be the largest if $r > 0$, $3r^3 > 2r^2$ and $3r^3 > 4r^4$

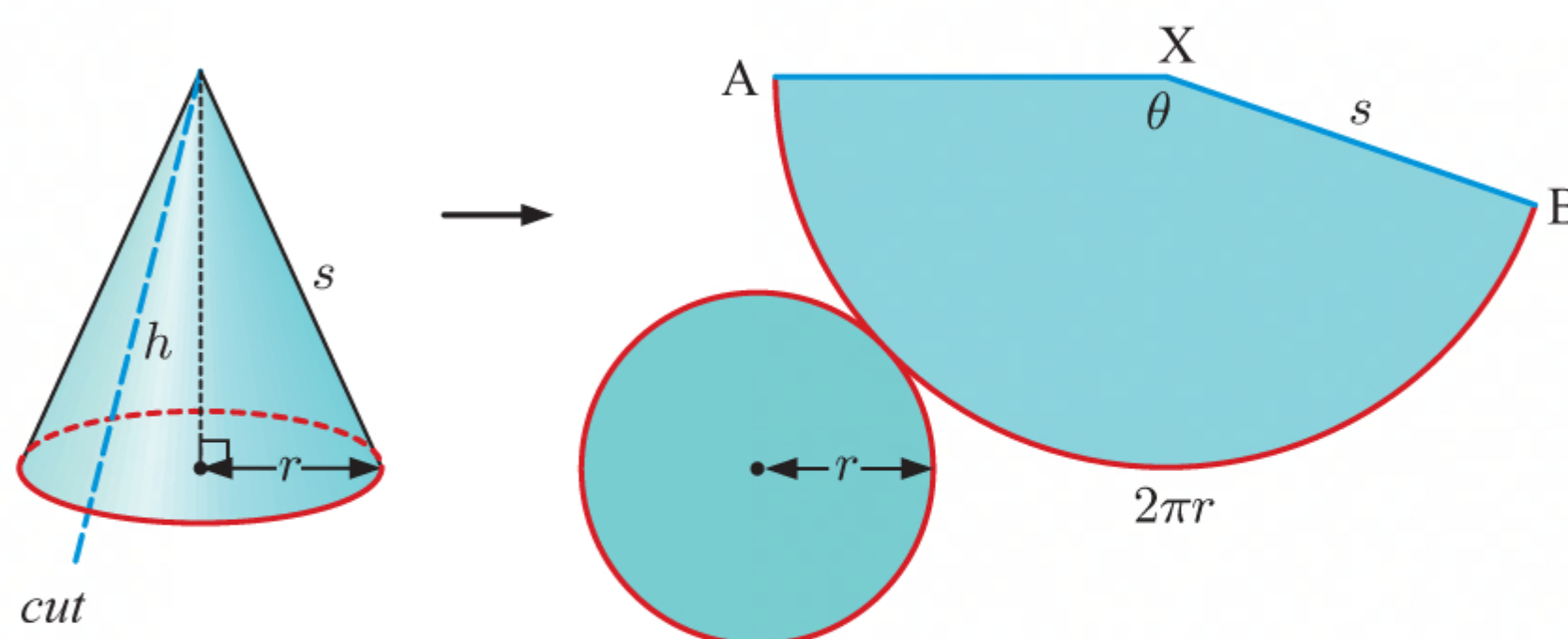
$$\therefore r > \frac{2}{3} \text{ and } r < \frac{3}{4}$$

$$\text{If } r = \frac{2}{3} \text{ then } \sum_{n=1}^{\infty} nr^n = \frac{\frac{2}{3}}{\left(1 - \frac{2}{3} \right)^2} = \frac{\frac{2}{3}}{\frac{1}{9}} = 6.$$

$$\text{If } r = \frac{3}{4} \text{ then } \sum_{n=1}^{\infty} nr^n = \frac{\frac{3}{4}}{\left(1 - \frac{3}{4} \right)^2} = \frac{\frac{3}{4}}{\frac{1}{16}} = 12.$$

$$\therefore 6 < \sum_{n=1}^{\infty} nr^n < 12$$

- 4 a The curved surface of a cone is made from a sector of a circle. The radius of the sector is equal to the slant height s of the cone. The arc length AB of the sector is equal to the circumference of the base.



$$\begin{aligned}
 \text{arc AB} &= 2\pi r \\
 \therefore \left(\frac{\theta}{2\pi} \right) 2\pi s &= 2\pi r \\
 \therefore \frac{\theta}{2\pi} &= \frac{r}{s}
 \end{aligned}$$

\therefore the area of the curved surface = the area of sector AXB

$$\begin{aligned}
 &= \left(\frac{\theta}{2\pi} \right) \pi s^2 \\
 &= \frac{r}{s} \times \pi s^2 \\
 &= \pi rs \\
 &= \pi r \sqrt{r^2 + h^2}
 \end{aligned}$$

- b i** The formula $A = \int_0^h 2\pi f(x) dx$ uses the curved surface area of infinitely many cylinders to approximate the surface area of the cone.

However, the cylinders do not take into account the slant of the cone's surface. Increasing the number of cylinders does not improve the accuracy of the approximation, as the sum of the heights of the cylinders remains fixed at h , whereas the slant height they should represent is $\sqrt{r^2 + h^2}$.

- ii** If the i th slice with radius r_i is made at s -coordinate s_i then by similar triangles,

$$\frac{r_i}{r} = \frac{s_i}{\sqrt{r^2 + h^2}}$$

$$\therefore r_i = r \frac{s_i}{\sqrt{r^2 + h^2}}$$

\therefore the area of the curved surface of the cone

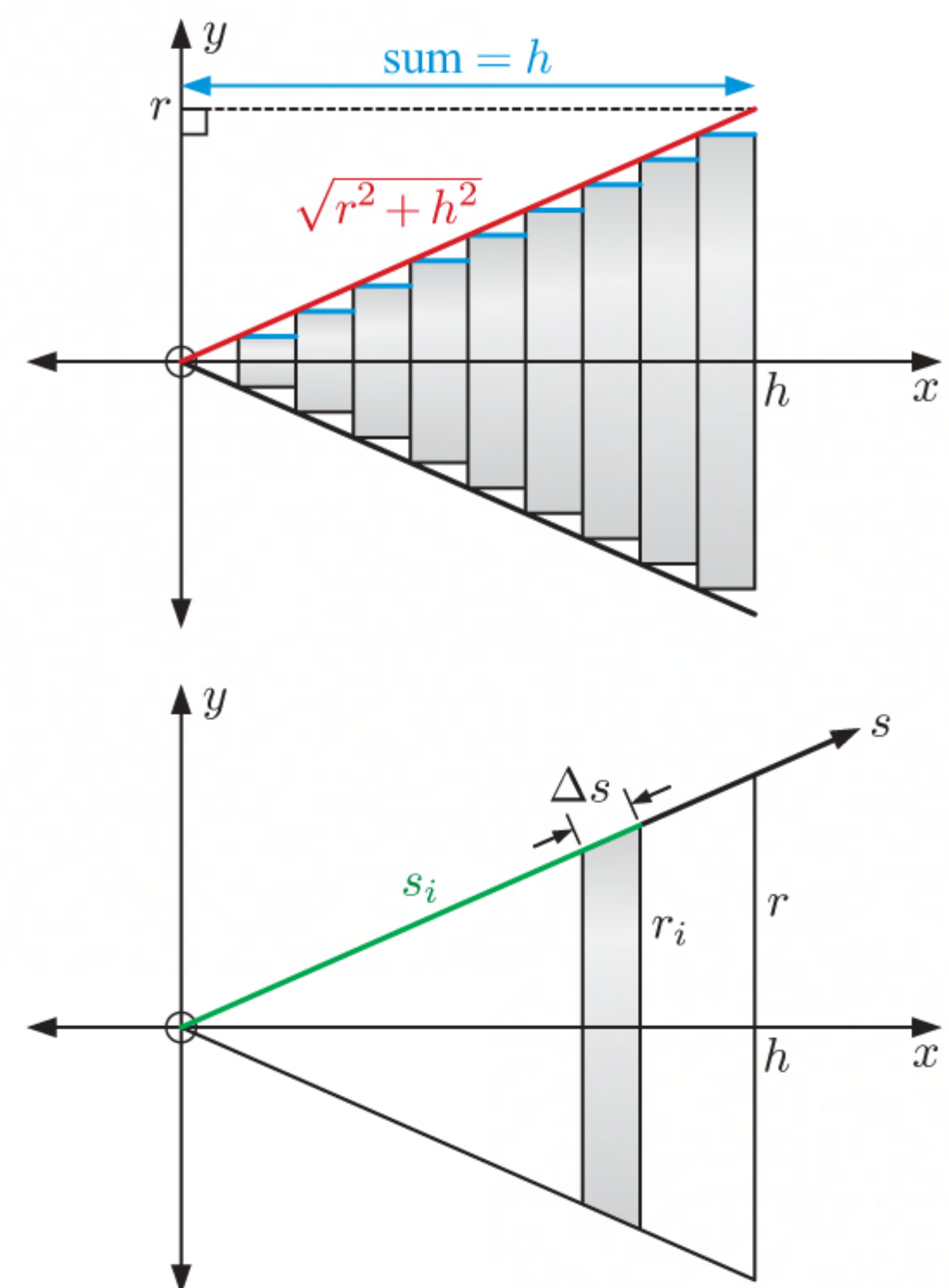
$$= \lim_{\Delta s \rightarrow 0} \sum_i 2\pi r \frac{s_i}{\sqrt{r^2 + h^2}} \Delta s$$

$$= \int_0^{\sqrt{r^2 + h^2}} 2\pi r \frac{s}{\sqrt{r^2 + h^2}} ds$$

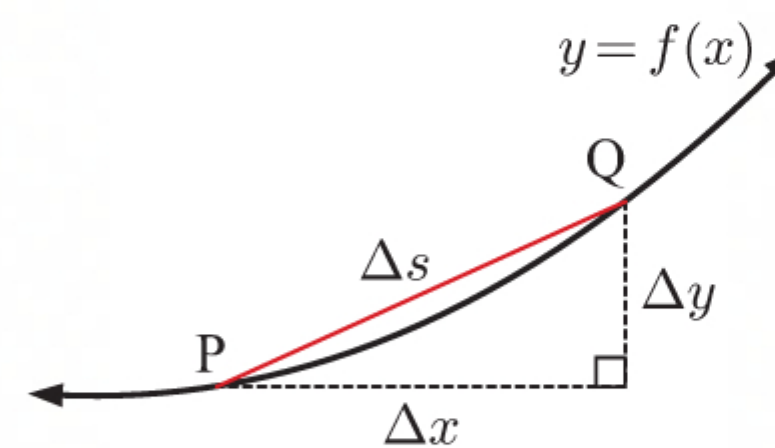
$$\text{Now } \int_0^{\sqrt{r^2 + h^2}} 2\pi r \frac{s}{\sqrt{r^2 + h^2}} ds = \frac{2\pi r}{\sqrt{r^2 + h^2}} \left[\frac{1}{2} s^2 \right]_0^{\sqrt{r^2 + h^2}}$$

$$= \frac{2\pi r}{\sqrt{r^2 + h^2}} \left(\frac{r^2 + h^2}{2} \right)$$

$$= \pi r \sqrt{r^2 + h^2} \quad \text{which agrees with a.}$$



- c i** Using Pythagoras, $\Delta s^2 = \Delta x^2 + \Delta y^2$
- $$\therefore \frac{\Delta s^2}{\Delta x^2} = 1 + \frac{\Delta y^2}{\Delta x^2}$$
- $$\therefore \Delta s^2 = \left(1 + \left(\frac{\Delta y}{\Delta x} \right)^2 \right) \Delta x^2$$
- $$\therefore \Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2} \Delta x$$



- ii** Suppose we approximate the length of $y = f(x)$ from $x = a$ to $x = b$ by dividing the interval $[a, b]$ into intervals of length Δx .

The sum of the chord lengths is $\sum_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x} \right)^2} \Delta x$

$$\therefore \text{the exact length of the curve is } L = \lim_{\Delta x \rightarrow 0} \sum_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x} \right)^2} \Delta x$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

- d i** $\left(\frac{x}{r} \right)^2 + \left(\frac{y}{kr} \right)^2 = 1$
- $$\therefore \left(\frac{y}{kr} \right)^2 = 1 - \frac{x^2}{r^2}$$
- $$\therefore y^2 = k^2(r^2 - x^2)$$
- $$\therefore y = k\sqrt{r^2 - x^2} \quad \{y \geq 0\}$$
- $$\therefore \frac{dy}{dx} = \frac{1}{2}k(r^2 - x^2)^{-\frac{1}{2}}(-2x)$$
- $$= -\frac{kx}{\sqrt{r^2 - x^2}}$$

$$\begin{aligned}
 \text{ii For } k = 1, \text{ arc length} &= \int_{-r}^r \sqrt{1 + \left(-\frac{x}{\sqrt{r^2 - x^2}}\right)^2} dx \\
 &= \int_{-r}^r \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx \\
 &= \int_{-r}^r \sqrt{\frac{r^2}{r^2 - x^2}} dx \\
 &= r \int_{-r}^r \frac{1}{\sqrt{r^2 - x^2}} dx \\
 &= r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r \cos \theta}{\sqrt{r^2 - r^2 \sin^2 \theta}} d\theta \quad \left\{ x = r \sin \theta, \frac{dx}{d\theta} = r \cos \theta \right\} \\
 &= r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r \cos \theta}{r \sqrt{1 - \sin^2 \theta}} d\theta \\
 &= r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \\
 &= r \left[\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= r \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\
 &= \pi r
 \end{aligned}$$

$$\begin{aligned}
 \text{iii For } k = 2, r = 1, \text{ arc length} &= \int_{-1}^1 \sqrt{1 + \left(-\frac{2x}{\sqrt{1 - x^2}}\right)^2} dx \\
 &= \int_{-1}^1 \sqrt{1 + \frac{4x^2}{1 - x^2}} dx \\
 &= \int_{-1}^1 \sqrt{\frac{1 - x^2 + 4x^2}{1 - x^2}} dx \\
 &= \int_{-1}^1 \sqrt{\frac{1 + 3x^2}{1 - x^2}} dx \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{1 + 3 \sin^2 \theta}{1 - \sin^2 \theta}} \cos \theta d\theta \quad \left\{ x = \sin \theta, \frac{dx}{d\theta} = \cos \theta \right\} \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + 3 \sin^2 \theta} d\theta \\
 &\approx 4.844 \quad \{\text{using technology}\}
 \end{aligned}$$

- e i** We divide the interval into i slices, each with width Δx .

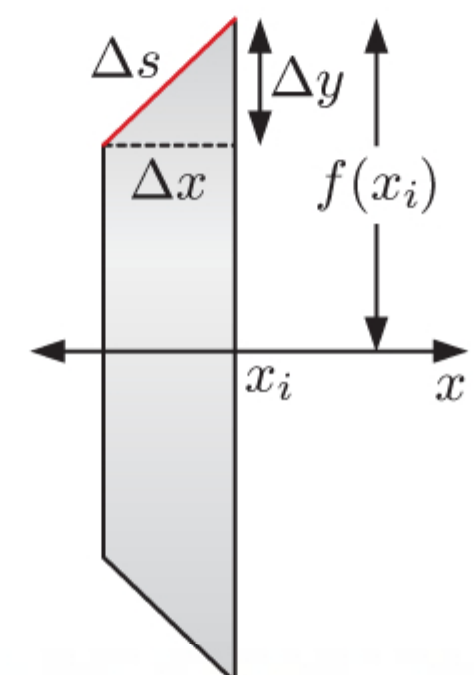
The contribution to the surface area made by the i th slice is $2\pi f(x_i) \Delta s$

$$= 2\pi f(x_i) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \Delta x \quad \{\text{using c i}\}$$

$$\therefore \text{ the total surface area} = \lim_{\Delta x \rightarrow 0} \sum_i 2\pi f(x_i) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \Delta x$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$



ii For the cone in **a**, $f(x) = \frac{r}{h}x$, $0 \leq x \leq h$

$$\begin{aligned}
 \therefore \text{area} &= \int_0^h 2\pi \left(\frac{r}{h}x\right) \sqrt{1 + \left(\frac{r}{h}\right)^2} dx \\
 &= \frac{2\pi r}{h} \int_0^h x \sqrt{\frac{h^2 + r^2}{h^2}} dx \\
 &= \frac{2\pi r}{h} \frac{\sqrt{h^2 + r^2}}{h} \int_0^h x dx \\
 &= \frac{2\pi r \sqrt{r^2 + h^2}}{h^2} \left[\frac{1}{2}x^2\right]_0^h \\
 &= \frac{2\pi r \sqrt{r^2 + h^2}}{h^2} \left(\frac{1}{2}h^2\right) \\
 &= \pi r \sqrt{r^2 + h^2} \quad \text{which agrees with **a**.}
 \end{aligned}$$

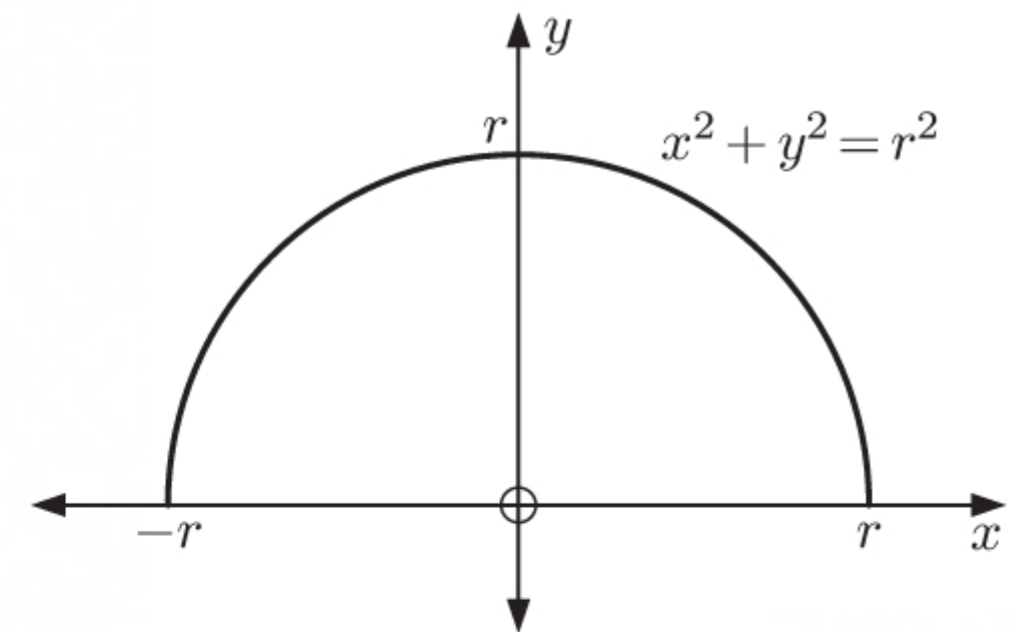
iii A sphere with radius r is formed by rotating the top half of the circle $x^2 + y^2 = r^2$ about the x -axis.

$$\therefore y^2 = r^2 - x^2$$

$$\therefore y = \sqrt{r^2 - x^2} \quad \{y \geq 0\}$$

$$\text{So, } f(x) = \sqrt{r^2 - x^2}$$

$$\text{and } f'(x) = -\frac{x}{\sqrt{r^2 - x^2}} \quad \{\text{from **d i**}\}$$



$$\begin{aligned}
 \therefore \text{the surface area of a sphere} &= \int_{-r}^r 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \left(-\frac{x}{\sqrt{r^2 - x^2}}\right)^2} dx \\
 &= 2\pi \int_{-r}^r \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx \\
 &= 2\pi \int_{-r}^r \sqrt{r^2 - x^2} \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}} dx \\
 &= 2\pi \int_{-r}^r r dx \\
 &= 2\pi [rx]_{-r}^r \\
 &= 2\pi(r^2 + r^2) \\
 &= 4\pi r^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5 \ a} \quad \text{For any } k \in \mathbb{R}, \quad &x^2 \frac{d^2}{dx^2}(kJ_\alpha(x)) + x \frac{d}{dx}(kJ_\alpha(x)) + (x^2 - \alpha^2)(kJ_\alpha(x)) \\
 &= k(x^2 J_\alpha''(x) + x J_\alpha'(x) + (x^2 - \alpha^2)J_\alpha(x)) \\
 &= 0
 \end{aligned}$$

\therefore any constant multiple of $J_\alpha(x)$ is a solution.

$$\mathbf{b} \quad \text{Let } y = J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} x^{-\frac{1}{2}} \sin x$$

$$\therefore \frac{dy}{dx} = \sqrt{\frac{2}{\pi}} \left(-\frac{1}{2}x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x\right)$$

$$\begin{aligned}
 \therefore \frac{d^2y}{dx^2} &= \sqrt{\frac{2}{\pi}} \left(\frac{3}{4}x^{-\frac{5}{2}} \sin x - \frac{1}{2}x^{-\frac{3}{2}} \cos x - \frac{1}{2}x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x\right) \\
 &= \sqrt{\frac{2}{\pi}} \left(\frac{3}{4}x^{-\frac{5}{2}} \sin x - x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x\right)
 \end{aligned}$$

$$\text{Now } x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y$$

$$\begin{aligned}
 &= \sqrt{\frac{2}{\pi}} \left(\frac{3}{4\sqrt{x}} \sin x - \sqrt{x} \cos x - x\sqrt{x} \sin x - \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x + x\sqrt{x} \sin x - \frac{1}{4\sqrt{x}} \sin x\right) \\
 &= 0 \quad \checkmark
 \end{aligned}$$

- c** Since $J_0(x)$ is a solution for $\alpha = 0$,

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0 \quad \dots (1)$$

We must show that $x^2(-J_0'(x))'' + x(-J_0'(x))' + (x^2 - 1)(-J_0'(x)) = 0$

Differentiating (1) with respect to x gives

$$\begin{aligned} 2x J_0''(x) + x^2 J_0'''(x) + J_0'(x) + x J_0''(x) + 2x J_0(x) + x^2 J_0'(x) &= 0 \\ \therefore [x^2 J_0'''(x) + x J_0''(x) + x^2 J_0'(x)] + 2x J_0''(x) + J_0'(x) + 2x J_0(x) &= 0 \\ \therefore [x^2 J_0'''(x) + x J_0''(x) + x^2 J_0'(x)] + \frac{2}{x} [x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x)] - J_0'(x) &= 0 \\ \therefore [x^2 J_0'''(x) + x J_0''(x) + x^2 J_0'(x)] - J_0'(x) &= 0 \quad \{\text{using (1)}\} \\ \therefore x^2(-J_0'''(x)) + x(-J_0''(x)) + x^2(-J_0'(x)) + J_0'(x) &= 0 \\ \therefore x^2(-J_0'(x))'' + x(-J_0'(x))' + (x^2 - 1)(-J_0'(x)) &= 0 \quad \text{as required.} \end{aligned}$$

- d** Let $J_n(x) = \sum_{k=0}^{\infty} a_k x^k$ for $n \in \mathbb{N}$

$$\therefore J_n'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$\therefore J_n''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Substituting into the differential equation:

$$\begin{aligned} x^2 \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} + (x^2 - n^2) \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \therefore \sum_{k=2}^{\infty} k(k-1) a_k x^k + a_1 x + \sum_{k=2}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^{k+2} & \\ - n^2 a_0 - n^2 a_1 x - \sum_{k=2}^{\infty} n^2 a_k x^k &= 0 \\ \therefore -n^2 a_0 + (1 - n^2) a_1 x + \sum_{k=2}^{\infty} (k(k-1) + k - n^2) a_k x^k + \sum_{m=2}^{\infty} a_{m-2} x^m &= 0 \quad \{\text{letting } m = k+2\} \\ \therefore -n^2 a_0 + (1 - n^2) a_1 x + \sum_{k=2}^{\infty} (k^2 - n^2) a_k x^k + \sum_{k=2}^{\infty} a_{k-2} x^k &= 0 \quad \{\text{letting } k = m\} \\ \therefore -n^2 a_0 + (1 - n^2) a_1 x + \sum_{k=2}^{\infty} [(k^2 - n^2) a_k + a_{k-2}] x^k &= 0 \quad \dots (2) \end{aligned}$$

- e** The result in **d** must hold for all x where the Maclaurin series converges.

\therefore every coefficient of x^k , $k \in \mathbb{N}$, in the LHS of (2) must be zero.

\therefore every constant a_k must be zero until we reach k such that $k^2 - n^2 = 0$.

\therefore the first non-zero a_k occurs when $k = n$.

\therefore the first (non-zero) term in the Maclaurin series expansion for $J_n(x)$ is the x^n term.

- f** The coefficient of x^k is $(k^2 - n^2) a_k + a_{k-2} = 0$

$$\therefore a_k = -\frac{1}{k^2 - n^2} a_{k-2} \quad \text{for all } k \neq n$$

If n is odd, the first (non-zero) term in the series is an odd power of x , and $a_n \neq 0$.

$\therefore a_k = 0$ for all even k , and for all odd k such that $k < n$, and $a_k \neq 0$ for all odd k such that $k \geq n$.

\therefore the Maclaurin series consists only of odd powers of x .

$\therefore J_n(x)$ is odd.

Using the same argument, if n is even then $a_k = 0$ for all odd k , and for all even k such that $k < n$, and $a_k \neq 0$ for all even k such that $k \geq n$.

\therefore the Maclaurin series consists only of even powers of x

$\therefore J_n(x)$ is even.

g i Let $J_0(x) = \sum_{k=0}^{\infty} a_k x^k$

Since $J_0(0) = 1$, $a_0 = 1$

Using **f**, $a_1 = 0$, and indeed $a_k = 0$ for all odd k .

For all even $k \geq 2$, $a_k = -\frac{1}{k^2} a_{k-2}$

$$\therefore a_2 = -\frac{1}{2^2} a_0 = \frac{(-1)^1}{4} \times \frac{1}{1^2}$$

$$\therefore a_4 = -\frac{1}{4^2} a_2 = \frac{(-1)^2}{4^2} \times \frac{1}{(1 \times 2)^2}$$

$$\therefore a_6 = -\frac{1}{6^2} a_4 = \frac{(-1)^3}{4^3} \times \frac{1}{(1 \times 2 \times 3)^2}$$

$$\therefore a_{2m} = \frac{(-1)^m}{4^m (m!)^2} \quad \text{for all } m \in \mathbb{Z}, m \geq 1$$

$$\therefore J_0(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m (m!)^2} x^{2m}$$

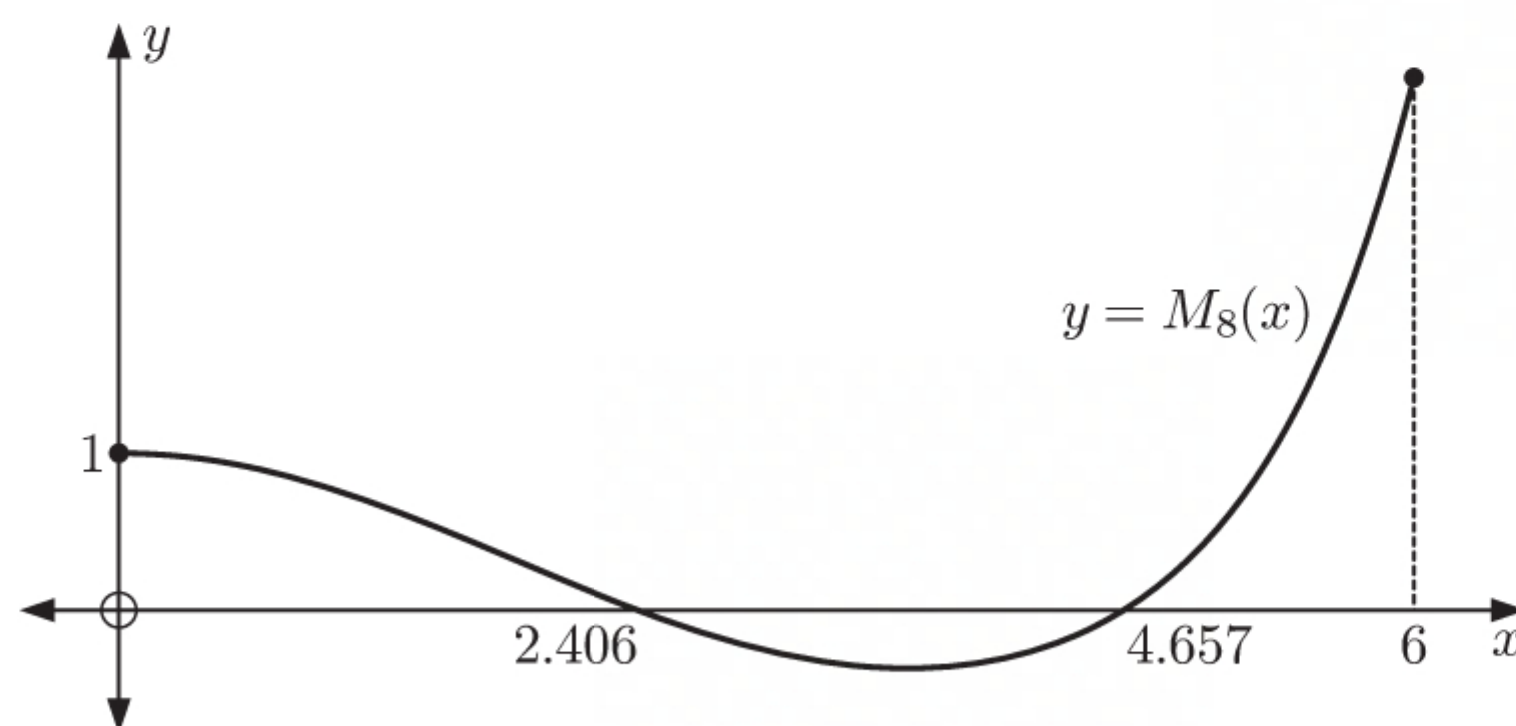
ii Using **c**, $J_1(x) = -\frac{d}{dx} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m (m!)^2} x^{2m} \right]$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \times 2m}{4^m (m!)^2} x^{2m-1}$$

h The 8th order Maclaurin series polynomial for $J_0(x)$ is

$$M_8(x) = 1 - \frac{1}{4}x^2 + \frac{1}{4^2(2!)^2}x^4 - \frac{1}{4^3(3!)^2}x^6 + \frac{1}{4^4(4!)^2}x^8$$

$$= 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8$$



The Maclaurin polynomial approximation is more accurate when we are closer to $x = 0$.